



HAL
open science

Specialised mathematical knowledge in the scope of geometric isometric transformation translation and its impact on teacher's levels of Interpretive Knowledge

Caroline Silva, Miguel Ribeiro

► To cite this version:

Caroline Silva, Miguel Ribeiro. Specialised mathematical knowledge in the scope of geometric isometric transformation translation and its impact on teacher's levels of Interpretive Knowledge. Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14), Free University of Bozen-Bolzano; ERME, Feb 2025, Bozen-Bolzano, Italy. <hal-05290966>

HAL Id: hal-05290966

<https://hal.science/hal-05290966v1>

Submitted on 30 Sep 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Copyright - All rights reserved

Specialised mathematical knowledge in the scope of geometric isometric transformation translation and its impact on teacher's levels of Interpretive Knowledge

Caroline Silva¹ and Miguel Ribeiro¹

¹State University of Campinas, Brazil; cmribas78@gmail.com

We discuss the specialisation of mathematics teachers' knowledge in the scope of geometric isometric transformation translation, considering the Mathematics Teacher's Specialised Knowledge and Interpretative Knowledge. Students and teachers reveal difficulties in such topic which makes it a good subject for discussion. 14 mathematics teachers participated in a Professional Development context focusing on developing teacher's knowledge and here we focus on the knowledge (written, oral and gestural) revealed when discussing an Interpretative Task. The results pinpoint to connections between teachers mathematical specialised knowledge in the context of translation and the interpretation performed (IK levels).

Keywords: Mathematics Teacher's Specialised Knowledge, Interpretative Knowledge, Interpretative Task, geometric isometric transformation translation.

Introduction

It is expected that mathematics teachers' practices allow developing students' mathematical thinking and understanding the diversity of topics they have to deal with, and the connections among them. Such practice, starting from what the students know and how they know it, requires a specialised knowledge which is perceived here in the sense of the Mathematics Teacher's Specialised Knowledge – MTSK conceptualization (Carrillo et al., 2018) and the Interpretative Knowledge – IK (Jakobsen et al., 2014). Among the diversity of mathematical topics in which students reveal difficulties, we focus on Geometric Isometric Transformations and here we give a special look at translation considering the students' (e.g., Flores & Yanik, 2016) and teachers' (e.g., Gomes, 2012) difficulties. In that sense we address the following research question: which mathematical specialised knowledge do mathematics teachers reveal when participating in a Professional Development context and how is it related to the interpretation performed (IK levels)?

Teacher's Specialised and Interpretative knowledge

Both students and teachers reveal difficulties in geometric isometric transformation translation. Students' difficulties involve: (i) perform the translation (especially with polygonal figures and when the translation vector is parallel to one of its sides), considering the length of the vector as the space between the figure and the image (Godino & Ruíz, 2002); (ii) establish the distance between the corresponding points of the figure and the image during the process (Sünker & Zembat, 2012) as equivalent to the magnitude of the translation vector (Flores & Yanik, 2016). Teachers' difficulties include determining the magnitude of the translation vector mistakenly associating (as students do) the magnitude of the translation vector with the space between the figure and its image (Gomes, 2012).

Considering that teacher's knowledge is related to student's learning and results, understanding the specificities of teachers' knowledge is fundamental to contribute to design and implement Tasks for Teacher Education and Professional Development contexts leading to improve students' understanding, knowledge and, in consequence, results. In order to develop such student's knowledge and understanding we perceive as fundamental to assume as a starting point what the students know and how they know it so that we can "move" them to a state of knowing and for doing so it requires from teachers a specialised mathematical knowledge grounding which we name the Interpretative Knowledge (Jakobsen et al., 2014). Such mathematical specialised knowledge (MK) is perceived here as part of the Mathematics Teacher's Specialised Knowledge – MTSK (Carrillo et al., 2018).

The MK is comprised by three subdomains, and even considering that all of them are equally important, here we focus on Knowledge of Topics (KoT) which is composed by four categories: (i) procedures; (ii) definitions, properties and foundations; (iii) registers of representation; and (iv) phenomenology and applications. From these we address special attention to procedures and foundations.

Procedures (which we term as KoTp) includes teachers' knowledge on the different ways of doing mathematics to solve a problem or calculation, considering: how it is done; when it can or cannot be done; why it is done that way; and the characteristics of the result obtained (e.g., Carrillo et al., 2018). In the topic of translation, it includes knowing the different steps for adding the translation vector to "each point of the figure" to obtain the corresponding point in the image. Foundations (KoTf) refers to the knowledge of the set of fundamental mathematical attributes that mathematically support the existence of the topic and connect concepts (Reyes & Sosa, 2019). It is, therefore, a subset of properties and it includes, for example, knowing that the translation vector and its properties is one of the foundations of translation (in Portuguese we use three words – direção, sentido, magnitude).

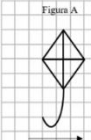
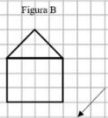

The content of teacher's mathematical specialised knowledge grounds the interpretation performed to student's productions and, thus, the teacher's interpretative practices. Such practices are related with teachers' Interpretative Knowledge – IK (Jakobsen et al., 2014) which refers to a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge starting from their own reasoning and productions, no matter how non-standard or incorrect they might be. IK completes the knowledge of typical errors or solution strategies, with the knowledge of possible source for typical or atypical error and the knowledge of possible use of errors (Di Martino et al., 2020). At the moment IK comprises three levels which are intertwined with the interpretative practices performed (Mellone et al., 2017): (i) Evaluative interpretation: a process through which the teacher determines congruence between student's productions and the mathematical scheme of correct answers (s)he has; (ii) Interpretation for the educational design: the manner in which the teacher designs educational steps based on the work produced by the students; (ii) Interpretation as research: teacher's willingness and ability to revise his/her mathematical formalization in order to ensure that it is coherent with students' productions (even when these seem in conflict with the traditional mathematics taught in school).

Teachers' Interpretative and Specialised knowledge does not naturally develop over the course of years of work experience, but it is something that requires a special attention from an educational

point of view (Ribeiro et al., 2013), and for doing so it is required to design and implement educational contexts with such an aim. Contributing for such development we have been developing Interpretative Tasks – IT (Mellone et al., 2020) aiming at assessing and developing teachers’ IK, and for each IT we devise a research question, focusing on a specific mathematical topic – in this specific case the geometric isometric transformation translation. Each IK is comprised by three parts: Preliminary Part, with focus on teachers’ mathematical specialised knowledge; Part I starts from a conceptualized task for students to develop their mathematical thinking and understanding on the topic (usually a problem), and a set of questions for teachers, where one of them is frequently related to “*anticipate possible correct and incorrect students’ productions to the task and which mathematical knowledge grounds the associated reasoning*”; Part II considers, usually, some students productions to the task (written, videos, transcripts) and includes some questions requiring teachers to reveal their IK.

Context and method

This paper is part of a broad research project on teachers’ knowledge and mathematical practices in which one of the focuses is Geometrical Thinking. Here we focus on interpretative and specialised knowledge of 14 mathematics teachers (upper and lower secondary) participating in the context of a professional development program (40 hours – nine meetings) aimed to access and develop teachers’ IK and particularly in one IT on the space of geometric isometric transformation translation. Participating teachers had between one and 26 years of experience teaching mathematics at lower and upper secondary (students aged 11 to 18 in Brazil). All teachers had a degree on mathematics teacher education (four years). Data collection includes audio and video recordings of teachers’ discussions in small groups (of three or four) and during the plenary discussion as well as teachers’ productions to the IT. Here we focus on Part I of the IT. The task for students pursues the aim of developing students understanding the set of procedures to perform translation of polygonal and non-polygonal figures as well as some properties of translation. The first question to teachers is to solve the students’ task without thinking in a teaching context. Part II includes three student’s productions directing the focus of attention to the mathematical thinking grounding the student’s errors and the teacher’s knowledge to devise ways to overcome such difficulties. Due to space constrains, we present part of the IT on translation with the questions on which we focus our attention (Figure 1).

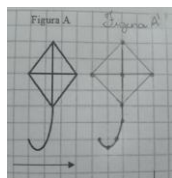
Teacher knowledge in the scope of translation	
Preliminary Part	
[...].	
Part I	
<p>Task: Changing places... (You should always explain your reasoning by describing the process you use to answer the question. For doing so you can use diagrams, words, calculations, ...)</p> <p>1. Represent the figure (image) in each situation, according to the arrow:</p>	
	
	

[...].

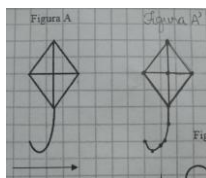
1. Consider the previous task:
 - i) Solve the task by yourself (without thinking on a teaching context).
- [...].

Part II

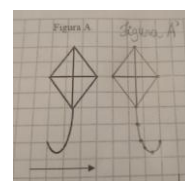
1. Teacher Josi implemented the previous task in her 7th grade classroom. She received different responses and as she was participating in one of the professional development contexts of CIEspMat she decided to take these productions to discuss them with the other participants.



Paula's production



Laura's production



Maria's production

- a) For each of the students' productions, indicate whether you consider them mathematically correct (adequate) or not, justifying the mathematical reasoning shown.
- [...].

Figure 1: Part of an Interpretative Task in the scope of translation

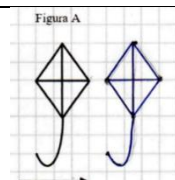
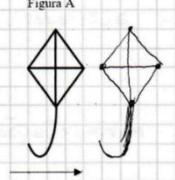
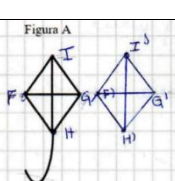
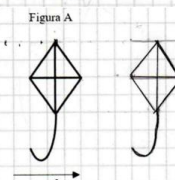
Each one of these productions were included as they allow discussing specific student's difficulties. Here, for space constraints, we focus on Laura's production because it has allowed a large span of knowledge revealed. Laura's production brings to front the students' difficulties in establishing the distance between each point of the figure and the corresponding point in image as equivalent to the magnitude of the translation vector. Preliminary Part was answered individually and Parts I and II (distributed separately), in groups of three or four (G1 to G4). After that, a collective discussion took place – having as a starting point the participants' production. Teachers' productions (written and oral) were transcribed, and the dialogues were complemented with teacher's gestures.

To identify teacher's specialised knowledge, the written productions were transcribed *ipsis litteris*, and for the analysis we used the MTSK categories (KoTpn and KoTfn are used to identify the knowledge revealed on procedures and foundations (n increases sequentially for each one separately)). For IK, after transcribing the productions, we have identified (colored red) keywords associated to each level of IK (incorrect; partial correct; correct) considering the three categories presented by Mellone et al., (2017) and from the analysis a new category for the interpretative practice emerged (misinterpretation, corresponding to interpret as correct a mathematical incorrect student's production). For identification we used a set of symbols after the acronymous (*; **; []) where * is associated to an inadequate mathematical knowledge, ** when it is incomplete and [] absence of evidence of knowledge required to interpret the student production.

Analysis and discussion

From the four teachers' groups only two (G1 and G2) performed the translation correctly. G3 correctly performed the translation of the polygonal component of Figure A, but didn't provide the translation of the curved component. G4 performed an incorrect translation using six distance length measurement units for the translation vector magnitude instead of four (Table 1).

Table 1: Teachers' productions for the question “Represent the figure (image) in each situation, according to the arrow” focusing on Figure A

Group	Teachers' Productions			
G1		<i>G1</i>	<i>Transcription</i>	
	117	Aline	This point here must move four.	
	118	Joice	That's it.	
	119		[Aline counts according to the squares of the grid].	
	120	Aline	One, two, three, four.	
G2		<i>G2</i>	<i>Transcription</i>	
	166	Marcia	So, is this one going to come here?	
	167	Jorge	His vertex there.	
	168	Marcia	Here? One, two, three, four. Here more or less?	
	169	Jorge	That.	
		[...]		
176	Jorge	That's it. You can now close the drawing above.		
G3		<i>G3</i>	<i>Transcription</i>	
	50	Igor	We can name the point, name each point	
	51	Igor	and place a “line”.	
	52	Telma	You have to make the stitches.	
	53	Igor	And do A', B'.	
G4		<i>G4</i>	<i>Transcription</i>	
	42	Luana	I think that only the distance has to be the same,	
	43	Luana	if we are going to transfer the kite here, how many? Two	
	44	Luana	little squares? And answer and put it right here.	
	45	Luana	[Luana points to figure A and the place where its image	
	46	Luana	would be located on the task sheet].	

G1 reveals [117 – 120] knowing that the translation procedures consist of choosing crucial point on the figure and moving them, counting the sides of the squares on the grid according to the unit of distance measurement indicated by the vector magnitude, obtaining the corresponding image points (KoTp1**). This is a particular case of procedures for translation requiring to know the need to consider the three properties of vectors (direction, sense and magnitude) and adding the vector to fundamental points of the figure to obtain its corresponding image points [(KoTp2)].

Teachers in G2 employ similar procedures as G1, but explicitly chooses the vertices of the figure to be translated [167], so that they can later be joined, and the image is obtained [176]. This involves knowing that in polygons it is enough to translate the vertices, taking into account the translation vector and its properties, and then join them by straight line segments (KoTp3). Similarly to the first two groups, G3 [50 – 53], reveals knowing that for translating a figure it is enough to carry out the transformation of some of its points, naming the points [vertices] of the figure by Latin capital letters and their correspondents in the image using that same letter added by an apostrophe (KoTp4).

At G4 teachers reveal they know that it is necessary to maintain the distance between all points in the figure and the corresponding points in the image [42], but they incorrectly use six length units for the translation vector magnitude instead of four (KoTf1*), not establishing a relationship between this distance and the magnitude of the translation vector provided. Translation is a topic for students at grade 7 (12–13 years old) in the Brazilian context, but G4 performed the same students' common mistake, disregarding the magnitude of the translation vector (Flores & Yanik, 2016). It reveals an

inadequate knowledge on the three foundational properties of the translation vector – direction, sense and magnitude [(KoTf2)].

When interpreting Laura's production (Part II), G1 and G2 considered the production incorrect, while G3 and G4 assumed it was correct.

- G1: Laura → incorrect, despite maintaining the size of the drawing, it did not correctly shift according to the vector.
- G2: Laura → Mathematically there was misconception because she used the vector as the distance between the figure and the image and not as the translation of the corresponding points.
- G3: Laura's production: Analysing the image, we noticed that the student used the limit on the right to start the entire transformation, (except) for this reason she performed the activity correctly.
- G4: Laura's production: We consider Laura's production to be correct, as the student created the image with sense, direction and distance.

G1 and G2 carry out an evaluative interpretation (IK level 1). G1 identifies some indication of the student understanding (*despite maintaining the size of the drawing*) associated with knowing that in translation the dimensions of the figure in the image are maintained, and G2 focus on describing the error in the production (*used the vector as the distance (...) points*).

G3 and G4 provided an interpretation characterized as a misinterpretation (IK level 0), since they assume a production that is mathematically inadequate as correct – reveal similar difficulties as those of students (Gomes, 2012). Even if G3 identify one of the student's errors (*the student used (...) transformation*), revealing knowing to identify the students' mistake of associating the translation vector to the oriented segment that joins a point in the figure with a point closer to the image (Flores & Yanik, 2016), they interpret the production as correct. G4 does not consider the three properties of the vector (direction, sense and magnitude) and that needs to be considered in the translation of a figure, as they consider the student's production correct (*as the student (...) distance*); however, Laura did not use the magnitude of the vector provided, being this – not using all the translation vector properties – a common students' difficulty (Sünker & Zembat, 2012).

In Table 2 we associate the content descriptors of revealed knowledge to the interpretative practices.

Table 2: Synthesis of the interpretative practice carried out associated with the revealed knowledge and the adequate knowledge based on the evidence of the groups

Evidence	Revealed knowledge	Adequate knowledge	Interpretative practice and level of IK
G1: incorrect , despite maintaining the size of the drawing, it did not correctly shift according to the vector.	Know that during translation the dimensions of the figure in the image are preserved and that the properties of the translation vector must be considered to carry out this transformation.		Evaluative interpretation: Interpret as incorrect a student's production that, despite maintaining the dimensions of the figure in the image, does not move it according to the vector provided, indicating the error in the production. (IK level 1)
G2: Mathematically she had a misconception	Know that in translation it is the magnitude of the translation vector that		Evaluative interpretation: Interpret as incorrect a student's production that

because she used the vector as the distance between the figure and the image and not as the translation of the corresponding points.	determines the distance between each point in the figure and its corresponding point in the image.		considers the vector as the distance between the figure and the image and not as the distance measurement between the original and transformed points, making the error in the production explicit. (IK level 1)
G3: the student used the limit on the right to start the entire transformation, (except) for this reason she performed the activity correctly .	Know that students make the mistake of identifying the translation vector as the oriented segment that joins a point in the figure with a point closer to the image.	Know that performing the translation disregarding the magnitude of the translation vector as corresponding to the distance between each point in the figure and its corresponding point in the image is mathematically inadequate.	Misinterpretation: Interpreting as correct a student's production that is mathematically inadequate. The error consists of using the right-hand limit of a figure to apply the vector and start the translation. (IK level 0)
G4: Laura's production was correct , as the student created the image with sense, direction and distance.	Know that when translating a figure, it is necessary to consider the same sense, direction and distance to translate all its points and obtain the image.	Know that when translating a figure, it is necessary to consider the same direction, sense and distance to translate all its points and obtain the image, these properties being determined by the properties of the translation vector and that they all need to be considered.	Misinterpretation: Interpreting as correct a student's production that is mathematically inadequate and that obtains an image through translation, disregarding the magnitude of the translation vector provided. (IK level 0)

Some final comments

Developing teachers' IK is fundamental for promoting interpretative practices beyond evaluative, requiring, thus, developing teachers specialised mathematical knowledge in each mathematical topic, such as translation. We found that the content of teachers' MK impacts the type and level of interpretative practice and knowledge – low level of MK constrains teacher's interpretations. A new category emerged related with the so-called mistaken interpretation, which was associated to a level 0 of IK (it's important to notice that this does not mean absence of knowledge), as it does not refer to the "common" evaluative interpretation, which at least points out the student's error (level 1). The mistaken interpretation grounds on teacher's MK and reflects his/her insufficient mathematical specialised knowledge to interpret a mathematically incorrect student's production, and so, even with a minimal level of MK, teachers assume they can interpret the students' productions, independently of the level of mathematical elaboration of such production. In that sense one open question to further discuss refers to identify and define the content of specialised MK levels. Another further research path to follow, considering the PD context in which data has been gathered concerns to focus the attention on the IK development and on the factor sustaining such development – not only by solving IK tasks but also due to the context of preparing, implementing tasks to their students and selecting student's productions to elaborate IT for the PD colleagues.

Acknowledgment

This work is part of the research projects funded by CNPq (404959/2021-0 and 317937/2023-5).

References

- Carrillo, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, Á., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The Mathematics Teacher's Specialised Knowledge (MTSK) model. *Research in Mathematics Education*, 20(3), 236–253. <https://doi.org/10.1080/14794802.2018.1479981>
- Di Martino, P., Mellone, M., & Ribeiro, M. (2020). Interpretative Knowledge. Em S. Lerman (Org.), *Encyclopedia of Mathematics Education* (pp. 424–428). Springer International Publishing. https://doi.org/10.1007/978-3-319-77487-9_100019-1
- Flores, A., & Yanik, H. B. (2016). Geometric Translations: An Interactive Approach Based on Students' Concept Images. *North American GeoGebra Journal*, 5(1). <https://mathed.miamioh.edu/index.php/ggbj/article/view/67/0>
- Godino, J. D., & Ruíz, F. (2002). *Geometría y su didáctica para maestros* [Geometry and its teaching for teachers]. Universidad de Granada, Departamento de Didáctica de la Matemática.
- Gomes, A. (2012). Transformações geométricas: Conhecimentos e dificuldades de futuros professores [Knowledge and difficulties of future teachers]. In H. Pinto, H. Jacinto, A. Henriques, A. Silvestre, & C. Nunes (Orgs.), *Atas do XXIII Seminário de Investigação em Educação Matemática* (pp. 233–243). Associação de Professores de Matemática. https://www.apm.pt/siem_atas
- Mellone, M., Ribeiro, M., Jakobsen, A., Carotenuto, G., Romano, P., & Pacelli, T. (2020). Mathematics teachers' interpretative knowledge of students' errors and non-standard reasoning. *Research in Mathematics Education*, 22(2), 154–167. <https://doi.org/10.1080/14794802.2019.1710557>
- Mellone, M., Tortora, R., Jakobsen, A., & Ribeiro, M. (2017). Prospective teachers interpret student responses: Between assessment, educational design and research. In T. Dooley & G. Gueudet (Eds.), *Proceedings of CERME 10* (pp. 2948–2955). DCU Institute of Education and ERME. <https://hal.science/hal-01949034>
- Reyes, A. M., & Sosa, L. (2019). Conocimiento especializado del profesor de primaria en formación: Un estudio de caso de la enseñanza de la noción de razón [Specialized knowledge of the pre-service primary school teacher: A case study of teaching the notion of reason]. *Cuadrante*, 28(2), 100–124. <https://doi.org/10.48489/quadrante.23029>
- Ribeiro, C. M., Mellone, M., & Jakobsen, A. (2013). *Characterizing Prospective Teachers' Knowledge in/for Interpreting Students' Solutions*. 4, 89–96. <https://www.iris.unina.it/handle/11588/586250>
- Sünker, S., & Zembat, İ. Ö. (2012). Teaching of translations through use of vectors in Wingeom-tr environment. *Elementary Education Online*, 11(1).