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Weak power domination

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Abstract

An extension of the concept of power domination, which is motivated by the context of the monitoring of electric networks, is introduced. Theoretical properties, models and algorithms are presented together with computational results.

Keywords: Power domination , integer linear programming, branch-and-cut, electric power network monitoring

1 Introduction

In order to monitor electric power networks, measurement devices called phasor measurement units (PMUs) can be positioned at several locations. Once installed at some node v , a PMU is able to measure (*directly*) the voltage at v but also the current phase for all the edges incident to v . Then, by making use of Ohm's and Kirchoff's laws, it may also be possible to measure (*indirectly*) the current phase and voltage at other places of the network. Given the high cost of PMUs, a natural problem consists in identifying a minimum cost node subset where to install PMUs so as to determine the measurements w.r.t. all the nodes and edges of the network (no matter whether this is done “directly” or “indirectly” as mentioned above).

This problem motivated the introduction of the *power domination* concept by Haynes et al. (2002) in graph theory, which has since received a lot of attention (see, e.g., (Dorbec, 2020; Zhao et al., 2006) and the references therein), namely from a complexity and optimization perspective. However, the power domination concept fails to properly simulate the application of Ohm’s and Kirchoff’s laws from the motivating application (more details are given in the next section). In this work, we investigate theoretical properties and methods to solve optimization problems related to an extension of the power domination concept that we call *weak power domination*, which overcomes the flaw mentioned above.

The paper is organized as follows. In Section 2 we provide formal definitions of the considered domination concepts and problems, together with a statement of our contributions. Theoretical results are presented in Section 3, before we describe a solution method in Section 4. Section 5 is dedicated to computational experiments, before we conclude in Section 6.

2 Preliminaries

Let $G = (V, E)$ denote an undirected simple graph having node set V and edge set E . Given any node $u \in V$, let $N(u)$ represent the set of its neighbors (also called the *open neighborhood* of the node u): $N(u) = \{v \in V : \{u, v\} \in E\}$ and let $N[u] = \{u\} \cup N(u)$ stand for its *closed neighborhood*. Given a node subset S , we define $N(S) = \{v \in V \setminus S : \exists u \in S \text{ such that } v \in N(u)\}$, i.e. the nodes out of S having a neighbor in S , and $N[S] = N(S) \cup S$. If $N[S] = V$, then S is called a *dominating set*. $G[S]$ stands for the subgraph of G that is induced by S , i.e. the graph $(S, E[S])$ with $E[S] = \{\{i, j\} : \{i, j\} \in E, (i, j) \in S \times S\}$. The node set of any graph G may also be denoted by $V(G)$. Given a node $v \in V$, $d_G(v)$ stands for its degree in G , i.e. $d_G(v) = |N(v)|$.

Given a set $S \subseteq V$ of nodes that are said to be *initially observed*, whereas all the others in $V \setminus S$ are *not observed*, we define the following three propagation rules for node *observation*:

- (R1) all the nodes in $N[S]$ become *observed*.
- (R2) if an unobserved node u is the neighbor of a node v such that all the nodes in $N[v] \setminus \{u\}$ are *observed*, then the node u becomes *observed* as well.
- (R3) if an unobserved node u has all its neighbors observed, then the node u becomes *observed*.

The rule (R1) is applied first and only once simultaneously to all the nodes in S , whereas the two rules (R2)-(R3) are repeatedly applied after (R1) until no additional node can be observed. In what follows, by “*a round*” we mean the application of either (R1) or the two rules (R2)-(R3), the latter ((R2) and (R3)) being applied simultaneously to all the currently unobserved nodes. The application of these rules is illustrated in Figure 1-(a) where the number inside each node corresponds to the iteration number of the round at which the node becomes observed (S corresponds to the node with the value zero inside).

Let $\Omega_{\text{WP}}(S)$ be the set of all the nodes, starting from S , that become observed by making use of the three rules, (R2) and (R3) being applied iteratively until no

additional node can be observed, whereas $(R1)$ is applied to the nodes of S only once before the other rules (and defines the first round). Let $\Omega_p(S)$ be defined similarly as $\Omega_{wp}(S)$ except that the rule $(R3)$ is never used. If $\Omega_{wp}(S) = V$ ($\Omega_p(S) = V$) then S is called a *weak power dominating set* (*power dominating set*), abbreviated as “wpds” (“pds”) also for its plural form. Figure 1-(b) (similar to the previous figure) illustrates the application of the rules $(R1)$ and $(R2)$ only, $(R1)$ being applied first and once, whereas $(R2)$ is repeatedly applied after $(R1)$. Note that, for every graph G , every pds in G is a wpds in G , but the converse is not true in general (Figure 1-(a) provides such an example).

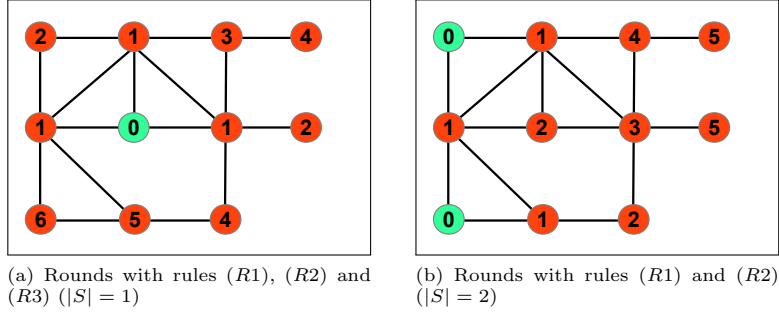


Fig. 1: Illustrations of the propagation rules

Assume that the graph G represents an electric network for which we want to determine the voltage at each node, making use of measurements provided by some PMUs, together with Ohm’s and Kirchoff’s laws. For our purposes, we always assume that the network we consider is composed of zero injection buses only, which means that there is no generation nor load, so that Ohm’s and Kirchoff’s laws are applicable at any location. This was the considered framework when the power domination concept was introduced, and that has been commonly used since.

Note that, while the power domination concept was motivated by the monitoring of electric networks, the rules $(R1)$ and $(R2)$ alone fail to account for the fact that if the voltage is unknown at some given node v , while it is known for all its neighbors, then the voltage at v can also be determined (using both Ohm’s and Kirchoff’s laws). This is the basic motivation for the *weak power domination* concept. To the best of our knowledge, though the rule $(R3)$ already appears in the literature, it has always been considered in addition to other technical constraints (see, e.g., [Carvalho et al. \(2018\)](#) and the references therein) and properties of the weak power domination concept as defined above have not been investigated yet.

In this paper, we focus on the *minimum cost weak power dominating set problem* denoted by *CWPDS* which we formulate as follows and is motivated by the fact that the monitoring facilities to be installed in an electric network may be very expensive, and their costs may depend on many parameters such as location and size of the neighborhood.

CWPDS: MINIMUM COST WEAK POWER DOMINATING SET PROBLEM

Input: An undirected simple graph $G = (V, E)$, a node-weight function $w \in \mathbb{R}_+^V$.

Objective: Find a wpds $S \subseteq V$ whose cost (or weight), given by $\sum_{v \in S} w_v$ is minimum.

The particular case of the problem CWPDS when $w_v = 1$ for all $v \in V$ (i.e. the minimum cardinality wpds problem) is denoted by WPDS. Replacing "wpds" by "pds" in the objective of CWPDS we get the minimum cost pds problem CPDS. And if, in addition, all the node weights are equal to one, we obtain the minimum cardinality pds problem denoted by PDS. The minimum cardinality pds (resp. wpds, dominating set) in the graph G is denoted by $\gamma_P(G)$ (resp. $\gamma_{WP}(G)$, $\gamma(G)$). Since any dominating set is a pds and any pds is also a wpds, for any graph G we have: $\gamma_{WP}(G) \leq \gamma_P(G) \leq \gamma(G)$.

Our contributions are as follows.

- We provide theoretical results related to the notion of wpds, pointing out similarities but also significant differences between the concepts of power domination and weak power domination.
- We present integer linear programming (ILP) models and methods to solve CWPDS.
- Computational results are reported for WPDS (and also PDS for comparison) on diverse types of graphs from the literature.

3 Theoretical results

In what follows we always assume that the graph $G = (V, E)$ has order $|V| \geq 3$ and is connected unless otherwise stated.

3.1 Characterization of weak power dominating sets

Given an undirected graph G , a nonempty node subset F is called a *fort* (Fast and Hicks, 2018) if every node out of F has either no or at least two neighbors in F . Equivalently, F is a fort if and only if $|N(v) \cap F| \neq 1$, for all $v \in V \setminus F$. If F is a fort then $N[F]$ is called a *fort neighborhood* (Smith and Hicks, 2022).

We now introduce an extension of the notion of fort for the concept of weak power domination. A nonempty node subset set F is called a *strong fort* if it is a fort such that $N(v) \cap F \neq \emptyset$, for all $v \in F$. If F is a strong fort, then $N[F]$ is called a *strong fort neighborhood*. (See Figure 2a for an illustration, where three strong fort neighborhoods are circled with grey lines. For each one of them, the set of its black nodes defines the corresponding strong fort.)

Proposition 3.1. *The following holds.*

- No node of a strong fort F is observed from an initially observed set of nodes S unless $N[F]$ contains a node of S , i.e., $F \cap \Omega_{WP}(S) = \emptyset$ if $S \cap N[F] = \emptyset$.*
- Let S denote any node subset such that $\Omega_{WP}(S) \neq V$. Then $V \setminus \Omega_{WP}(S)$ is a strong fort.*
- Let S denote any nonempty node subset. Then $S \setminus \Omega_{WP}(N(S)) = S \setminus \Omega_{WP}(X)$ for any node subset X satisfying $N(S) \subseteq X \subseteq V \setminus S$.*

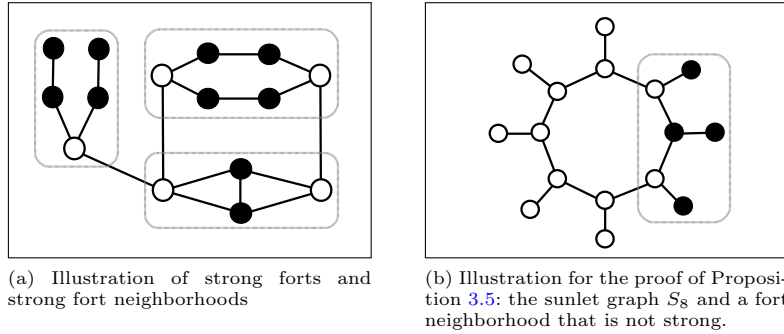


Fig. 2: Illustrations of forts and strong fort neighborhoods

(iv) If \widehat{F} is a minimal (inclusionwise) strong fort neighborhood, then $G[\widehat{F}]$ is connected.

Proof. See A.1. □

Let $\mathfrak{F}(G)$ stand for the set of all the minimal strong fort neighborhoods in G . Given a node subset $S \subseteq V$, $\chi^S \in \{0, 1\}^V$ represents its incidence vector, i.e. $\chi_v^S = 1$ if $v \in S$ and $\chi_v^S = 0$ otherwise. The next proposition gives a simple characterization of wpds.

Proposition 3.2. *A node subset $S \subseteq V$ is a wpds if and only if χ^S satisfies the inequalities*

$$\sum_{v \in W} x_v \geq 1, \forall W \in \mathfrak{F}(G). \quad (1)$$

Proof. Sufficiency comes from Proposition 3.1(i). Necessity follows from the fact that if $\Omega_{\text{WP}}(S) \neq V$, then, by Proposition 3.1(ii), $V \setminus \Omega_{\text{WP}}(S)$ is a strong fort. But the latter implies (by Proposition 3.1(i)) $S \cap N[V \setminus \Omega_{\text{WP}}(S)] = \emptyset$, a contradiction with (1). □

Necessary and sufficient conditions for inequalities (1) to be facet defining appear in Balas and Ng's work (Balas and Ng, 1989).

3.2 Connections between the problems PDS and WPDS

Similarly to the concept of power domination (Haynes et al., 2002), there is no forbidden subgraph characterization of the graphs G for which $\gamma(G) = \gamma_{\text{WP}}(G)$. Indeed, any graph $H = (V(H), E(H))$ can be shown to be an induced subgraph of a graph G satisfying $\gamma(G) = \gamma_{\text{WP}}(G)$. To see this, it is sufficient to consider the graph $G = H \circ K_2$ which is obtained from H by adding, for each node $v \in V(H)$ two nodes v_1, v_2 , together with the edges $\{v, v_1\}$, $\{v, v_2\}$ and $\{v_1, v_2\}$. We can verify that $\gamma(G) = \gamma_{\text{WP}}(G) = \gamma_{\text{P}}(G) = |V(H)|$.

Also, given any connected graph G having at least one node with degree at least three, similarly to PDS, there exists an optimal solution of WPDS containing only nodes with degree at least three. In fact, we can prove a stronger property w.r.t. WPDS on some graph family that we state in the next proposition. A node having degree one in a graph is also called a *leaf*. Given any graph $G = (V, E)$, $\overset{\circ}{G} = (\overset{\circ}{V}, \overset{\circ}{E})$ denotes

the graph obtained from G by removing all the leaves, i.e. $\mathring{G} = G[V \setminus L]$, where L represents all the leaves in G .

Proposition 3.3. *For any connected graph $G = (V, E)$ such that \mathring{G} contains at least one node with degree at least three, there exists an optimal solution of WPDS on G that is composed only of nodes with degree at least three in \mathring{G} .*

Proof. See A.2. □

Proposition 3.4. *For any connected graph G with at least three nodes, any pds in \mathring{G} is a wpds in G ; thus, $\gamma_{\text{WP}}(G) \leq \gamma_{\text{P}}(\mathring{G})$.*

Proof. See A.3. □

Proposition 3.5. *There exists a graph family composed of graphs $G = (V, E)$ for which the ratio $\frac{\gamma_{\text{P}}(G)}{\gamma_{\text{WP}}(G)}$ and the gap $\gamma_{\text{P}}(G) - \gamma_{\text{WP}}(G)$ is $\Omega(|V|)$.*

Proof. We exhibit a graph family for which the statement of the proposition holds. For some positive integer $n \geq 3$, let S_n denote the sunlet graph of order $2n$, i.e., the graph obtained from an elementary cycle on n nodes by adding a leaf node adjacent to each node of the cycle. Figure 2b gives an illustration for S_8 . On that figure, the black nodes correspond to a fort whose closed neighborhood is the closed area delimited by a grey line. One can easily check that $\gamma_{\text{WP}}(S_n) = 1$ (any node of the cycle is a wpds). And since any three consecutive nodes on the cycle together with their neighbors out of the cycle define a fort neighborhood, we can deduce $\gamma_{\text{P}}(S_n) \geq \left\lceil \frac{|V|}{6} \right\rceil = \left\lceil \frac{n}{3} \right\rceil$. □

3.3 Properties of WPDS in trees

Let $K_{1,m}$ denote a star with m leaves. A graph G is called a *starlike tree*¹ (Watanabe and Schwenk, 1979) if it is homeomorphic to a star $K_{1,m}$. If $m \leq 2$, G is a path. And if $m \geq 3$, G is a tree with exactly one node having a degree larger than two. We define a *leavy starlike tree* as a tree $G = (V, E)$ such that $G[V \setminus L]$ is a starlike tree, where L represents the set of all the leaves in G . (See Figure 3a where the white nodes correspond to the set L .)

Similarly to the case of PDS (Haynes et al., 2002, Theorem 7), we can establish a simple characterization of the trees for which an optimal solution of WPDS is composed of a single node.

Proposition 3.6. *A tree T satisfies $\gamma_{\text{WP}}(T) = 1$ if and only if it is a leavy starlike tree.*

Proof. See A.4. □

Let us recall the following theorem from (Haynes et al., 2002).

Theorem 3.7. (Haynes et al., 2002, Theorem 9) *For any tree T of order at least 3, $\gamma_{\text{P}}(T) = \gamma(T)$ if and only if T has a unique minimum cardinality dominating set S and every node in S is adjacent to two or more leaves.*

¹The notion of *starlike tree*, as defined in (Watanabe and Schwenk, 1979) and used here, coincides with that of *spider* as defined in (Haynes et al., 2002). However, the latter conflicts with the usual notion of spider as defined in (Koebe, 1992). This motivates our use of the expression *starlike tree* in lieu of *spider*. Recall that two graphs are homeomorphic if they can be transformed into each other by a series of edge subdivisions and edge contractions, see e.g., (Diestel, 2017).

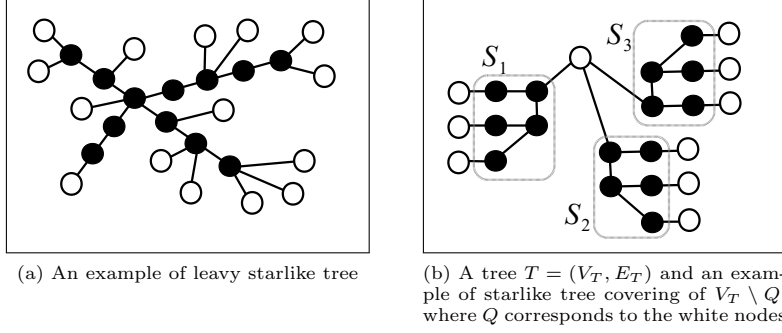


Fig. 3

By contrast with Theorem 3.7, the next result shows that $\gamma_{\text{WP}}(T) = \gamma(T)$ holds for a much more restricted class of trees.

Proposition 3.8. *A tree T satisfies $\gamma_{\text{WP}}(T) = \gamma(T)$ if and only if it is a star.*

Proof. See A.5. □

We now give a characterization of $\gamma_{\text{WP}}(T)$ for any tree T . Given a graph $G = (V, E)$, and a node subset $F \subseteq V$, let $\mathcal{S}(G, F)$ denote the set of all the stable sets in G that are contained in F . Given a tree $T = (V_T, E_T)$, let L denote the set of its leaves. A set of node subsets $\{S_1, S_2, \dots, S_p\}$ is called a *starlike tree covering* of $W \subseteq V_T$ if $W = \cup_{i=1}^p S_i$ and $T[S_i]$ is a starlike tree, for all $i \in \{1, 2, \dots, p\}$. We then define the quantity

$$sc(T) = \min_{Q \in \mathcal{S}(T, V_T \setminus N(L))} \{p : \exists \{S_1, S_2, \dots, S_p\} \text{ starlike tree covering of } V_T \setminus Q\}. \quad (2)$$

Figure 3b gives an example of starlike tree covering (with size $p = 3$) of $V_T \setminus Q$, where Q corresponds to the white nodes.

Theorem 3.9. *For any tree T , $\gamma_{\text{WP}}(T) = sc(T)$.*

Proof. See A.6. □

3.4 Complexity

The decision version of the WPDS problem can be stated as follows.

WPDS_DECISION

Input: An undirected simple graph $G = (V, E)$, a positive integer k .

Question: Does there exist a wpds S with $|S| \leq k$?

The decision version of PDS was shown to be NP-complete for bipartite and chordal graphs by (Haynes et al., 2002). A simpler proof was then given by (Guo et al., 2008), extending this result for circle graphs (i.e. intersection graphs of chords in a cycle) and planar graphs. Their work can be extended to get the next two theorems.

Theorem 3.10. *WPDS_DECISION is NP-complete for bipartite, chordal, circle and planar graphs.*

Proof. See [A.7](#). □

Theorem 3.11. *For any tree T , WPDS can be solved in linear time.*

Proof. See [A.8](#). □

It follows from the proof of [Theorem 3.10](#) that the inapproximability and parameterized tractability results with respect to the dominating set problem transfer to WPDS. In particular, WPDS is $W[2]$ -hard with the size of the wpds as parameter.

4 Solution method

In this section we present a simple formulation for CWPDS as an ILP ([§4.1](#)). Given that the latter involves a large number of constraints, an iterative constraint generation scheme is designed for its solution ([§4.2](#)). Two additional ingredients which turn out to potentially impact significantly the computational performance are then described: one is a method to generate an initial set of inequalities of the form [\(1\)](#) ([§4.3](#)), whereas the other is a pre-processing procedure ([§4.4](#)) that aims to reduce the size of the graph of the original instance and to strengthen ILP relaxations in some special way.

4.1 Formulation (WF) for CWPDS

Using [Proposition 3.2](#), CWPDS can be reformulated as the ILP

$$\min_{x \in \{0,1\}^V} \sum_{v \in V} w_v x_v : x \text{ satisfies } (1). \tag{WF}$$

By replacing in [\(WF\)](#) the set $\mathfrak{F}(G)$ by the set of all the inclusionwise minimal fort neighborhoods in G , we obtain the formulation from [Smith and Hicks \(2022\)](#) for PDS, which we denote by [\(PF\)](#).

Next we introduce a particular solution strategy to deal with the large number of constraints involved in [\(WF\)](#) (and [\(PF\)](#)).

4.2 Solving (WF) and (PF) with constraint generation

In what follows, we focus our presentation for [\(WF\)](#) as it can be easily adjusted for [\(PF\)](#). Both formulations [\(WF\)](#) and [\(PF\)](#) are solved by branch-and-cut with lazy constraint generation.

Finding a violated inequality of the form [\(1\)](#) in that context requires identifying a strong fort neighborhood that is not intersected by some given integer solution. Two options are considered for that purpose. One identifies a minimum cardinality strong fort neighborhood by solving an ILP ([§4.2.1](#)), the other is a combinatorial algorithm determining an inclusionwise minimal strong fort neighborhood ([§4.2.2](#)).

4.2.1 Minimum cardinality strong fort neighborhood

An ILP to find a minimum cardinality fort neighborhood is introduced in ([Smith and Hicks, 2022](#)). We describe hereafter an adaptation of the latter to generate violated inequalities to solve [\(WF\)](#).

Given a graph $G = (V, E)$, let $J(G)$ denote the set of the nodes with degree at least three in G , and let \mathcal{P}_G denote the set of the node subsets of the connected components in $G[V \setminus J(G)]$. A minimum cardinality strong fort neighborhood of the form $N[F]$, where F is a strong fort not intersecting $\Omega_{\text{WP}}(S)$ for some given node subset S , can be obtained by solving the following ILP.

$$\begin{aligned}
& \min_{F, M} \sum_{v \in J(G)} M_v + \sum_{P \in \mathcal{P}_G} |P| F_P \\
& \text{s.t.} \quad \sum_{v \in J(G)} M_v \geq 1 \\
& F_v \leq M_u \quad \forall u, v \in J(G) : \{u, v\} \in E \\
& F_v \leq F_P \quad \forall P \in \mathcal{P}_G, \forall v \in N(P) \\
& F_P \leq M_v \quad \forall P \in \mathcal{P}_G, \forall v \in N(P) \\
& 2(M_v - F_v) \leq \sum_{\substack{u \in J(G): \\ v \in N(u)}} F_u + \sum_{P \in \mathcal{P}_G} |N(v) \cap P| F_P \quad \forall v \in J(G) \\
& F_v \leq M_v \quad \forall v \in J(G) \\
& F_v \leq \sum_{w \in N(v)} F_w \quad \forall v \in J(G) : N(v) \subset J(G) \\
& F_P \leq \sum_{w \in N(P)} F_w \quad \forall P \in \mathcal{P}_G : |P| = 1 \\
& F_v = 0 \quad \forall v \in J(G) : v \in \Omega_{\text{WP}}(S) \\
& F_P = 0 \quad \forall P \in \mathcal{P}_G : P \subseteq \Omega_{\text{WP}}(S) \\
& F \in \{0, 1\}^{J(G) \cup \mathcal{P}_G}, M \in \{0, 1\}^{J(G)}.
\end{aligned} \tag{MSF}$$

In the formulation (MSF) above, the variable F_v (F_P) indicates if the node $v \in J(G)$ (the node subset $P \in \mathcal{P}_G$) is contained in the strong fort F defining the strong fort neighborhood $N[F]$. The variable M_v indicates if the node $v \in J(G)$ belongs to the strong fort neighborhood $N[F]$. By comparison with the formulation from (Smith and Hicks, 2022) (for minimum cardinality fort neighborhood), (MSF) contains the additional constraints

$$\begin{aligned}
F_v &\leq \sum_{w \in N(v)} F_w \quad \forall v \in J(G) : N(v) \subset J(G) \\
F_P &\leq \sum_{w \in N(P)} F_w \quad \forall P \in \mathcal{P}_G : |P| = 1
\end{aligned}$$

They ensure that the node set $Z = \{v \in J(G) : F_v^* = 1\} \cup \left(\bigcup_{P \in \mathcal{P}_G} \{P : F_P^* = 1\} \right)$ verifies the property that $G[Z]$ has no isolated node (i.e. each node in Z has at least one neighbor in G that belongs to Z), where (F^*, M^*) represents an optimal solution of (MSF).

4.2.2 Minimal strong fort neighborhood

A simple combinatorial alternative to the previous separation procedure, is the following method which determines an inclusionwise minimal strong fort neighborhood that is not intersected by the current solution.

Given a node subset S that is not a wpds, we iteratively consider each node $v \notin S$. If $S \cup \{v\}$ is not a wpds, then node v is added to S . After all the nodes have been processed, a minimal (inclusionwise) strong fort neighborhood is given by $N[V \setminus \Omega_{\text{WP}}(S)]$.

In our implementation, we also make use of the observation that if S is not a wpds, then the same holds for W where W is the set of the nodes whose closed neighborhood is contained in $\Omega_{\text{WP}}(S)$. Formally, $W = \{v \in V : N[v] \subseteq \Omega_{\text{WP}}(S)\}$. This may help S grow faster and reduce the number of calls to compute the set of nodes that are observed.

The approach described above can be easily adapted for (PF) by considering "pds" and " Ω_{P} " instead of "wpds" and " Ω_{WP} ", respectively.

4.3 Initialization of the formulations (WF) and (PF)

Though it is possible to initialize the relaxations of (WF) and (PF) with an empty set of constraints, it turns out that starting with some *suitably selected and moderate number* of constraints may greatly affect the computational performance (see Section 5). We describe the method we designed for (WF), followed by a brief description of its adaptation for (PF).

The procedure works in three steps. The first one consists in computing a wpds S^* by solving an auxiliary LP. Then, an inclusionwise minimal wpds S_{\min} contained in S^* is determined. Finally, for each node $v \in S_{\min}$, we identify an inequality (1) violated by $S_{\min} \setminus \{v\}$ that we keep for the initial relaxation. We now describe each step more precisely.

- **Step 1:** determine a wpds S^* . Firstly, a linear relaxation of the minimum weight dominating set problem is solved.

$$\min_{x \in [0,1]^V} \sum_{v \in V} w_v x_v : \sum_{u \in N[v]} x_u \geq 1, v \in V. \quad (\text{MDS})$$

S^* is initialized with the empty set. Then nodes are processed by decreasing values according to the optimal solution found for MDS. They are iteratively added to S^* until a wpds is obtained.

- **Step 2:** determine a minimal wpds S_{\min} contained in S^* . Firstly, we set $F \leftarrow S^*$. Then, for each node v of S^* (in an arbitrary order): if $F \setminus \{v\}$ is a wpds in G , then we set $F \leftarrow F \setminus \{v\}$, and otherwise F is left unchanged. (Note that $\Omega_{\text{WP}}(F)$ can be computed in linear time, similar to $\Omega_{\text{P}}(F)$, see, e.g., (Brimkov et al., 2019).)

The outcome of this procedure is an inclusionwise minimal wpds S_{\min} in G .

- **Step 3:** generate inequalities. For each node $v \in S_{\min}$, $S_{\min} \setminus \{v\}$ is not a wpds and thus the set of the nodes that cannot be observed by $S_{\min} \setminus \{v\}$ defines a strong fort. For each node $v \in S_{\min}$ a minimal strong fort neighborhood contained in $N[V(G) \setminus \Omega_{\text{WP}}(S_{\min} \setminus \{v\})]$ is computed (using one of the two methods described in §4.2). The constraint (1) corresponding to this strong fort neighborhood is then added to the initial relaxation.

The initialization procedure presented above for (WF) can be adapted for (PF) as follows. In the first step, we consider addition of nodes to S^* (in the same order as before) until a pds is obtained. In the second step, a minimal pds S_{\min} contained in

S^* is determined: for each node v of S^* (processed in an arbitrary order), if $F \setminus \{v\}$ is a pds in G , then we set $F \leftarrow F \setminus \{v\}$ (and otherwise F is left unchanged). In the third step: for each node $v \in S_{\min}$ a minimal fort neighborhood contained in $N[V(G) \setminus \Omega_{\mathcal{P}}(S_{\min} \setminus \{v\})]$ is computed.

4.4 Pre-processing

In this section we describe two pre-processing procedures designed to speedup resolution of CWPDS. The first one (§4.4.1) looks for a potential reduction of the size of the input graph so as to compute a solution of the original problem working with a smaller graph. The second one (§4.4.2) aims to strengthen the continuous relaxation of the ILP formulations presented in §4 with additional inequalities. Adaptations of these procedures for CPDS are reported in B.

4.4.1 Graph reduction

We describe two procedures denoted by GR1 and GR2 to reduce the size of the original graph. It can be easily verified that any of these transformations does not change the optimal objective value of the original instance and that an optimal solution for the original problem can be inferred from an optimal solution of the reduced one.

GR1. If ℓ_1, \dots, ℓ_q are leaves in G , all neighbors of some node z , then they are replaced by a single leaf $\hat{\ell}$ that is adjacent to z and having for weight $w_{\hat{\ell}} = \min_{i=1, \dots, q} w_{\ell_i}$.

GR2. Assume that $\mathcal{P} = (v_1, v_2, \dots, v_p)$, with $p \geq 4$, is an induced path in G such that $d_G(v_2) = \dots = d_G(v_{p-1}) = 2$, and we may have $v_1 = v_p$ but $\{v_1, \dots, v_{p-1}\}$ are pairwise distinct. It is clear that if $G = \mathcal{P}$ then \mathcal{P} can be replaced by an isolated node of minimum weight. So assume from now on that G is not a path nor a cycle. Then,

- if one of the endpoints of \mathcal{P} , say v_p , is a leaf in G , \mathcal{P} is replaced by the path (v_1, v_2, z) , where z is a new node of weight $w_z = \min_{i=3, \dots, p} w_{v_i}$.
- otherwise (i.e. none of the endpoints of \mathcal{P} is a leaf), \mathcal{P} is replaced by the path (or cycle in case $v_1 = v_p$) (v_1, v_2, z, v_p) , where z is a new node with $w_z = \min_{i=3, \dots, p-1} w_{v_i}$.

4.4.2 ILP strengthening

We now describe a procedure denoted by ST to add inequalities to the initial relaxations. Although it may reduce the set of optimal solutions of the original instance, one can easily check that there always exists an optimal solution of the original instance satisfying the added constraints. The basic idea is the following. Let S_1 and S_2 represent two distinct node subsets in G such that $\Omega_{\text{WP}}(S_2) \subseteq \Omega_{\text{WP}}(S_1)$, $S_2 \setminus S_1 \neq \emptyset$ and $\sum_{v \in S_2} w_v \geq \sum_{v \in S_1} w_v$. Then there exists an optimal solution S^* satisfying $|S^* \cap S_2| \leq |S_2| - 1$. This means that the inequality $\sum_{v \in S_2} x_v \leq |S_2| - 1$ can be added to (WF). Remark that in the particular case of two distinct nodes u and v such that $\Omega_{\text{WP}}(\{u\}) \subseteq \Omega_{\text{WP}}(\{v\})$, and $w_u \geq w_v$, we can set $x_u = 0$. More specifically, for the case of unit weights, setting to value zero variables corresponding to nodes of degree at most two (recall Proposition 3.3) can arise as a byproduct of ST considering sets S_1, S_2 having cardinality one.

In order to apply the above idea iteratively without cutting off all optimal solutions

of the original problem, we consider some node ordering (arbitrary in the case of our implementation) ensuring that, whenever a constraint of the form $\sum_{v \in S} x_v \leq |S| - 1$ is generated, then, there exists a node subset $W \subseteq V \setminus S$ which is composed of nodes, none of which appears in the support of any of the previously generated constraints, and such that $\Omega_{\text{WP}}(S) \subseteq \Omega_{\text{WP}}(W)$, and $\sum_{v \in S} w_v \geq \sum_{v \in W} w_v$. The basic algorithmic scheme we used in our implementation is presented in Algorithm 1.

Algorithm 1 Simple generic ST

Input: an instance of CWPDS, a set \mathcal{S} of ordered node subsets: $(S_i)_{i=1}^q$.
Output: a list \mathcal{L} of inequalities to be added to the initial relaxation of (WF)

```

1:  $\mathcal{L} \leftarrow \emptyset$  ▷ Empty list of constraints
2:  $\mathcal{N} \leftarrow \emptyset$  ▷ Nodes in the support of a constraint in  $\mathcal{L}$ 
3: for  $i$  from 1 to  $q$  step 1 do ▷ iteration over node subsets
4:   if  $S_i \cap \mathcal{N} = \emptyset$  then ▷ No node of  $S_i$  appears in constraints of  $\mathcal{L}$ 
5:      $\mathcal{W} \leftarrow \left\{ Q \in \mathcal{S} : N[Q] \subseteq \Omega_{\text{WP}}(Q), Q \cap (S_i \cup \mathcal{N}) = \emptyset, \sum_{v \in Q} w_v \geq \sum_{v \in S_i} w_v \right\}$ 
6:     for all  $W \in \mathcal{W}$  do
7:        $\mathcal{L} \leftarrow \mathcal{L} \cup \left\{ \sum_{v \in W} x_v \leq |W| - 1 \right\}$  ▷ Add constraint to  $\mathcal{L}$ 
8:        $\mathcal{N} \leftarrow \mathcal{N} \cup W$  ▷ Update  $\mathcal{N}$ 
9:     end for
10:   end if
11: end for
12: return  $\mathcal{L}$ 

```

5 Computational experiments

Including instances beyond classical ones from the literature, our computational study is designed to evaluate the performance of the diverse approaches and procedures presented in the previous section and compare the results obtained on both problems WPDS and PDS.

5.1 Experimental setting and instances

All the experiments have been performed on a laptop with an i5 2.30GHz processor, and 15.5 GB of RAM. Our code is written in C. LPs and ILPs were solved using Gurobi 9.5.1 [Gurobi Optimization, LLC \(2022\)](#) (with a single-thread and default settings otherwise). The time limit was set to 1800 seconds. The models (PF) and (WF) are solved using branch-and-cut and lazy constraint separation (callback function). The graphs used in our experiments include:

- instances from the IEEE benchmark dataset ² the PEGASE (Pan European Grid Advanced Simulation and state Estimation) project ([Fliscounakis et al., 2013](#)) and the Network Repository ([Rossi and Ahmed, 2015](#)). These instances represent power systems and their use is thus motivated by the application from which PDS originates, and the fact that they have been already widely used in the literature.

²<http://www.ee.washington.edu/research/pstca/>

- the sunlet graph S_{1000} . Though it is a very specific structure it is motivated by Proposition 3.5 and its proof, and is intended to illustrate further (now computationally) a case when there is a large gap between the optimal objective values of PDS and WPDS.
- grid graphs ('GRID d_xxx ' where d is the dimension and 'xxx' gives the number of nodes in each dimension). They correspond to a commonly used basic model in the context of electrical grid monitoring.
- randomly generated 3 and 4-regular graphs (' k -reg. n ' where n denotes the order and $k \in \{3, 4\}$). For such graphs Proposition 3.3 does not allow fixing x_v to value zero for any node $v \in V$. Also, similarly to Watts-Strogatz graphs considered in the next item, cubic graphs appear to be quite challenging for the zero-forcing problem (which may be viewed as a domination concept involving the rule (R2) only) (Brimkov et al., 2019).
- randomly generated connected Watts-Strogatz graphs ('WS- n_k ', where n stands for the number of nodes and $k \in \{4; 10\}$ is the mean degree). The probability of rewiring edges is set to value 0.3. Graphs in this family have *small world* network properties, including small average path length and high clustering (Watts and Strogatz, 1998).
- randomly generated connected Erdős-Rényi graphs (Erdős and Rényi, 1959) ('ER- n_p ' where n stands for the order of the graph and p is the probability for edge creation).

For the graphs that are randomly generated, the results are averaged over five instances for each class, and each instance was generated using NetworkX ³.

5.2 Computational results

To ease presentation, in this section we only report the detailed results obtained for the case of unit node weights (i.e. problems WPDS and PDS). Detailed results for the weighted case (leading to similar comments) are given in C.

Tables 1 and 2 provide information about initialization and pre-processing together with instances characteristics (number of nodes and edges). With respect to pre-processing, we indicate in the field 'Post-GR graph' the number of nodes and edges of the graph obtained with the procedure GR together with the time needed (in seconds). Value zero for time means less than one millisecond. Also in parentheses and close to the number of nodes we indicate the number of nodes with degree two in the reduced graph. Then, with respect to the strengthening procedure ST - which in our case handles all singletons first and then pairs of nodes - we distinguish between the part dealing with singletons ('ST1') and the one dealing with node pairs ('ST2'). For each, we indicate the number of constraints generated and the time required (in seconds). The following part of the tables is concerned with the initialization procedure, considering either the use of an ILP or a combinatorial algorithm ('comb') to generate constraints, after pre-processing has been applied. (To ease presentation we do not detail the results of initialization without pre-processing since they lead to similar observations.) We report the number of generated constraints ('# cn') and

³<https://networkx.org/documentation/stable/reference/index.html>

Table 1: Pre-processing and initialization for PDS

Input graph	PDS pre-processing										PDS initialization (after pre-pro)		
	Input graph		Post-GR graph				ST1		ST2		PDS initialization (after pre-pro)		
	Name	[V]	[E]	[V]	[E]	time	# cn	time	# cn	time	# cn	ILP	comb
IEEE_14_bus	14	20	13 (6)	19	0	11	0	0	0	0	0	0	0
IEEE_30_bus	30	41	25 (13)	36	0	18	0	1	0	3	0.014	0.007	0.007
IEEE_57_bus	57	78	42 (18)	63	0	27	0	6	0	4	0.008	0.003	0.003
IEEE_118_bus	118	179	101 (46)	162	0	67	0	11	0.001	10	0.014	0.004	0.004
IEEE_300_bus	300	409	274 (119)	383	0.003	166	0	43	0.014	32	0.092	0.011	0.011
power-685-bus	685	1282	671 (160)	1268	0.010	329	0.003	357	0.266	50	0.349	0.127	0.127
power-eris	1174	8687	976 (375)	8489	0.020	820	0.003	26	0.209	101	1.363	0.122	0.122
pegase_1354	1354	1710	1157 (661)	1513	0.049	757	0.005	75	0.776	181	0.659	0.111	0.111
bcspr99	1723	2394	1514 (698)	2185	0.081	877	0.012	257	2.115	177	1.042	0.255	0.255
pegase_2869	2869	3968	2474 (1288)	3573	0.220	1596	0.024	207	7.151	318	2.771	0.660	0.660
polish_2383	2383	2886	1778 (1002)	2281	0.139	1062	0.013	120	4.250	222	1.287	0.337	0.337
us_west_4941	4941	6594	4167 (2108)	5820	0.659	2562	0.069	446	41.788	525	8.074	1.839	1.839
pegase_9241	9241	14207	7639 (3839)	12605	2.398	4972	0.232	682	200.144	879	31.191	6.494	6.494
pegase_13659	13659	18625	10268 (5772)	15234	4.196	6703	0.377	541	577.946	1711	76.591	9.680	9.680
S_1000	2000	2000	2000 (1000)	2000	0.025	1000	0.016	333	3.421	334	1.508	0.274	0.274
GRD2_10_10	100	180	100 (4)	180	0	8	0	193	0.001	4	0.460	0.007	0.007
GRD2_20_20	400	760	400 (4)	760	0.001	8	0.001	514	0.087	8	1800.000	0.142	0.142
GRD2_30_30	900	1740	900 (4)	1740	0.006	8	0.007	753	0.849	13	1800.000	0.849	0.849
GRD3_4_4_4	64	144	64 (0)	144	0	0	0	59	0.001	4	0.360	0.013	0.013
GRD3_5_5_5	125	300	125 (0)	300	0	0	0	51	0.009	7	0.729	0.012	0.012
3-reg_50	50	75	50 (0)	75	0	4	0	38.6	0	4.4	0.027	0.005	0.005
4-reg_50	50	100	50 (0)	100	0	0.2	0	8.8	0.001	4.4	0.110	0.006	0.006
3-reg_80	80	120	80 (0)	120	0	2.4	0	38	0.001	6	0.150	0.013	0.013
4-reg_80	80	160	80 (0)	160	0	0.2	0	6.8	0.005	6.4	0.336	0.013	0.013
WS_30_4	80	160	80 (3.6)	160	0	24.8	0	22.4	0.002	5.2	0.177	0.009	0.009
WS_30_10	80	400	80 (0)	400	0	2.4	0	12.2	0.008	4.2	0.668	0.014	0.014
WS_100_4	100	200	100 (5.4)	200	0	32	0	20	0.003	6.4	0.151	0.012	0.012
WS_100_10	100	500	100 (0)	500	0	3.2	0	21	0.013	4.6	1.738	0.019	0.019
ER_30_0.1	80	313.4	80 (0.6)	313.4	0	2.2	0	4.8	0.010	4.8	0.875	0.014	0.014
ER_30_0.2	80	640	80 (0)	640	0	0.2	0	0.2	0.019	4.6	0.562	0.016	0.016
ER_30_0.3	80	947.6	80 (0)	947.6	0	0	0	0.4	0.030	3.6	2.606	0.017	0.017
ER_30_0.4	80	1265.2	80 (0)	1265.2	0	0	0	3.8	0.034	2.4	1.811	0.016	0.016
ER_100_0.1	100	495	100 (0.6)	495	0	0.8	0	1.8	0.019	6.4	1.081	0.020	0.020
ER_100_0.2	100	1001.4	100 (0)	1001.4	0	0	0	0	0.040	4.2	17.414	0.023	0.023
ER_100_0.3	100	1491.4	100 (0)	1491.4	0	0	0	0	0.066	4	4.113	0.025	0.025
ER_100_0.4	100	1986	100 (0)	1986	0	0	0	0	0.094	3	5.316	0.026	0.026

the time (in seconds). Remark that when the time reported for one of the two methods ('ILP' or 'comb') reaches the time limit (1800s) then no constraint is generated by the latter, and the number of constraints generated refers to the method with a lower computational time. A star ('*') indicates that a feasible solution was found by the pre-processing procedure. For the special case of unit weights our implementation allows for early termination of the overall algorithm if pre-processing identifies a feasible solution with size at most two (which must also be optimal from the way node subsets are handled).

One first important observation (for both PDS and WPDS) with respect to pre-processing is that it seems relevant essentially for the instances taken from the literature (i.e. the first 14 graphs): their special structure may allow for a moderate reduction of the original graph, the generation of many constraints with ST1 and to a lesser extent with ST2. The computation times for GR and ST1 are negligible throughout all instances whereas ST2 may hamper performance for large and specially structured graphs (such as `pegase_9241` or `pegase_13659`). Except for the largest instances taken from the literature and largest grid graph, the initialization takes less than one second with the combinatorial procedure, whereas the ILP approach cannot handle the largest grid graphs within our time limit, and takes more time on the overall. For PDS the average values for the number of constraints stemming from ST1, ST2 and initialization are approximately: 585, 134 and 129 respectively. For WPDS we get 687, 132, and 53 respectively. So, on average, the number of constraints generated by the pre-processing and initialization is close for both problems but with more constraints of type (1) for PDS.

Table 2: Pre-processing and initialization for WPDS

Input graph			PDS pre-processing						PDS initialization			
			Post-GR graph		ST1		ST2		(after pre-pro)			
Name	V	E	V	E	time	# cn	time	# cn	time	# cn	ILP	comb
IEEE_14_bus	14	20	14 (7)	20	0	12	0	0 *	0 *	*	*	*
IEEE_30_bus	30	41	29 (17)	40	0	26	0	0	0	3	0.010	0.004
IEEE_57_bus	57	78	51 (27)	72	0	37	0	19	0	4	0.013	0.004
IEEE_118_bus	118	179	116 (61)	177	0	87	0	11	0.001	8	0.022	0.009
IEEE_300_bus	300	409	279 (126)	388	0.004	222	0.001	28	0.006	17	0.067	0.014
power-685-bus	685	1282	682 (171)	1279	0.015	378	0.003	447	0.228	38	0.408	0.195
power-eris	1174	8687	1170 (569)	8683	0.040	1035	0.003	21	0.222	83	1.915	0.215
pegase_1354	1354	1710	1142 (667)	1498	0.041	893	0.007	70	0.438	73	0.382	0.126
bcspwr09	1723	2394	1559 (757)	2230	0.079	1122	0.024	260	1.284	79	0.887	0.407
pegase_2869	2869	3968	2531 (1370)	3630	0.197	1910	0.031	230	4.639	145	1.862	0.991
polish_2383	2383	2886	2026 (1269)	2529	0.141	1577	0.019	127	2.768	93	0.940	0.999
us_west_4941	4941	6594	4389 (2396)	6042	0.632	3313	0.088	436	26.926	268	5.670	4.220
pegase_9241	9241	14207	8247 (4482)	13213	2.353	6277	0.315	827	143.479	448	21.930	9.905
pegase_13659	13659	18625	9839 (5707)	14805	3.692	7732	0.422	763	222.727	543	29.104	13.959
S_1000	2000	2000	2000 (1000)	2000	0.025	*	0.006 *	*	*	*	*	*
GRID2_10_10	100	180	100(4)	180	0	20	0	154	0.001	4	0.493	0.015
GRID2_20_20	400	760	400(4)	760	0.001	20	0.002	409	0.078	8	1800.000	0.152
GRID2_30_30	900	1740	900(4)	1740	0.008	20	0.007	689	0.954	15	1800.000	1.516
GRID3_4_4_4	64	144	64(0)	144	0	0	0	38	0.001	4	0.513	0.006
GRID3_5_5_5	125	300	125(0)	300	0	0	0	37	0.016	7	0.492	0.021
3-reg_50	50	75	50(0)	75	0	4	0	54.6	0	4.2	0.039	0.043
4-reg_50	50	100	50(0)	100	0	0.2	0	8	0.001	4.4	0.343	0.069
3-reg_80	80	120	80(0)	120	0	2.4	0	45	0.002	5.4	0.167	0.113
4-reg_80	80	160	80(0)	160	0	0.2	0	7.4	0.006	6	0.503	0.486
WS_80_4	80	160	80(3.6)	160	0	24.8	0	27.4	0.001	5	0.162	0.162
WS_80_10	80	400	80(0)	400	0	2.4	0	11.8	0.011	4.2	0.673	0.662
WS_100_4	100	200	100(5.4)	200	0	32.4	0	22.6	0.003	5.8	0.222	0.010
WS_100_10	100	500	100(0)	500	0	3.2	0	21	0.013	4.6	1.366	0.019
ER_80_0_1	80	313.4	80(0.6)	313.4	0	2.6	0	4.8	0.009	4.8	0.707	0.713
ER_80_0_2	80	640	80(0)	640	0	0.2	0	0.2	0.018	4.6	0.480	0.016
ER_80_0_3	80	947.6	80(0)	947.6	0	0	0	0.4	0.031	3.6	3.401	0.018
ER_80_0_4	80	1265.2	80(0)	1265.2	0	0	0.001	3.8	0.037	2.4	2.240	0.016
ER_100_0_1	100	495	100(0.6)	495	0	1	0	2	0.018	6.2	1.452	0.021
ER_100_0_2	100	1001.4	100(0)	1001.4	0	0	0	0	0.041	4.2	14.243	0.023
ER_100_0_3	100	1491.4	100(0)	1491.4	0	0	0.001	0	0.066	4	4.08	0.026
ER_100_0_4	100	1986	100(0)	1986	0	0	0.001	0	0.096	3	6.547	0.023

We now turn our attention to the results obtained for PDS and WPDS by considering or not the pre-processing and/or initialization, but also the use of either an ILP or a combinatorial algorithm ('comb') for separation. Recall that for the special case of unit weights variables corresponding to nodes with degree at most two can be fixed to zero (for both problems). This property is used in all the methods. The results obtained using ILP ('comb') for separation are reported in Table 3 (Table 4) for PDS and Table 5 (Table 6) for WPDS. We indicate the optimal objective value ('opt'), and for each method: the number of violated constraints generated (*not* including the ones stemming from pre-processing or initialization when these procedures are used), the number of nodes explored reported by the solver ('# nd'), the total time in seconds including pre-processing and initialization when they are used ('time') and the gap (in percentage) reported by the solver ('gap'). The notation '-' in any entry of a table means that the corresponding information is not available (this may concern the objective value if some instance could not be solved to optimality by any method, or the gap if the solver could not find a feasible solution). For each instance and in each table we indicate in bold the best time associated with a zero gap (when applicable).

One first observation we may point out is the difference of the objective values obtained with the instances taken from the literature for PDS and WPDS, illustrating a potentially significant impact of the rule (R3): for example we have a factor larger than three for **pegase_13659**. Another interesting point is the fact that some instances with a small order (such as **GRID3_5_5_5** and **ER_100_0_2**) are challenging in comparison with other much larger graphs taken from the literature. Overall the 'best' solution

strategies (i.e. the ones providing the best solution times most often) correspond to the two options including the initialization procedure (i.e., 'Preprocess+Init' and 'Init').

Looking at instances that can be solved to optimality both using ILP and 'comb' (for PDS or WPDS) we observe that for any fixed strategy the number of constraints generated tends to be higher with 'comb'. This suggests that inequalities generated with ILP are stronger than 'comb'. However the computation times per iteration with the former are higher than with the latter in particular for the largest instances. This feature permits the combinatorial approach to solve a higher number of instances than with IPL, in particular among the randomly generated instances. For example, considering the strategy using both pre-processing and initialization ('Preprocess+Init+(PF)') together with combinatorial separation, there are four groups of randomly generated graphs for which no instance (recall that there are five instances per group) could be solved to optimality within our time limit. This number increases to ten using ILP separation. More strikingly, considering the group of instances ER_100_0.4 and the method including pre-processing and initialization (for both PDS and WPDS): the average solving time lies under 30s with combinatorial separation whereas no instance could be solved within our time limit using ILP separation.

6 Conclusion and perspectives

In this paper we investigated an extension of power domination that we named *weak power domination* and that is derived from fundamental laws in electric circuit analysis. Theoretical and computational results have been presented, illustrating similarities but also major contrasts between the two domination concepts. Further research work may be directed, e.g. towards the development of efficient combinatorial algorithms for special graph families, and the study of alternative constraint generation schemes.

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Table 3: Results for PDS with ILP separation

Instance	opt	Preprocess+Intr+(PF)			Preprocess+(PF)			Intr+(PF)			(PF)						
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap				
IEEE14.bus	2	*	*	0	0	*	*	0	0	1	1	0.012	0	6	1	0.017	0
IEEE30.bus	3	1	1	0.015	0	7	1	0.023	0	4	1	0.010	0	8	1	0.015	0
IEEE57.bus	3	5	1	0.019	0	10	1	0.018	0	1	1	0.013	0	11	1	0.033	0
IEEE118.bus	8	4	1	0.028	0	14	1	0.097	0	7	1	0.040	0	18	1	0.250	0
IEEE300.bus	30	13	1	0.190	0	43	1	1.573	0	14	1	0.173	0	47	1	1.195	0
power-685-bus	37	125	4033	25.829	0	226	642	95.127	0	159	557	45.953	0	258	550	114.181	0
power-eris	99	8	1	1.767	0	109	1	32.947	0	11	2	2.846	0	116	3	58.021	0
pegase.1354	176	12	1	1.589	0	189	1	16.984	0	18	1	0.886	0	206	1	15.725	0
bcsp909	167	30	1	3.608	0	197	1	46.555	0	49	1	2.278	0	204	1	38.691	0
pegase.2869	305	48	10	12.552	0	362	1	61.480	0	101	73	9.406	0	380	1	55.170	0
polish.2383	203	89	62	8.320	0	309	16	53.236	0	114	146	6.088	0	307	12	65.881	0
us-west.4941	494	134	152	60.886	0	636	80	326.335	0	182	297	24.994	0	686	254	258.667	0
pegase.9241	811	309	528	279.814	0	1176	535	1327.996	0	420	539	119.670	0	1313	614	1446.332	0
pegase.13659	1677	80	12	664.060	0	507	1	1800.000	-	106	15	92.251	0	1827	17	1462.864	0
S1000	334	668	1	8.410	0	226	1	1800.000	-	670	1	5.015	0	101	1	1800.000	-
GR12.10.10	3	19	1	25.350	0	27	7	67.503	0	23	1	29.366	0	27	10	58.228	0
GR12.20.20	-	0	0	1800.000	-	0	0	1800.000	-	0	0	1800.000	-	0	0	1800.000	-
GR12.30.30	-	0	0	1800.000	-	0	0	1800.000	-	0	0	1800.000	-	0	0	1800.000	-
GR13.4.4.4	4	117	717	14.249	0	133	758	19.480	0	128	820	14.918	0	133	511	21.105	0
GR13.5.5.5	5	275	2624	1800.000	-	255	1474	1800.000	-	329	2111	1800.000	40	280	1403	1800.000	-
3-reg-50	3.6	85.8	289.4	3.035	0	84.6	280.4	4.447	0	88.4	289.6	3.160	0	85.2	368.8	4.729	0
4-reg-50	4	245.6	889	25.583	0	255.6	940.8	31.722	0	251.4	1125.8	26.619	0	253	1059.2	31.671	0
3-reg-80	5	471.6	12734.2	116.686	0	482	12703	128.632	0	475.8	13015.8	118.004	0	477	13008.4	136.629	0
4-reg-80	4.8	1396.2	28905.2	1074.982	0	1449.6	28363.4	1251.303	0	1542.2	38120	1150.085	0	1385.2	33332.8	1343.473	0
NS-80.4	4	74	199.8	21.345	0	101.8	225.6	29.946	0	86	332	29.704	0	130.6	468.6	38.563	0
NS-80.10	3	465	1501	1065.144	0	403.2	1467	1510.271	0	465.6	1452.8	1142.054	0	352.6	1068	1591.355	0
NS-100.4	4.6	188.8	1242.8	99.583	0	212	1332.8	125.158	0	262.6	2162.6	133.392	0	242.6	2228.4	129.188	0
NS-100.10	4	30.4	2.4	1800.000	-	5.6	0.6	1800.000	-	34.2	29.2	1800.000	-	5.6	0.6	1800.000	-
ER-80.0.1	3.8	576.4	2041	1731.08	-	190.8	527	1800.000	-	577.4	2296.4	1728.389	-	150.8	379.8	1800.000	-
ER-80.0.2	3	10	1	1800.000	-	1.2	0	1800.000	-	10.4	1	1800.000	-	1.2	0	1800.000	-
ER-80.0.3	3	70	66.2	1800.000	-	1.8	0	1800.000	-	73	72	1800.000	-	1.8	0	1800.000	-
ER-80.0.4	2.8	509.6	150.4	1350.993	-	5.2	0.4	1440.002	-	537.4	197	1522.16	-	3.8	0	1800.000	-
ER-100.0.1	-	1	0	1800.000	-	0	0	1800.000	-	1.2	0	1800.000	-	0	0	1800.000	-
ER-100.0.2	-	0.2	0	1800.000	-	0	0	1800.000	-	0.2	0	1800.000	-	0	0	1800.000	-
ER-100.0.3	3	1.6	0.2	1800.000	-	0	0	1800.000	-	1.6	0.2	1800.000	-	0	0	1800.000	-
ER-100.0.4	3	5.2	0.8	1800.000	-	0	0	1800.000	-	5.2	0.8	1800.000	-	0	0	1800.000	-

Table 4: Results for PDS with combinatorial separation

Instance	opt	Preprocess+Intra(PF)			Preprocess+(PF)			Intra+(PF)			(PF)						
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap				
IEEE14.bus	2	*	*	0	0	*	*	0	0	1	1	0.006	0	7	1	0.003	0
IEEE30.bus	3	0	1	0.012	0	6	1	0.007	0	4	1	0.004	0	8	1	0.003	0
IEEE57.bus	3	3	1	0.005	0	12	1	0.006	0	12	1	0.006	0	12	1	0.006	0
IEEE118.bus	8	4	1	0.008	0	17	1	0.011	0	6	1	0.008	0	19	1	0.013	0
IEEE300.bus	30	17	4	0.060	0	47	1	0.141	0	21	1	0.065	0	55	1	0.135	0
power-695-bus	37	270	551	5.876	0	359	555	7.462	0	331	582	6.928	0	394	661	8.550	0
power-eris	99	12	1	0.427	0	112	1	5.187	0	8	1	0.306	0	110	1	6.479	0
pegase1364	176	23	1	1.048	0	220	1	7.922	0	33	1	0.369	0	230	1	10.395	0
bcspr09	167	90	43	4.420	0	395	253	33.843	0	134	84	5.506	0	432	239	48.838	0
pegase2869	305	84	48	12.086	0	420	13	92.189	0	157	184	12.100	0	451	43	127.046	0
polish2383	203	159	456	11.124	0	386	716	37.541	0	208	679	21.221	0	418	180	88.355	0
us-west4941	494	251	446	75.512	0	813	513	393.583	0	304	522	67.344	0	946	522	530.045	0
pegase9241	811	513	573	436.706	0	546	7	1800.000	-	570	1026	600.464	0	665	17	1800.000	-
pegase13659	1677	173	28	654.063	0	680	1	1800.000	-	147	32	101.808	0	875	1	1800.000	-
S1000	334	668	1	4.654	0	1004	1	33.873	0	670	1	0.993	0	1005	1	30.886	0
GR102.10.10	3	34	1	0.047	0	31	1	0.044	0	31	4	0.047	0	35	1	0.049	0
GR102.20.20	-	3392	102827	1800.000	17	3974	79129	1800.000	17	3850	82271	1800	17	4015	84320	1800.000	17
GR102.30.30	-	4146	32847	1800.000	44	3610	43613	1800.000	33	3060	36370	1800	40	3275	41642	1800.000	25
GR103.4.4.4	4	250	993	0.606	0	256	991	0.509	0	382	1898	1.476	0	346	827	0.666	0
GR103.5.5.5	5	2810	50754	404.826	0	3515	59843	745.435	0	2965	37877	383.786	0	2450	35032	205.581	0
3-reg-50	3.6	138.6	344.6	0.252	0	146.6	341.2	0.216	0	130.4	334.6	0.233	0	139.8	376.2	0.202	0
4-reg-50	4	531.8	1245.2	1.425	0	551.6	1492.6	2.059	0	1555.8	35283.2	1.574	0	551.4	1208.6	1.987	0
3-reg-80	5	1535.8	26277.6	66.952	0	1541.2	33753.4	106.763	0	5466	75291.6	1134.762	0	1596.4	33597	107.083	0
4-reg-80	4.8	5384	65858.8	977.400	6.6	6370.6	77678.2	1322.671	13.2	241.8	789.2	1.964	0	223.2	603.8	1.086	0
NS-80.4	4	223.2	423.4	0.569	0	239.6	468.4	0.646	0	97.4	1503.4	5.730	0	1498.8	2213.6	10.455	0
NS-80.10	3	1857.4	3979.4	18.832	0	1464	2708.4	14.833	0	773	7643.8	33.566	0	812.4	6846.4	23.606	0
NS-100.4	4.6	720.4	5159.2	19.368	0	624.6	3711.8	11.201	0	3693.4	12513.4	101.503	0	3748.8	13594.8	122.663	0
NS-100.10	4	3739.8	12381.6	100.803	0	3793.2	10754.2	97.133	0	3738.6	7729	67.185	0	3956.6	7718	95.698	0
ER-80.0.1	3.8	3545.4	8728.8	67.317	0	3174.8	6620	57.468	0	5086.4	3510.4	117.868	0	5116.8	3770.6	120.230	0
ER-80.0.2	3	4817.6	3476	87.201	0	5903.4	4538.4	175.639	0	1384.4	629	7.051	0	1489.6	642.6	7.673	0
ER-80.0.3	3	1383.6	653.6	6.994	0	1599.2	744.2	8.993	0	1292.8	181.4	6.894	0	1351.6	294.4	7.142	0
ER-80.0.4	2.8	1228.6	149.6	6.813	0	1232.6	148	6.805	0	14113.8	31247.6	1800	40	14413.8	30159	1800	44
ER-100.0.1	-	13727.4	30877.6	1800.000	42	14009.4	26692.8	1800	25	19207	11692.8	1800	25	19201	12785.6	1800	25
ER-100.0.2	-	19221.8	11604	1800.000	25	19399	12822	1800	25	4884.4	1870.6	165.395	0	4327.6	1646.6	115.051	0
ER-100.0.3	3	4884.4	1870.6	165.395	0	4327.6	1646.6	114.764	0	2473.6	458.6	26.377	0	2461.4	536.2	24.565	0
ER-100.0.4	3	2473.6	458.6	26.377	0	2461.4	536.2	26.634	0	2473.6	458.6	26.427	0	2461.4	536.2	24.565	0

Table 5: Results for WPDS with ILP separation

Instance	opt	Preprocess+Init+(WF)				Preprocess+(WF)				Init+(WF)				(WF)			
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap
IEEE14.bus	2	*	*	0	0	*	*	0	0	0	0.015	0	0	6	1	0.017	0
IEEE30.bus	3	0	0	0.015	0	7	1	0.021	0	1	0.008	0	0	8	1	0.014	0
IEEE57.bus	3	1	1	0.020	0	9	1	0.039	0	6	0	0	10	1	0.031	0	
IEEE118.bus	7	10	1	0.051	0	14	1	0.215	0	14	1	0	20	1	0.234	0	
IEEE300.bus	15	7	1	0.168	0	29	1	2.485	0	17	1	0	33	1	3.743	0	
power-685-bus	25	57	105	37.094	0	119	134	96.541	0	142	387	0	259	564	201.682	0	
power-eris	80	9	1	2.427	0	91	1	51.447	0	12	1	0	96	3	51.433	0	
pegase.1354	61	9	1	1.018	0	87	1	4.275	0	44	8	2.407	102	1	10.504	0	
bcspart9	62	108	380	18.562	0	131	50	84.965	0	272	520	217.337	237	168	393.886	0	
pegase.2869	124	57	59	9.911	0	189	1	78.160	0	94	143	21.795	265	1289	177.259	0	
polish.2383	70	117	527	13.500	0	210	522	71.056	0	181	527	35.171	189	167	112.338	0	
us-west.4941	223	132	2513	63.017	0	388	237	414.864	0	326	517	181.852	479	516	448.622	0	
pegase.9241	381	343	1483	414.692	0	751	1165	1675.785	0	639	37004	879.184	445	1	1800.000	-	
pegase.13659	469	229	517	368.201	-0	760	584	1607.981	0	549	565	804.145	557	29	1800.000	-	
S1000	1	*	*	0.031	0	37	27	160.765	0	0	1	0.148	6	1	0.242	0	
GRID2.10.10	3	37	1	70.306	0	2	0	1800.000	-	29	1	81.102	34	11	124.092	0	
GRID2.20.20	5	0	0	1800.000	-	2	0	1800.000	-	0	0	1800.000	0	0	1800.000	-	
GRID2.30.30	-	0	0	1800.000	-	3	0	1800.000	-	0	0	1800.000	0	0	1800.000	-	
GRID3.4.4.4	4	222	2155	34.196	0	225	1919	40.136	0	230	1185	35.362	229	1303	42.476	0	
GRID3.5.5.5	-	128	238	1800.000	-	56	27	1800.000	-	123	543	1800.000	44	125	1800.000	-	
3-reg-50	3.4	171	585.2	3.860	0	175.2	434.4	5.946	0	168	544.8	4.015	173.6	550.8	6.586	0	
4-reg-50	3.8	388.2	1602.2	38.103	0	393	1421.8	45.175	0	367.2	1131	36.15	384.8	1500.8	45.793	0	
3-reg-80	4.8	1541.8	31495.2	366.799	0	1516	43674.2	467.031	0	1554.8	32792.4	376.489	1541.8	39450.2	453.776	0	
4-reg-80	-	1230.6	15863.6	1620.263	24	1214.8	11232.2	1660.287	28	1194.8	16931.6	1609.734	1200.4	7984.4	1800.000	32	
NS-80.4	3.4	126.8	363.6	33.898	0	89.4	285.2	41.466	0	143.6	710.2	47.950	136.6	773.8	57.732	0	
NS-80.10	3	334.8	875.8	1120.808	0	304	1020	1476.825	-	336.2	1709	1296.900	6.6	318.6	972.4	1664.114	-
NS-100.4	4	252.4	1241.4	180.476	0	217.8	1076	215.047	0	225.4	1082.6	170.073	0	228.8	2038.6	227.400	0
NS-100.10	4	18.8	5.8	1800.000	-	4.4	0.6	1800.000	-	1.4	1	1800.000	0	4.4	1800	-	-
ER-80.0.1	3.8	266.2	970.2	1800.000	-	13.2	0.8	1800.000	-	302.4	607	1800.000	0	7.2	0.4	1800	-
ER-80.0.2	3	10.2	1.0	1800.000	-	0.8	0	1800.000	-	10.2	1	1800.000	0	0.8	0	1800	-
ER-80.0.3	3	28.2	17.6	1800.000	-	1.4	0	1800.000	-	28.4	17.6	1800	0	1.4	0	1800	-
ER-80.0.4	3	447.4	165.6	1420.861	-	2.4	0	1440.002	-	548.2	156.2	1721.614	0	3	0	1800	-
ER-100.0.1	-	3	0	1800.000	-	0	0	1800.000	-	0.4	0	1800.000	0	0	0	1800	-
ER-100.0.2	-	2.6	0	1800.000	-	0	0	1800.000	-	0	0	1800.000	0	0	0	1800	-
ER-100.0.3	3	3.4	0	1800.000	-	0	0	1800.000	-	1.4	0	1800.000	0	0	0	1800	-
ER-100.0.4	3	3.8	0.4	1800.000	-	0	0	1800.000	-	3.8	0.4	1800.000	0	0	0	1800	-

Table 6: Results for WPDS with combinatorial separation

Instance	opt	Preprocess+Init+(WF)				Preprocess+(WF)				Init+(WF)				(WF)			
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap
IEEE14.bus	2	*	*	0	0	*	*	0	0	0	0	0	0	7	1	0.009	0
IEEE30.bus	3	0	0	0.006	0	7	1	0.008	0	0	0	0.006	0	8	1	0.003	0
IEEE57.bus	3	1	1	0.006	0	9	1	0.004	0	6	1	0.006	0	10	1	0.006	0
IEEE118.bus	7	14	1	0.025	0	16	1	0.011	0	12	1	0.019	0	22	1	0.016	0
IEEE300.bus	15	15	1	0.063	0	34	12	0.135	0	18	1	0.074	0	64	10	0.231	0
power-695-bus	25	351	547	8.500	0	296	517	6.974	0	326	548	9.390	0	494	795	11.123	0
power-eris	80	8	1	0.517	0	93	1	5.950	0	8	1	0.305	0	88	1	5.286	0
pegase.1364	61	24	5	0.936	0	109	1	5.487	0	62	3	2.654	0	138	90	9.219	0
bcsp99	62	245	1457	17.650	0	367	521	38.132	0	614	560	74.088	0	723	660	106.572	0
pegase.2869	124	71	181	12.926	0	223	106	58.823	0	187	4912	34.634	0	361	17244	111.990	0
polish.2383	70	181	1221	21.211	0	283	528	51.521	0	287	535	69.457	0	499	575	142.304	0
us-west.4941	223	374	747	177.091	0	607	514	594.015	0	571	655	429.357	0	837	562	978.514	0
pegase.9241	381	609	346191	1090.053	0	497	64	1800	-	1285	199464	1800.000	0.5	458	69	1800.000	-
pegase.13659	469	451	561	629.568	0	496	73	1800.000	-	429	264	1800.000	-	475	37	1800.000	-
S1000	1	*	*	0.031	0	*	*	0.031	0	*	*	8.066	0	6	1	47.784	0
GR102.10.10	3	57	69	0.114	0	66	56	0.270	0	58	40	0.087	0	78	74	0.121	0
GR102.20.20	5	2252	111242	1694.143	0	2710	83853	897.032	0	2359	106777	1800.000	20	2299	71322	816.472	0
GR102.30.30	-	3883	34083	1800.000	50	3710	32149	1800.000	50	2931	29453	1800.000	37.5	3475	25618	1800.000	37.5
GR103.4.4.4	4	553	2301	4.584	0	523	2283	4.223	0	561	3167	3.104	0	549	1261	1.322	0
GR103.5.5.5	-	6835	102718	1800.000	20	8307	83002	1800.000	20	5933	112383	1800.000	20	7784	93992	1800.000	20
3-reg.50	3.4	256.2	588.2	3.312	0	268.4	445.8	0.450	0	250.4	445	0.448	0	257.6	556.4	0.540	0
4-reg.50	3.8	711.8	1471	3.299	0	727.8	1460	3.031	0	795	1289	2.320	0	722.2	1493.4	2.601	0
3-reg.80	4.8	4209.6	53235.6	557.381	0	4230.2	60526.2	735.19	4	4191.8	72278	766.074	0	4293.4	71831.2	696.603	0
4-reg.80	-	8439.2	74613.4	1459.342	16	8461.8	73351.2	1443.637	16	8607.4	78842.8	1446.978	16	9126.8	78203.2	1459.054	16
NS.80.4	3.4	340.2	779	0.900	0	326.2	802.4	1.138	0	392	1478.4	1.755	0	356.2	1336.2	2.639	0
NS.80.10	3	1250.2	2126.8	10.801	0	1814.4	2970.8	14.810	0	823.6	1356.8	4.831	0	1598.4	2446	16.021	0
NS.100.4	4	420.8	1225	3.256	0	549.8	1747	5.540	0	575.6	2555.4	6.110	0	578.4	2211.4	9.239	0
NS.100.10	4	3914.8	14480.2	170.715	0	4043	12154.4	140.360	0	4107.4	12600	124.624	0	4054.4	15073.4	159.370	0
ER.80.0.1	3.8	4652.4	9712.2	117.061	0	4602.2	9266.4	104.616	0	4315.8	7438.4	83.238	0	4390.8	8620	91.189	0
ER.80.0.2	3	5036.2	4237.2	185.526	0	5419	3718	130.617	0	5344	4558.2	186.144	0	4059.6	3349.4	97.262	0
ER.80.0.3	3	1355.8	519	7.025	0	1479	597	7.065	0	1338	456.4	6.394	0	1465.6	886.4	6.743	0
ER.80.0.4	3	1253.8	180	7.234	0	1243.4	153.6	6.904	0	1400	254.2	7.424	0	1306.2	191.8	7.172	0
ER.100.0.1	-	14680	28730.2	1745.570	32	14360.4	23678	1580.846	32	15735.6	30280.4	1646.009	25	15497	24907.6	1800.000	40
ER.100.0.2	-	19064.2	12242	1800.000	25	19251.4	12856.6	1800	25	18998.6	12190.2	1800	25	19765.8	12850	1800	25
ER.100.0.3	3	6500.3	2619	298.360	0	4232	1951.2	95.390	0	6500	2619	298.220	0	4232	1951.2	95.991	0
ER.100.0.4	3	2458.2	493.8	24.712	0	2567.2	519.2	27.910	0	2458.2	493.8	123.116	0	2567.2	519.2	27.756	0

Appendix A Omitted proofs from Section 3

A.1 Proof of Proposition 3.1

Proof. (i) Let $S \subseteq V \setminus N[F]$. Since $\Omega_{\text{WP}}(S) \subseteq \Omega_{\text{WP}}(V \setminus N[F])$, it is sufficient to prove that $\Omega_{\text{WP}}(V \setminus N[F]) \cap F = \emptyset$. Taking $V \setminus N[F]$ for the set of initially observed nodes, in the first round, among the nodes of $N[F]$ only nodes contained in $N(F)$ can be observed (by application of the rule (R1)). This also holds with respect to the subsequent rounds (during which only the two rules (R2) and (R3) are applied) given that the definition of strong fort does not allow the observation of any node in F with (R2) nor with (R3), even for the case that all the nodes in $N(F)$ are observed. Thus, we have $F \cap \Omega_{\text{FP}}(V \setminus N[F]) = \emptyset$.

(ii) Let $W = V \setminus \Omega_{\text{WP}}(S)$. Since no node of W can be observed with the rule (R2) we can deduce that any node in $N(W)$ has at least two neighbors in W and thus W is a fort. Also, since no node of W can be observed with the rule (R3), it follows that no node z of W can have all its neighbors in $N(W)$. This, together with the fact that G is assumed to be connected implies that each node of W has at least one neighbor in W .

So, we deduce that W is a strong fort.

(iii) At each round, the observation of any non observed node v only depends on the set of observed nodes at distance at most two from v . Given that the nodes in $N[N(S)]$ are observed in the first round (with the rule (R1)) adding further nodes of the set $V \setminus S$ to $N(S)$ for the initially observed nodes has no implication on the observation of the nodes in S , hence the result.

(iv) Assume for a contradiction that $G[\widehat{F}]$ is not connected, and let C_1, C_2, \dots, C_p denote the node subsets of its connected components ($p \geq 2$). Let F denote a strong fort such that $\widehat{F} = N[F]$ and let $j \in \{1, 2, \dots, p\}$. Using the result of the previous item, we have $C_j \setminus \Omega_{\text{WP}}(V \setminus N[C_j]) = C_j \setminus \Omega_{\text{WP}}(V \setminus \widehat{F})$. Since $C_j \cap F \neq \emptyset$ and $F \cap \Omega_{\text{FP}}(V \setminus \widehat{F}) = \emptyset$ (by the result of the first item) we have $C_j \setminus \Omega_{\text{WP}}(V \setminus N[C_j]) \neq \emptyset$. Using the result of the second item we deduce that $V \setminus \Omega_{\text{WP}}(V \setminus N[C_j])$ is a strong fort. Thus, $N[C_j]$ contains a strong fort neighborhood, which contradicts the minimality of \widehat{F} . □

A.2 Proof of Proposition 3.3

Proof. Let F^* denote an optimal solution of WPDS on G and assume that it contains at least one node u that is either a leaf in G or has degree at most two in \check{G} . Then replacing u by a closest node w in G that belongs to \check{V} and has degree at least three in \check{G} , the set of the observed nodes cannot get smaller. To prove this it is sufficient to show that replacing node u by w in F^* , the node u and all its neighbors get observed. This follows from the fact that

- all the nodes in the path p_{uw} joining u and w get observed by application of the rule (R2)
- all the leaves adjacent to nodes in p_{uw} get observed by application of the rule (R3)

- if node u has a neighbor z which does not belong to p_{uw} and is not a leaf (i.e. u has degree two in \hat{G}), then z can be observed with the rule (R2). □

A.3 Proof of Theorem 3.4

Proof. Let S denote a pds in \hat{G} and assume that S is not an fpds in G . Let W denote the nodes of \hat{G} that are not observed in G (using the rules (R1) – (R3), starting with S). Then, necessarily $N[W] \cap S = \emptyset$ (since otherwise at least one node of W could be observed with the rule (R1)) and $N[W]$ is a fort neighborhood in \hat{G} (since otherwise the rule (R2) could be applied in G to observe a node of W), a contradiction with S being a pds in \hat{G} . □

A.4 Proof of Proposition 3.6

Before we give the proof of Proposition 3.6, we state and prove a useful auxiliary result.

Proposition A.7.1 *Let G denote an undirected graph with node set V such that $|V| \geq 2$ and G contains a leaf w . Then,*

$$\gamma_{\mathbb{P}}(G[V \setminus \{w\}]) \leq \gamma_{\mathbb{P}}(G) \quad \text{and} \quad \gamma_{\text{WP}}(G[V \setminus \{w\}]) \leq \gamma_{\text{WP}}(G).$$

Proof. We do the proof for the second inequality. The proof of the other inequality can be done similarly and is thus omitted.

First notice that there exists an optimal solution of WPDS not containing w . (If w belongs to an optimal solution, replacing it by its neighbor node gives another optimal solution.) Then, let S denote a wpds in G not containing w and let u denote the neighbor of w . Note that w can be observed by application of the rule (R1) (if $u \in S$) or (R3) (after node u gets observed), and that w is not involved for the observation of its neighbor u . Also, if the node u is involved in the application of (R2) to observe a node in $N(u) \setminus \{w\}$, then this remains possible in $G[V \setminus \{w\}]$. So removing w from G , the set of observed nodes cannot get smaller and we deduce that S is a wpds in $G[V \setminus \{w\}]$. Thus, $\gamma_{\text{WP}}(G[V \setminus \{w\}]) \leq \gamma_{\text{WP}}(G)$. □

Proof of Proposition 3.6.

Proof. [\Leftarrow] Let T denote a leavy starlike tree with node set V and set of leaves L such that $|V \setminus L| \geq 3$ (since otherwise the result trivially holds). Then, $\gamma_{\mathbb{P}}(T[V \setminus L]) = 1$ because $T[V \setminus L]$ is a starlike tree (see Haynes et al. (2002)). The result holds by Theorem 3.4.

[\Rightarrow] Assume for a contradiction that there exists a tree T that is not a leavy starlike tree and satisfies $\gamma_{\text{WP}}(T) = 1$. Let \hat{T} denote such a tree having a minimum number of nodes. Note that \hat{T} necessarily contains two nodes with degree at least 3 (since otherwise it would be a starlike tree, and thus also a leavy starlike tree), and using Proposition A.8.1 we have the property that removing any leaf from \hat{T} we get a leavy starlike tree (since otherwise \hat{T} would not be a smallest counterexample). Let w be a leaf in \hat{T} and let Q denote the tree obtained from \hat{T} by removing w and its incident edge. So Q is a leavy starlike tree, and now let $(V_{Q,1}, V_{Q,2})$ denote the bipartition of

the nodes of Q such that $Q[V_{Q,1}]$ is a starlike tree and $V_{Q,2}$ consists of all the leaves in Q satisfying: for each $z \in V_{Q,2}$ its neighbor node has degree at least three in Q . Since \hat{T} is not a leavy starlike tree, w cannot be adjacent to a node in $V_{Q,1}$. Let u denote the neighbor of w in $V_{Q,2}$ and z denote the neighbor of u in $V_{Q,1}$. Let W denote the nodes in $V_{Q,1}$ that are not leaves in Q . If z is a node of maximum degree in $Q[W]$ or of degree one in $Q[W]$, then \hat{T} is a leavy starlike tree and we get a contradiction. So z must have degree two in $Q[W]$ and there must exist another node x in W with degree at least three in $Q[W]$. Let y denote the neighbor of z on the path joining z and x in Q . (Possibly, $y = x$.) In particular, note that, in addition to y , the node z has at least two other neighbors that are not leaves in \hat{T} (at least one neighbor in W different from y , and u). Then, let C_1 and C_2 denote the connected components of \hat{T} obtained after removing the edge $\{z, y\}$. (See Figure A1 for an illustration.)

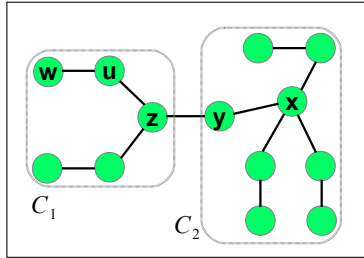


Fig. A1: Illustration for the proof of Proposition 3.6

Since each of these components contains at least one strong fort neighborhood, we must have $\gamma_{\text{WP}}(\hat{T}) \geq 2$, a contradiction. (Consider the component containing the node z . If it does not contain any node of the wpds, z can only be observed either with the rule (R1) or (R2) using its neighbor y . But then, the neighbors of z that are not leaves and different from y -there are at least two such neighbors- cannot be observed. A similar reasoning can be applied for the component containing x .) \square

A.5 Proof of Proposition 3.8

Proof. [\Leftarrow] Trivially, if T is a star, then $\gamma_{\text{WP}}(T) = \gamma(T) = 1$.

[\Rightarrow] Let T denote a tree satisfying $\gamma_{\text{WP}}(T) = \gamma(T)$, and assume for a contradiction that in addition we have $(\gamma(T) =) \gamma_{\text{WP}}(T) \geq 2$. Note that since $\gamma_{\text{WP}}(T) \leq \gamma_{\text{P}}(T) \leq \gamma(T)$, we must have $\gamma_{\text{WP}}(T) = \gamma_{\text{P}}(T) = \gamma(T)$, and the Theorem 9 from Haynes et al. (2002) (recalled above) applies. The latter namely implies T does not contain a leaf adjacent to a node with degree two.

Let S denote the unique (by (Haynes et al., 2002, Theorem 9)) minimum cardinality dominating set in T .

Let \hat{T} denote the tree obtained from T by removing its leaves, and let v_0 denote a leaf of \hat{T} . Note that, from the properties mentioned above for T , v_0 must be adjacent to at least two leaves in T .

Since S is a dominating set it must intersect the closed neighborhood of each leaf in T . In addition, from (Haynes et al., 2002, Theorem 9), S does not contain any leaf of T . Thus we must have $v_0 \in S$.

The unique neighbor v_1 of v_0 in \mathring{T} cannot belong to S , since otherwise $S \setminus \{v_0\}$ is a wpds. Also, v_1 cannot be adjacent to any leaf, since otherwise S could not be a dominating set (recall that S does not contain any leaf and it must intersect the closed neighborhood of each leaf).

From what precedes we can deduce that $S \setminus \{v_0\}$ is a dominating set of $H = G[V(T) \setminus N[v_0]]$. This implies that if we take $S \setminus \{v_0\}$ for the set of initially observed nodes in T , then, by applying the rule (R1) all the nodes in $V(T) \setminus N[v_0]$ are observed. The application of the rule (R2) (using a neighbor of v_1 different from v_0) then leads to the observation of v_1 and of v_0 . Then the rule (R3) permits the observation of the leaves adjacent to v_0 . So, in the end, we just proved that $S \setminus \{v_0\}$ is in fact a wpds in T , contradicting the assumption that S is a minimum cardinality wpds in T .

So we must have the property that the tree T verifies $\gamma(T) = 1$, i.e. T is a star. \square

A.6 Proof of Theorem 3.9

Theorem 3.9 directly follows from the next two lemmas.

Lemma A.9.1 *For any tree T , $sc(T) \leq \gamma_{\text{WP}}(T)$.*

Proof. We do the proof by induction on $q = \gamma_{\text{WP}}(T)$. Consider firstly the case when $\gamma_{\text{WP}}(T) = 1$. By Proposition 3.6, T must be a leavy starlike tree, and we can deduce $sc(T) = 1$ (taking for Q all the leaves of T in formula (2)).

Assume now that $\gamma_{\text{WP}}(T) = q + 1 \geq 2$ and that the proposition holds for any tree T' such that $\gamma_{\text{WP}}(T') \leq q$. Let $S = \{v_1, v_2, \dots, v_{q+1}\}$ denote a wpds in T such that all the nodes in S have degree at least three in \mathring{T} . (Such a set exists by Proposition 3.3).

With no loss of generality assume that $T = (V(T), E(T))$ is rooted at the node v_{q+1} and that v_1 is the node at maximum distance from v_{q+1} in T . Necessarily v_1 has at least two children and each descendant of v_1 has degree at most two in \mathring{T} .

Let u_1 be the ancestor of v_1 that is at minimum distance from v_1 and with degree at least three in \mathring{T} ; so either v_1 is a child of u_1 or all the nodes in the path joining u_1 and v_1 have degree two in \mathring{T} . We then consider two cases.

Case 1. $u_1 \in S$.

Let w_1 be the child of u_1 on the unique path joining u_1 and v_1 in T . Let T_1 be the subtree of T rooted at w_1 and containing all the descendants of w_1 in T . So, T_1 is a leavy starlike tree for which v_1 is the single node with degree larger than two in \mathring{T}_1 . This namely implies that $\{v_1\}$ is a wpds of T_1 . Now let T_2 be the subtree of T that is induced by the node subset $V(T) \setminus V(T_1)$, i.e. $T_2 = T[V(T) \setminus V(T_1)]$. Note that $S \setminus \{v_1\}$ is a wpds of T_2 , thus $\gamma_{\text{WP}}(T_2) \leq q$. If we had $\gamma_{\text{WP}}(T_2) < q$, then this would lead to $\gamma_{\text{WP}}(T) < q + 1$ (since adding the node $\{v_1\}$ to a minimum cardinality wpds of T_2 results in a wpds of T), a contradiction. So we deduce that $\gamma_{\text{WP}}(T_2) = q$. Using the inductive assumption, there exists a stable set B of T_2 and a starlike tree covering $\{X_1, X_2, \dots, X_q\}$ of $V(T_2) \setminus B$. Let L_1 denote the set of all the leaves of T_1 . Then, $B \cup L_1$ is a stable set in T , and $\left\{X_1, X_2, \dots, X_q, V(\mathring{T}_1)\right\}$ is a starlike tree covering of $V(T) \setminus (B \cup L_1)$, proving that $sc(T) \leq q + 1 = \gamma_{\text{WP}}(T)$.

Case 2. $u_1 \notin S$.

Let w_1, w_2, \dots, w_k denote the children of u_1 in \hat{T} , with w_1 being the child of u_1 on the path joining u_1 and v_1 in T . For $i = 1, 2, \dots, k$, let W_i stand for all the descendants of node w_i in T , including the node w_i . Since $u_1 \notin S$ and S is a wpds in T , all of the sets W_1, W_2, \dots, W_k except possibly one must contain a vertex in S .

Assume that $W_2 \cap S = \emptyset$, and let $T_2 = T[W_2]$. Then, necessarily \hat{T}_2 consists of a path (possibly composed of a single node), and $|W_i \cap S| \geq 1$, for $i = 3, \dots, k$. We can then deduce that the subgraph \hat{T}_1 defined as

$$\hat{T}_1 = T[W_1 \cup W_2 \cup \{u_1\} \cup L(u_1)]$$

is a leavy starlike tree, where $L(u_1)$ represents all the leaves adjacent to the node u_1 . Define then the subgraph \hat{T}_2 as

$$\hat{T}_2 = T[V(T) \setminus (\cup_{i=1}^k W_i \cup \{u_1\} \cup L(u_1))],$$

and define $\hat{T}_i = T[W_i]$ for $i = 3, \dots, k$. For $i = 1, 2, \dots, k$ the set S_i defined as $S_i = S \cap V(\hat{T}_i)$ is a wpds of \hat{T}_i , implying $\gamma_{\text{WP}}(\hat{T}_i) \leq |S_i|$. If we had $\gamma_{\text{WP}}(\hat{T}_i) < |S_i|$, then, adding to $S \setminus S_i$ a minimum cardinality wpds of \hat{T}_i , we would obtain a wpds of T with cardinality strictly inferior to $|S|$, a contradiction. Thus $\gamma_{\text{WP}}(\hat{T}_i) = |S_i| \leq q$, for all $i \in \{1, 2, \dots, k\}$. Applying the inductive assumption, we deduce $sc(\hat{T}_i) \leq \gamma_{\text{WP}}(\hat{T}_i) = |S_i|$. It follows that, for each $i \in \{1, 2, \dots, k\}$ there exists a stable set B_i of \hat{T}_i , and a starlike tree covering of $V(\hat{T}_i) \setminus B_i$ with size at most $\gamma_{\text{WP}}(\hat{T}_i)$. In particular, note that \hat{T}_1 is a leavy starlike tree and we can take for B_1 the leaves of \hat{T}_1 (note that by construction u_1 is not a leaf of \hat{T}_1). Then, the set $B = \cup_{i=1}^k B_i$ is a stable set of T . So we can deduce from what precedes that there exists a starlike tree covering of $V(T) \setminus B$ with size $\sum_{i=1}^k sc(\hat{T}_i) \leq \sum_{i=1}^k |S_i| = \gamma_{\text{WP}}(T)$, thus leading to $sc(T) \leq \gamma_{\text{WP}}(T)$.

Consider now the case when $S \cap W_i \neq \emptyset$ for all $i \in \{1, 2, \dots, k\}$. Let $T_1 = T[W_1 \cup \{u_1\} \cup L(u_1)]$, $T_i = T[W_i]$ for $i = 2, \dots, k$, $T_{k+1} = T[V(T) \setminus (\cup_{i=1}^k W_i \cup \{u_1\})]$ and $S_i = S \cap V(T_i)$ for all $i \in \{1, 2, \dots, k+1\}$. Similarly to the case considered above (i.e. when one of the sets W_2, \dots, W_k does not intersect S), we can prove that S_i is a minimum cardinality wpds of T_i . Then, using the inductive assumption (which is possible since $|S_i| = \gamma_{\text{WP}}(T_i) \leq q$) for each $i \in \{1, 2, \dots, k\}$ there exists a stable set B_i of T_i , and a starlike tree covering of $V(T_i) \setminus B_i$ with size at most $\gamma_{\text{WP}}(T_i)$. In particular, it is possible to take for B_1 the leaves of T_1 to the exception of u_1 (for the case u_1 is not adjacent to a leaf in T). Then, the set $B = \cup_{i=1}^k B_i$ is a stable set of T and we can deduce the existence of a starlike tree covering of $V(T) \setminus B$ with size $\sum_{i=1}^k sc(T_i) \leq \sum_{i=1}^k |S_i| = \gamma_{\text{WP}}(T)$, thus leading to $sc(T) \leq \gamma_{\text{WP}}(T)$. □

Lemma A.9.2 For any tree T , $\gamma_{\text{WP}}(T) \leq sc(T)$.

Proof. We do the proof by induction on $q = sc(T)$. Consider firstly the case when $sc(T) = 1$. Then, T is a leafy starlike tree and by Proposition 3.6 we have $\gamma_{\text{WP}}(T) = 1 = sc(T)$.

Assume then that all the trees T' with $sc(T') = q$ for some integer $q \geq 1$ verify $\gamma_{\text{WP}}(T') \leq sc(T')$. Let $T = (V(T), E(T))$ be a tree satisfying $sc(T) = q + 1$ and consider a starlike tree covering $\{S_1, S_2, \dots, S_{q+1}\}$ of $V(T) \setminus Q$ where Q stands for a stable set in T . Let $Z = \{z_1, z_2, \dots, z_p\}$ denote the set of the nodes in Q having degree at least two in T , i.e. $Z = Q \cap V(\hat{T})$. For each $i \in \{1, 2, \dots, q + 1\}$ let L_i denote the set of all the leaves of T that are contained in Q and adjacent to a node in S_i .

Then let us define nodes subsets $V_1, V_2, \dots, V_{q+1+p}$ as follows: $V_i = S_i$ for all $i \in \{1, 2, \dots, q + 1\}$, and $V_{q+1+i} = \{z_i\}$ for all $i \in \{1, 2, \dots, p\}$. Let H be the graph having for node set $\bigcup_{i=1}^{q+1+p} V_i$, and be such that two nodes V_i and V_j are adjacent if and only if there exists an edge in T having one endpoint in V_i and the other in V_j . Since T is a tree, the same holds for H by construction. With no loss of generality (possibly renaming some subsets) assume that V_1 is a leaf of H having V_2 for (unique) neighbor. Let T' be the subtree of T that is induced by the node subset $V(T) \setminus (V_1 \cup L_1)$. If $sc(T') < q$, then by adding V_1 to a starlike tree covering of $V(T) \setminus (V_1 \cup L_1)$, we would obtain a starlike tree covering of $V(T)$ with size strictly less than $q + 1$, a contradiction. Thus, $sc(T') \geq q$ and given that $\{V_2, V_3, \dots, V_{q+1}\}$ is a starlike tree covering of $V(T') \setminus Q$, we deduce that $sc(T') = q$. Applying the inductive assumption, we know there exists a wpds S' of T' such that $|S'| \leq q$. Let v_1 denote a node of maximum degree in $T[V_1 \cup L_1]$. Then $S' \cup \{v_1\}$ is a wpds of T and we deduce $\gamma_{\text{WP}}(T) \leq q + 1 = sc(T)$. \square

A.7 Proof of Theorem 3.10

Proof. Deciding whether some given node subset S is a wpds can be decided in polynomial time. Thus WPDS_DECISION belongs to the class NP. We now prove that WPDS_DECISION is NP-complete by reduction from the DS_DECISION problem which is obtained from WPDS_DECISION replacing "wpds" by "dominating set" in the statement of the question.

Given an instance of DS_DECISION with a graph $G = (V, E)$ and parameter k , we consider the instance of WPDS_DECISION with the graph $G' = (V' = V \cup A, E')$ and the same parameter k , where G' is the graph obtained from G as follows. For each node $v \in V$, we add two nodes: v_1, v_2 together with the edges $\{v, v_1\}$ and $\{v_1, v_2\}$. The set of all the added nodes is denoted by A , and the union of E with all the edges generated in this way is denoted by E' .

Let S_D be a dominating set in G . Consider then the application of the observation rules (R1)-(R3) in G' , taking S_D for the set of initially observed nodes. The application of the rule (R1) leads to the observation of all the nodes in $N[S_D]$ (which includes V). The application of the rule (R2) enables then the observation of the nodes in $A \setminus N[S_D]$. It follows that S_D is a wpds in G' . In particular, S_D is a pds in G' .

Now, let S_F stand for a minimum cardinality wpds in G' and suppose for nontriviality that G' has at least one node with degree three (otherwise take for S_F any node

of G). Then, by Proposition 3.3, we can assume that all the nodes in S_F have degree at least three in G' . And since all the nodes in A have degree at most two, it follows that $S_F \subseteq V' \setminus A = V$.

Note that any node $u \notin S_F$ cannot be used to observe any (not observed) neighbor $w \in V \cap N(u)$ using (R2) because the application of this rule would namely require that the neighbor v_1 of u in A is observed. But v_1 can only be observed with (R2) (not (R1) because $u \notin S_F$, and not (R3) because v_1 has degree two and its neighbor $v_2 \in A$ is not observed initially). So, all the nodes in $V \setminus S_F$ can only be observed with the rule (R1), implying that S_F is a dominating set in G . (After all the nodes in V get observed by application of the rule (R1), the rule (R2) can be used to observe all the nodes in A having a neighbor in V (i.e. the node of degree two in A). Finally, all the nodes in A with degree one can be observed by application of (R3).)

The property that a graph is bipartite (chordal, circle, planar) is preserved by the reduction described above. Since DS_DECISION is NP-complete for all these graph classes, the same holds for WPDS_DECISION. \square

A.8 Proof of Theorem 3.11

Proof. We show that Algorithm 1 given hereafter solves WPDS in trees in linear time. We firstly prove that the set S_F that is output by Algorithm 1 is indeed a wpds for T .

Algorithm 2 Combinatorial algorithm for WPDS in trees

Input: a tree $T = (V, E)$ rooted at a non-leaf node r of $\overset{\circ}{T}$.

Output: a minimum size wpds S_F of T .

```

1: Initialize the list  $\mathcal{L}$  with the non-leaf nodes of  $\overset{\circ}{T}$  sorted according to a post-order
   traversal of  $\overset{\circ}{T}$ 
2:  $S_F \leftarrow \emptyset$ ;  $B \leftarrow \emptyset$ ;
3: while  $\mathcal{L} \neq \{r\}$  do
4:    $v \leftarrow$  the first node in  $\mathcal{L}$ ;  $\mathcal{L} \leftarrow \mathcal{L} \setminus \{v\}$ ;
5:   if  $v$  has at least two unobserved children not in  $B$  then
6:      $S_F \leftarrow S_F \cup \{v\}$ ;
7:     Exhaustively apply the two observation rules (R1) and (R2) to  $\overset{\circ}{T}$ ;
8:   else if  $v$  has all its children observed and  $v$  is not adjacent to a leaf of  $T$  then
9:      $B \leftarrow B \cup \{v\}$ ;
10:  end if
11: end while
12: if  $r$  is unobserved then
13:    $S_F \leftarrow S_F \cup \{r\}$ ;
14: end if return  $S_F$ ;

```

To do so, it is sufficient to prove that all the nodes are observed by S_L with the rules (R1)-(R3) in $\overset{\circ}{T}$, but restricting the use of (R3) to nodes that are not adjacent to any leaf of T .

Then, we proceed by induction on the depth k_{depth} of the nodes in $\overset{\circ}{T}$, where the depth of any node v of $\overset{\circ}{T}$, denoted by $depth(v)$, is defined as the length (i.e. the number of edges) of the path joining v and r . Firstly, it is clear that the node with $k_{depth} = 0$ (i.e. the root node r) is observed due to instructions 12-14. Assume now that all the nodes with depth strictly less than k are observed, for some $k > 0$, and consider some node v satisfying $depth(v) = k$. If $v \in B$ (with B as defined in Algorithm 1), then by definition of B , all the children of v are observed, and using the inductive assumption, the father of v is also observed. So all the neighbors of v are observed, and by applying the rule (R3), we get that the node v is observed as well. If the father of v belongs to S_F then v gets observed with the rule (R1). If v has at least two unobserved children not in B , then $v \in S_F$ and v is trivially observed. So consider now the case when v has exactly one unobserved child not in B and the father u of v does not belong to S_F . Since $u \notin S_F$ it has at most one unobserved child not in B . Using the inductive assumption, we know that u is observed but also its father. It follows that the children of u in B are observed (with (R3)) and that this is also the case for the single unobserved child of u with (R2). So all the children of u are observed (among which v). This terminates the proof that all the nodes in $\overset{\circ}{T}$ are observed by S_F with (R1), (R2) and the restriction of (R3) to nodes that are not adjacent to any leaf of T . Thus, we get that all the nodes in T get observed by S_F with (R1)-(R3).

To prove that S_F is an optimal solution to WPDS we restrict our attention to the case when $\overset{\circ}{T} = (\overset{\circ}{V}, \overset{\circ}{E})$ contains at least one node with degree three since otherwise the result is obvious. We proceed in a similar way to Guo et al. Guo et al. (2008), proving the next claim. Recall that by Proposition 3.3 there always exists an optimal solution of WPDS having all its nodes contained in $\overset{\circ}{V}$ and all having degree at least three in $\overset{\circ}{T}$.

Claim. Given any tree $T = (V, E)$ and $\overset{\circ}{T} = (\overset{\circ}{V}, \overset{\circ}{E})$ rooted at a non-leaf node r , Algorithm 1 finds a wpds S_F of T such that $|S_F \cap \overset{\circ}{V}_u| \leq |F^* \cap \overset{\circ}{V}_u|$ for any optimal solution F^* of WPDS that is contained in the node set of $\overset{\circ}{T}$, and for any node u of $\overset{\circ}{T}$, where $\overset{\circ}{V}_u$ denotes the subtree of $\overset{\circ}{T}$ rooted at u .

Proof of the claim. Let $\overset{\circ}{T}_u = (\overset{\circ}{V}_u, \overset{\circ}{E}_u)$ be the subtree of $\overset{\circ}{T}$ rooted at node u . Let $l := \max_{v \in \overset{\circ}{V}} \{depth(v)\}$. For any node $u \in \overset{\circ}{V}$, we define $p_u := |S_F \cap \overset{\circ}{V}_u|$ and $q_u := |F^* \cap \overset{\circ}{V}_u|$. We proceed by induction on the depth of the nodes of $\overset{\circ}{T}$, starting with the depth value l and decreasing to value 0.

Note that the Algorithm 1 adds no leaf of $\overset{\circ}{T}$ with depth l to the set S_F . Thus, $0 = p_u \leq q_u$ for any node $u \in \overset{\circ}{V}$ with $depth(u) = l$.

Assume now that $p_u \leq q_u$ holds for any node $u \in \overset{\circ}{V}$ with $depth(u) > k$ for some $k < l$. Consider then a node v with depth k . Due to the inductive assumption, we have $p_u \leq q_u$ for all $u \in C_v$, where C_v denotes the set of all v 's children in $\overset{\circ}{T}$. In order to prove that $p_v \leq q_v$, it is sufficient to consider the case when the following items are all satisfied:

1. $v \in S_F$,
2. $v \notin F^*$, and
3. $\sum_{u \in C_v} p_u = \sum_{u \in C_v} q_u$.

Indeed, in all other cases we have $p_v \leq q_v$. This can be proved as follows for the case $v \neq r$. (The case $v = r$ can be handled similarly.) Using the inductive assumption together with the third item above, we deduce $p_u = q_u$ for all $u \in C_v$. For the first item to hold ($v \in S_F$), v must have two unobserved children u_1 and u_2 not in B . We then prove that node v must belong to F^* for F^* to be a wpds. To show this, we firstly prove the following:

$$\forall x \in \left(F^* \cap \mathring{V}_{u_i} \right), i \in \{1, 2\}: \exists y \in P_{u_i, x} \cap S_F \quad (\text{A1})$$

where $P_{u_i, x}$ denotes the path joining u_i and x in T (including the nodes u_i and x). With no loss of generality consider the case $i = 1$ and assume that (A1) is not satisfied by some node $x \in F^* \cap \mathring{V}_{u_1}$. Since $x \notin S_F$ we must have $p_x < q_x$. Using the inductive assumption, since $P_{u_i, x}$ contains no node of S_F , we deduce $p_z < q_z$ for all the nodes $z \in P_{u_i, x}$. In particular we get $p_{u_1} < q_{u_1}$ contradicting $p_{u_1} = q_{u_1}$. This terminates the proof of (A1).

From (A1), for each node $x \in F^* \cap \mathring{V}_{u_i}$ ($i \in \{1, 2\}$) there must exist a vertex $y \in P_{u_i, x}$ that observes u_i . This implies that if u_i is observed by a node in $F^* \cap \mathring{V}_{u_i}$, then there is a node $y \in P_{u_i, x}$ that observes u_i . But since we have $v \in S_F$, u_1 and u_2 cannot be observed by the nodes in $S_F \cap \left(\mathring{V}_{u_1} \cup \mathring{V}_{u_2} \right)$. Thus, u_1 and u_2 cannot be observed by the nodes in $F^* \cap \left(\mathring{V}_{u_1} \cup \mathring{V}_{u_2} \right)$. Since F^* is a solution to WPDS, node v must belong to F^* (since otherwise u_1 and u_2 cannot be observed). Thus, we get a contradiction with the assumption $v \notin F^*$ and this concludes the proof of the claim.

The proof of the (linear) running time complexity is completely similar to (Guo et al., 2008, Theorem 3) and is thus omitted. \square

Appendix B Preprocessing for CPDS

We present hereafter adaptations of the transformations GR1 and GR2 (see §4.4.1) for the problem CPDS

GR1_{CPDS}. If ℓ_1, \dots, ℓ_q are leaves in G , all neighbors of some node z and satisfying $\min_{i=1, \dots, q} w_{\ell_i} \geq w_z$ then all but two leaves are removed, keeping the original weights for the remaining leaves.

GR2_{CPDS}. Assume that $\mathcal{P} = (v_1, v_2, \dots, v_p)$, with $p \geq 3$, is an induced path in G such that the nodes v_1, \dots, v_{p-1} are pairwise distinct and $d_G(v_2) = \dots = d_G(v_{p-1}) = 2$. (Assuming the graph G to be simple we must have $p \geq 4$ if $v_1 = v_p$.) It is clear that if G is a path or a cycle, then it can be reduced to an isolated node of minimum weight. So assume from now on that G is not a path nor a cycle. Then,

- if one of the endpoints of \mathcal{P} , say v_p , is a leaf in G , \mathcal{P} is replaced by the edge $\{v_1, z\}$, where z is a node of weight $w_z = \min_{i=2, \dots, p} w_{v_i}$.
- if $v_1 = v_p$, \mathcal{P} is replaced by the cycle (v_1, v_2, z, v_p) with $w_z = \min_{i=3, \dots, p-1} w_{v_i}$.
- otherwise (i.e. none of the endpoints of \mathcal{P} is a leaf and $v_1 \neq v_p$), \mathcal{P} is replaced by the path (v_1, z, v_p) with $w_z = \min_{i=2, \dots, p-1} w_{v_i}$.

The procedure ST can be used for CPDS as described in §4.4.2 replacing Ω_{WP} by Ω_{P} .

Appendix C Computational experiments with randomly generated node weights

Similarly to the computational experiments carried out for PDS and WPDS (see §5.2), we report hereafter results obtained for instances of CWPDS and CPDS with randomly and uniformly generated node weights in $\{1, 2, \dots, 100\}$. Tables C1 and C2 provide information about pre-processing and initialization for CPDS and CWPDS, respectively. Tables C3 and C4 (C5 and C6) provide the results obtained for CPDS (CWPDS) with variants of the proposed solution method using ILP and combinatorial separation, respectively.

Each row corresponds to averaged results over five instances:

- for the case of graphs from the literature: for each such instance the graph is fixed and we considered five node-weight functions.
- for the case of randomly generated graphs: all five instances corresponding to one row differ in the graph and the node-weight function (both are randomly generated).

Table C1: Pre-processing and initialization for CPDS

Input graph			PDS pre-processing								PDS initialization (after pre-pro)				
Name	V	E	Post-GR graph		ST1		ST2		# cn	time	# cn	time	# cn	ILP	comb
			V	E	time	# cn	time	# cn							
IEEE_14_bus	14	20	13 (6)	19	0	8.2	0	0	0	2	0.005	0.003			
IEEE_30_bus	30	41	25 (13)	36	0	10.4	0	2.4	0	3.8	0.007	0.003			
IEEE_57_bus	57	78	42 (18)	63	0	16.0	0	5.8	0	5.0	0.010	0.003			
IEEE_118_bus	118	179	101 (46)	162	0	39.8	0	17.4	0.002	11.4	0.020	0.005			
IEEE_300_bus	300	409	282.0 (127.0)	391.0	0.002	105.6	0.001	43.2	0.032	35.4	0.078	0.015			
power-685-bus	685	1282	671 (160)	1268	0.009	218.8	0.004	161.2	0.392	53.2	0.409	0.146			
power-eris	1174	8687	976 (375)	8489	0.022	617.4	0.005	83.4	0.483	102.6	1.468	0.151			
pegase_1354	1354	1710	1218.4 (722.4)	1574.4	0.052	447.4	0.011	186.4	1.731	182.8	0.689	0.141			
bcsppr09	1723	2394	1540.0 (724.0)	2211.0	0.082	518.6	0.02	186.8	4.475	190.0	1.136	0.289			
pegase_2869	2869	3968	2565.8 (1379.8)	3664.8	0.222	975.8	0.048	325.6	15.737	336.2	3.191	0.919			
polish_2383	2383	2886	1790.6 (1014.6)	2293.6	0.125	565.6	0.025	208.2	8.193	250.0	1.533	0.515			
us-west_4941	4941	6594	4246.4 (2187.4)	5899.4	0.664	1495.8	0.124	590.4	87.103	568.2	9.041	2.482			
pegase_9241	9241	14207	7766.2 (3966.2)	12732.2	2.416	3115.6	0.426	977.6	409.994	944.2	36.052	9.135			
pegase_13659	13659	18625	12010.8 (7514.8)	16976.8	5.193	4550.0	0.992	2027.6	1523.147	1756.8	84.506	13.813			
S1000	2000	2000	2000 (1000)	2000	0.025	503.0	0.024	396.0	7.960	445.2	2.273	0.357			
GRID2_10_10	100	180	100 (4)	180	0	6.2	0	92.0	0.001	3.6	3.988	0.007			
GRID2_20_20	400	760	400 (4)	760	0.001	7.0	0.002	158.0	0.110	7.8	1687.029	0.109			
GRID2_30_30	900	1740	900 (4)	1740	0.007	6.6	0.007	225.2	1.290	12.0	1800.000	0.935			
GRID3_4_4_4	64	144	64 (0)	144	0	0	0	18.8	0.001	5.4	0.070	0.005			
GRID3_5_5_5	125	300	125 (0)	300	0	0	0	16.0	0.011	7.4	0.858	0.01			
3-reg_50	50	75	50 (0)	75	0	4	0	14.8	0	4.4	0.030	0.003			
4-reg_50	50	100	50 (0)	100	0	0.2	0	1.8	0.001	4.6	0.133	0.004			
3-reg_80	80	120	80 (0)	120	0	2.4	0	16.8	0.001	6.4	0.176	0.005			
4-reg_80	80	160	80 (0)	160	0	0.2	0	2	0.004	6.8	0.189	0.006			
WS_80_4	80	160	80 (3.6)	160	0	16.8	0	11.8	0.002	5.6	0.130	0.005			
WS_80_10	80	400	80 (0)	400	0	1.6	0	5.2	0.008	4.8	0.916	0.007			
WS_100_4	100	200	100 (5.4)	200	0	22.2	0	9.6	0.004	7.0	0.264	0.007			
WS_100_10	100	500	100 (0)	500	0	1.6	0	4.4	0.015	5.6	1.247	0.010			
ER_80_0.1	80	313.4	80 (0.6)	313.4	0	1.4	0	1.0	0.008	5.6	0.671	0.007			
ER_80_0.2	80	640	80 (0)	640.0	0	0.2	0	0.2	0.021	4.4	1.456	0.008			
ER_80_0.3	80	947.6	80 (0)	947.6	0	0	0.001	0	0.037	4.2	0.313	0.010			
ER_80_0.4	80	1265.2	80 (0)	1265.2	0	0	0.001	2.4	0.044	3.4	2.296	0.010			
ER_100_0.1	100	495	100 (0.6)	495.0	0	0.8	0	0.4	0.018	6.8	2.163	0.012			
ER_100_0.2	100	1001.4	100 (0)	1001.4	0	0	0.001	0	0.045	5.0	7.510	0.016			
ER_100_0.3	100	1491.4	100 (0)	1491.4	0	0	0.001	0	0.070	4.6	2.719	0.014			
ER_100_0.4	100	1986	100 (0)	1986.0	0	0	0.002	0	0.102	3.8	2.940	0.016			

Table C2: Pre-processing and initialization for CWPDS

Input graph			PDS pre-processing						PDS initialization (after pre-pro)			
Name	V	E	Post-GR graph			ST1		ST2		# cn	ILP	comb
			V	E	time	# cn	time	# cn	time			
IEEE14_bus	14	20	14 (7)	20	0	9.2	0	2.4	0	2	0.006	0.003
IEEE30_bus	30	41	29 (17.0)	40.0	0	18.6	0	4.6	0	3.6	0.009	0.002
IEEE57_bus	57	78	51.0 (27.0)	72.0	0	27.4	0	15.4	0	4.2	0.011	0.004
IEEE118_bus	118	179	116.0 (61.0)	177.0	0	62.0	0	24.8	0.002	10.4	0.024	0.006
IEEE300_bus	300	409	279.0 (126.0)	388.0	0.003	169.0	0.001	58.2	0.015	20.2	0.083	0.022
power-685_bus	685	1282	682.0 (171.0)	1279.0	0.015	278.6	0.004	277.0	0.321	39.8	0.796	0.255
power-eris	1174	8687	1170.0 (569.0)	8683.0	0.040	857.2	0.008	119.2	0.556	98.2	1.913	0.232
pegase_1354	1354	1710	1142.0 (667.0)	1498.0	0.042	672.2	0.012	185.6	0.982	95.0	0.601	0.355
bcspwr09	1723	2394	1559.0 (757.0)	2230.0	0.082	826.2	0.029	385.8	2.601	104.4	1.468	0.905
pegase_2869	2869	3968	2531.0 (1370.0)	3630.0	0.198	1413.8	0.057	475.2	10.058	197.0	2.716	1.900
polish_2383	2383	2886	2026.0 (1269.0)	2529.0	0.143	1183.8	0.037	319.4	6.423	142.6	1.978	4.548
us_west_4941	4941	6594	4389.0 (2396.0)	6042.0	0.634	2459.4	0.157	823.4	62.415	346.4	8.617	14.680
pegase_9241	9241	14207	8247.0 (4482.0)	13213.0	2.338	4776	0.574	1515.8	304.456	568.2	29.422	23.483
pegase_13659	13659	18625	9839.0 (5707.0)	14805.0	3.630	5701.4	0.822	1715.6	554.404	787.0	44.529	35.062
S1000	2000	2000	2000.0 (1000.0)	2000.0	0.025	1999.0	0.024	0	0	1.2	0.180	9.670
GRID2_10_10	100	180	100.0 (4.0)	180.0	0	12.8	0	78.4	0.001	3.6	1.009	0.005
GRID2_20_20	400	760	400.0 (4.0)	760.0	0.001	11.2	0.002	282.8	0.115	7.6	1800	0.139
GRID2_30_30	900	1740	900.0 (4.0)	1740.0	0.007	13.4	0.008	540.4	1.363	10.0	1800	0.923
GRID3_4_4_4	64	144	64.0 (0)	144.0	0	0	0	33.0	0.001	5.2	0.108	0.005
GRID3_5_5_5	125	300	125.0 (0)	300.0	0	0	0	37.4	0.012	7.0	1.279	0.011
3-reg-50	50	75	50.0 (0)	75.0	0	4.0	0	28.2	0	3.8	0.039	0.003
4-reg-50	50	100	50.0 (0)	100.0	0	0.2	0	13.2	0.001	4.4	0.143	0.003
3-reg-80	80	120	80.0 (0)	120.0	0	2.4	0	43.0	0.001	6.0	0.147	0.005
4-reg-80	80	160	80.0 (0)	160.0	0	0.2	0	15.6	0.004	6.2	0.303	0.005
WS_80_4	80	160	80.0 (3.6)	160.0	0	17.0	0	27.6	0.002	5.4	0.144	0.005
WS_80_10	80	400	80.0 (0)	400.0	0	1.6	0	14.8	0.008	4.8	0.799	0.008
WS_100_4	100	200	100.0 (5.4)	200.0	0	22.6	0	31.2	0.003	7.0	0.2047	0.009
WS_100_10	100	500	100.0 (0)	500.0	0	1.6	0	20.0	0.014	5.8	1.174	0.010
ER_80_0_1	80	313.4	80.0 (0.6)	313.4	0	1.6	0	8.4	0.009	5.2	0.569	0.007
ER_80_0_2	80	640	80.0 (0)	640.0	0	0.2	0	1.8	0.020	4.4	1.458	0.007
ER_80_0_3	80	947.6	80.0 (0)	947.6	0	0	0	2.6	0.033	4.2	0.296	0.010
ER_80_0_4	80	1265.2	80.0 (0)	1265.2	0	0	0.001	239.4	0.038	3.4	3.225	0.010
ER_100_0_1	100	495	100.0 (0.6)	495.0	0	1.0	0	5.8	0.021	6.6	2.016	0.013
ER_100_0_2	100	1001.4	100.0 (0)	1001.4	0	0	0.001	0.6	0.045	5.0	7.363	0.014
ER_100_0_3	100	1491.4	100 (0)	1491.4	0	0	0.001	0.2	0.070	4.4	6.937	0.014
ER_100_0_4	100	1986	100.0 (0)	1986.0	0	0	0.001	0	0.109	3.8	3.240	0.019

Compliance with Ethical Standards

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References

Brimkov, B., Fast, C.C., Hicks, I.V.: Computational approaches for zero forcing and related problems. *European Journal of Operational Research* **273**(3), 889–903 (2019) <https://doi.org/10.1016/j.ejor.2018.09.030>

Balas, E., Ng, S.M.: On the set covering polytope: I. all the facets with coefficients in $\{0, 1, 2\}$. *Mathematical Programming* **43**, 57–69 (1989) <https://doi.org/10.1007/BF01582278>

Carvalho, M., Klimentova, X., Viana, A.: Observability of power systems with optimal pmu placement. *Computers and Operations Research* **96**, 330–349 (2018) <https://doi.org/10.1016/j.cor.2017.10.012>

Diestel, R.: *Graph Theory*. Springer, Heidelberg (2017). <https://doi.org/10.1007/978-3-662-53622-3>

Table C3: Results for CPDS with ILP separation

Instance	opt	Preprocess+Intr+(PF)				Preprocess+(PF)				Intr+(PF)				(PF)				
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap	
IEEE14.bus	25.8	0	0.8	0.009	0	6.0	1.0	0.010	0	0	1.0	0.009	0	7.0	1.0	1.0	0.011	0
IEEE30.bus	68.2	4.0	1.0	0.013	0	9.8	1.0	0.019	0	6.0	1.0	0.017	0	11.2	1.0	1.0	0.032	0
IEEE57.bus	81.2	3.0	1.0	0.015	0	10.0	1.0	0.037	0	6.2	1.0	0.028	0	11.2	1.0	1.0	0.043	0
IEEE118.bus	195.4	9.4	1.0	0.089	0	20.4	1.0	0.327	0	11.6	1.0	0.088	0	22.2	1.0	1.0	0.312	0
IEEE300.bus	679.4	24.6	4.6	0.667	0	59.2	6.6	2.937	0	26.8	7.2	0.708	0	60.8	1.0	1.0	2.581	0
power-695-bus	765.6	77.0	258.2	21.104	0	128.4	214.8	101.020	0	82.8	249.4	20.856	0	119.0	252.8	1.0	100.947	0
power-eris	2624.8	69.8	6.8	6.360	0	173.0	3.4	205.482	0	53.0	3.0	4.175	0	171.2	2.6	1.0	76.887	0
pegase1364	5870.2	139.4	220.6	6.014	0	324.2	129.4	33.295	0	140.2	90.0	4.103	0	322.2	272.8	1.0	31.218	0
bcspr09	4691.0	151.2	767.0	15.582	0	337.4	1775.8	110.255	0	154.0	786.0	11.160	0	339.4	1919.6	1.0	97.278	0
pegase2869	9609.0	302.8	2772.2	56.379	0	626.6	2294.6	160.169	0	286.8	2170.8	39.213	0	641.0	1625.0	1.0	144.791	0
polish2383	5704.8	522.6	12104.0	87.973	0	758.0	2185.2	218.877	0	470.4	6716.2	78.723	0	761.8	310415.6	1.0	509.051	0
us-west4941	14325.4	592.4	14442.8	243.172	0	1134.8	368364.2	1075.380	0.04	583.2	668.8	166.106	0	1154.2	1973.2	1.0	776.790	0
pegase9241	22335.6	917.8	19855.2	1001.858	0	842.6	72.0	1800	-	875.2	5146.0	540.001	0	1028.6	261.0	1.0	1800	-
S1000	60135.0	411.8	428.8	1800	-	64.2	1.0	1800	-	1588.2	7322.2	579.845	0	761.4	44.4	1.0	1800	-
GR100	8298.4	276.8	521.2	17.825	0	48.0	1.0	1800	-	288.8	529.4	10.642	0	49.4	1.0	1.0	1800	-
GR102.10.10	20.6	13.0	1.0	25.828	0	14.8	1.0	39.870	0	14.0	1.0	23.926	0	15.6	1.0	1.0	43.293	0
GR102.20.20	29.2	0	0	1800	-	0	0	1800	-	0	0	1800	-	0	0	0	1800	-
GR102.30.30	-	0	0	1800	-	0	0	1800	-	0	0	1800	-	0	0	0	1800	-
GR103.4.4.4	28.4	16.4	1.0	2.880	0	20.6	1.0	7.573	0	17.2	1.0	3.482	0	21.0	1.0	1.0	8.452	0
GR103.5.5.5	40.2	55.4	49.2	18.6577	0	75.6	44.0	577.191	0	64.6	49.2	204.008	0	62.4	41.4	1.0	593.482	0
3-reg50	32.0	10.6	1.0	0.639	0.0	13.2	1.0	2.098	0	11.2	1.0	0.82	0	14.2	1.0	1.0	2.358	0
4-reg50	31.0	21.6	1.0	3.636	0	23.4	4.0	7.179	0	22.0	1.0	3.945	0	31.4	3.4	1.0	8.262	0
3-reg80	41.4	34.6	8.4	10.617	0	42.2	9.0	21.471	0	42.4	16.4	11.735	0	42.0	7.8	1.0	22.901	0
4-reg80	51.4	119.8	96.8	138.657	0	117.8	109.4	275.794	0	109.6	90.2	165.512	0	137.4	137.0	1.0	319.232	0
NS80.4	40.8	29.6	11.6	5.400	0	37.4	5.2	16.071	0	34.8	10.6	7.406	0	42.4	10.6	1.0	17.292	0
NS100.10	24.2	36.2	4.8	22.874	0	41.2	5.0	850.565	0	36.4	8.4	258.355	0	43.0	6.6	1.0	947.951	0
NS100.4	60.2	41.4	20.8	1081.908	0	56.6	21.6	33.599	0	39.2	23.4	26.245	0	55.4	23.4	1.0	37.169	0
NS100.10	37.4	47.2	14.4	550.635	0	17.0	0.6	1800	-	45.6	24.6	1140.691	-	10.0	0.6	1.0	1800	-
ER80.0.1	37.6	103.2	64.8	632.892	0	90.6	43.2	1400.237	-	104.2	65.4	514.121	0	83.6	45.8	1.0	1508.807	-
ER80.0.2	23.8	48.8	8.6	1581.170	-	1.2	0	1800	-	45.6	9.8	1662.822	-	1.2	0	1.0	1800	-
ER80.0.3	14.4	26.6	5.2	632.892	0	7.8	0	1800	-	25.6	5.2	624.186	0	1.8	0	1.0	1800	-
ER80.0.4	12.4	17.6	3.6	247.142	0	7.8	0.4	1797.963	-	19.4	1.6	278.636	0	3.6	0	1.0	1800	-
ER100.0.1	36.0	1.2	0.2	1800	-	0	0	1800	-	1.2	0.2	1800	-	0	0	1.0	1800	-
ER100.0.2	26.8	0.8	0	1800	-	0	0	1800	-	0.8	0	1800	-	0	0	1.0	1800	-
ER100.0.3	19.0	1.0	0	1800	-	0	0	1800	-	1.0	0	1800	-	0	0	1.0	1800	-
ER100.0.4	14.6	6.0	0.8	1800	-	0	0	1800	-	6.0	0.8	1800	-	0	0	1.0	1800	-

Table C4: Results for CPDS with combinatorial separation

Instance	opt	Preprocess+Init+(PF)			Preprocess+(PF)			Init+(PF)			(PF)						
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap				
IEEE14.bus	25.8	2.4	0.8	0.004	0	6.4	1.0	0.003	0	4.0	1.0	0.006	0	7.4	1.0	0.003	0
IEEE30.bus	68.2	4.2	1.0	0.007	0	10.2	1.0	0.004	0	5.8	1.0	0.006	0	11.2	1.0	0.005	0
IEEE57.bus	81.2	3.6	1.0	0.006	0	11.4	1.0	0.005	0	6.2	1.0	0.008	0	12.4	1.0	0.006	0
IEEE118.bus	195.4	11.0	1.0	0.015	0	24.8	1.0	0.016	0	14.0	1.0	0.015	0	26.0	1.0	0.018	0
IEEE300.bus	679.4	27.0	6.8	0.108	0	62.8	18.2	0.187	0	29.2	8.6	0.091	0	68.6	19.6	0.183	0
power-685-bus	765.6	129.8	549.6	2.883	0	208.4	667.4	4.698	0	143.4	715.4	2.885	0	212.4	693.4	4.491	0
power-eris	2624.8	82.0	12.6	2.159	0	184.8	2.2	8.119	0	66.2	8.2	1.842	0	186.8	9	10.457	0
power1354	5870.2	169.8	242.2	5.637	0	373.2	551.8	17.731	0	174.4	464.2	4.960	0	377.2	465.8	19.367	0
bcspart9	4691.0	303.4	4091.0	20.553	0	639.0	1542.0	60.570	0	321.8	3138.2	21.451	0	658.4	2721.4	79.431	0
pegase.2869	9609.0	403.4	1576.2	68.633	0	806.8	1486.8	193.930	0	402.0	553.6	68.754	0	801.6	537.6	231.752	0
polish.2383	5704.8	722.8	983981.6	487.568	0.05	1021.6	1587970.2	799.035	0.37	728.4	639141.4	513.809	0.11	1045.4	41398.8	282.756	0
us_west.4941	14325.4	787.6	424728.4	666.017	0.04	1431	6200.2	826.160	0	772.0	3365.0	404.118	0	1465.8	4172.2	1044.255	0
pegase.9241	-	733.4	270.2	1800	-	438.2	20.8	1800	-	680.8	231.0	1800	-	337.0	5.8	1800	-
pegase.13659	-	90.6	8.4	1800	-	199.4	4.2	1800	-	293.8	4.8	1800	-	244.6	2.6	1800	-
S1000	8298.4	331.2	882930.0	375.250	0.11	788.4	906843.8	395.343	0.132	343.8	522.0	11.807	0	794.2	523.0	36.699	0
GRID2.10.10	20.6	17.2	1.0	0.029	0	23.0	1.0	0.032	0	21.4	1.0	0.036	0	27.0	1.6	0.033	0
GRID2.20.20	29.2	267.8	297.2	4.367	0	376.8	509.0	6.847	0	354.2	452.4	6.594	0	396.6	464.6	6.988	0
GRID2.30.30	37.0	3890.4	28334.2	773.966	0	4371.8	32915.6	1110.122	2.694	3874.0	24152.8	840.898	2.79	4299.8	19664.2	813.537	0
GRID3.4.4.4	28.4	19.4	1.0	0.02	0	28.8	1.0	0.023	0	19.8	1.0	0.018	0	28.4	2.4	0.021	0
GRID3.5.5.5	40.2	160.4	157.2	0.372	0	190.4	191.0	0.391	0	195.4	215.8	0.505	0	188.8	201.8	0.362	0
3-reg-50	32.0	17.2	1.0	0.015	0	21.4	1.0	0.011	0	17.0	1.0	0.013	0	23.8	1.6	0.013	0
4-reg-50	31.0	29.2	1.0	0.038	0	35.8	1.8	0.032	0	23.0	1.0	0.025	0	32.0	1.8	0.021	0
3-reg-80	41.4	62.4	22.4	0.089	0	85.4	26.4	0.108	0	68.0	0.098	0	84.6	40.8	0.098	0	
4-reg-80	51.4	250.0	258.6	0.369	0	287.8	249.8	0.352	0	246.8	249.2	0.330	0	295.6	273.6	0.390	0
NS-80.4	40.8	51.6	23.4	0.054	0	55.2	11.2	0.088	0	50.8	25.2	0.075	0	53.8	16.6	0.060	0
NS-80.10	24.2	55.6	19.0	0.099	0	68.8	10.0	0.119	0	53.6	25.2	0.075	0	65.6	8.4	0.106	0
NS-100.4	60.2	85.0	50.0	0.127	0	104.2	84.2	0.146	0	90.4	63.2	0.136	0	126.4	93.0	0.169	0
NS-100.10	27.4	94.0	34.2	0.294	0	105.6	42.4	0.245	0	93.0	49.6	0.177	0	171.2	116.8	0.332	0
ER-80.0.1	37.6	208.0	181.4	0.305	0	205.6	153.8	0.311	0	190.6	162.4	0.274	0	233.8	163.6	0.377	0
ER-80.0.2	23.8	72.6	33.4	0.158	0	78.4	22.6	0.153	0	84.6	40.8	0.148	0	89.6	26.4	0.148	0
ER-80.0.3	14.4	32.0	5.8	0.114	0	38.4	5.6	0.105	0	33.6	4.8	0.073	0	38.2	4.6	0.082	0
ER-80.0.4	12.4	26.8	3.6	0.119	0	38.6	1.0	0.128	0	30.4	5.2	0.074	0	40.8	1.0	0.099	0
ER-100.0.1	36.0	571.8	481.8	1.850	0	547.4	477.2	1.607	0	621.8	531.8	2.133	0	552.8	475.0	1.473	0
ER-100.0.2	26.8	245.0	149.0	0.691	0	316.4	175.4	0.848	0	277.6	164.6	0.746	0	295.0	94.6	0.716	0
ER-100.0.3	19.0	83.6	32.4	0.365	0	113.8	43.8	0.437	0	81.0	28.8	0.305	0	117.0	45.0	0.388	0
ER-100.0.4	14.6	48.0	15.8	0.326	0	59.8	7.4	0.340	0	48.0	15.8	0.238	0	59.8	7.4	0.246	0

Table C5: Results for cWPDs with ILP separation

Instance	opt	Preprocess+init+ (WF)				Preprocess+ (WF)				Init+ (WF)				(WF)			
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap
IEEE14.bus	25.8	0	0.6	0.009	0	5.4	1.0	0.024	0	0	1.0	0.008	0	7	1.0	0.012	0
IEEE30.bus	59.2	1.0	1.0	0.014	0	7.6	1.0	0.020	0	5.2	1.0	0.015	0	9.0	1.0	0.022	0
IEEE57.bus	51.4	2.8	1.0	0.020	0	8.4	1.0	0.061	0	5.8	1.0	0.025	0	9.4	1.0	0.076	0
IEEE118.bus	135.4	7.0	1.0	0.085	0	17.6	1.0	0.353	0	9.0	1.0	0.058	0	18.6	1.0	0.331	0
IEEE300.bus	221.2	15.2	3.8	0.650	0	38.0	3.4	5.018	0	19.4	5.0	0.888	0	41.0	4.8	5.313	0
power-695-bus	428.4	95.4	190.4	55.493	0	123.0	249.0	156.956	0	108.2	266.4	66.409	0	122.6	195.6	174.321	0
power-eris	1911.0	39.4	3.8	4.723	0	141.4	9.6	95.875	0	42.6	2.6	5.115	0	143.2	10.6	93.324	0
pegase1364	1161.2	78.8	162.4	10.926	0	167.2	176.4	41.562	0	86.2	219.2	13.079	0	182.8	295.4	45.167	0
bcsp99	988.8	183.8	820.8	80.952	0	295.4	822.4	457.421	0	187.0	827.2	98.850	0	270.0	844.8	459.119	0
pegase2869	2361.4	203.2	1545.8	100.694	0	392.0	558.8	492.526	0	213	551.2	101.963	0	398.4	551.2	457.914	0
polish2383	-	794.4	31251.6	1025.532	1.44	799.6	17582.0	1348.787	1.72	803.2	26919.2	1056.207	0.614	826.8	11835.4	1496.23	1.33
us-west4941	3951.2	483.8	2952.4	603.850	0	399.8	115.8	1800	-	497.8	2561.4	594.505	0	457.6	311.4	1800	-
pegase9241	-	422.0	3457.4	1524.485	1.46	385.8	1.6	1800	-	280.8	2875.0	1421.632	1.65	432.0	2.6	1800	-
pegase13659	-	231.0	181.6	1800	9.86	285.2	34.8	1800	-	484.0	492.0	1800	6.23	474.4	381.4	1800	-
S1000	1.0	0	0	0.276	0	5.0	1.0	0.283	0	2.4	1.0	0.324	0	6.0	1.0	0.287	0
GR1D2.10.10	15.4	12.0	1.0	24.184	0	14.0	1.0	80.044	0	12.4	1.0	24.726	0	14.8	1.0	87.639	0
GR1D2.20.20	22.6	0	0	1800	-	0	0	1800	-	0	0	1800	-	0	0	1800	-
GR1D2.30.30	-	0	0	1800	-	0	0	1800	-	0	0	1800	-	0	0	1800	-
GR1D3.4.4.4	25.4	13.8	1.0	4.595	0	19.4	1.0	13.262	0	16.4	1.0	5.676	0	21.4	1.8	14.769	0
GR1D3.5.5.5	33.4	73.6	70.4	424.643	0	66.8	73.4	1213.739	0	76.8	103.4	417.906	0	68.0	60.4	1410.696	-
3-reg-50	29.0	10.8	1.0	0.779	0	15.4	1.0	3.082	0	12.8	1.0	0.972	0	16.4	1.0	3.512	0
4-reg-50	27.2	18.2	4.0	4.421	0	21	1.6	12.476	0	19.6	1.8	5.413	0	24.2	2.4	13.966	0
3-reg-80	33.2	34.6	5.8	21.078	0	37.0	13.8	63.794	0	36.4	5.8	24.55	0	41.6	1.4	74.168	0
4-reg-80	41.8	91.4	84.6	171.935	0	112.6	80.6	534.690	0	89.2	79.4	202.488	0	130.2	80.8	619.353	0
NS-80.4	33.0	17.2	2.0	8.659	0	31.0	3.4	19.765	0	22.6	9.4	10.012	0	32.4	2.8	21.010	0
NS-80.10	22.2	31.4	1.8	253.891	0	39.2	1.6	914.913	0	35.2	2.8	285.706	0	32.2	2.6	1003.474	0
NS-100.4	41.8	47.2	30.4	49.136	0	56.8	35.6	84.055	0	56.4	23.6	1213.751	0	56.6	50.0	87.35	0
NS-100.10	27.4	58.2	24	1188.250	-	5.2	0.6	1800	-	56.4	23.6	1213.751	0	4.4	0.4	1800	-
ER-80.0.1	36.0	101.6	79.0	762.842	0	21.2	3.4	1652.802	-	77.8	47.8	858.373	0	17.2	0.4	1686.217	-
ER-80.0.2	23.4	41.2	11.4	1652.721	-	0.8	0	1800	-	39.8	10.6	1607.960	-	0.8	0	1800	-
ER-80.0.3	14.4	27.8	2.6	834.276	0	1.4	0	1800	-	27.8	2.4	917.700	0	1.4	0	1800	-
ER-80.0.4	12.4	22.0	1.8	292.242	0	3.0	0	1800	-	23.0	1.8	348.250	0	3.0	0	1800	-
ER-100.0.1	33.4	0.6	0	1800	-	0	0	1800	-	0.6	0	1800	-	0	0	1800	-
ER-100.0.2	26.4	1.2	0	1800	-	0	0	1800	-	1.2	0	1800	-	0	0	1800	-
ER-100.0.3	19.0	1.0	0	1800	-	0	0	1800	-	1.0	0	1800	-	0	0	1800	-
ER-100.0.4	14.6	8.0	0.6	1800	-	0	0	1800	-	8.0	0.6	1800	-	0	0	1800	-

Table C6: Results for cWPDS with combinatorial separation

Instance	opt	Preprocess+Init+ (WF)			Preprocess+ (WF)			Init+ (WF)			(WF)						
		# cn	# nd	time	gap	# cn	# nd	time	gap	# cn	# nd	time	gap				
IEEE14.bus	25.8	1.4	0.6	0.005	0	5.8	1.0	0.004	0	4.0	1.0	0.005	0	7.4	1.0	0.005	0
IEEE30.bus	59.2	1.8	1.0	0.006	0	8.2	1.0	0.005	0	1800	1.0	0.006	0	10.0	1.0	0.008	0
IEEE57.bus	51.4	2.8	1.0	0.008	0	9.4	1.0	0.005	0	226.0	1.0	0.007	0	10.4	1.0	0.006	0
IEEE118.bus	135.4	11.0	1.0	0.020	0	20.4	1.0	0.020	0	15.8	1.0	0.015	0	21.6	1.0	0.016	0
IEEE300.bus	221.2	24.8	8.0	0.126	0	53.8	18.2	0.221	0	473.4	13.0	0.121	0	53.4	16.4	0.203	0
power-685-bus	428.4	181.4	499.6	5.127	0	232.6	413.0	6.911	0	193.2	541.8	5.055	0	220.6	684.4	5.797	0
power-eris	1911.0	43.0	5.6	1.625	0	145.8	1.0	8.833	0	46.0	3.0	1.251	0	146.2	1.0	8.079	0
pegase.1354	1161.2	119.8	440.6	4.907	0	220.0	1071.4	11.235	0	142.0	624.2	5.938	0	251.6	1255.2	14.825	0
pegase.2869	988.8	542.4	3496.0	55.631	0	751.6	1560.8	89.54	0	561.8	1578.0	66.84	0	775.6	2686.8	110.715	0
pegase.2869	2361.4	354.8	4838.4	69.452	0	601.2	581.8	157.256	0	343.0	592.8	71.872	0	583.0	597.2	183.641	0
polish.2383	-	2582.0	102559.6	1129.276	0.26	2535.4	217732.6	1250.470	0.29	2387.0	82596.8	1416.517	0.29	2641.2	70327.2	1477.36	0.80
us-west.4941	3951.2	992.6	15199.4	920.593	0	1349.0	19837.6	1570.215	0.66	1081.8	19585.0	1261.193	0	1205.4	5349.0	1773.714	2.54
pegase.9241	-	830.8	1002.4	1731.496	4.84	381.8	15.2	1800	-	743.2	656.0	1785.938	9.06	316.8	9.0	1800	-
pegase.12659	-	328.6	149.6	1800	-	323.4	82.6	1800	-	226.0	92.8	1800	-	226.8	4.2	1800	-
S1000	1.0	0	0	9.739	0	5.0	1.0	39.931	0	3.0	1.0	33.049	0	6.0	1.0	48.146	0
GR12.10.10	15.4	12.4	1.0	0.022	0	14.2	2.0	0.018	0	15.8	1.6	0.027	0	17.4	1.0	0.023	0
GR12.20.20	22.6	427.6	1024.8	10.081	0	503.8	1078.2	10.947	0	473.4	1149.0	11.489	0	520.0	1197.0	12.007	0
GR12.30.30	28.0	3677.2	18940.2	679.626	3.43	3883.0	29296.4	842.606	0	3547.6	24756.2	839.487	3.43	3946.2	18287.2	835.013	3.89
GR13.4.4.4	25.4	18.2	1.0	0.019	0	28.0	3.2	0.023	0	18.8	1.2	0.019	0	30.0	1.0	0.024	0
GR13.5.5.5	33.4	180.4	187.4	0.423	0	160.6	132.6	0.399	0	185.6	202.4	0.430	0	196.2	172.0	0.435	0
3-reg-50	29.0	16.8	1.0	0.015	0	20.8	1.0	0.015	0	17.8	1.0	0.013	0	23.8	2.0	0.012	0
4-reg-50	27.2	27.8	4.4	0.029	0	30.4	2.4	0.029	0	31.4	2.6	0.038	0	31.6	3.2	0.022	0
3-reg-80	33.2	47.0	9.2	0.085	0	84.8	29.8	0.127	0	63.2	14.8	0.115	0	80.0	22.2	0.116	0
4-reg-80	41.8	239.8	210.2	0.349	0	278.2	273.0	0.474	0	239.0	219.2	0.337	0	260.0	242.6	0.479	0
NS-80.4	33.0	33.2	14.8	0.043	0	51.0	12.8	0.065	0	35.0	8.4	0.041	0	54.2	13.4	0.059	0
NS-80.10	22.2	50.2	9.6	0.108	0	56.0	6.8	0.087	0	43.2	17.4	0.062	0	65.2	5.6	0.099	0
NS-100.4	41.8	97.6	81.6	0.256	0	108.2	87.8	0.171	0	111.4	84.2	0.181	0	116.0	82.4	0.176	0
NS-100.10	27.4	94.8	44.0	0.236	0	132.4	65.8	0.322	0	101.8	59.2	0.217	0	139.4	63.2	0.284	0
ER-80.0.1	36.0	194.6	122.4	0.279	0	183.2	129.4	0.270	0	172.4	125.0	0.245	0	216.8	132.0	0.334	0
ER-80.0.2	23.4	66.2	17.6	0.156	0	83.2	18.4	0.176	0	78.6	28.8	0.145	0	109.2	37.2	0.194	0
ER-80.0.3	14.4	31.8	3.4	0.108	0	39.2	5.4	0.108	0	33.0	4.0	0.077	0	39.2	4.2	0.072	0
ER-80.0.4	12.4	31.0	3.0	0.132	0	34.4	2.0	0.120	0	34.0	4.4	0.090	0	35.2	2.0	0.087	0
ER-100.0.1	33.4	335.8	326.8	0.951	0	356.6	254.4	0.781	0	349.4	317.6	0.896	0	391.8	309.4	0.950	0
ER-100.0.2	26.4	248.8	142.6	0.689	0	289.2	140.8	0.793	0	275.4	158.8	0.872	0	287.2	142.4	0.741	0
ER-100.0.3	19.0	114.4	45.4	0.451	0	108.2	41.6	0.416	0	114.2	43.8	0.391	0	110.2	42.2	0.365	0
ER-100.0.4	14.6	43.6	14.0	0.287	0	59.0	7.4	0.335	0	43.6	14.0	0.188	0	59.0	7.4	0.234	0

- Dorbec, P.: In: Haynes, T.W., Hedetniemi, S.T., Henning, M.A. (eds.) *Power Domination in Graphs*, pp. 521–545. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-51117-3_16
- Erdős, P., Rényi, A.: On random graphs I. *Publicationes Mathematicae* **6**(3-4), 290–297 (1959)
- Fast, C.C., Hicks, I.V.: Effects of vertex degrees on the zero-forcing number and propagation time of a graph. *Discrete Applied Mathematics* **250**, 215–226 (2018) <https://doi.org/10.1016/j.dam.2018.05.002>
- Fliscounakis, S., Panciatici, P., Capitanescu, F., Wehenkel, L.: Contingency ranking with respect to overloads in very large power systems taking into account uncertainty, preventive, and corrective actions. *IEEE Trans. Power Syst.* **28**, 4909–4917 (2013) <https://doi.org/10.1109/TPWRS.2013.2251015>
- Guo, J., Niedermeier, R., Raible, D.: Improved algorithms and complexity results for power domination in graphs. *Algorithmica* **52**(2), 177–202 (2008) <https://doi.org/10.1007/s00453-007-9147-x>
- Gurobi Optimization, LLC: Gurobi Optimizer Reference Manual (2022). <https://www.gurobi.com>
- Haynes, T.W., Hedetniemi, S.M., Hedetniemi, S.T., Henning, M.A.: Domination in graphs applied to electric power networks. *SIAM J. Discrete Math.* **15**(4), 519–529 (2002) <https://doi.org/10.1137/S0895480100375831>
- Koebe, M.: On a new class of intersection graphs. In: Nešetřil, J., Fiedler, M. (eds.) *Proceedings 4th Czechoslovakian Symposium on Combinatorics, Graphs and Complexity, Prachatice 1990*. *Annals of Discrete Mathematics*, vol. 51, pp. 141–143. Elsevier, North Holland, Amsterdam (1992). [https://doi.org/10.1016/S0167-5060\(08\)70618-6](https://doi.org/10.1016/S0167-5060(08)70618-6)
- Rossi, R.A., Ahmed, N.K.: The network data repository with interactive graph analytics and visualization. In: *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 29 (2015). <https://doi.org/10.1609/aaai.v29i1.9277>
- Smith, L.A., Hicks, I.V.: New computational approaches for the power dominating set problem: Set covering and the neighborhoods of zero forcing forts. *Networks* **79**(2), 202–219 (2022) <https://doi.org/10.1002/net.22056>
- Watanabe, M., Schwenk, A.J.: Integral starlike trees. *Journal of the Australian Mathematical Society* **28**(01), 120–128 (1979)
- Watts, D., Strogatz, S.: Collective dynamics of ‘small-world’ networks. *Nature* **393**, 440–442 (1998) <https://doi.org/10.1038/30918>

Zhao, M., Kang, L., Chang, G.J.: Power domination in graphs. *Discrete Mathematics* **306**(15), 1812–1816 (2006) <https://doi.org/10.1016/j.disc.2006.03.037>