



HAL
open science

Reachability for multiagent control systems via Lyapunov functions

Giulia Cavagnari, Marc Quincampoix

► **To cite this version:**

Giulia Cavagnari, Marc Quincampoix. Reachability for multiagent control systems via Lyapunov functions. Communications on Pure and Applied Analysis, 2025, <10.3934/cpaa.2025100>. <hal-05272166>

HAL Id: hal-05272166

<https://hal.science/hal-05272166v1>

Submitted on 22 Sep 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

REACHABILITY FOR MULTIAGENT CONTROL SYSTEMS VIA LYAPUNOV FUNCTIONS

GIULIA CAVAGNARI AND MARC QUINCAMPOIX

ABSTRACT. This paper concerns the problem of reachability of a given state for a multiagent control system in \mathbb{R}^d . In such a system, at every time each agent can choose his/her velocity which depends both on his/her position and on the position of the whole crowd of agents (modeled by a probability measure on \mathbb{R}^d). The main contribution of the paper is to study the above reachability problem with a given rate of attainability through a Lyapunov method adapted to the Wasserstein space of probability measures. As a byproduct we obtain a new comparison result for viscosity solutions of Hamilton Jacobi equations in the Wasserstein space.

1. INTRODUCTION

During the last years, there has been a large interest on controlled multiagent systems modeled on the Wasserstein space of probability measures on \mathbb{R}^d . In such a system, the number of agents is so huge that only a statistical description is possible. Namely the state variable μ is a probability measure on \mathbb{R}^d such that for any (Borel) set $A \subset \mathbb{R}^d$, $\mu(A)$ is the fraction of the total amount of agents that are present in A . The time evolution $t \mapsto \mu_t$ of the state variable occurs according to a two-level dynamics, as follows.

- *Microscopic level.* Each agent at position $x(t)$ at time t follows

$$\dot{x}(t) \in F(x(t), \mu_t) \text{ for almost every } t \geq 0,$$

where $F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightrightarrows \mathbb{R}^d$ is a set-valued map and $\mathcal{P}_2(\mathbb{R}^d)$ is the set of probability measures on \mathbb{R}^d with finite second moment. Namely, the velocity of each agent depends both on his own position and on the position of the whole crowd of agents. It is worth pointing out that we make an indistinguishability assumption meaning that an agent does not interact individually with every other agents but only with the whole crowd of agents.

- *Macroscopic level.* The dynamics of the crowd of agents is given by the following so-called *continuity equation*

$$(1.1) \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \text{ for a.e. } t \text{ and } \mu_t\text{-a.e. } x \in \mathbb{R}^d,$$

where $v_t(\cdot)$ is a Borel time dependent vector field on \mathbb{R}^d . This equation holds true in the sense of distributions on $\mathbb{R} \times \mathbb{R}^d$ (cf. Definition 2.2 later on).

The coupling between microscopic and macroscopic dynamics is given by

$$(1.2) \quad v_t(x) \in F(x, \mu_t) \text{ for a.e. } t \text{ and } \mu_t\text{-a.e. } x \in \mathbb{R}^d.$$

Such bi-level dynamics have been widely studied in the literature. For the continuity equation we refer to [1], the control dynamics was studied in [10, 13–15, 21, 22] (cf. also [6–9]).

Our main goal is to study a reachability problem: the controller aims to drive the system to a given probability measure $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ following an admissible solution of the dynamical system (1.1),(1.2). Some results in this direction have been obtained in [17], where the authors study the

2020 *Mathematics Subject Classification.* 49L25, 49J52, 49K27, 49J15, 34A60, 93C15.

Key words and phrases. Reachability, Hamilton-Jacobi-Bellman equation, Lyapunov function, optimal control in Wasserstein spaces, multi-agent.

Acknowledgments. This research is supported by INdAM-GNAMPA under the Project 2024 ‘‘Controllo ottimo infinito dimensionale: aspetti teorici ed applicazioni’’, CUP E53C23001670001.

controllability problem in the case where the Lipschitz control is localized to act in a prescribed region. However, in addition to the required regularity, the authors address the problem in the case of local dynamics, thus without interactions.

Analogously to the classical case [2], we aim to provide sufficient conditions for reachability using a Lyapunov function approach under more general assumptions on the dynamics, including non-locality. A recent contribution for Measure Differential Equations in this direction is given in [16]. We assume that there exists a Lipschitz continuous function $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}_+$ such that

$$\{\bar{\mu}\} = \{\mu \in \mathcal{P}_2(\mathbb{R}^d) : V(\mu) = 0\}.$$

It is well known that the requirement for V to serve as a Lyapunov function is closely linked to the Hamiltonian of the given dynamical system. Of course, since the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ is not a vector space and the Lyapunov functions could not be regular enough, we need a suitable notion of solution for stating the corresponding partial differential inequality satisfied by the Lyapunov function. We will use the notion of viscosity (super) solution in the Wasserstein space adopted in [21, 22]. Various notions of viscosity solution for Hamilton Jacobi equations have been investigated in [5, 11, 18–23].

The main results of the paper are contained in Theorems 4.10 and 4.16. We investigate two rates of convergence of the trajectory: either asymptotically with an exponential decay or linearly in finite time horizon. A key ingredient in our approach is a new comparison result for Hamilton Jacobi equations in Wasserstein spaces (a slightly different version of the result developed in [21]), which is a byproduct of our article.

The paper is organized as follows: a preliminary section is devoted to notations, essential recalls on tools of optimal transport theory and introduction of the dynamical setting under investigation. In Section 3, we recall basic facts on Hamilton Jacobi Equations and we state an adapted version of the comparison principle for viscosity solutions on the Wasserstein space, whose proof is postponed to Appendix B. Section 4 contains the core of the paper. Finally, Appendix A is devoted to some technical proofs.

2. PRELIMINARIES AND SETTINGS

In this section, we first recall standard definitions and results of measure theory and optimal transport theory for which we refer to the survey [1]. Then, we introduce the dynamical setting under consideration.

2.1. The Wasserstein metric. Let X be a separable Banach space (here X will be either \mathbb{R}^d or $C([a, b]; \mathbb{R}^d)$, the space of \mathbb{R}^d -valued continuous curves on $[a, b] \subset \mathbb{R}$ endowed with the uniform norm). We denote by $\mathcal{P}(X)$ the space of Borel probability measures on X , endowed with the topology of narrow convergence¹.

Given Y separable Banach space, $r : X \rightarrow Y$ a Borel measurable map and $\nu \in \mathcal{P}(X)$, we denote by $r_{\#}\nu \in \mathcal{P}(Y)$ the *push-forward* measure defined by

$$\int_Y f(y) d(r_{\#}\nu)(y) := \int_X f(r(x)) d\nu(x),$$

for any $f : Y \rightarrow [0, +\infty)$ Borel measurable function. Given $\nu \in \mathcal{P}(X)$, we denote by $m_2(\nu)$ its *2-moment*

$$m_2(\nu) := \left(\int_X |x|^2 d\nu(x) \right)^{\frac{1}{2}}$$

and by $\mathcal{P}_2(X)$ the set of $\nu \in \mathcal{P}(X)$ such that $m_2(\nu) < +\infty$. On this space, we consider the Wasserstein metric W_2 defined by

$$(2.1) \quad W_2(\nu_1, \nu_2) := \left(\inf_{\sigma \in \Gamma(\nu_1, \nu_2)} \int_{X \times X} |x_1 - x_2|^2 d\sigma(x_1, x_2) \right)^{1/2},$$

¹We recall that a sequence $\{\nu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ *narrowly converges* to $\nu \in \mathcal{P}(X)$, and we write $\nu_n \rightarrow \nu$ as $n \rightarrow +\infty$, if for any $f \in C([a, b]; \mathbb{R})$ bounded $\int_X f(x) d\nu_n(x) \rightarrow \int_X f(x) d\nu(x)$, as $n \rightarrow +\infty$.

where $\Gamma(\nu_1, \nu_2)$ denotes the set of admissible couplings/plans between ν_1 and ν_2 , given by

$$\Gamma(\nu_1, \nu_2) := \{ \sigma \in \mathcal{P}(X \times X) : \pi_{\#}^1 \sigma = \nu_1, \pi_{\#}^2 \sigma = \nu_2 \},$$

and $\pi^i : X \times X \rightarrow X$, $i = 1, 2$, are the projections maps defined by $\pi^i(x_1, x_2) = x_i$, $i = 1, 2$. The set of optimal couplings realizing the infimum in (2.1) is denoted by $\Gamma_o(\nu_1, \nu_2)$. We recall that the topology induced by the W_2 metric on $\mathcal{P}_2(X)$ is stronger than the narrow topology. (cf. e.g. [1, Remark 7.1.11]).

We also recall that a map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an *optimal transport map* between $\nu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ if $\nu_2 = T_{\#} \nu_1$ and the induced plan is optimal, i.e. $(\text{id}_{\mathbb{R}^d}, T)_{\#} \nu_1 \in \Gamma_o(\nu_1, \nu_2)$.

2.2. Trajectories of the control system. We introduce the controlled non-local dynamics under consideration in the form of a differential inclusion. In particular, in this section we provide assumptions on the involved set-valued map $F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightrightarrows \mathbb{R}^d$ that will be required later on, we give the definition of admissible trajectories in the Wasserstein space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ and of associated Hamiltonian. This is the setting considered in [21] (cf. also [13–15, 22]).

Denote by $F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightrightarrows \mathbb{R}^d$ the set-valued vector field involved in the controlled dynamics under study. Whenever specified, we work under the following assumptions, where on the space $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ we consider the metric

$$d((x_1, \nu_1), (x_2, \nu_2)) := |x_1 - x_2| + W_2(\nu_1, \nu_2), \quad (x_i, \nu_i) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \quad i = 1, 2.$$

Hypotesis 2.1.

- (1) F has convex, compact and nonempty images;
- (2) F is L -Lipschitz continuous, i.e. there exists $L > 0$ such that

$$F(x_1, \nu_1) \subset F(x_2, \nu_2) + L d((x_1, \nu_1), (x_2, \nu_2)) \overline{B(0, 1)},$$

for any $(x_i, \nu_i) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $i = 1, 2$.

Denoted by

$$(2.2) \quad K_F := \max_{v \in F(0, \delta_0)} |v|,$$

thanks to (2), we get the following estimate

$$(2.3) \quad F(x, \nu) \subset F(0, \delta_0) + L (m_2(\nu) + |x|) \overline{B(0, 1)} \subset C(1 + m_2(\nu)) (1 + |x|) \overline{B(0, 1)},$$

for any $(x, \nu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where $C := \max\{1, L K_F\}$.

We use the notation $\mu = (\mu_t)_{t \in I}$ to denote a curve $\mu : I \rightarrow \mathcal{P}_2(\mathbb{R}^d)$, where I is a given interval.

Definition 2.2 (Admissible trajectories). Let $I = [a, b] \subset \mathbb{R}$, $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mu = (\mu_t)_{t \in I} \subset \mathcal{P}_2(\mathbb{R}^d)$ be an absolutely continuous curve. We say that μ is an *admissible trajectory starting from $\bar{\mu}$* , and we write $\mu \in \mathcal{A}_I(\bar{\mu})$, if there exists a Borel measurable vector field $v : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $v_t \in L^2_{\mu_t}(\mathbb{R}^d; \mathbb{R}^d)$ for a.e. $t \in I$, satisfying

- $v_t(x) \in F(x, \mu_t)$ for a.e. $t \in I$ and μ_t -a.e. $x \in \mathbb{R}^d$;
- $\partial_t \mu_t + \text{div}(v_t \mu_t) = 0$ in the sense of distributions on $I \times \mathbb{R}^d$, equivalently

$$\frac{d}{dt} \int_{\mathbb{R}^d} \phi(x) d\mu_t(x) = \int_{\mathbb{R}^d} \langle \nabla \phi(x), v_t(x) \rangle d\mu_t(x),$$

for a.e. $t \in I$ and all $\phi \in C_c^1(\mathbb{R}^d)$.

We say that a locally absolutely continuous curve $\mu = (\mu_t)_{t \in [0, +\infty)}$ is admissible starting from $\bar{\mu}$, and we write $\mu \in \mathcal{A}_{[0, +\infty)}(\bar{\mu})$, if the restriction $\mu|_{[0, T]}$ of μ to the compact interval $[0, T]$ is admissible starting from $\bar{\mu}$ for any $T > 0$.

We postpone to Appendix A the computation of standard a priori estimates on the 2-moments along admissible trajectories.

3. HAMILTON JACOBI BELLMAN EQUATION

The definition of Lyapunov function requires a suitable notion of variational inequality related to Hamilton-Jacobi equation. Such equation is solved by the value function of the following Mayer-type optimal control problem [21]. Given a Lipschitz continuous final cost $g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, we define the value function $U_g : [a, b] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$U_g(t, \nu) := \inf \{g(\mu_b) : \mu \in \mathcal{A}_{[t,b]}(\nu)\}.$$

Under Hypothesis 2.1, it is possible to prove the existence of optimal trajectories (cf. [21, Corollary 2]) and a Dynamic Programming Principle (cf. [21, Proposition 3]), which states in particular that the map

$$t \mapsto U_g(t, \mu_t)$$

is non decreasing in $[a, b]$ for any $\mu = (\mu_t)_{t \in [a,b]} \in \mathcal{A}_{[a,b]}(\bar{\mu})$, and it is constant along optimal trajectories.

We introduce the following *Hamilton-Jacobi-Bellman* equation in $[a, b] \times \mathcal{P}_2(\mathbb{R}^d)$ (with $a < b$),

$$(3.1) \quad \partial_t U(t, \nu) + \mathcal{H}(\nu, D_\nu U(t, \nu)) = 0,$$

where $\mathcal{H}(\nu, p)$ is defined for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L_\nu^2(\mathbb{R}^d)$. Since $U : [a, b] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ may not be regular enough, equation (3.1) has to be interpreted in a viscosity sense which we will specify later on in Section 3.2.

In the case of the Mayer problem, we will consider the following Hamiltonian associated to the set-valued map F

$$(3.2) \quad \mathcal{H}_F(\nu, p) := \int_{\mathbb{R}^d} \inf_{v \in F(x, \nu)} \langle p(x), v \rangle d\nu(x), \quad p \in L_\nu^2(\mathbb{R}^d), \nu \in \mathcal{P}_2(\mathbb{R}^d).$$

3.1. Sub/super-differentials. Before giving the definition of sub/super-differential in $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ or $\mathcal{P}_2(\mathbb{R}^d)$, taken from [21, 22] (cf. also [1, 20]), we need to recall the following.

Definition 3.1 (Optimal displacement). Let $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$. A function $p \in L_{\bar{\nu}}^2(\mathbb{R}^d)$ is called *optimal displacement from $\bar{\nu}$* if there exists $\lambda > 0$ and an optimal transport map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ between $\bar{\nu}$ and $T_{\#}\bar{\nu}$ such that

$$p = \lambda(T - \text{id}_{\mathbb{R}^d}).$$

Remark 3.2. We recall the following equivalent characterization of optimal displacement ([21, Lemma 4]): p is an *optimal displacement from $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$* if there exist $\lambda > 0$, $\xi \in \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma \in \Gamma_o(\bar{\nu}, \xi)$ such that

$$p(x) = \lambda \left(\int_{\mathbb{R}^d} y d\sigma_x(y) - x \right),$$

where $\{\sigma_x\}_{x \in \mathbb{R}^d} \subset \mathcal{P}_2(\mathbb{R}^d)$ is the family obtained by disintegrating σ w.r.t. the projection on the first marginal $\bar{\nu}$ (cf. [1, Theorem 5.3.1]).

Definition 3.3 (Sub/super-differential in $\mathbb{R} \times \mathcal{P}_2(\mathbb{R}^d)$). Let $a, b \in \mathbb{R}$, $a < b$, and consider the map $U : [a, b] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$. Given $(\bar{t}, \bar{\nu}) \in [a, b] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\delta \geq 0$, we say that the pair $(p_{\bar{t}}, p_{\bar{\nu}}) \in \mathbb{R} \times L_{\bar{\nu}}^2(\mathbb{R}^d)$ is a *(total) δ -viscosity subdifferential* of U at $(\bar{t}, \bar{\nu})$, and we write $(p_{\bar{t}}, p_{\bar{\nu}}) \in \mathbf{D}_{\delta}^{-} U(\bar{t}, \bar{\nu})$, if

- (1) $p_{\bar{\nu}}$ is an optimal displacement from $\bar{\nu}$;
- (2) for all $(t, \nu) \in [a, b] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma \in \Gamma(\bar{\nu}, \nu)$,

$$U(t, \nu) - U(\bar{t}, \bar{\nu}) \geq p_{\bar{t}}(t - \bar{t}) + \int_{\mathbb{R}^d} \langle p_{\bar{\nu}}(x), y - x \rangle d\sigma(x, y) - \delta \cdot \Delta_{t, \sigma} + o(\Delta_{t, \sigma}),$$

$$\text{where } \Delta_{t, \sigma} := \sqrt{|t - \bar{t}|^2 + \int |x - y|^2 d\sigma(x, y)}.$$

Analogously, the set of *(total) δ -viscosity superdifferentials* of U at $(\bar{t}, \bar{\nu})$ is given by $\mathbf{D}_{\delta}^{+} U(\bar{t}, \bar{\nu}) := -\mathbf{D}_{\delta}^{-}(-U)(\bar{t}, \bar{\nu})$.

If the map U does not depend on t or it is constant w.r.t. t the above definition becomes

Definition 3.4 (Sub/super-differential in $\mathcal{P}_2(\mathbb{R}^d)$). Let $U : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$. Given $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$ and $\delta \geq 0$, we say that $p \in L^2_{\bar{\nu}}(\mathbb{R}^d)$ is a δ -viscosity subdifferential of U at $\bar{\nu}$, and we write $p \in \partial_{\delta}^{-} U(\bar{\nu})$, if

- (1) p is an optimal displacement from $\bar{\nu}$;
- (2) for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma \in \Gamma(\bar{\nu}, \nu)$,

$$U(\nu) - U(\bar{\nu}) \geq \int_{\mathbb{R}^d} \langle p(x), y - x \rangle d\sigma(x, y) - \delta \cdot \Delta_{\sigma} + o(\Delta_{\sigma}),$$

$$\text{where } \Delta_{\sigma} := \left(\int |x - y|^2 d\sigma(x, y) \right)^{1/2}.$$

Analogously, the set of δ -viscosity superdifferentials of U at $\bar{\nu}$ is given by $\partial_{\delta}^{+} U(\bar{\nu}) := -\partial_{\delta}^{-}(-U)(\bar{\nu})$.

In case $U : [a, b] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is partially differentiable w.r.t. t , in the following proposition we prove a relation between the pair composed by the (partial) derivative in (a, b) and the (partial) δ -differential in $\mathcal{P}_2(\mathbb{R}^d)$, with the pair of total δ -differential in $(a, b) \times \mathcal{P}_2(\mathbb{R}^d)$.

Proposition 3.5. Let $a, b \in \mathbb{R}$, $a < b$, $(\bar{t}, \bar{\nu}) \in (a, b) \times \mathcal{P}_2(\mathbb{R}^d)$ and $\delta \geq 0$. Let $U : [a, b] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be partially differentiable w.r.t. $t \in (a, b)$ at $(\bar{t}, \bar{\nu})$, so that there exists $\partial_t U(\bar{t}, \bar{\nu})$. Then

$$(3.3) \quad (p_{\bar{t}}, p_{\bar{\nu}}) \in \mathbf{D}_{\delta}^{-} U(\bar{t}, \bar{\nu}) \Rightarrow \begin{cases} p_{\bar{t}} \in \overline{B(\partial_t U(\bar{t}, \bar{\nu}), \delta)}, \\ p_{\bar{\nu}} \in \partial_{\delta}^{-} U(\bar{t}, \bar{\nu}). \end{cases}$$

Analogously, we have

$$(3.4) \quad (p_{\bar{t}}, p_{\bar{\nu}}) \in \mathbf{D}_{\delta}^{+} U(\bar{t}, \bar{\nu}) \Rightarrow \begin{cases} p_{\bar{t}} \in \overline{B(\partial_t U(\bar{t}, \bar{\nu}), \delta)}, \\ p_{\bar{\nu}} \in \partial_{\delta}^{+} U(\bar{t}, \bar{\nu}). \end{cases}$$

Proof. We prove (3.3) since the proof of (3.4) is analogous. Let $(p_{\bar{t}}, p_{\bar{\nu}}) \in \mathbf{D}_{\delta}^{-} U(\bar{t}, \bar{\nu})$. Then, by Definition 3.3, $p_{\bar{\nu}}$ is an optimal displacement from $\bar{\nu}$ and for any $(t, \nu) \in [a, b] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma \in \Gamma(\bar{\nu}, \nu)$, we have

$$(3.5) \quad U(t, \nu) - U(\bar{t}, \bar{\nu}) \geq p_{\bar{t}}(t - \bar{t}) + \int_{\mathbb{R}^d} \langle p_{\bar{\nu}}(x), y - x \rangle d\sigma(x, y) - \delta \cdot \Delta_{t, \sigma} + o(\Delta_{t, \sigma}).$$

We first prove that $p_{\bar{t}} \in \overline{B(\partial_t U(\bar{t}, \bar{\nu}), \delta)}$: by choosing $\nu = \bar{\nu}$ and $\sigma = (\text{id}, \text{id})_{\#} \bar{\nu}$ in (3.5), we get

$$U(t, \bar{\nu}) - U(\bar{t}, \bar{\nu}) \geq p_{\bar{t}}(t - \bar{t}) - \delta|t - \bar{t}| + o(|t - \bar{t}|),$$

for any $t \in [a, b]$. By partial differentiability of U w.r.t. t , we can expand by Taylor's formula the left-hand side and get

$$\partial_t U(\bar{t}, \bar{\nu}) \cdot (t - \bar{t}) + o(|t - \bar{t}|) \geq p_{\bar{t}}(t - \bar{t}) - \delta|t - \bar{t}| + o(|t - \bar{t}|),$$

for any $t \in [a, b]$. Thus, dividing by $|t - \bar{t}|$ and letting $|t - \bar{t}| \rightarrow 0$, we finally obtain

$$\partial_t U(\bar{t}, \bar{\nu}) - \delta \leq p_{\bar{t}} \leq \partial_t U(\bar{t}, \bar{\nu}) + \delta.$$

It remains to prove that $p_{\bar{\nu}} \in \partial_{\delta}^{-} U(\bar{t}, \bar{\nu})$. By choosing $t = \bar{t}$ in (3.5), we have

$$U(\bar{t}, \nu) - U(\bar{t}, \bar{\nu}) \geq \int_{\mathbb{R}^d} \langle p_{\bar{\nu}}(x), y - x \rangle d\sigma(x, y) - \delta \cdot \Delta_{\sigma} + o(\Delta_{\sigma}),$$

for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\sigma \in \Gamma(\bar{\nu}, \nu)$. By Definition 3.4, we conclude. \square

Remark 3.6. In particular, if $\delta = 0$, by Proposition 3.5, we have

$$\mathbf{D}_0^{-} U(\bar{t}, \bar{\nu}) \subset \{(\partial_t U(\bar{t}, \bar{\nu}), p) : p \in \partial_0^{-} U(\bar{t}, \bar{\nu})\},$$

and the inclusion may be strict.

3.2. Viscosity solutions and the Comparison principle. Using the above notion of sub/super differential, we are able to give a meaning of viscosity solution to the Hamilton Jacobi Bellman equation (3.1).

Definition 3.7 (Viscosity solutions). A function $U : [a, b] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is

- (1) a *subsolution* of (3.1) if U is u.s.c. and there exists a map $C : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$, $C(\cdot)$ bounded on bounded sets, such that

$$p_t + \mathcal{H}(\nu, p_\nu) \geq -C(\nu)\delta,$$

for all $(t, \nu) \in [a, b] \times \mathcal{P}_2(\mathbb{R}^d)$, $(p_t, p_\nu) \in \mathbf{D}_\delta^+ U(t, \nu)$ and $\delta > 0$;

- (2) a *supersolution* of (3.1) if U is l.s.c. and there exists a map $C : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$, $C(\cdot)$ bounded on bounded sets, such that

$$p_t + \mathcal{H}(\nu, p_\nu) \leq C(\nu)\delta,$$

for all $(t, \nu) \in [a, b] \times \mathcal{P}_2(\mathbb{R}^d)$, $(p_t, p_\nu) \in \mathbf{D}_\delta^- U(t, \nu)$ and $\delta > 0$;

- (3) a *solution* of (3.1) if U is both a supersolution and a subsolution.

We state here the Comparison Principle for (3.1) that will be used in this paper. Its proof is postponed to Appendix B.

Theorem 3.8 (Comparison). *Let $u_i : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, $i = 1, 2$, be such that*

- (1) $t \mapsto u_i(t, \nu)$ is continuous in $[0, T]$ for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $i = 1, 2$;
(2) $\nu \mapsto u_i(t, \nu)$ is \mathbf{m}_2 -locally Lipschitz in $\mathcal{P}_2(\mathbb{R}^d)$, uniformly w.r.t. t , i.e. for any $R > 0$ there exists $L_{R,i} > 0$ such that

$$|u_i(t, \nu_1) - u_i(t, \nu_2)| \leq L_{R,i} W_2(\nu_1, \nu_2),$$

for any $\nu_i \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathbf{m}_2(\nu_i) \leq R$, $i = 1, 2$, for any $t \in [0, T]$.

- (3) u_1 and u_2 are respectively a viscosity subsolution and supersolution of

$$(3.6) \quad \partial_t U(t, \nu) + \mathcal{H}(\nu, D_\nu U(t, \nu)) = 0$$

according with Definition 3.7, where $\mathcal{H}(\nu, p)$ is defined for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L_\nu^2(\mathbb{R}^d)$.

Assume that there exists a continuous nondecreasing map $\omega_{\mathcal{H}} : \mathbb{R}^2 \rightarrow [0, +\infty)$ such that $\omega_{\mathcal{H}}(0, 0) = 0$ and

$$(3.7) \quad |\mathcal{H}(\nu_1, \lambda p_{\nu_1, \nu_2}) - \mathcal{H}(\nu_2, \lambda q_{\nu_1, \nu_2})| \leq \omega_{\mathcal{H}}(W_2(\nu_1, \nu_2), \lambda W_2^2(\nu_1, \nu_2)),$$

for all $\lambda > 0$, $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$, with p_{ν_1, ν_2} and q_{ν_1, ν_2} as in (B.1). Then

$$\inf_{(t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)} \{u_2(t, \nu) - u_1(t, \nu)\} = \inf_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} \{u_2(T, \nu) - u_1(T, \nu)\}.$$

4. REACHABILITY THROUGH LYAPUNOV FUNCTIONS

4.1. Lyapunov function and asymptotic reachability. We introduce Lyapunov functions for a suitable Lipschitz continuity notion that we call \mathbf{m}_2 -local Lipschitzianity.

Definition 4.1 (\mathbf{m}_2 -local Lipschitzianity). Consider a map $\phi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$. We say that

- (1) ϕ is \mathbf{m}_2 -locally Lipschitz if for any $R > 0$ there exists $L_R > 0$ such that

$$(4.1) \quad |\phi(\nu_1) - \phi(\nu_2)| \leq L_R W_2(\nu_1, \nu_2),$$

for any $\nu_i \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathbf{m}_2(\nu_i) \leq R$, $i = 1, 2$;

- (2) ϕ is locally Lipschitz continuous if for any $\nu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ there exist $R_{\nu_1} > 0$ and $L_{\nu_1} > 0$ such that

$$|\phi(\nu_1) - \phi(\nu_2)| \leq L_{\nu_1} W_2(\nu_1, \nu_2),$$

for any $\nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $W_2(\nu_1, \nu_2) \leq R_{\nu_1}$;

(3) ϕ is *Lipschitz continuous* if there exists $L > 0$ such that

$$|\phi(\nu_1) - \phi(\nu_2)| \leq L W_2(\nu_1, \nu_2),$$

for any $\nu_i \in \mathcal{P}_2(\mathbb{R}^d)$, $i = 1, 2$.

Remark 4.2. Notice that, according to Definition 4.1, if ϕ satisfies either (1) or (2), then ϕ is continuous in the W_2 -metric. Furthermore, if $\phi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is Lipschitz continuous, then it is trivially \mathfrak{m}_2 -locally Lipschitz. The viceversa does not hold. As an example, the function $\tilde{\phi} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, defined by $\tilde{\phi}(\nu) := \frac{1}{2}W_2^2(\nu, \delta_0) = \frac{1}{2}\mathfrak{m}_2^2(\nu)$, is not Lipschitz continuous, however it is \mathfrak{m}_2 -locally Lipschitz. Indeed, by triangular inequality we have for any $\nu_i \in \mathcal{P}_2(\mathbb{R}^d)$, $i = 1, 2$,

$$\begin{aligned} |\tilde{\phi}(\nu_1) - \tilde{\phi}(\nu_2)| &\leq \frac{1}{2} |\mathfrak{m}_2(\nu_1) - \mathfrak{m}_2(\nu_2)| (\mathfrak{m}_2(\nu_1) + \mathfrak{m}_2(\nu_2)) \\ &\leq \frac{1}{2} W_2(\nu_1, \nu_2) (\mathfrak{m}_2(\nu_1) + \mathfrak{m}_2(\nu_2)). \end{aligned}$$

We have also the following relation.

Lemma 4.3. *If $\phi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is \mathfrak{m}_2 -locally Lipschitz, then it is locally Lipschitz continuous.*

Proof. Let $\nu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and fix $R_1 > 0$. Define $R := R_1 + \mathfrak{m}_2(\nu_1)$ and L_R be as in Definition 4.1(1). Then, taken $\nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $W_2(\nu_1, \nu_2) \leq R_1$, we have

$$\mathfrak{m}_2(\nu_2) = W_2(\nu_2, \delta_0) \leq W_2(\nu_1, \nu_2) + \mathfrak{m}_2(\nu_1) \leq R_1 + \mathfrak{m}_2(\nu_1) = R.$$

We can thus apply the \mathfrak{m}_2 -local Lipschitzianity of ϕ to conclude

$$|\phi(\nu_1) - \phi(\nu_2)| \leq L_R W_2(\nu_1, \nu_2).$$

□

In the following, F is the set-valued map of Section 2.2 and \mathcal{H}_F is given by (3.2).

Definition 4.4 (Lyapunov function). Given $\varphi : [0, +\infty) \rightarrow \mathbb{R}$, we say that a \mathfrak{m}_2 -locally Lipschitz map $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ is a *Lyapunov function for F with rate φ* if there exists a map $C : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$ bounded on bounded sets such that the following *Hamilton-Jacobi inequality* holds

$$(4.2) \quad \varphi(V(\nu)) + \mathcal{H}_F(\nu, p) \leq C(\nu)\delta, \quad \forall \nu \in \mathcal{P}_2(\mathbb{R}^d), \forall p \in \partial_{\delta}^- V(\nu), \forall \delta > 0,$$

where \mathcal{H}_F is the Hamiltonian given in (3.2).

To carry on our main study of Section 4.2, we are interested in the following function $\varphi : [0, +\infty) \rightarrow \mathbb{R}$, that will be associated with the characterization of asymptotic reachability (see next Definition 4.5):

$$(4.3) \quad \varphi(y) := \alpha y,$$

where $\alpha > 0$ is a given parameter. Analogously, in Section 4.3 we will present the case

$$(4.4) \quad \varphi(y) := \alpha,$$

with $\alpha > 0$ given; this one will be involved, instead, in the characterization of reachability.

Definition 4.5. Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, we say that

- (1) ν is *reachable* if for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exist $T > 0$ and $\mu \in \mathcal{A}_{[0, T]}(\mu_0)$ such that $\mu_{t=T} = \nu$.
- (2) ν is *weakly asymptotically reachable* if for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists $\mu \in \mathcal{A}_{[0, +\infty)}(\mu_0)$ such that $\mu_t \rightarrow \nu$ narrowly as $t \rightarrow +\infty$;
- (3) ν is *strongly asymptotically reachable* if for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists $\mu \in \mathcal{A}_{[0, +\infty)}(\mu_0)$ such that $\mu_t \xrightarrow{W_2} \nu$ as $t \rightarrow +\infty$, i.e. $W_2(\mu_t, \nu) \rightarrow 0$.

4.2. Main results. Inspired by the classical finite-dimensional setting presented in [2], we give sufficient conditions for asymptotic reachability via Lyapunov functions using a viscosity-type approach. A viscosity approach for control problems in Wasserstein spaces is used also in [13] concerning the problem of viability/invariancy.

In the whole section, we consider φ as in (4.3) with $\alpha > 0$ a given parameter.

Lemma 4.6. *Let $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a \mathfrak{m}_2 -locally Lipschitz map and $T > 0$. The map V is a Lyapunov function for F with rate φ if and only if the function $S : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$, defined by $S(t, \nu) := e^{\alpha t} V(\nu)$, is a partial-viscosity supersolution of*

$$(4.5) \quad \partial_t U(t, \nu) + \mathcal{H}_F(\nu, D_\nu U(t, \nu)) = 0,$$

i.e. there exists a map $C_T : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$, $C_T(\cdot)$ bounded on bounded sets, such that

$$(4.6) \quad \partial_t S(t, \nu) + \mathcal{H}_F(\nu, \xi) \leq C_T(\nu) \delta, \quad \forall (t, \nu) \in (0, T) \times \mathcal{P}_2(\mathbb{R}^d), \forall \xi \in \partial_\delta^- S(t, \nu), \forall \delta > 0.$$

Proof. Notice that $\partial_t S(t, \nu) = \alpha e^{\alpha t} V(\nu)$ and, by Definition 3.4, we have

$$(4.7) \quad \partial_\delta^- S(t, \nu) = e^{\alpha t} \partial_{\delta e^{-\alpha t}}^- V(\nu), \quad \forall (t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d).$$

Then the relation (4.6) is equivalent to the following condition

$$\alpha e^{\alpha t} V(\nu) + e^{\alpha t} \mathcal{H}_F(\nu, p) \leq C_T(\nu) \delta, \quad \forall (t, \nu) \in (0, T) \times \mathcal{P}_2(\mathbb{R}^d), \forall p \in \partial_{\delta e^{-\alpha t}}^- V(\nu), \forall \delta > 0.$$

Dividing by $e^{\alpha t}$ and since the condition holds for any $\delta > 0$, we get the equivalence with

$$\alpha V(\nu) + \mathcal{H}_F(\nu, p) \leq C_T(\nu) \tilde{\delta}, \quad \forall \nu \in \mathcal{P}_2(\mathbb{R}^d), \forall p \in \partial_{\tilde{\delta}}^- V(\nu), \forall \tilde{\delta} > 0,$$

i.e. V is a Lyapunov function for F with rate φ . □

We point out that, even if U is partially differentiable w.r.t. $t \in (a, b)$, there is no apparent relation in general between being a viscosity supersolution according with Definition 3.7 and being a partial viscosity supersolution as in Lemma 4.6. Of course, if we modify the conditions in Definitions 3.7 and Lemma 4.6 to hold only for $\delta = 0$, then by Remark 3.6 we have that being a partial supersolution is stronger than being a supersolution of (3.1).

However, even without this requirement on δ , we can still prove the following result.

Lemma 4.7. *Let $T > 0$ and $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a Lyapunov function for F with rate φ , namely V is \mathfrak{m}_2 -locally Lipschitz and there exists a map $C : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$ bounded on bounded sets such that the following holds*

$$(4.8) \quad \alpha V(\nu) + \mathcal{H}_F(\nu, p) \leq C(\nu) \delta, \quad \forall \nu \in \mathcal{P}_2(\mathbb{R}^d), \forall p \in \partial_\delta^- V(\nu), \forall \delta > 0.$$

Then the function $S : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$, defined by $S(t, \nu) := e^{\alpha t} V(\nu)$, is a viscosity supersolution of

$$(4.9) \quad \partial_t U(t, \nu) + \mathcal{H}_F(\nu, D_\nu U(t, \nu)) = 0,$$

in the sense of Definition 3.7.

Proof. Let us fix $\delta > 0$, $(t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $(p_t, p_\nu) \in \mathbf{D}_\delta^- S(t, \nu)$; we will prove that

$$(4.10) \quad p_t + \mathcal{H}_F(\nu, p_\nu) \leq (C(\nu) e^{\alpha T} + 1) \delta,$$

which implies that S is a viscosity supersolution to (4.24). By Proposition 3.5 and recalling the equivalence (4.7), there exist $a_t \in [-1, 1]$ and $\bar{p}_\nu \in \partial_{e^{-\alpha t} \delta}^- V(\nu) \subset \partial_\delta^- V(\nu)$ such that

$$\begin{cases} p_t = \alpha e^{\alpha t} V(\nu) + \delta a_t \\ p_\nu = e^{\alpha t} \bar{p}_\nu. \end{cases}$$

So that

$$\begin{aligned} p_t + \mathcal{H}_F(\nu, p_\nu) &= \delta a_t + \alpha e^{\alpha t} V(\nu) + \mathcal{H}_F(\nu, e^{\alpha t} \bar{p}_\nu) \\ &\leq \delta + e^{\alpha t} C(\nu) \delta \leq (C(\nu) e^{\alpha T} + 1) \delta, \end{aligned}$$

since (4.8) holds for \bar{p}_ν . Hence we get (4.10) and the proof is complete. □

Proposition 4.8. *Assume Hypothesis 2.1. Let $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be \mathfrak{m}_2 -locally Lipschitz. The map V is a Lyapunov function for F with rate φ if and only if for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists $\mu \in \mathcal{A}_{[0, +\infty)}(\mu_0)$ such that the function $\mathcal{S} : [0, +\infty) \rightarrow [0, +\infty)$, defined by $\mathcal{S}(t) := e^{\alpha t} V(\mu_t)$, is nonincreasing.*

Proof. First, assume that V is a Lyapunov function for F with rate φ . Fix $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $T > 0$ and consider the functional $g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ defined by

$$g(\nu) := e^{\alpha T} V(\nu), \quad \text{for any } \nu \in \mathcal{P}_2(\mathbb{R}^d).$$

Let $U_g : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, defined by

$$(4.11) \quad U_g(t, \nu) := \inf \{g(\mu_T) : \mu \in \mathcal{A}_{[t, T]}(\nu)\}.$$

Since g is \mathfrak{m}_2 -locally Lipschitz, by Lemma B.2 the function U_g is continuous in $[0, T]$, \mathfrak{m}_2 -locally Lipschitz in $\mathcal{P}_2(\mathbb{R}^d)$ and it is a viscosity solution of

$$(4.12) \quad \begin{cases} \partial_t U(t, \nu) + \mathcal{H}_F(\nu, D_\nu U(t, \nu)) = 0, \\ U(T, \nu) = g(\nu), \end{cases}$$

according with Definition 3.7. Moreover, by Lemma 4.7, the function S is a viscosity supersolution of (4.12). Recalling Lemma B.1, by the Comparison Principle (Theorem 3.8) we get

$$U_g(t, \nu) \leq S(t, \nu), \quad \forall (t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d).$$

As recalled in Section 2.2, by the Dynamic Programming Principle (cf. [21, Corollary 2 and Proposition 3]), there exists $\mu \in \mathcal{A}_{[0, T]}(\mu_0)$ such that $U_g(t, \mu_t) = U_g(s, \mu_s)$ for all $0 \leq s \leq t \leq T$. Thus, we have

$$(4.13) \quad e^{\alpha T} V(\mu_T) = U_g(T, \mu_T) = U_g(s, \mu_s) \leq S(s, \mu_s) = e^{\alpha s} V(\mu_s), \quad \forall s \in [0, T].$$

By the arbitrariness of $T > 0$ and since the gluing of admissible trajectories is still an admissible trajectory, we conclude that there exists $\mu \in \mathcal{A}_{[0, +\infty)}(\mu_0)$ such that the map \mathcal{S} , is nonincreasing in $[0, +\infty)$. This proves the left-to-right implication in Proposition 4.8.

We now prove the reverse implication. Let $T > 0$ be arbitrarily fixed. In view of the equivalence result of Lemma 4.6, in order to prove that V is a Lyapunov function for F with rate φ , we proceed as in the proof of [21, Theorem 3, Claim 2]. Fix $(\bar{t}, \bar{\nu}) \in (0, T) \times \mathcal{P}_2(\mathbb{R}^d)$, $\delta > 0$, and take $p_{\bar{\nu}} \in \partial_{\bar{\delta}}^- V(\bar{\nu})$. By assumption, up to a time variable change, there exists $\mu \in \mathcal{A}_{[\bar{t}, +\infty)}(\bar{\nu})$ such that $S(t, \mu_t) \leq S(s, \mu_s)$ for all $\bar{t} \leq s \leq t \leq T$.

By Definition 3.4, for all $\pi_t \in \Gamma(\bar{\nu}, \mu_t)$ we have

$$\begin{aligned} 0 &\geq S(t, \mu_t) - S(\bar{t}, \bar{\nu}) = e^{\alpha t} V(\mu_t) - e^{\alpha \bar{t}} V(\bar{\nu}) \\ &= (e^{\alpha t} - e^{\alpha \bar{t}}) V(\mu_t) + e^{\alpha \bar{t}} (V(\mu_t) - V(\bar{\nu})) \\ &\geq (e^{\alpha t} - e^{\alpha \bar{t}}) V(\mu_t) + e^{\alpha \bar{t}} \left(\int \langle p_{\bar{\nu}}(x), y - x \rangle d\pi_t(x, y) - \delta \cdot \Delta_{\pi_t} + o(\Delta_{\pi_t}) \right) \end{aligned}$$

where $\Delta_{\pi_t} := \sqrt{\int |x - y|^2 d\pi_t}$. We divide by $|t - \bar{t}|$ and pass to the $\liminf_{t \rightarrow \bar{t}^+}$; using the continuity of $s \mapsto \mu_s$ and of V , we get

$$0 \geq \alpha e^{\alpha \bar{t}} V(\bar{\nu}) + e^{\alpha \bar{t}} \liminf_{t \rightarrow \bar{t}^+} \frac{1}{t - \bar{t}} \left(\int \langle p_{\bar{\nu}}(x), y - x \rangle d\pi_t(x, y) - \delta \cdot \Delta_{\pi_t} + o(\Delta_{\pi_t}) \right),$$

hence

$$(4.14) \quad \liminf_{t \rightarrow \bar{t}^+} \frac{\Delta_{\pi_t}}{t - \bar{t}} \left(\delta - \frac{o(\Delta_{\pi_t})}{\Delta_{\pi_t}} \right) \geq \alpha V(\bar{\nu}) + \liminf_{t \rightarrow \bar{t}^+} \frac{1}{t - \bar{t}} \int \langle p_{\bar{\nu}}(x), y - x \rangle d\pi_t(x, y).$$

Let $\mathbf{e}_t : \mathbb{R}^d \times C([\bar{t}, T]) \rightarrow \mathbb{R}^d$ be the evaluation operator at time $t \in [\bar{t}, T]$, defined by $\mathbf{e}_t(x, \gamma) := \gamma(t)$. Let $\boldsymbol{\eta} \in \mathcal{P}(\mathbb{R}^d \times C([\bar{t}, T]))$ be such that $\mu_t = \mathbf{e}_{t\#} \boldsymbol{\eta}$ and such that for $\boldsymbol{\eta}$ -a.e. $(x, \gamma) \in \mathbb{R}^d \times C([\bar{t}, T])$ we have $\gamma(\bar{t}) = x$ and γ is an absolutely continuous solution of $\dot{\gamma}(t) = v_t(\gamma(t))$, where v_t is an admissible Borel vector field driving μ , i.e. $v_t \in L_{\mu_t}^2(\mathbb{R}^d; \mathbb{R}^d)$, $v_t(x) \in F(x, \mu_t)$ and

$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$ for $t > \bar{t}$. We recall that such a measure $\boldsymbol{\eta}$ exists in view of the Superposition Principle [1, Theorem 8.2.1] and thanks to the growth property of F in (2.3). Define the plan $\tilde{\boldsymbol{\pi}}_t := (\mathbf{e}_{\bar{t}}, \mathbf{e}_t)_{\#} \boldsymbol{\eta} \in \Gamma(\bar{\nu}, \mu_t)$, then by Jensen's inequality we have

$$\begin{aligned} \frac{1}{(t-\bar{t})^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\tilde{\boldsymbol{\pi}}_t(x, y) &= \int_{\mathbb{R}^d \times C([\bar{t}, T])} \left| \frac{\gamma(t) - \gamma(\bar{t})}{t-\bar{t}} \right|^2 d\boldsymbol{\eta}(x, \gamma) \\ &= \int_{\mathbb{R}^d \times C([\bar{t}, T])} \left| \frac{1}{t-\bar{t}} \int_{\bar{t}}^t \dot{\gamma}(s) ds \right|^2 d\boldsymbol{\eta}(x, \gamma) \\ &\leq \int_{\mathbb{R}^d \times C([\bar{t}, T])} \frac{1}{t-\bar{t}} \int_{\bar{t}}^t |\dot{\gamma}(s)|^2 ds d\boldsymbol{\eta}(x, \gamma) \\ &\leq \int_{\mathbb{R}^d \times C([\bar{t}, T])} \|\dot{\gamma}\|_{L^\infty([\bar{t}, T])}^2 d\boldsymbol{\eta} \leq D(\bar{\nu}), \end{aligned}$$

where $D(\bar{\nu})$ comes from Lemma A.1. We then get

$$(4.15) \quad \frac{\Delta \tilde{\boldsymbol{\pi}}_t}{t-\bar{t}} = \sqrt{\frac{1}{(t-\bar{t})^2} \int |x-y|^2 d\tilde{\boldsymbol{\pi}}_t} \leq \sqrt{D(\bar{\nu})} =: C(\bar{\nu}),$$

where $C : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$ is bounded on bounded sets.

We now analyze the right-hand side in (4.14) with the choice of $\tilde{\boldsymbol{\pi}}_t$. We have

$$\begin{aligned} \frac{1}{t-\bar{t}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle p_{\bar{\nu}}(x), y-x \rangle d\tilde{\boldsymbol{\pi}}_t(x, y) &= \int_{\mathbb{R}^d \times C([\bar{t}, T])} \langle p_{\bar{\nu}}(x), \frac{\gamma(t) - \gamma(\bar{t})}{t-\bar{t}} \rangle d\boldsymbol{\eta}(x, \gamma) \\ &= \int_{\mathbb{R}^d \times C([\bar{t}, T])} \langle p_{\bar{\nu}}(x), \frac{1}{t-\bar{t}} \int_{\bar{t}}^t \dot{\gamma}(s) ds \rangle d\boldsymbol{\eta}(x, \gamma) \\ &\geq \int_{\mathbb{R}^d \times C([\bar{t}, T])} \frac{1}{t-\bar{t}} \int_{\bar{t}}^t \inf_{v \in F(\gamma(s), \mu_s)} \langle p_{\bar{\nu}}(x), v \rangle ds d\boldsymbol{\eta}(x, \gamma) \\ &\geq \int_{\mathbb{R}^d \times C([\bar{t}, T])} \frac{1}{t-\bar{t}} \int_{\bar{t}}^t \left(\inf_{v \in F(x, \bar{\nu})} \langle p_{\bar{\nu}}(x), v \rangle - \delta_s |p_{\bar{\nu}}(x)| \right) ds d\boldsymbol{\eta}(x, \gamma), \end{aligned}$$

where $\delta_s = L(W_2(\mu_s, \bar{\nu}) + |\gamma(s) - x|)$. By taking the $\liminf_{t \rightarrow \bar{t}^+}$ and applying Fatou's Lemma (recall the estimate in Lemma A.1, that $p_{\bar{\nu}} \in L^2_{\bar{\nu}}$ and the growth property of F in (2.3)), we have

$$\begin{aligned} \liminf_{t \rightarrow \bar{t}^+} \frac{1}{t-\bar{t}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle p_{\bar{\nu}}(x), y-x \rangle d\tilde{\boldsymbol{\pi}}_t(x, y) &\geq \\ &\geq \int_{\mathbb{R}^d \times C([\bar{t}, T])} \liminf_{t \rightarrow \bar{t}^+} \frac{1}{t-\bar{t}} \int_{\bar{t}}^t \left(\inf_{v \in F(x, \bar{\nu})} \langle p_{\bar{\nu}}(x), v \rangle - \delta_s |p_{\bar{\nu}}(x)| \right) ds d\boldsymbol{\eta}(x, \gamma) \\ &= \int_{\mathbb{R}^d} \inf_{v \in F(x, \bar{\nu})} \langle p_{\bar{\nu}}(x), v \rangle d\bar{\nu}(x), \end{aligned}$$

where the last equality follows by the continuity of $s \mapsto \delta_s$. Thus, by (4.14) applied to the plan $\tilde{\boldsymbol{\pi}}_t$, using (4.15) and the estimate above, we get

$$(4.16) \quad C(\bar{\nu})\delta \geq \alpha V(\bar{\nu}) + \int_{\mathbb{R}^d} \inf_{v \in F(x, \bar{\nu})} \langle p_{\bar{\nu}}(x), v \rangle d\bar{\nu}(x) = \varphi(V(\bar{\nu})) + \mathcal{H}_F(\bar{\nu}, p_{\bar{\nu}}).$$

We conclude that V is a Lyapunov function for F with rate φ . \square

Following [2, 4], we can give another characterization for Lyapunov functions in terms of viability conditions over their epigraph, as follows.

We denote by $\operatorname{epi}(U)$ the *epigraph* of a function $U : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, i.e. the set

$$\operatorname{epi}(U) := \left\{ (\nu, r) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} : r \geq U(\nu) \right\}.$$

Corollary 4.9. *Assume Hypothesis 2.1. Let $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a \mathfrak{m}_2 -locally Lipschitz map. Then V is a Lyapunov function for F with rate φ if and only if $\text{epi}(V)$ is viable for the coupled dynamics*

$$(4.17) \quad \begin{cases} \partial_t \mu_t + \text{div}(v_t \mu_t) = 0, & t > 0, & \text{with } v_t(x) \in F(x, \mu_t), \\ y'(t) = -\varphi(y(t)), & t > 0. \end{cases}$$

Proof. Recall that, by definition, $\text{epi}(V)$ is viable for the coupled dynamics (4.17) if and only if the following condition holds

$$(V1) \quad \begin{aligned} & \forall (\mu_0, y_0) \in \text{epi}(V) \text{ there exists a solution } (\mu, y) \text{ of (4.17) with initial condition } (\mu_0, y_0), \\ & \text{i.e. } \mu \in \mathcal{A}_{[0, +\infty)}(\mu_0) \text{ and } y(t; y_0) = e^{-\alpha t} y_0, \text{ such that } (\mu_t, y(t; y_0)) \in \text{epi}(V) \text{ for all } t \geq 0, \\ & \text{i.e. } V(\mu_t) \leq e^{-\alpha t} V(\mu_0) \text{ for all } t \geq 0. \end{aligned}$$

Notice that condition (V1) is equivalent to the following one, where the initial condition is taken only at the boundary of $\text{epi}(V)$:

$$(V2) \quad \begin{aligned} & \forall \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \text{ there exists } \mu \in \mathcal{A}_{[0, +\infty)}(\mu_0) \text{ such that} \\ & (\mu_t, y(t; V(\mu_0))) = (\mu_t, e^{-\alpha t} V(\mu_0)) \in \text{epi}(V) \text{ for all } t \geq 0, \\ & \text{i.e. } e^{\alpha t} V(\mu_t) \leq V(\mu_0) \text{ for all } t \geq 0. \end{aligned}$$

Thanks to Proposition 4.8, to conclude we only need to prove that (V2) is equivalent to

$$(4.18) \quad \begin{aligned} & \forall \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \text{ there exists } \mu \in \mathcal{A}_{[0, +\infty)}(\mu_0) \text{ such that} \\ & \mathcal{S}(t) := e^{\alpha t} V(\mu_t), \text{ is nonincreasing.} \end{aligned}$$

Clearly (4.18) implies (V2). Conversely, let us assume that (V2) holds true. Fix $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $T > 0$. Consider for any integer $n > 0$ the subdivision of $[0, T]$ given by $k\frac{T}{n}$, for $k = 0, 1, \dots, n$. By (V2) there exists $\mu^n \in \mathcal{A}_{[0, \frac{T}{n}]}(\mu_0)$ such that

$$e^{\alpha t} V(\mu_t^n) \leq V(\mu_0), \quad \forall t \in \left[0, \frac{T}{n}\right].$$

Applying again (V2) with the initial condition $\mu_{\frac{T}{n}}^n$, we can extend μ^n on $[\frac{T}{n}, 2\frac{T}{n}]$ such that

$$e^{\alpha(t - \frac{T}{n})} V(\mu_t^n) \leq V(\mu_{\frac{T}{n}}^n), \quad \forall t \in \left[\frac{T}{n}, 2\frac{T}{n}\right].$$

By a direct iteration, we have obtained $\mu^n \in \mathcal{A}_{[0, T]}(\mu_0)$ such that

$$(4.19) \quad e^{\alpha(t - k\frac{T}{n})} V(\mu_t^n) \leq V(\mu_{k\frac{T}{n}}^n), \quad \forall t \in \left[k\frac{T}{n}, (k+1)\frac{T}{n}\right], \quad \forall k = 0, 1, \dots, (n-1).$$

By relative compactness of $\mathcal{A}_{[0, T]}(\mu_0)$ (cf. [21, Corollary 1]), we have that $\{\mu^n\}_n \subset \mathcal{A}_{[0, T]}(\mu_0)$ converges to some $\mu \in \mathcal{A}_{[0, T]}(\mu_0)$ uniformly on $[0, T]$, up to subsequences.

Fix $0 < \sigma < \tau < T$, we claim that $e^{\alpha\sigma} V(\mu_\sigma) \geq e^{\alpha\tau} V(\mu_\tau)$. Indeed, for n large enough, we can find integers $k_\sigma^n < k_\tau^n \leq T$ such that

$$0 < k_\sigma^n \frac{T}{n} \leq \sigma < (1 + k_\sigma^n) \frac{T}{n} < k_\tau^n \frac{T}{n} \leq \tau < (1 + k_\tau^n) \frac{T}{n} < T.$$

Denoting $\sigma_n := k_\sigma^n \frac{T}{n}$ and $\tau_n := k_\tau^n \frac{T}{n}$, a straightforward use of (4.19) gives

$$e^{\alpha(\tau_n - \sigma_n)} V(\mu_{\tau_n}^n) \leq V(\mu_{\sigma_n}^n).$$

Passing to the limit as $n \rightarrow +\infty$, the above inequality gives our claim $e^{\alpha\sigma} V(\mu_\sigma) \geq e^{\alpha\tau} V(\mu_\tau)$. Thus $t \mapsto e^{\alpha t} V(\mu_t)$, is nonincreasing on $[0, T]$. Applying the same argument on the intervals $[iT, (i+1)T]$, for all $i \in \mathbb{N}$, we obtain (4.18). \square

In the following, we use Proposition 4.8 to obtain the main result of the paper: a sufficient condition for asymptotic reachability via Lyapunov functions.

Theorem 4.10 (Asymptotic reachability). *Assume Hypothesis 2.1. Let $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$ and assume there exists a Lyapunov function V for F with rate φ such that $V(\bar{\nu}) = 0$, $V(\nu) > 0$ for any $\nu \neq \bar{\nu}$ and satisfying either one of the following conditions:*

- (1) V is narrowly lower semicontinuous and if $\{\nu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ s.t. $m_2(\nu_n) \rightarrow +\infty$, then $V(\nu_n) \rightarrow +\infty$, as $n \rightarrow +\infty$;
- (2) V has compact sublevels in W_2 -topology;
- (3) there exists $\varepsilon > 0$ such that, for any $\{\nu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$ with $m_{2+\varepsilon}(\nu_n) \rightarrow +\infty$, we have $V(\nu_n) \rightarrow +\infty$, as $n \rightarrow +\infty$.

If (1) holds, then $\bar{\nu}$ is weakly asymptotically reachable; while if either (2) or (3) holds, then $\bar{\nu}$ is strongly asymptotically reachable.

Proof. Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. By Proposition 4.8, there exists $\mu \in \mathcal{A}_{[0,+\infty)}(\mu_0)$ such that $0 \leq V(\mu_t) \leq e^{-\alpha t} V(\mu_0)$, for all $t \geq 0$. In particular,

$$(4.20) \quad V(\mu_t) \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

i.e. given $K > 0$ there exists $\bar{t} > 0$ such that $V(\mu_t) \leq K$ for any $t \geq \bar{t}$.

- (1) Assume (1) holds. Then, by (4.20) there exists $\tilde{K} > 0$ such that $m_2(\mu_t) \leq \tilde{K}$ for all $t \geq 0$, hence $\{\mu_t\}_{t \geq 0}$ is tight thanks to [1, Remark 5.1.5] with $\phi(x) = |x|^2$. By Prokhorov's compactness theorem [1, Theorem 5.1.3], there exists $\{t_n\}_{n \in \mathbb{N}}$ and $\bar{\mu} \in \mathcal{P}(\mathbb{R}^d)$ such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\mu_{t_n} \rightarrow \bar{\mu}$ narrowly as $n \rightarrow +\infty$. Moreover, since $\mathcal{P}(\mathbb{R}^d) \ni \nu \mapsto m_2(\nu)$ is l.s.c. with respect to narrow convergence (see [1, Lemma 5.1.7]), then in particular $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. Since V is narrowly lower semicontinuous, by (4.20) we get that

$$0 = \lim_{n \rightarrow +\infty} V(\mu_{t_n}) \geq V(\bar{\mu}) \geq 0,$$

hence $\bar{\mu} = \bar{\nu}$ and we conclude that $\bar{\nu}$ is weakly asymptotically reachable.

- (2) Assume (2) holds. By (4.20), $\{\mu_t\}_{t \geq \bar{t}}$ is a W_2 -closed subset of the W_2 -compact set $\{\nu \in \mathcal{P}_2(\mathbb{R}^d) : V(\nu) \leq K\}$, hence it is compact as well. Thus, by continuity of V in W_2 and reasoning as in the conclusion of the previous item, there exists $\{t_n\}_{n \in \mathbb{N}}$ such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\mu_{t_n} \xrightarrow{W_2} \bar{\nu}$ as $n \rightarrow +\infty$. This proves that $\bar{\nu}$ is strongly asymptotically reachable.
- (3) Assume (3) holds. Then, (4.20) implies that there exists $\tilde{K} > 0$ such that $m_{2+\varepsilon}(\mu_t) \leq \tilde{K}$ for all $t \geq 0$. Reasoning as in item (1), there exists $\{t_n\}_{n \in \mathbb{N}}$ and $\bar{\mu} \in \mathcal{P}(\mathbb{R}^d)$ such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\mu_{t_n} \rightarrow \bar{\mu}$ narrowly as $n \rightarrow +\infty$. Moreover, $\{\mu_t\}_{t \geq 0}$ has uniformly integrable 2-moments (see [1, Equation (5.1.20)]), hence $\mu_{t_n} \xrightarrow{W_2} \bar{\mu}$ as $n \rightarrow +\infty$. We conclude by the continuity of V in W_2 -metric. □

4.3. The case $\varphi(y) = \alpha$. In this section, similarly as for Section 4.2, we study sufficient conditions for (finite-time) reachability via Lyapunov functions. Differently from Section 4.2, throughout this section we consider φ as in (4.4), i.e. $\varphi(y) := \alpha$, with $\alpha > 0$ a given parameter.

Unless otherwise explicitly stated, the proofs follow the same track as the corresponding results in Section 4.2, therefore they are omitted.

Lemma 4.11. *Let $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a m_2 -locally Lipschitz map and $T > 0$. The map V is a Lyapunov function for F with rate φ if and only if the function $S : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$, defined by $S(t, \nu) := \alpha t + V(\nu)$, is a partial-viscosity supersolution of*

$$(4.21) \quad \partial_t U(t, \nu) + \mathcal{H}_F(\nu, D_\nu U(t, \nu)) = 0,$$

i.e. there exists a map $C_T : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$, $C_T(\cdot)$ bounded on bounded sets, such that

$$(4.22) \quad \partial_t S(t, \nu) + \mathcal{H}_F(\nu, \xi) \leq C_T(\nu) \delta, \quad \forall (t, \nu) \in (0, T) \times \mathcal{P}_2(\mathbb{R}^d), \forall \xi \in \partial_\delta^- S(t, \nu), \forall \delta > 0.$$

Proof. It is sufficient to notice that $\partial_\delta^- S(t, \nu) = \partial_\delta^- V(\nu)$ for any $(t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. □

Lemma 4.12. *Let $T > 0$ and $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a Lyapunov function for F with rate φ , namely V is m_2 -locally Lipschitz and there exists a map $C : \mathcal{P}_2(\mathbb{R}^d) \rightarrow (0, +\infty)$ bounded on bounded sets such that the following holds*

$$(4.23) \quad \alpha + \mathcal{H}_F(\nu, p) \leq C(\nu)\delta, \quad \forall \nu \in \mathcal{P}_2(\mathbb{R}^d), \forall p \in \partial_\delta^- V(\nu), \forall \delta > 0.$$

Then the function $S : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$, defined by $S(t, \nu) := \alpha t + V(\nu)$, is a viscosity supersolution of

$$(4.24) \quad \partial_t U(t, \nu) + \mathcal{H}_F(\nu, D_\nu U(t, \nu)) = 0,$$

in the sense of Definition 3.7.

Proof. Proceeding as in the proof of Lemma 4.7, we easily get that

$$p_t + \mathcal{H}_F(\nu, p_\nu) \leq (C(\nu) + 1)\delta,$$

for any $(t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, $(p_t, p_\nu) \in D_\delta^- S(t, \nu)$, $\delta > 0$, which gives the thesis. \square

Proposition 4.13. *Assume Hypothesis 2.1 and $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a Lyapunov function for F with rate φ . Then, for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists $\mu \in \mathcal{A}_{[0, T_0]}(\mu_0)$, with $T_0 := \frac{V(\mu_0)}{\alpha}$, such that the function $\mathcal{S} : [0, T_0] \rightarrow [0, +\infty)$, defined by $\mathcal{S}(t) := \alpha t + V(\mu_t)$, is nonincreasing.*

Proof. The proof of this result is similar to the proof of the left-to-right implication in Proposition 4.8. Fix $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $T > 0$. Due to the different rate function (4.4), in this case we consider the functional $g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ defined by $g(\nu) := \alpha T + V(\nu)$, and U_g as in (4.11). By the Dynamic Programming Principle (cf. [21, Corollary 2 and Proposition 3]), there exists $\mu \in \mathcal{A}_{[0, T]}(\mu_0)$ such that $U_g(t, \mu_t) = U_g(s, \mu_s)$ for all $0 \leq s \leq t \leq T$. By Lemma 4.12 and Theorem 3.8, we get

$$\alpha T + V(\mu_T) = U_g(T, \mu_T) = U_g(s, \mu_s) \leq S(s, \mu_s) = \alpha s + V(\mu_s), \quad \forall s \in [0, T].$$

By the arbitrariness of $T > 0$ and since the gluing of admissible trajectories is still an admissible trajectory, we conclude as follows. There exists $\mu \in \mathcal{A}_{[0, +\infty)}(\mu_0)$ such that

$$(4.25) \quad \alpha t + V(\mu_t) \leq \alpha s + V(\mu_s), \quad \forall 0 \leq s \leq t,$$

in particular this holds for $s = 0$, leading to $V(\mu_t) \leq V(\mu_0) - \alpha t$. Hence, since $V \geq 0$, we deduce that (4.25) holds with $s, t \in [0, \frac{V(\mu_0)}{\alpha}]$, thus concluding the proof. \square

Remark 4.14. Notice that, differently from Proposition 4.8, the reverse implication in Proposition 4.13 does not hold in general. Indeed, by Lemma 4.11, proving that V is a Lyapunov function for F with rate φ is equivalent to prove that $S : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ is a partial-viscosity supersolution of (4.21) for some $T > 0$ independent on the state measure μ_0 where S is evaluated.

Corollary 4.15. *Assume Hypothesis 2.1. Let $V : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a Lyapunov function for F with rate φ . Then $\text{epi}(V)$ is finite-time viable for the coupled dynamics*

$$(4.26) \quad \begin{cases} \partial_t \mu_t + \text{div}(v_t \mu_t) = 0, & t > 0, \quad \text{with } v_t(x) \in F(x, \mu_t), \\ y'(t) = -\varphi(y(t)), & t > 0; \end{cases}$$

in particular, for any $(\mu_0, y_0) \in \text{epi}(V)$ there exists a solution (μ, y) of (4.26) defined in the time interval $[0, T_0]$, with $T_0 := \frac{V(\mu_0)}{\alpha}$ and with initial condition (μ_0, y_0) , i.e. $\mu \in \mathcal{A}_{[0, T_0]}(\mu_0)$ and $y(t; y_0) = -\alpha t + y_0$ for $t \in [0, T_0]$, such that

$$(\mu_t, y(t; y_0)) \in \text{epi}(V), \quad \forall t \in [0, T_0].$$

Theorem 4.16. *Assume Hypothesis 2.1. Let $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$ and assume there exists a Lyapunov function V for F with rate φ such that $V(\bar{\nu}) = 0$, $V(\nu) > 0$ for any $\nu \neq \bar{\nu}$. Then $\bar{\nu}$ is reachable; in particular, given $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists $\mu \in \mathcal{A}_{[0, T]}(\mu_0)$ such that $\mu_T = \bar{\nu}$, with $T \leq \frac{V(\mu_0)}{\alpha}$.*

Proof. Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Thanks to Proposition 4.13, there exists $\mu \in \mathcal{A}_{[0, T_0]}(\mu_0)$, with $T_0 := \frac{V(\mu_0)}{\alpha}$, such that $0 \leq V(\mu_t) \leq -\alpha t + V(\mu_0)$, for all $t \in [0, T_0]$. In particular, the relation holds with $t = T_0$, leading to the equality $V(\mu_{T_0}) = 0$. Hence, $\mu_{T_0} = \bar{\nu}$, by the assumptions on V . \square

APPENDIX A. A PRIORI ESTIMATES

In the following, we recall standard a priori estimates (cf. e.g. [12, Lemma 3.5], [14, Appendix A]) that we used in the proof of Proposition 4.8 and in Lemma B.2. We stress that the existence of a representation $\boldsymbol{\eta}$ for a curve $\mu \in \mathcal{A}_{[a, b]}(\bar{\mu})$, as in the following statement, is guaranteed by the Superposition Principle [1, Theorem 8.2.1] and the growth property of F (2.3).

In the following, L is the Lipschitz constant of F and K_F is given by (2.2). The map $\mathbf{e}_t : \mathbb{R}^d \times C([a, b]) \rightarrow \mathbb{R}^d$ denotes the evaluation operator at time $t \in [a, b]$, defined by $\mathbf{e}_t(x, \gamma) := \gamma(t)$.

Lemma A.1. *Assume Hypothesis 2.1. Let $0 \leq a < b$, $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mu \in \mathcal{A}_{[a, b]}(\bar{\mu})$. Let $\boldsymbol{\eta} \in \mathcal{P}(\mathbb{R}^d \times C([a, b]))$ be such that*

- (1) $\mathbf{e}_{t\#}\boldsymbol{\eta} = \mu_t$ for all $t \in [a, b]$;
- (2) for $\boldsymbol{\eta}$ -a.e. $(x, \gamma) \in \mathbb{R}^d \times C([a, b])$ and a.e. $t \in [a, b]$, γ is an absolutely continuous curve such that $\gamma(a) = x$ and $\dot{\gamma}(t) \in F(\gamma(t), \mu_t)$.

Then

- (1) $\mathbf{m}_2(\mu_t) \leq e^{L(b-a)e^{L(b-a)}} C_{[a, b]}(\bar{\mu})$, for any $t \in [a, b]$;
- (2) $\int_{\mathbb{R}^d \times C([a, b])} \|\dot{\gamma}\|_{L^\infty([a, b])}^2 d\boldsymbol{\eta}(x, \gamma) \leq D_{[a, b]}(\bar{\mu})$,

where

$$(A.1) \quad \begin{aligned} C_{[a, b]}(\bar{\mu}) &:= e^{L(b-a)} (\mathbf{m}_2(\bar{\mu}) + (K_F(b-a))); \\ D_{[a, b]}(\bar{\mu}) &:= \left(K_F + L \left[e^{L(b-a)e^{L(b-a)}} C(\bar{\mu}) + C(\bar{\mu}) + L(b-a)e^{L(b-a)e^{L(b-a)}} C(\bar{\mu}) \right] \right)^2. \end{aligned}$$

Proof. For $\boldsymbol{\eta}$ -a.e. (x, γ) and a.e. $t \in [a, b]$ we have

$$\dot{\gamma}(t) \in F(\gamma(t), \mu_t) \subset F(0, \delta_0) + L(|\gamma(t)| + \mathbf{m}_2(\mu_t)) \overline{B(0, 1)},$$

hence

$$(A.2) \quad |\dot{\gamma}(t)| \leq K_F + L(|\gamma(t)| + \mathbf{m}_2(\mu_t)).$$

Moreover, for any $t \in [a, b]$,

$$|\gamma(t)| - |\gamma(a)| \leq \int_a^t |\dot{\gamma}(\tau)| d\tau \leq K_F(b-a) + L \left(\int_a^t (|\gamma(\tau)| + \mathbf{m}_2(\mu_\tau)) d\tau \right).$$

By Grönwall's inequality, this gives

$$(A.3) \quad |\gamma(t)| \leq e^{L(b-a)} \left(|\gamma(a)| + K_F(b-a) + L \int_a^t \mathbf{m}_2(\mu_\tau) d\tau \right),$$

and together with (A.2), we have

$$(A.4) \quad \|\dot{\gamma}\|_{L^\infty([a, b])} \leq K_F + L \left[\sup_{t \in [a, b]} \mathbf{m}_2(\mu_t) + e^{L(b-a)} \left(|\gamma(a)| + K_F(b-a) + L \int_a^b \mathbf{m}_2(\mu_\tau) d\tau \right) \right].$$

By taking the $L^2_{\boldsymbol{\eta}}$ -norm of (A.3) and using triangular inequality, we get

$$(A.5) \quad \mathbf{m}_2(\mu_t) \leq e^{L(b-a)} \left(\mathbf{m}_2(\bar{\mu}) + K_F(b-a) + L \int_a^t \mathbf{m}_2(\mu_\tau) d\tau \right),$$

thus, by Grönwall's inequality,

$$\mathbf{m}_2(\mu_t) \leq e^{L(b-a)e^{L(b-a)}} \left[e^{L(b-a)} (\mathbf{m}_2(\bar{\mu}) + K_F(b-a)) \right],$$

which proves item (1). By taking the L^2_η -norm of (A.4) and using triangular inequality we get item (2). \square

APPENDIX B. PROOF OF COMPARISON PRINCIPLE

In this section, devoted to the proof of Theorem 3.8, we recall some results proved in [21] which have been used in the proof of Propositions 4.8,4.13. We mention that the Comparison Principle in [21, Theorem 2] for (3.1) is proved in the context of bounded and uniformly continuous viscosity solutions $U : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$. Due to the lack of boundedness in our framework, we readapt the result in Theorem 3.8.

We recall the following notation taken from [21, Section 5.1]. Given $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$, we consider the unique optimal plan $\pi \in \Gamma_o(\nu_1, \nu_2)$ satisfying

$$\pi = \arg \min \left\{ \int_{\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y \, d\pi_x(y) \right|^2 \, d\nu_1(x) : \pi = \nu_1 \otimes \pi_x \in \Gamma_o(\nu_1, \nu_2) \right\},$$

where the notation $\{\pi_x\}_x \subset \mathcal{P}_2(\mathbb{R}^d)$ stands for the family obtained by disintegrating π w.r.t. the projection on the first marginal ν_1 (cf. [1, Theorem 5.3.1]).

Denote by $\pi^{-1} \in \Gamma_o(\nu_2, \nu_1)$ the inverse of π , i.e. $\pi^{-1} = \mathfrak{s}_\# \pi$ where $\mathfrak{s}(x, y) = (y, x)$, and by $\{\pi_y^{-1}\}_y \subset \mathcal{P}_2(\mathbb{R}^d)$ the family obtained by disintegration of π^{-1} w.r.t. the projection on the first marginal ν_2 . We define $p_{\nu_1, \nu_2} \in L^2_{\nu_1}$ and $q_{\nu_1, \nu_2} \in L^2_{\nu_2}$ by

$$(B.1) \quad \begin{cases} p_{\nu_1, \nu_2}(x) := x - \int_{\mathbb{R}^d} y \, d\pi_x(y), \\ q_{\nu_1, \nu_2}(y) := y - \int_{\mathbb{R}^d} x \, d\pi_y^{-1}(x) \end{cases}$$

In what follows, we denote by (X, d_X) and $(X \times X, d_{X \times X})$ the metric spaces given by

$$\begin{aligned} X &:= [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\ d_X(r_1, r_2) &:= \sqrt{(s_1 - s_2)^2 + W_2^2(\nu_1, \nu_2)}, \\ d_{X \times X}(z_1, z_2) &:= d_X(r_1, r_2) + d_X(\hat{r}_1, \hat{r}_2), \end{aligned}$$

with $r_i := (s_i, \nu_i) \in X$, $z_i := (r_i, \hat{r}_i) \in X \times X$. Notice that both (X, d_X) and $(X \times X, d_{X \times X})$ are complete.

Proof of Theorem 3.8. The proof follows mainly the structure of [21, proof of Theorem 2] with minor variations due to the lack of boundedness assumption on u_i .

Let

$$A := \inf_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} \{u_2(T, \nu) - u_1(T, \nu)\}.$$

If $A = -\infty$, the statement is trivially verified. So let $A \neq -\infty$; since $u_1 - A$ is still a subsolution of (3.6), we can assume $A = 0$ with no loss of generality. Let

$$-\xi := \inf_{(t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)} \{u_2(t, \nu) - u_1(t, \nu)\}$$

and notice that by assumptions Theorem 3.8(1),(2) we have $-\xi \neq -\infty$. Assume by contradiction that $-\xi < 0$ and fix $(t_0, \nu_0) \in X$ such that

$$u_2(t_0, \nu_0) - u_1(t_0, \nu_0) < -\frac{\xi}{2}.$$

By (1) we can assume $t_0 \neq 0$; fix $\sigma > 0$ such that $\frac{2\sigma}{t_0} \leq \frac{\xi}{8}$, and let $R > 0$. Given $\varepsilon, \eta > 0$ we define the functional $\Phi_{\varepsilon, \eta} : X \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} \Phi_{\varepsilon, \eta}(s, \nu_1, t, \nu_2) &= u_2(t, \nu_2) - u_1(s, \nu_1) + \frac{1}{2\varepsilon} d_X^2((s, \nu_1), (t, \nu_2)) + \\ &\quad - \eta s + \frac{\sigma}{s} + \frac{\sigma}{t}, \end{aligned}$$

if $st \neq 0$ and $W_2(\nu_0, \nu_i) \leq R$ for $i = 1, 2$; otherwise we set

$$\Phi_{\varepsilon, \eta}(s, \nu_1, t, \nu_2) = +\infty.$$

By triangular inequality, notice that if $W_2(\nu_0, \nu_i) \leq R$, then

$$(B.2) \quad m_2(\nu_i) = W_2(\nu_i, \delta_0) \leq R + m_2(\nu_0) =: \bar{R}, \quad i = 1, 2.$$

Hence, if $st \neq 0$ and $W_2(\nu_0, \nu_i) \leq R$ for $i = 1, 2$, we have

$$\begin{aligned} u_2(t, \nu_2) - u_1(s, \nu_1) &= u_2(t, \nu_2) - u_2(t, \nu_0) + u_2(t, \nu_0) - u_1(s, \nu_0) + u_1(s, \nu_0) - u_1(s, \nu_1) \\ &\geq -L_{\bar{R}, 2}W_2(\nu_2, \nu_0) + u_2(t, \nu_0) - u_1(s, \nu_0) - L_{\bar{R}, 1}W_2(\nu_0, \nu_1) \\ &\geq -(L_{\bar{R}, 2} + L_{\bar{R}, 1})R + m_2 - M_1, \end{aligned}$$

where we used assumption 2 for the first inequality and 1 for the existence of $m_i, M_i \in \mathbb{R}$ such that $m_i \leq u_i(r, \nu_0) \leq M_i$, for any $r \in [0, T]$. Hence, $\Phi_{\varepsilon, \eta}$ is bounded from below and lower semicontinuous in the complete metric space $(X \times X, d_{X \times X})$. Define $z_0 = (t_0, \nu_0, t_0, \nu_0)$; by Ekeland Variational Principle, for any $\delta > 0$ there exists $z_{\varepsilon\eta\delta} = (s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1, t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) \in X \times X$ such that, for any $z = (s, \nu_1, t, \nu_2)$ we have

$$(B.3) \quad \begin{cases} \Phi_{\varepsilon, \eta}(z_{\varepsilon\eta\delta}) \leq \Phi_{\varepsilon, \eta}(z_0), \\ \Phi_{\varepsilon, \eta}(z_{\varepsilon\eta\delta}) \leq \Phi_{\varepsilon, \eta}(z) + \delta d_{X \times X}(z, z_{\varepsilon\eta\delta}). \end{cases}$$

In particular, $t_{\varepsilon\eta\delta}, s_{\varepsilon\eta\delta} \neq 0$ and $W_2(\nu_0, \nu_{\varepsilon\eta\delta}^i) \leq R$, $i = 1, 2$.

Let $z_1 = (s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1, s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1)$ and $z_2 = (t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2, t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2)$. By summing up the second inequality in (B.3) with $z = z_1$ and with the choice $z = z_2$, we get

$$(B.4) \quad u_2(t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) - u_2(s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1) + u_1(t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) - u_1(s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1) + \frac{1}{\varepsilon}\rho_{\varepsilon\eta\delta}^2 \leq \rho_{\varepsilon\eta\delta}(2\delta + \eta),$$

where $\rho_{\varepsilon\eta\delta} := d_X((s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1), (t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2))$.

Recall that, thanks to assumptions 1, 2 and (B.2), we have

$$(B.5) \quad \begin{aligned} &u_i(t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) - u_i(s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1) \\ &= u_i(t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) - u_i(t_{\varepsilon\eta\delta}, \nu_0) + u_i(t_{\varepsilon\eta\delta}, \nu_0) - u_i(s_{\varepsilon\eta\delta}, \nu_0) + u_i(s_{\varepsilon\eta\delta}, \nu_0) - u_i(s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1) \\ &\geq -L_{\bar{R}, i}W_2(\nu_{\varepsilon\eta\delta}^2, \nu_0) + m_i - M_i - L_{\bar{R}, i}W_2(\nu_0, \nu_{\varepsilon\eta\delta}^1) \\ &\geq -2L_{\bar{R}, i}R + m_i - M_i, \end{aligned}$$

for $i = 1, 2$.

Fix $0 < \eta < 1$; we aim to prove that

$$(B.6) \quad \lim_{\varepsilon, \delta \rightarrow 0^+} \rho_{\varepsilon\eta\delta} = 0 \quad \text{and} \quad \lim_{\varepsilon, \delta \rightarrow 0^+} \frac{1}{\varepsilon}\rho_{\varepsilon\eta\delta}^2 = 0.$$

We start by showing that the first limit cannot be $+\infty$. By contradiction, assume that there exist sequences $\{\varepsilon_n\}_n, \{\delta_n\}_n$ with $\varepsilon_n, \delta_n \rightarrow 0^+$ such that $\lim_{n \rightarrow +\infty} \rho_{\varepsilon_n \eta \delta_n} = +\infty$. Then, for n sufficiently large we have $\varepsilon_n, \delta_n \leq \frac{1}{2}$ and, by (B.4) and (B.5), recalling that $\eta < 1$ we get

$$-2R(L_{\bar{R}, 1} + L_{\bar{R}, 2}) + m_1 + m_2 - (M_1 + M_2) + 2\rho_{\varepsilon_n \eta \delta_n}^2 < 2\rho_{\varepsilon_n \eta \delta_n}.$$

This gives a contradiction, so the first limit in (B.6) is finite.

Assume by contradiction that there exist $\alpha > 0$ and sequences $\{\varepsilon_n\}_n, \{\delta_n\}_n$ with $\varepsilon_n, \delta_n \rightarrow 0^+$ such that $\lim_{n \rightarrow +\infty} \rho_{\varepsilon_n \eta \delta_n} = 2\alpha$. Then, for n sufficiently large we have $\alpha < \rho_{\varepsilon_n \eta \delta_n} < 3\alpha$, so that from (B.4) and (B.5), we have

$$-2R(L_{\bar{R}, 1} + L_{\bar{R}, 2}) + m_1 + m_2 - (M_1 + M_2) + \frac{\alpha^2}{\varepsilon_n} \leq 3\alpha(2\delta_n + \eta).$$

Since the right hand side is bounded, while the left hand side tends to $+\infty$, we get a contradiction. This proves the first relation in (B.6).

In order to prove the second relation in (B.6), notice that from (B.4) we get

$$\begin{aligned} \frac{1}{\varepsilon} \rho_{\varepsilon\eta\delta}^2 &\leq \rho_{\varepsilon\eta\delta}(2\delta + \eta) + |u_2(t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) - u_2(s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1)| + \\ &\quad + |u_1(t_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^2) - u_1(s_{\varepsilon\eta\delta}, \nu_{\varepsilon\eta\delta}^1)|. \end{aligned}$$

By continuity of u_1, u_2 , the right hand side tends to 0 as $\varepsilon, \delta \rightarrow 0^+$, and this proves the second relation in (B.6).

The identities in (B.6) are used in the remaining part of the proof to get that for $\varepsilon, \delta, \eta > 0$ sufficiently small, we have $t_{\varepsilon\eta\delta}, s_{\varepsilon\eta\delta} \notin (0, T)$. This is done arguing by contradiction using [21, Lemma 6] together with assumption (3) in Theorem 3.8 and (3.7). Finally, we can prove that $t_{\varepsilon\eta\delta}, s_{\varepsilon\eta\delta} \neq T$ by using the definition of ξ , (B.3) and assumption (2) in Theorem 3.8. This part of the proof, being identical to [21, Theorem 2, claims 2,3], is not reported here.

Since we finally get $t_{\varepsilon\eta\delta}, s_{\varepsilon\eta\delta} \notin [0, T]$, this implies $\xi = 0$, hence the conclusion. \square

The proof of the following result is provided in [21, Theorem 3 - step1], with running cost $\mathcal{L} = 0$.

Lemma B.1. *Assume Hypothesis 2.1. The Hamiltonian \mathcal{H}_F in (3.2) satisfies the assumptions of Theorem 3.8 with $\omega_{\mathcal{H}_F}(r, s) = Ls$, where L is the Lipschitz constant of F .*

The following is an easy adaptation of [21, Proposition 4] and [21, Theorem 3 - step2] to the case of \mathfrak{m}_2 -locally Lipschitz final cost g in place of bounded and uniformly continuous. We report only the essential steps of the proof for the reader's convenience.

Lemma B.2. *Assume Hypothesis 2.1. Let $g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ be a \mathfrak{m}_2 -locally Lipschitz map and $U_g : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be the value function of the associated Mayer problem, defined by*

$$(B.7) \quad U_g(t, \nu) := \inf \{g(\mu_T) : \mu \in \mathcal{A}_{[t, T]}(\nu)\}.$$

Then the following hold

- (1) the map $t \mapsto U_g(t, \nu)$ is continuous in $[0, T]$ for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$;
- (2) the map $\nu \mapsto U_g(t, \nu)$ is \mathfrak{m}_2 -locally Lipschitz in $\mathcal{P}_2(\mathbb{R}^d)$, uniformly w.r.t. t , i.e. for any $R > 0$ there exists $L_{R, U_g} > 0$ such that

$$|U_g(t, \nu_1) - U_g(t, \nu_2)| \leq L_{R, U_g} W_2(\nu_1, \nu_2),$$

for any $\nu_i \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathfrak{m}_2(\nu_i) \leq R$, $i = 1, 2$, for any $t \in [0, T]$.

Moreover, U_g is a viscosity solution of

$$\begin{cases} \partial_t U(t, \nu) + \mathcal{H}_F(\nu, D_\nu U(t, \nu)) = 0, \\ U(T, \nu) = g(\nu) \end{cases}$$

according with Definition 3.7.

Proof. Let $R > 0$ and denote by $L_{R, g} > 0$ the local Lipschitz constant of g as in (4.1), let L be the Lipschitz constant of F as in Hypothesis 2.1. We prove item (2). Let $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ be such that $\mathfrak{m}_2(\nu_i) \leq R$ for $i = 1, 2$, let $s \in [0, T]$ and $\mu = (\mu_t)_t \in \mathcal{A}_{[s, T]}(\nu_1)$ be an optimal trajectory for the optimization problem (B.7), whose existence is granted by [21, Corollary 2]. By the Gronwall estimate in [21, Proposition 2], there exists an admissible trajectory $\theta = (\theta_t)_t \in \mathcal{A}_{[s, T]}(\nu_2)$ such that

$$W_2(\mu_t, \theta_t) \leq e^{LT+Te^{LT}} W_2(\nu_1, \nu_2), \quad \forall t \in [s, T].$$

We get

$$(B.8) \quad U_g(s, \nu_2) - U_g(s, \nu_1) \leq g(\theta_T) - g(\mu_T) \leq L_{\tilde{R}, g} e^{LT+Te^{LT}} W_2(\nu_1, \nu_2),$$

where

$$\tilde{R} = e^{LT} e^{LT+LT} (R + K_F T),$$

comes from Lemma A.1(1). By reversing the roles of ν_1 and ν_2 we get that $U_g(s, \cdot)$ is \mathfrak{m}_2 -locally Lipschitz in $\mathcal{P}_2(\mathbb{R}^d)$ uniformly w.r.t. $s \in [0, T]$.

We now prove item (1). Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $0 \leq s_1 \leq s_2 \leq T$ and $\mu = (\mu_t)_t \in \mathcal{A}_{[s_1, T]}(\nu)$ be an optimal trajectory. Since gluing of admissible trajectories is still admissible and $\nu = \mu_{s_1}$, we have

$$U_g(s_1, \nu) - U_g(s_2, \nu) = U_g(s_2, \mu_{s_2}) - U_g(s_2, \mu_{s_1}) \leq L_{R,g} e^{LT+Te^{LT}} W_2(\mu_{s_2}, \mu_{s_1}),$$

thanks to (B.8), and recalling again that, by Lemma A.1(1), there exists $R > 0$ such that

$$\sup_{t \in [0, T]} \mathfrak{m}_2(\mu_t) \leq R.$$

By absolute continuity of $t \mapsto \mu_t$ and reversing the roles of s_1, s_2 we conclude.

The proof of the last part of the statement is provided in [21, Theorem 3 - step2] with no variations. \square

REFERENCES

- [1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré, *Gradient Flows In Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics. ETH Zürich, Birkhäuser Basel, 2008.
- [2] Jean-Pierre Aubin, *Viability theory*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1991.
- [3] Jean-Pierre Aubin and Arrigo Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag Berlin Heidelberg, Germany, 1984.
- [4] Jean-Pierre Aubin and Hélène Frankowska, *Set-valued analysis*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2009.
- [5] A. Aussedat, O. Jerhaoui, and H. Zidani, *Viscosity solutions of centralized control problems in measure spaces*, ESAIM, Control Optim. Calc. Var. **30** (2024), 37. Id/No 91.
- [6] Yurii Averboukh, Antonio Marigonda, and Marc Quincampoix, *Extremal shift rule and viability property for mean field-type control systems*, J. Optim. Theory Appl. **189** (2021), no. 1, 244–270.
- [7] Zeinab Badreddine and Hélène Frankowska, *Viability and invariance of systems on metric spaces*, Nonlinear Analysis **225** (2022).
- [8] Benoît Bonnet and Hélène Frankowska, *Viability and Exponentially Stable Trajectories for Differential Inclusions in Wasserstein Spaces*, 2022 IEEE 61st Conference on Decision and Control (CDC), 2022, pp. 5086–5091.
- [9] Benoît Bonnet-Weill and Hélène Frankowska, *On the viability and invariance of proper sets under continuity inclusions in Wasserstein spaces*, SIAM J. Math. Anal. **56** (2024), no. 3, 2863–2914.
- [10] Rossana Capuani, Antonio Marigonda, and Michele Ricciardi, *Random lift of set valued maps and applications to multiagent dynamics*, Set-Valued Var. Anal. **31** (2023), no. 3, Paper No. 28, 41.
- [11] P. Cardaliaguet and M. Quincampoix, *Deterministic differential games under probability knowledge of initial condition*, International Game Theory Review **10** (2008), no. 1, 1–16.
- [12] Giulia Cavagnari and Antonio Marigonda, *Attainability property for a probabilistic target in Wasserstein spaces*, Discrete Contin. Dyn. Syst. **41** (2021), no. 2, 777–812.
- [13] Giulia Cavagnari, Antonio Marigonda, and Marc Quincampoix, *Compatibility of state constraints and dynamics for multiagent control systems*, J. Evol. Equ. **21** (2021), no. 4, 4491–4537.
- [14] Giulia Cavagnari, Stefano Lisini, Carlo Orrieri, and Giuseppe Savaré, *Lagrangian, Eulerian and Kantorovich formulations of multi-agent optimal control problems: equivalence and gamma-convergence*, J. Differential Equations **322** (2022), 268–364.
- [15] Giulia Cavagnari, Antonio Marigonda, and Benedetto Piccoli, *Averaged time-optimal control problem in the space of positive Borel measures*, ESAIM Control Optim. Calc. Var. **24** (2018), no. 2, 721–740.
- [16] C. D’Apice, R. Manzo, B. Piccoli, and V. Vespri, *Lyapunov stability for measure differential equations*, Math. Control Relat. Fields **14** (2024), no. 4, 1391–1407.
- [17] Michel Duprez, Morgan Morancey, and Francesco Rossi, *Approximate and exact controllability of the continuity equation with a localized vector field*, SIAM J. Control Optim. **57** (2019), no. 2, 1284–1311.
- [18] Wilfrid Gangbo, Truyen Nguyen, and Adrian Tudorascu, *Hamilton-Jacobi equations in the Wasserstein space*, Methods and Applications of Analysis **15** (2008), no. 2, 155–184.
- [19] Wilfrid Gangbo and Adrian Tudorascu, *On differentiability in the Wasserstein space and well-posedness for Hamilton-Jacobi equations*, Journal de Mathématiques Pures et Appliquées. **125** (2019), 119–174.
- [20] Chloé Jimenez, *Equivalence between strict viscosity solution and viscosity solution in the Wasserstein space and regular extension of the Hamiltonian in $L^2_{\mathbb{P}}$* , J. Convex Anal. **31** (2024), no. 2, 619–670.
- [21] Chloé Jimenez, Antonio Marigonda, and Marc Quincampoix, *Optimal control of multiagent systems in the Wasserstein space*, Calc. Var. Partial Differential Equations **59** (2020), no. 2, Paper No. 58.

- [22] ———, *Dynamical systems and Hamilton-Jacobi-Bellman equations on the Wasserstein space and their L^2 representations*, SIAM J. Math. Anal. **55** (2023), no. 5, 5919–5966.
- [23] Antonio Marigonda and Marc Quincampoix, *Mayer control problem with probabilistic uncertainty on initial positions*, Journal of Differential Equations **264** (2018), no. 5, 3212–3252.
- [24] Filippo Santambrogio, *Optimal transport for applied mathematicians* **87** (2015). Calculus of variations, PDEs, and modeling.
- [25] Cédric Villani, *Optimal transport* **338** (2009).

GIULIA CAVAGNARI: POLITECNICO DI MILANO, DIPARTIMENTO DI MATEMATICA, PIAZZA LEONARDO DA VINCI 32, 20133 MILANO (ITALY)

Email address: giulia.cavagnari@polimi.it

MARC QUINCAMPOIX: UMR CNRS 6205, UNIV BREST, LABORATOIRE DE MATHÉMATIQUES DE BRETAGNE ATLANTIQUE, 6 AVENUE VICTOR LE GORGEU, 29200 BREST (FRANCE)

Email address: marc.quincampoix@univ-brest.fr