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Dispersion of First Sound in a Weakly Interacting Ultracold Fermi Liquid

Thomas Repplinger, Songtao Huang, Yunpeng Ji, Nir Navon, and Hadrien Kurkjian*

At low temperature, a normal gas of unpaired spin-1/2 fermions is one of the cleanest realizations of a Fermi liquid. It is described by Landau's theory, where no phenomenological parameters are needed as the quasiparticle interaction function can be computed perturbatively in powers of the scattering length a , the sole parameter of the short-range interparticle interactions. Obtaining an accurate solution of the transport equation nevertheless requires a careful treatment of the collision kernel, as the uncontrolled error made by the relaxation time approximations increases when the temperature T drops below the Fermi temperature. Here, sound waves in the hydrodynamic regime are studied up to second order in the Chapman-Enskog's expansion. It is found that the frequency ω_q of the sound wave is shifted above its linear departure as $\omega_q = c_1 q (1 + \alpha q^2 \tau^2)$ where c_1 and q are the speed and wavenumber of the sound wave and the typical collision time τ scales as $1/a^2 T^2$. Besides the shear viscosity, the coefficient α is described by a single second-order collision time, which is computed exactly from an analytical solution of the transport equation, resulting in a positive dispersion $\alpha > 0$. The results suggest that ultracold atomic Fermi gases are an ideal experimental system for quantitative tests of second-order hydrodynamics.

1. Introduction

Landau's Fermi liquid theory is an effective theory which, when it is applicable, greatly simplifies the description of the many-body dynamics of a system of fermions, breaking it down to a single kinetic equation on a distribution of dilute quasiparticles. It is a very successful theory in describing the phenomenology of a wide class of fermionic systems, such as liquid ^3He ,^[1,2] electron gases,^[3–5] quantum gases,^[6] down to nuclear/neutron matter.^[7,8] Nevertheless, if both the quasiparticle interaction function and the collision probability^[9] are known, either from measurements^[10,11] or from a microscopic calculation,^[12] Fermi liquid theory provides quantitative predictions on dynamical properties, such as the transport coefficients.^[13–15]

In normal ^3He , despite decades of research, there is still a discrepancy between theory and experiment on the

value of the transport coefficients. Consider for instance the shear viscosity η , which scales with temperature as T^{-2} : the theory still underestimates the product ηT^2 from the measurement by more than 20%.^[1,16–19] This is due to a limited knowledge of the quasiparticle interaction function and collision probability,^[1] which are not computed from a microscopic theory, and whose experimental determination is limited to the lowest spherical harmonics. As a consequence, the exact solutions of the transport equation^[13–15] were never validated experimentally, and relaxation time approximations^[1,20] remain in use today.^[18]

Ultracold gases of fermionic gases provide exciting opportunities to quantitatively test those transport calculations.^[21] These gases behave as Fermi liquids when the s-wave scattering length a is negative and sufficiently small to open a regime of temperatures $T_c \leq T \ll T_F$ where T_c is the superfluid critical temperature and T_F the Fermi temperature. A Fermi liquid regime may also exist when a is positive and small enough to suppress three-body recombination.^[22] The quasiparticle dispersion and interaction function can be computed perturbatively in powers of $k_F |a|$ (with k_F the Fermi wavenumber).^[12] Experimentally, both the interaction strength and the temperature can be varied such that the typical collision time τ can be adjusted over several orders of magnitude,^[23,24] to explore both collisionless and hydrodynamic regimes.^[21] Using flat-bottom potentials,^[25,26] sound can be excited at very low wavevector q in homogeneous samples, and the

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propagation or attenuation of sound waves can be observed in a very controlled environment.^[24]

Theoretically, great efforts were devoted to the calculation of the viscosity at strong coupling, in particular in the unitary regime $|a| = +\infty$ ^[27–29] and exact results are available in the high temperature virial regime.^[30–34] At intermediate temperatures however, a controlled approach has not been found due to the absence of a separation of timescales between the collisional and kinetic dynamics.^[35] The temperature range of the Fermi liquid regime shrinks as the quasiparticle cross-section increases with the interaction strength,^[36] and it is eventually hidden by the onset of a superfluid phase at a critical temperature $T_c \approx 0.17T_F$ at unitarity. At temperatures low compared to T_c , sound attenuation is dominated by phonon–phonon interactions.^[27,29,37] In the weakly interacting normal phase, the transport coefficients were computed using relaxation time approximations^[6,38–40] (either in the “variational”,^[38,40] or in the original Abrikosov-Khalatnikov^[6,39] formulation), even though their uncontrolled error increases toward low temperatures. In this article, we perform an exact calculation of the transport coefficients^[13,14] to lowest order in $k_F|a|$, and show that the error of the relaxation time approximations is significant, up to 25%.

For negative values of a , the quasiparticle interactions are attractive,^[41] which prevents the emergence of a zero sound mode as in liquid ^3He . We thus lack a parameter similar to $c_0 - c_1$, where c_0, c_1 are the speed of zero and first sound respectively, to characterize the dispersion of sound, as was done in ^3He .^[16] In this work, we derive the leading-order deviation of the frequency ω_q from its linear departure $c_1 q$. To do so, we solve the transport equation to the order τ^2 of the Chapman-Enskog’s expansion. This is the so-called second-order hydrodynamics^[42–44] often used in a relativistic context to cure the acausality of the diffusion equations characteristic of dissipative hydrodynamics.^[45] We find the exact solution of the transport equation by decomposing the 3D quasiparticle distribution function on a basis of orthogonal polynomials adapted to the low-temperature limit.^[46] Remarkably, the frequency shift involves only two parameters of the collision kernel: the viscous relaxation time τ_η and a second-order viscous time t_η , which we both compute exactly. As for the dispersion of the sound branch,^[37] we find that ω_q is above its linear departure $c_1 q$, the deviation being proportional to $q^3 \tau^2$ with τ scaling as $1/a^2 T^2$.

2. Transport Equation at Low Temperature

Landau’s theory postulates that a Fermi liquid is described by a local quasiparticle distribution $n_\sigma(\mathbf{p}, \mathbf{r}, t)$, which is the number of quasiparticles of spin σ having momentum \mathbf{p} at position \mathbf{r} and time t . This distribution deviates slightly on average from its value in the Fermi sea $n_\sigma^0 = \Theta(p_F - p)$ (where p_F is the Fermi momentum and Θ the Heaviside function). The energy of an arbitrary quasiparticle configuration is expanded to second order in $\delta n_\sigma^0 = n_\sigma - n_\sigma^0$:

$$E = E_0 + \sum_{\mathbf{p}, \mathbf{r}, \sigma} \epsilon_\sigma^0(\mathbf{p}) \delta n_\sigma^0(\mathbf{p}, \mathbf{r}) + \frac{1}{2} \sum_{\mathbf{p}, \sigma, \mathbf{p}', \sigma', \mathbf{r}} f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}') \delta n_\sigma^0(\mathbf{p}, \mathbf{r}) \delta n_{\sigma'}^0(\mathbf{p}', \mathbf{r}) \quad (1)$$

In the general case, the dispersion relation ϵ_σ^0 and interaction function $f_{\sigma\sigma'}$ are phenomenological parameters usually re-expressed in terms of an effective mass and Landau parameters. In the case of a weakly interacting gas with contact interactions, these quantities can be calculated perturbatively in powers of the coupling constant $g = 4\pi a/m$ ^[12] (we use $\hbar = k_B = 1$ throughout this work). To first order in perturbation theory, we have:

$$\epsilon_\sigma^0(\mathbf{p}) = \frac{p^2}{2m} + g\rho_{\sigma'}, \quad f_{\uparrow\downarrow} = g/V, \quad f_{\sigma\sigma} = 0 \quad (2)$$

where V is the volume of the gas and ρ_σ the density of fermions of spin σ .

The time evolution of the quasiparticle distribution is described by a transport equation:

$$\frac{\partial n_\sigma}{\partial t} + \frac{\partial \epsilon_\sigma}{\partial \mathbf{p}} \cdot \frac{\partial n_\sigma}{\partial \mathbf{r}} - \frac{\partial(\epsilon_\sigma + U_\sigma)}{\partial \mathbf{r}} \cdot \frac{\partial n_\sigma}{\partial \mathbf{p}} = I_\sigma \quad (3)$$

where U_σ is an external driving field, $\epsilon_\sigma(\mathbf{p}, \mathbf{r}) = \epsilon_\sigma^0(\mathbf{p}) + \sum_{\mathbf{p}', \sigma'} f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}') \delta n_{\sigma'}^0(\mathbf{p}', \mathbf{r})$ is the local energy of the quasiparticles, and the collision integral is given in this weakly interacting limit by Fermi’s golden rule

$$I_\sigma(\mathbf{p}) = \frac{2\pi g^2}{V^2} \sum_{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4} \delta_{\mathbf{p}+\mathbf{p}_2, \mathbf{p}_3+\mathbf{p}_4} \delta(\epsilon_\sigma(\mathbf{p}) + \epsilon_{\sigma'}(\mathbf{p}_2) - \epsilon_\sigma(\mathbf{p}_3) - \epsilon_{\sigma'}(\mathbf{p}_4)) \times \left[(1 - n_\sigma(\mathbf{p}))(1 - n_{\sigma'}(\mathbf{p}_2))n_\sigma(\mathbf{p}_3)n_{\sigma'}(\mathbf{p}_4) - n_\sigma(\mathbf{p})n_{\sigma'}(\mathbf{p}_2)(1 - n_\sigma(\mathbf{p}_3))(1 - n_{\sigma'}(\mathbf{p}_4)) \right] \quad (4)$$

In this expression, all the quasiparticle distributions n are evaluated at position \mathbf{r} and time t , which reflects the assumption that collisions are local and instantaneous.

As we seek the eigenmodes of the transport equation, we assume that the drive is weak, slowly varying in both time and space, and we linearize Equation (3) around the thermal equilibrium distribution $n_{\text{eq}}(\epsilon_0) = 1/(1 + e^{(\epsilon_0 - \mu)/T})$. We focus here on the unpolarized case so μ is the common chemical potential of the two spin species and $\epsilon_0 \equiv \epsilon_\uparrow^0 = \epsilon_\downarrow^0$. We also restrict ourselves to excitations of the total density, and define the total driving field $U_{\text{tot}} = U_\uparrow + U_\downarrow$, that we decompose in Fourier space $U_{\text{tot}}(\mathbf{r}, t) = \text{Re}(U \sum_{\mathbf{q}} e^{i(\mathbf{q}\cdot\mathbf{r} - \omega t)})$. Restricting to terms of first order in temperature and drive intensity U , the transport equation obeyed by $\delta n = n_\uparrow + n_\downarrow - 2n_{\text{eq}}(\epsilon_0(\mathbf{p}))$ is, in Fourier space:

$$\left(\omega - \frac{\mathbf{p} \cdot \mathbf{q}}{m}\right) \delta n(\mathbf{p}, \mathbf{q}, \omega) + \frac{\partial n_{\text{eq}}}{\partial \epsilon_0} \frac{\mathbf{p} \cdot \mathbf{q}}{m} (g\delta\rho(\mathbf{q}, \omega) + U) = iI_{\text{lin}} \quad (5)$$

where $\delta\rho(\mathbf{q}, \omega) = (1/V) \sum_{\mathbf{p}'} \delta n(\mathbf{p}', \mathbf{q}, \omega)$ is the fluctuation of the total density about $\rho_{\text{eq}} = \rho_\uparrow + \rho_\downarrow$ and the linearized collision integral I_{lin} is given in Appendix.

For $T \ll T_F$, transport occurs in a energy shell of typical depth T around the Fermi energy ϵ_F .^[1,47] We thus reparametrize the quasiparticle distribution as

$$\delta n(\mathbf{p}) = U \frac{\partial n_{\text{eq}}}{\partial \epsilon_0} \Big|_{\epsilon_0 = \epsilon_0(\mathbf{p})} v(\epsilon, \theta)$$

$$\text{with } \epsilon = \frac{\epsilon_0(\mathbf{p}) - \mu}{T} = \frac{p^2/2m - \epsilon_F}{T} \quad (6)$$

where θ is the angle between \mathbf{p} and \mathbf{q} , and we have used the equation of state $\mu = \epsilon_F + g\rho_{\text{eq}}/2$ of the weakly interacting Fermi gas. At low temperatures the reduced energy ϵ varies from $-\infty$ to $+\infty$ as the momentum p varies from 0 to $+\infty$.

Restricting the transport equation to momenta p lying in the relevant energy shell around the Fermi surface, we obtain, in the limit $T/T_F \rightarrow 0$ at fixed ϵ and ϵ' :

$$\begin{aligned} (c - \cos \theta)v(\epsilon, \theta) - \frac{k_F a}{\pi} \cos \theta \int_{-\infty}^{+\infty} d\epsilon' g(\epsilon') \int_0^\pi \sin \theta' d\theta' v(\epsilon', \theta') \\ + \frac{i}{\omega_0 \tau} \left[\Gamma(\epsilon)v(\epsilon, \theta) + \int_{-\infty}^{+\infty} d\epsilon' \right. \\ \left. \times \int \sin \theta' d\theta' \frac{d\phi'}{2\pi} \mathcal{N}_{\text{od}}(\epsilon, \epsilon', u)v(\epsilon', \theta') \right] = -\cos \theta \end{aligned} \quad (7)$$

where $g(\epsilon) = 1/(4\cosh^2(\epsilon/2))$ is the dimensionless density of states, $\omega_0 = v_F q$ is a typical excitation frequency (with $v_F = \sqrt{2\epsilon_F/m}$ the Fermi velocity), $c = \omega/\omega_0$ is the dimensionless excitation frequency, and

$$\tau = \frac{\pi}{2ma^2 T^2} \quad (8)$$

is a typical collision time. For numerical applications, we replace τ by:

$$\tau_\sigma = \frac{\pi}{2m\sigma T^2} = \frac{\tau}{4\pi} \quad (9)$$

where $\sigma = 4\pi a^2$ is the scattering cross-section. This is because the center of the hydrodynamic to collisionless crossover occurs for $\omega_0 \tau_\sigma \simeq 1$ rather than $\omega_0 \tau \simeq 1$. The diagonal part of the collision kernel in Equation (7) is given by

$$\Gamma(\epsilon) = \pi^2 + \epsilon^2 \quad (10)$$

This dimensionless function sets the physical quasiparticle lifetime to $\tau_{\text{qp}}(\epsilon) = \tau/\Gamma(\epsilon)$. The off-diagonal part of the kernel reads

$$\begin{aligned} \mathcal{N}_{\text{od}}(\epsilon, \epsilon', u) = \frac{S(\epsilon, -\epsilon')}{\sqrt{2(1+u)}} - 2 \frac{S(\epsilon, \epsilon')}{\sqrt{2(1-u)}} \\ \text{with } S(\epsilon, \epsilon') = \frac{\epsilon - \epsilon'}{2} \frac{\cosh \frac{\epsilon}{2}}{\cosh \frac{\epsilon'}{2} \sinh \frac{\epsilon - \epsilon'}{2}} \end{aligned} \quad (11)$$

where the angular dependence comes through $u = \cos(\mathbf{p}, \mathbf{p}') = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$.

The conservation of the number of quasiparticle in a collision provides a relation between Γ and S :

$$\int_{-\infty}^{+\infty} d\epsilon' S(\epsilon, \epsilon') = \int_{-\infty}^{+\infty} d\epsilon' \frac{g(\epsilon')}{g(\epsilon)} S(\epsilon', \epsilon) = \frac{\Gamma(\epsilon)}{2} \quad (12)$$

3. Hydrodynamic Limit

The hydrodynamic limit is the regime of short collision times

$$\omega_0 \tau \ll 1 \quad (13)$$

In this regime, collisions bring the quasiparticle distribution back to equilibrium much faster than the typical time $1/\omega_0$ at which the sound wave evolves. Only the few components of the distribution that are not affected by collisions (i.e., those that belong to the zero-energy space of the collision kernel) remain significantly excited. Equation (13) defines a dynamical regime, accessible from any equilibrium state of the phase diagram (i.e., for any value of a and T). Here, we have sent a and T to 0 at fixed $\omega_0 \tau$ and finally expanded the transport equation in powers of $\omega_0 \tau$. To guarantee that the system remains in the hydrodynamic regime, it is required to set $q/k_F = 2(T/T_F)^2 (k_F a)^2 \omega_0 \tau_\sigma$ in such a way that $\omega_0 \tau_\sigma \ll 1$. For a finite-size system (of size L), q is limited to $q = \pi/L$ which in typical experiments is $\approx 0.01 k_F$.

The usual expansion to order $O(\omega_0 \tau)^1$ yields the transport coefficients entering the Fermi-liquid version of the Navier-Stokes equations. Here, we will push to second-order in the hydrodynamic expansion, that is $O(\omega_0 \tau)^2$, and retrieve the sound dispersion from the quantum analogue of Burnett's hydrodynamics.^[43,44]

To obtain the exact solution of the transport equation in the hydrodynamic limit, we expand the distribution v over a basis of orthogonal polynomials

$$v(\epsilon, \theta) = \sum_{n,l=0}^{+\infty} v_n^l P_l(\cos \theta) Q_n(\epsilon) \quad (14)$$

where P_l are the Legendre polynomials, and Q_n are orthogonal for the scalar product weighted by the density of states:

$$\int_{-\infty}^{+\infty} g(\epsilon) Q_n(\epsilon) Q_m(\epsilon) d\epsilon = \|Q_n\|^2 \delta_{nm} \quad (15)$$

The polynomials Q_n are obtained by the usual recurrence relation:

$$\epsilon Q_n = Q_{n+1} + \xi_n Q_{n-1} \quad \text{with} \quad \xi_n \equiv \frac{\|Q_n\|^2}{\|Q_{n-1}\|^2} \quad (16)$$

We choose $Q_0 = 1$ and $Q_1 = \epsilon$ as the initial condition. Note that even and odd polynomials are respectively symmetric and anti-symmetric about the Fermi surface: $Q_n(-\epsilon) = (-1)^n Q_n(\epsilon)$.

By contrast to our exact approach, the relaxation time approximations^[1,20] truncate the expansion in Equation (14) to $n = 0$, thereby neglecting the energy dependence of the quasiparticle distribution. The difference between the Abrikosov-Khalatnikov^[20] and the variational formulation^[1,30,38] lies in the treatment of the remaining ϵ dependence of the collision kernel: Abrikosov and Khalatnikov replaced it by its value in $\epsilon = 0$ (in particular they approximate the quasiparticle lifetime by its value at the Fermi level $\Gamma(\epsilon) \approx \Gamma(0)$), while the variational formulation averages it over ϵ .

Note that polynomials Q_n used here differ from the orthogonal polynomials of the momentum p used at nonvanish-

ing temperature.^[21,30] The replacement Equation (6) converts even/odd powers of p^2 into even/odd powers of ϵ .

The matrix elements of the collision kernel in the orthogonal basis $\{Q_n\}$ are given by

$$\Gamma_{nn'} = \int_{-\infty}^{+\infty} d\epsilon g(\epsilon) \Gamma(\epsilon) \frac{Q_n(\epsilon)}{\|Q_n\|^2} Q_{n'}(\epsilon) \quad (17)$$

and

$$\begin{aligned} \mathcal{N}_{nn'}^l &= \int_{-\infty}^{+\infty} d\epsilon' d\epsilon g(\epsilon) \int_{-1}^1 du P_l(u) \frac{Q_n(\epsilon)}{\|Q_n\|^2} \mathcal{N}_{\text{od}}(\epsilon, \epsilon', u) Q_{n'}(\epsilon') \\ &= \frac{2}{2l+1} S_{nn'} ((-1)^{l+n'} - 2) \end{aligned} \quad (18)$$

with

$$S_{nn'} = \int_{-\infty}^{+\infty} d\epsilon d\epsilon' g(\epsilon) S(\epsilon, \epsilon') \frac{Q_n(\epsilon)}{\|Q_n\|^2} Q_{n'}(\epsilon') \quad (19)$$

Note that the subspaces of symmetric and antisymmetric functions of ϵ (even and odd n respectively) are decoupled: $\Gamma_{nn'} = \mathcal{N}_{nn'}^l = 0$ if $n + n'$ is odd.^[48] This property is specific to the low-temperature limit where both the density of state $g(\epsilon)$ and energy integration domain are symmetric about the Fermi surface. Treating separately the odd and even orders, Γ and S are tridiagonal matrices and can be expressed analytically as

$$\begin{aligned} \Gamma_{nn'} &= (\pi^2 + \xi_{n+1} + \xi_n) \delta_{nn'} + \delta_{n-2, n'} + \delta_{n+2, n'} \xi_{n+2} \xi_{n+1} \\ S_{nn'} &= 2\pi^2 \frac{n^2 + n - 1}{4n^2 + 4n - 3} \delta_{nn'} + \frac{\delta_{n-2, n'}}{n(n-1)} + \frac{\delta_{n+2, n'} \xi_{n+2} \xi_{n+1}}{(n+2)(n+1)} \end{aligned} \quad (20)$$

with $\xi_n = \frac{\pi^2 n^4}{(2n+1)(2n-1)}$ as can be shown recursively.

The transport equation projected on the orthogonal basis reads now

$$\begin{aligned} cv_n^l - \left[\frac{l+1}{2l+3} v_n^{l+1} + \frac{l}{2l-1} v_n^{l-1} \right] - \frac{2k_F a}{\pi} \delta_{l,1} \delta_{n,0} v_0^0 \\ + \frac{i}{\omega_0 \tau} \sum_{n'} \mathcal{M}_{nn'}^l v_{n'}^l = -\delta_{l,1} \delta_{n,0} \end{aligned} \quad (21)$$

where we introduce the complete collision tensor

$$\mathcal{M}_{nn'}^l = \Gamma_{nn'} + \mathcal{N}_{nn'}^l \quad (22)$$

The conservation of the number of quasiparticles, of the energy, and momentum in a collision (see Equation (4)) generates zero-energy eigenfunctions of the collision kernel. In our orthogonal basis, this translates respectively into $\mathcal{M}_{0n'}^0 = \mathcal{M}_{n0}^0 = 0$, $\mathcal{M}_{1n'}^0 = \mathcal{M}_{n1}^0 = 0$ and $\mathcal{M}_{0n'}^1 = \mathcal{M}_{n0}^1 = 0$, for all n, n' . The corresponding equations of motion on the conserved quantities v_0^0 , v_0^1 and v_1^0 are

$$cv_0^0 - \frac{v_0^1}{3} = 0 \quad (23)$$

$$cv_0^1 - \left(1 + \frac{2k_F a}{\pi} \right) v_0^0 - \frac{2}{5} v_0^2 = -1 \quad (24)$$

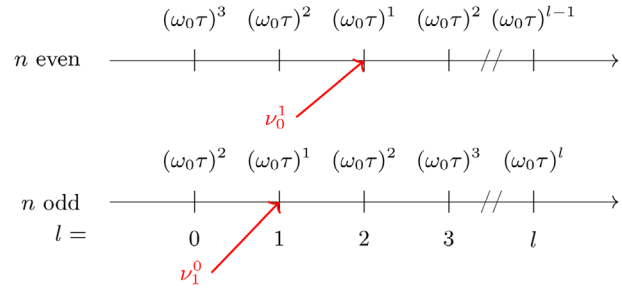


Figure 1. Schematic of the order in $\omega_0 \tau$ of the non conserved components v_n^l of the quasiparticle distribution. The red arrows represent the couplings to the conserved quantities v_0^1 and v_1^0 .

$$cv_1^0 - \frac{v_1^1}{3} = 0 \quad (25)$$

To give a physical interpretation to the three conserved components of v , we relate them to the fluctuations of the density, longitudinal velocity and energy density of the gas:

$$\delta\rho(\mathbf{q}, \omega) = \frac{1}{V} \sum_{\mathbf{p}} \delta n(\mathbf{p}, \mathbf{q}, \omega) \quad (26)$$

$$\rho_{\text{eq}} v_{\parallel}(\mathbf{q}, \omega) = \frac{1}{V} \sum_{\mathbf{p}} \frac{\mathbf{p} \cdot \mathbf{q}}{mq} \delta n(\mathbf{p}, \mathbf{q}, \omega) \quad (27)$$

$$\delta\epsilon(\mathbf{q}, \omega) = \frac{1}{V} \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2m} - \mu \right) \delta n(\mathbf{p}, \mathbf{q}, \omega) \quad (28)$$

Remarking that v is scaled to the intensity of the drive, one expresses the conserved components in terms of the linear response functions of the system: $v_0^0 = -2\pi^2 \chi_{\rho} / mk_F$ is proportional to the density response $\chi_{\rho} = \delta\rho/U$, $v_0^1 = -2k_F \chi_v$ is proportional to the response of the velocity $\chi_v = v_{\parallel}/U$, and $v_1^0 = -6\chi_e / mTk_F$ is proportional to the response of the energy density $\chi_e = \delta\epsilon/U$.

In the hydrodynamic limit, only these conserved quantities remain of order unity, while all the other components pick up one or several factors $\omega_0 \tau$. To evaluate the power of a given component v_n^l in $\omega_0 \tau$, one should count the number of projected transport equations needed to reach a conserved quantity using the couplings appearing in Equation (21), that is, $v_n^l \rightarrow v_n^{l\pm 1}$ and, via the collision kernel, $v_n^l \rightarrow v_{n'}^l$ (with n' having the same parity as n). The components v_n^l with n even and v_n^1 with n odd, which are directly coupled to the conserved quantities v_0^1 and v_1^0 respectively, are of order $O(\omega_0 \tau)^1$, and the other components are subleading, as depicted by **Figure 1**. In particular, the large l components decay exponentially as $O(\omega_0 \tau)^{l-1}$ or $O(\omega_0 \tau)^l$, depending on the parity of n .

4. First-Sound Waves

The drive on the right-hand-side of Equation (21) is coupled to v_0^1 . The perturbation it generates is therefore symmetric in ϵ , that is, $v_n^l = 0$ for all n odd. In particular the sound wave does not generate fluctuations of the energy density: $v_1^0 = 0$. To leading order in $\omega_0 \tau$ the system of Equations (23)–(24) describes an ideal hydro-

dynamic behavior, that is, an undamped resonance at $\omega_q = c_1 q$ with the first sound velocity

$$\frac{c_1}{v_F} = \sqrt{\frac{1 + \frac{2k_F a}{\pi}}{3}} \quad (29)$$

To study how the resonance deviates from c_1 , one must compute the set $\vec{v}^2 = (v_n^2)_{n \in \mathbb{N}}$ of the non-conserved components in the $l = 2$ subspace, that we write as an infinite-dimension vector. Keeping the leading and subleading terms in Equation (21) for $l = 2$, we obtain

$$\left(c + \frac{i}{\omega_0 \tau} \mathcal{M}^2 \right) \vec{v}^2 = \frac{2}{3} v_0^1 \vec{u}_0 + O(\omega_0 \tau)^2 \quad (30)$$

where we have introduced the matrix $\mathcal{M}^2 = (\mathcal{M}_{nn'}^{l=2})_{nn' \in \mathbb{N}}$ and the unit vectors $(\vec{u}_n)_{n' \in \mathbb{N}} = (\delta_{nn'})_{n'}$. We have neglected in Equation (30) the vector \vec{v}^3 and the components of \vec{v}^1 orthogonal to the zero eigenvector $v_0^1 \vec{u}_0$: both are of order $O(\omega_0 \tau)^2$ (see Figure 1), and hence negligible compared to the subleading term $c \vec{v}^2$ in Equation (30). As is noted in Ref. [48], the expansion in powers of $\omega_0 \tau$ becomes very tedious as soon as higher Legendre components have to be taken into account. Fortunately this is not the case in our calculation to order $O(\omega_0 \tau)^2$: the transport equation can be truncated to $l = 2$, and only two collision times are needed to express v_0^2 from Equation (30):

$$v_0^2 = -\frac{2i\omega_0}{3} v_0^1 \left[\tau_\eta + i\omega_0 c t_\eta^2 \right] + O(\omega_0 \tau)^3 \quad (31)$$

where we introduce the viscous collision time $\tau_\eta / \tau = \bar{u}_0 \frac{1}{\mathcal{M}^2} \bar{u}_0$ and the second-order viscous time $t_\eta / \tau = \sqrt{\bar{u}_0 \frac{1}{(\mathcal{M}^2)^2} \bar{u}_0}$. These two parameters characterize the Burnett hydrodynamic equations on the density and parallel velocity; a relaxation time approximation would not distinguish them since it amounts to replacing \mathcal{M}^2 by a number. In previous exact calculations, τ_η has been expressed as an infinite serie^[13,14] or as continued fraction,^[49] which converge rather slowly when truncated at $n = n' = n_{\max}$: $\tau_\eta - \tau_\eta^{(n_{\max})} = O(1/n_{\max}^2)$. Numerically, we obtain

$$\tau_\eta \simeq 1.079 \tau_\eta^{(0)} \simeq 1.288 \tau_\sigma \quad \text{and} \quad t_\eta \simeq 1.098 \tau_\eta^{(0)} \simeq 1.311 \tau_\sigma \quad (32)$$

where we have introduced the viscous collision time truncated at $n = 0$: $\tau_\eta^{(0)} / \tau_\sigma = 4\pi / \mathcal{M}_{00}^{l=2} = 15/4\pi$.

Plugging Equation (31) in Equations (23)–(24), we obtain a quasi-Lorentzian shape for the density response:

$$\chi_\rho = \frac{mk_F}{6\pi^2} \frac{1}{c^2 - \bar{c}_1^2 + \frac{4i\omega_0 c}{15} \left(\tau_\eta + i\omega_0 c t_\eta^2 \right)} \quad (33)$$

The pole z_1 of the denominator is located at:

$$z_1 = \bar{c}_1 - \frac{2i}{15} \omega_0 \tau_\eta + \frac{2\omega_0^2}{225\bar{c}_1} (15\bar{c}_1^2 t_\eta^2 - \tau_\eta^2) \quad (34)$$

where $\bar{c}_1 = c_1 / v_F$. Going back to the physical units, we write the resonance frequency $\omega_q = \text{Re}(z_1) \omega_0$, the main result of this work, as

$$\omega_q = c_1 q \left[1 + \theta(\bar{c}_1) \epsilon_F^2 \tau_\sigma^2 \left(\frac{q}{mc_1} \right)^2 + O(\omega_0 \tau)^3 \right] \quad (35)$$

with $\theta(c) \simeq 0.917c^2 - 0.059$. The first deviation from the linear spectrum $c_1 q$ is thus proportional to q^3 . The dispersion is positive at weak coupling since $\theta(\bar{c}_1) \simeq 0.247$ for $\bar{c}_1 = 1/\sqrt{3}$. The function θ vanishes and changes sign at $c_{\text{inv}} \simeq 0.25$ such that an inversion of the sign of the dispersion may occur in settings where the ratio c_1 / v_F is lower than c_{inv} .

The damping rate $\Gamma_q = -\text{Im}(z_1) \omega_0$ of the sound wave is determined only by the shear viscosity η in this low-temperature regime:

$$\Gamma_q = \frac{2}{3m\rho} \eta q^2 \quad \text{with} \quad \eta = \frac{2}{5} \rho \epsilon_F \tau_\eta \simeq 0.5153 \rho \epsilon_F \tau_\sigma \quad (36)$$

Our value coincides with the exact calculations of η done in the context of ${}^3\text{He}$ ^[15] or neutron matter.^[8] This is a factor $\tau_\eta / \tau_\eta^{(0)} \simeq 1.08$ above the value in the relaxation time approximation of Refs. [28, 30, 38] (called the ‘‘variational approximation’’ therein). We note that the underestimation consecutive to the approximation is larger than stated before.^[28,30] Conversely, in the Abrikosov-Khalatnikov approximation,^[20,39,47] the viscosity is overestimated by a factor $\tau_\eta^{(\text{AK})} / \tau_\eta = 4\tau_\eta^{(0)} / 3\tau_\eta \simeq 1.24$.

5. Conclusion

Dispersion of sound was extensively studied in classical gases, where it is very well captured by second-order (Burnett) hydrodynamics.^[50] We have shown here that a gas of fermions obeys a very similar kind of Burnett hydrodynamics although the presence of a Fermi sea translates into a $1/T^2$ behavior of the collision time, as opposed to $\tau \propto \sqrt{T}$ in a classical gas. A quantum gas is also affected by a quantum Vlasov force on the left-hand side of the transport equation. To leading order in a , this force is the gradient of $g\rho$ and therefore independent of the quasiparticle momentum \mathbf{p} . This may not be true to higher order in the interaction strength, and the effect on the hydrodynamic transport equations remains to be investigated.

Experiments on ultracold gases have extensively measured transport coefficients,^[23,51–55] and thus seem mature to investigate departures from standard Navier-Stokes hydrodynamics. The dispersion of first sound could be measured by tracking the variations of the resonance frequency in the density-density response function with the excitation wavenumber q . Alternatively, this resonance frequency could be compared to the speed of sound known from the equation of state. Dispersive effects also affect the real-time dynamics, such as the equilibration after a quench,^[56] or the propagation of shock waves.^[42,57,58]

Appendix A: Collision Kernel

We write the linearized collision integral as

$$I_{\text{lin}} = I_{\text{lin},\uparrow} + I_{\text{lin},\downarrow} = -\Gamma(\mathbf{p})\delta n(\mathbf{p}) - \frac{1}{L^3} \sum_{\mathbf{p}'} [E(\mathbf{p}', \mathbf{p}) - 2S(\mathbf{p}', \mathbf{p})] \delta n(\mathbf{p}') \quad (\text{A1})$$

with the collision kernels E and S (notice the swap of arguments $\mathbf{p} \leftrightarrow \mathbf{p}'$)

$$E(\mathbf{p}, \mathbf{p}') = \frac{2\pi g^2}{L^3} \sum_{\mathbf{p}_3, \mathbf{p}_4} \delta_{\mathbf{p}_3 + \mathbf{p}_4}^{\mathbf{p} + \mathbf{p}'} \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}'} - \epsilon_{\mathbf{p}_3} - \epsilon_{\mathbf{p}_4}) \times [n_{\text{eq}}(\epsilon_{\mathbf{p}'}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_3}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_4}) + \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}'}) n_{\text{eq}}(\epsilon_{\mathbf{p}_3}) n_{\text{eq}}(\epsilon_{\mathbf{p}_4})] \quad (\text{A2})$$

$$S(\mathbf{p}, \mathbf{p}') = \frac{2\pi g^2}{L^3} \sum_{\mathbf{p}_2, \mathbf{p}_4} \delta_{\mathbf{p}' + \mathbf{p}_4}^{\mathbf{p} + \mathbf{p}_2} \delta(\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_2} - \epsilon_{\mathbf{p}'} - \epsilon_{\mathbf{p}_4}) \times [n_{\text{eq}}(\epsilon_{\mathbf{p}_2}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}'}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_4}) + \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_2}) n_{\text{eq}}(\epsilon_{\mathbf{p}'}) n_{\text{eq}}(\epsilon_{\mathbf{p}_4})] \quad (\text{A3})$$

with $\bar{n}_{\text{eq}} = 1 - n_{\text{eq}}$, and the short-hand notation $\epsilon_{\mathbf{p}'}$ for $\epsilon_0(\mathbf{p}')$. The quasi-particle lifetime Γ is related to E and S by $\Gamma(\mathbf{p}) = \sum_{\mathbf{p}'} E(\mathbf{p}, \mathbf{p}')/L^3 = \sum_{\mathbf{p}'} S(\mathbf{p}, \mathbf{p}')/L^3$, and the total collision kernel obeys the conservation laws

$$\Gamma(\mathbf{p})\mathbf{p}^k + \frac{1}{L^3} \sum_{\mathbf{p}'} [E(\mathbf{p}', \mathbf{p}) - 2S(\mathbf{p}', \mathbf{p})] \mathbf{p}'^k = 0, \quad k = 0, 1, 2 \quad (\text{A4})$$

Using the property of thermal equilibrium

$$n_{\text{eq}}(\epsilon_{\mathbf{p}_1}) n_{\text{eq}}(\epsilon_{\mathbf{p}_2}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_3}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_4}) = \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_1}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}_2}) n_{\text{eq}}(\epsilon_{\mathbf{p}_3}) n_{\text{eq}}(\epsilon_{\mathbf{p}_4}) \quad (\text{A5})$$

(for four momenta on the energy shell $\epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4$), together with the change of variable Equation (6), we swap the arguments of E

$$E(\mathbf{p}', \mathbf{p}) \frac{n_{\text{eq}}(\epsilon_{\mathbf{p}'}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}'})}{n_{\text{eq}}(\epsilon_{\mathbf{p}}) \bar{n}_{\text{eq}}(\epsilon_{\mathbf{p}})} = E(\mathbf{p}, \mathbf{p}') \quad (\text{A6})$$

and similarly for S . We used here

$$T \frac{\partial n_{\text{eq}}}{\partial \epsilon_0} \Big|_{\epsilon_0 = \epsilon_0(\mathbf{p})} = -n_{\text{eq}}(\epsilon_{\mathbf{p}}) (1 - n_{\text{eq}}(\epsilon_{\mathbf{p}})) \quad (\text{A7})$$

In terms of the variable v the transport equation (5) becomes

$$\left(\omega - \frac{\mathbf{p} \cdot \mathbf{q}}{m} \right) v(\mathbf{p}) + \frac{\mathbf{p} \cdot \mathbf{q}}{m} (g\delta\rho + U) = -i \left(\Gamma(\mathbf{p}) v(\mathbf{p}) + \frac{1}{L^3} \sum_{\mathbf{p}'} [E(\mathbf{p}, \mathbf{p}') - 2S(\mathbf{p}, \mathbf{p}')] v(\mathbf{p}') \right) \quad (\text{A8})$$

where the fluctuations of the total density are rewritten as $\delta\rho = (1/V) \sum_{\mathbf{p}'} (\partial n_{\text{eq}} / \partial \epsilon_0)_{\epsilon_0 = \epsilon_0(\mathbf{p}')} v(\mathbf{p}')$.

In Equations (A2)–(A3), we eliminate \mathbf{p}_4 using momentum conservation, and convert the sums into integrals using

$$\frac{1}{L^3} \sum_{\mathbf{p}_i} \rightarrow \frac{m T p_F}{(2\pi)^2} \int_{-\infty}^{+\infty} d\epsilon_i \int_0^\pi \sin \vartheta_i d\vartheta_i \int_0^{2\pi} \frac{d\varphi_i}{2\pi} \quad (\text{A9})$$

where $i = 2, 3$ for E and S respectively, $\epsilon_i = (\epsilon_0(\mathbf{p}_i) - \mu)/T$, θ_i is the polar angle between \mathbf{p}_3 and $\mathbf{p} + \mathbf{p}'$ or \mathbf{p}_2 and $\mathbf{p} - \mathbf{p}'$. Such choices of spherical frames, together with the isotropic collision probability g^2 , guarantees that the integrand is independent of the azimuthal angles φ_i . Note that the conversion Equation (A9) approximates $p_i^2 dp_i \simeq m T p_F d\epsilon_i$, which is valid up to corrections in $\mathcal{O}(T)$. To leading order in T , the energy-conservation constraints can be replaced by their $T = 0$ version, namely

$$\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}'} - \epsilon_{\mathbf{p}_3} - \epsilon_{\mathbf{p}_4} = \frac{2p_F^2}{m} \left[\cos \frac{\alpha}{2} \left(\cos \theta_3 - \cos \frac{\alpha}{2} \right) \right] \quad (\text{A10})$$

$$\epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}_2} - \epsilon_{\mathbf{p}'} - \epsilon_{\mathbf{p}_4} = \frac{2p_F^2}{m} \left[\sin \frac{\alpha}{2} \left(\cos \theta_2 - \sin \frac{\alpha}{2} \right) \right] \quad (\text{A11})$$

The Dirac functions can thus be used to perform the angular integration over θ_i . There remains to integrate over the reduced energy ϵ_i , using:

$$\int_{-\infty}^{\infty} d\epsilon_3 [n(\epsilon') (1 - n(\epsilon_3) - n(\epsilon + \epsilon' - \epsilon_3)) + n(\epsilon_3) n(\epsilon + \epsilon' - \epsilon_3)] = S(\epsilon, -\epsilon') \quad (\text{A12})$$

$$\int_{-\infty}^{\infty} d\epsilon_2 [n(\epsilon_2) (1 - n(\epsilon') - n(\epsilon + \epsilon_2 - \epsilon')) + n(\epsilon') n(\epsilon + \epsilon_2 - \epsilon')] = S(\epsilon, \epsilon') \quad (\text{A13})$$

The result are collision kernels whose angular and energy dependence are factorized:

$$E(\mathbf{p}, \mathbf{p}') = \frac{m^2 T}{2\pi p_F} \frac{g^2}{2 \cos(\theta/2)} S(\epsilon, -\epsilon') \quad (\text{A14})$$

$$S(\mathbf{p}, \mathbf{p}') = \frac{m^2 T}{2\pi p_F} \frac{g^2}{2 \sin(\theta/2)} S(\epsilon, \epsilon') \quad (\text{A15})$$

This remarkable property is specific to Fermi liquids and distinguishes them from classical gases. It is a direct consequence of the freezing of the collisions to the vicinity of the Fermi surface when $T \rightarrow 0$.

To derive the transport equation in the low-temperature limit Equation (7), there remains to divide Equation (5) by $\omega_0 (\partial n_{\text{eq}} / \partial \epsilon_0)_{\epsilon_0 = \epsilon_0(\mathbf{p})}$ and expand each term in powers of T at fixed ϵ and ϵ' (which amounts to replacing p and p' by p_F outside functions that are peaked about the Fermi sphere).

Appendix B: Observability of the Spectrum in the Density Response Function

In this section, we verify that the attenuation and dispersion coefficients appearing in Equations (35)–(36) can be accurately recovered from the density response function $\bar{\chi}_\rho(c) \equiv v_0^0(c)$, which is the main experimental observable for the propagation of sound.^[21,24,59] To do so, we solve Equation (21) numerically by truncating it to $n_{\text{max}} = 100$ and l_{max} ranging from 20 in the hydrodynamic regime to 8500 in the collisionless regime. Examples of numerically computed spectra are shown in Figure B1; we observe that the shift $\omega_q - c_1 q$ is positive everywhere in the hydrodynamic to collisionless crossover.

On the hydrodynamic side of the crossover ($\omega_0 \tau_\sigma \lesssim 0.5$), and for $k_F a = 0$ ($c_1 = \nu_F / \sqrt{3}$), we fit $\text{Im} \bar{\chi}_\rho(c + i0^+)$ to a Lorentzian function:

$$f_\rho^{\text{fit}}(c) = \frac{Z}{(c - \bar{c}_1 - B\omega_0^2 \tau_\sigma^2)^2 + A^2 \omega_0^2 \tau_\sigma^2} \quad (\text{B1})$$

where A , B , and Z are real fitting parameters.

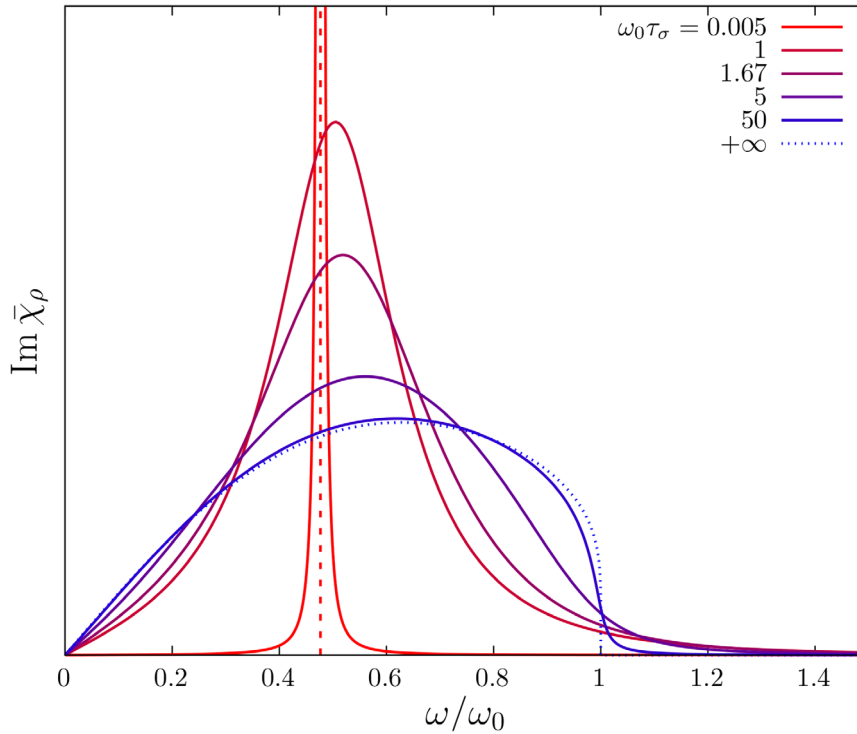


Figure B1. The transition from hydrodynamic (red curves) to collisionless regime (blue curves) in the density response of a weakly interacting Fermi gas with $k_F a = -0.5$. The broadening and the positive shift of the resonance frequency from the first sound velocity (vertical dashed line, Equation (29)) is clearly visible at small $\omega_0 \tau_\sigma$. The curves are obtained by numerically solving Equation (21) truncated to $n_{\max} = 100$ and l_{\max} ranging from 20 in the hydrodynamic regime to 8500 in the collisionless regime.

Figure B2 compares the fitted values at non-vanishing $\omega_0 \tau$ to the expected limit when $\omega_0 \tau \rightarrow 0$, revealing a very good convergence to

$$B \xrightarrow{\omega_0 \tau \rightarrow 0} \frac{\text{Re}(z_1 - \bar{c}_1)}{(\omega_0 \tau_\sigma)^2} = \frac{2\sqrt{3}}{225} \frac{5\tau_\eta^2 - \tau_\eta^2}{\tau_\sigma^2} \quad (\text{B2})$$

$$A \xrightarrow{\omega_0 \tau \rightarrow 0} -\frac{\text{Im}(z_1)}{\omega_0 \tau_\sigma} = \frac{2}{15} \frac{\tau_\eta}{\tau_\sigma} \quad (\text{B3})$$

To estimate the order of magnitude of the second-order hydrodynamical effects, we use the parameters $k_F a = -0.67$, $T/T_F = 0$, $q/k_F = 0.01$ taken from the experimental Ref. [21] (see Figure 4 therein). This corresponds to $\omega_0 \tau_\sigma \simeq 0.42$ for which a first sound resonance is still clearly visible in Figure 4a of Ref. [21]. Plugging these experimental parameters in Equation (34), we estimate the relative frequency shift and damping rate to be

$$\frac{\omega_q}{c_1 q} - 1 \simeq 3\% \quad (\text{B4})$$

$$\frac{\Gamma_q}{c_1 q} \simeq 13\% \quad (\text{B5})$$

The small value of $\omega_q - c_1 q$ is challenging to measure but it could be within experimental reach in the near future. This would allow to distinguish between supersonic $\omega_q > c_1 q$ and subsonic $\omega_q < c_1 q$ dispersion.

Appendix C: Other Transport Coefficients

Equation (25) on the energy density allows us to compute the thermal conductivity of the gas. At low temperature, the energy current

v_1^1 exists only if the quasiparticle distribution has components odd in n , that is if there is an energy-imbalance across the Fermi surface, which is not the case in a sound wave. We compute this energy current by writing down the transport equation for n odd and $l = 1$:

$$c v_n^1 + \frac{i}{\omega_0 \tau} \sum_{n'} \mathcal{M}_{nn'}^1 v_{n'}^1 - v_n^0 - \frac{2}{5} v_n^2 = 0 \quad (\text{C1})$$

To leading order in $\omega_0 \tau$, this gives

$$\frac{i}{\omega_0 \tau} \mathcal{M}_{\text{odd}}^1 \bar{v}_{\text{odd}}^1 = v_1^0 \bar{u}_1 + \mathcal{O}(\omega_0 \tau) \quad (\text{C2})$$

where $\bar{v}_{\text{odd}}^1 = (0, v_1^1, 0, v_3^1, 0, \dots)$ is the odd component of \bar{v}^1 , and we recall that \mathcal{M} does not couple the even and odd components. We solve Equation (C2) for v_1^1 , which yields

$$v_1^1 = -i \omega_0 \tau_\kappa v_1^0 \quad \text{with} \quad \frac{\tau_\kappa}{\tau_\sigma} = \bar{u}_1 \frac{4\pi}{\mathcal{M}^1} \bar{u}_1 \simeq 0.7478 \quad (\text{C3})$$

Going back to Equation (25) on the energy density, we obtain the heat diffusion equation

$$(\omega + i D_\kappa q^2) v_1^0 = 0 \quad \text{with} \quad D_\kappa = \frac{1}{3} v_F^2 \tau_\kappa \simeq 0.249 v_F^2 \tau_\sigma. \quad (\text{C4})$$

The thermal conductivity κ is related to D_κ by $\kappa = c_V D_\kappa$ where $c_V = m p_F T / 3$ is the heat capacity. The relaxation time approximation $\tau_\kappa^{(0)} / \tau_\sigma = 15 / 8\pi$ turns out much worse for the thermal conductivity than for the viscosity, with the error reaching 25%. We are thus in neat disagreement with the claim of Ref. [30] that $|\kappa - \kappa_{(0)}| / \kappa < 2\%$.

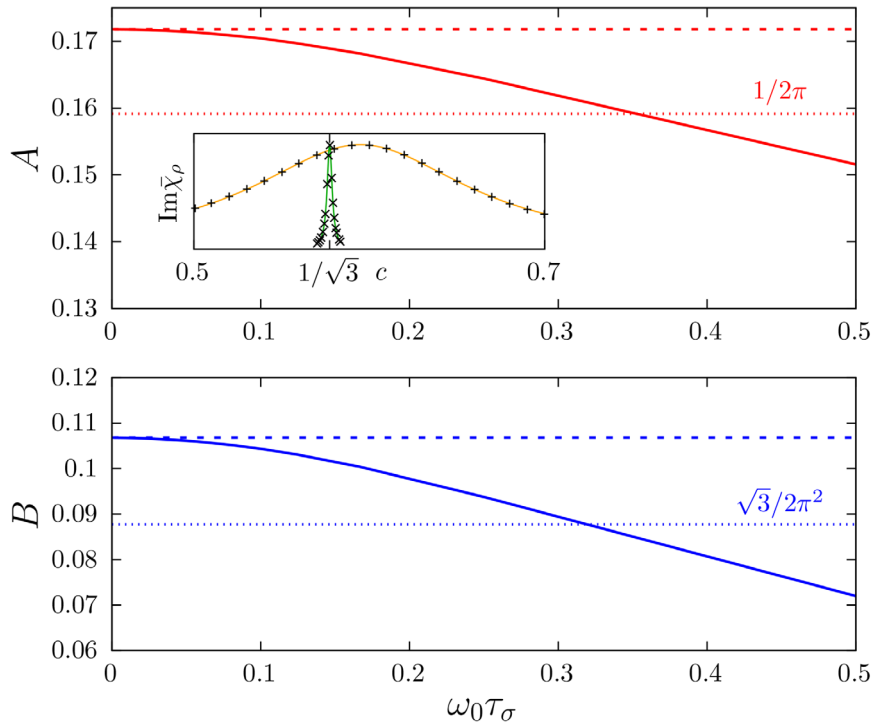


Figure B2. The sound attenuation and dispersion coefficients γ and Δc obtained by fitting a Lorentzian function Equation (B1) (green and orange lines in inset, respectively for $\omega_0\tau_\sigma = 0.01$ and $\omega_0\tau_\sigma = 0.5$) to the density response $\bar{\chi}_\rho$ obtained by numerically solving Equation (21) at non vanishing $\omega_0\tau_\sigma$ (symbols in the inset). The horizontal dashed and dotted line show respectively the exact hydrodynamic limits Equations (B2) and (B3), and the relaxation time approximation ($\tau_\eta, \tau_\eta \rightarrow \tau_\eta^{(0)}$). This figure is drawn in the limit $k_F a \rightarrow 0$ ($c_1/\nu_F = 1/\sqrt{3}$).

A Fermi liquid, just like a nearly ideal gas (at arbitrary temperatures) has no bulk viscosity.^[1,47] We attribute this to the effective quadratic behavior of the dispersion relation $\epsilon_p - \mu \simeq (p^2 - p_F^2)/2m$. After decomposition onto the Q_n polynomials, this properties decouples the components $v_{n>0}^0$ from the conserved v_0^1 , thereby removing a dissipative term in the velocity equation. Finally the spin diffusivity is outside the scope of this article, which does not consider fluctuations of the polarization $\delta n_\uparrow - \delta n_\downarrow$.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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collective modes, fermi gases, quantum transport, sound waves, ultracold atoms

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