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► To cite this version:

Olivier Piller, Sylvan Elhay, Jochen Deuerlein, Angus Simpson. How to use the first and second order sensitivities of steady-state solutions to water distribution systems. CCWI 2025 - 21st Computing & Control for the Water Industry Conference, The University of Sheffield, Sep 2025, Sheffield, United Kingdom. <10.15131/shef.data.29921123.v1>. <hal-05242733>

HAL Id: hal-05242733

<https://hal.science/hal-05242733v1>

Submitted on 5 Sep 2025

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HOW TO USE THE FIRST AND SECOND ORDER SENSITIVITIES OF STEADY-STATE SOLUTIONS TO WATER DISTRIBUTION SYSTEMS

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ABSTRACT

A hydraulic model can be of considerable benefit to those responsible for the design and operation of drinking water systems. In the contemporary era, it is imperative that water distribution systems are operated in a more sober manner, in a sustainable and resilient fashion, to combat the effects of global warming and to mitigate natural and anthropogenic risks.

However, this may necessitate the resolution of optimisation problems through the utilisation of extensive hydraulic models, a process that could encounter intractable challenges due to the curse of dimensionality. One such example is real-time demand calibration in conjunction with research into countermeasures for critical events in a large water distribution system. Quantifying uncertainty is imperative for robust countermeasures and decisions.

To address these significant challenges and potentially reduce computing time, a novel approach to deriving first and second order sensitivities was proposed at the 3rd International Joint Conference WDSA/CCWI (2024) [1]. This method involves solving a series of linear systems with the same matrix and Cholesky factors, but with different right-hand sides. This provides an efficient, low-cost way of calculating sensitivities.

The objective of this study is to ascertain the potential of this innovative approach to medium and large networks. Calculating the sensitivities of many state variables with respect to many parameters is not feasible and is fortunately not useful. In hydraulic models, there are a limited number of demand patterns, and we are often interested in just a few specific outputs.

The application will concentrate on two significant applications: the variation in pressure head according to local demand or the flow rate at a critical node in a large network, and the application to real-time calibration for a large network. Finally, we will discuss other potential applications for large networks and how to avoid certain pitfalls.

Keywords: Calibration, local state estimation, criticality

INTRODUCTION

Water distribution systems (WDSs) are often large, interconnected, complex and ageing subnetworks. They need to be protected and made more resilient in the event of natural and man-made disasters and terrorist attacks [2]. Proper

management and understanding of the operation of such systems is crucial for preserving health and safety, ensuring sustainability, and addressing a range of other issues. Examples of such issues include real-time calibration of water demand models, a prerequisite for water quality, quantification of uncertainty in pipe velocities, and representation of pollution propagation.

Modelling tools can provide significant benefits in the management of complex systems, facilitating the development of sustainable strategies and the implementation of effective crisis response measures. One potential avenue for achieving this could be the resolution of optimization problems by utilizing large hydraulic digital models. However, it should be noted that this approach may encounter challenges due to the so-called curse of dimensionality.

In response to this challenge, some authors have proposed two approaches. Firstly, linear sensitivity methods can be used to approximately quantify the impact of pipe failures or for leakage localization [3,4]. Secondly, graph partitioning, complex network theory and search-space reduction [5,6] can be used to make these problems manageable. Depending on the problem under consideration, sub-optimality or some kind of limitation may be reported, particularly when precision is required for decision-making and non-linearity is important.

In this research study, we first derived the sensitivities with respect to demand. We will then provide guidance on how to effectively calculate the first- and second-order sensitivity of one or a few head and flow rate variables with respect to a particular nodal demand or multiple demand patterns. Finally, we will illustrate the method on two significant applications in medium and large-size WDNs.

METHODS

The current practice is to categorize consumer behaviors and regroup nodes in the same category within one demand time pattern. Nodal demands within the same category will vary in a similar proportion over time. (e.g., a domestic, industrial 1, etc.). Here, we are expanding the sensitivities regarding demand pattern formulation derived in [7] for first- to second order. The notations are given in the appendix Table 1 and are the same as in [1, 7].

DIFFERENTIATION WRT DEMAND PATTERN COEFFICIENTS

We assume the nodal demand $\mathbf{d} \in \mathbb{R}^{n_j}$ can be rebuilt with a small number of coefficients that change with time t ; more specifically:

$$\mathbf{d}(t) = \mathbf{d}_{\text{known}}(t) + \mathbf{K}\boldsymbol{\alpha}(t). \quad (1)$$

Where some rows of matrix \mathbf{K} can be zeros, $\boldsymbol{\alpha} \in \mathbb{R}^{n_a}$ is the uncertain demand pattern (eventually to calibrate), and $n_a \leq n_j$.

Using the multivariate chain rule:

$$\frac{\partial h_i}{\partial \alpha_m} = \sum_{j=1}^{n_j} \frac{\partial h_i}{\partial d_j} \frac{\partial d_j}{\partial \alpha_m} = \sum_{j=1}^{n_j} \frac{\partial h_i}{\partial d_j} K_{jm}, \quad (2)$$

Then,

$$\frac{\partial h_i}{\partial \alpha_m} = \nabla_d h_i^T (\mathbf{K} \mathbf{e}_m),$$

which takes the vectorial form:

$$\nabla_\alpha \mathbf{h} = (\nabla_d \mathbf{h}) \mathbf{K},$$

Let \mathbf{S}_h be the selection matrix of some key components (in general in small number), we obtain:

$$\nabla_\alpha \mathbf{h}_{\text{key}} = \mathbf{S}_h (\nabla_d \mathbf{h}) \mathbf{K}.$$

Similarly, we obtain for flow rate observations:

$$\nabla_\alpha \mathbf{q}_{\text{key}} = \mathbf{S}_q (\nabla_d \mathbf{q}) \mathbf{K}.$$

More specifically we get:

$$\begin{aligned} \nabla_\alpha \mathbf{h}^{\text{key}} &= -\mathbf{S}_h (\mathbf{A}^T \mathbf{F}^{-1} \mathbf{A} + \mathbf{E})^{-1} \nabla_d \mathbf{c} \mathbf{K}, \\ \nabla_\alpha \mathbf{q}^{\text{key}} &= -\mathbf{S}_q \mathbf{F}^{-1} \mathbf{A} (\mathbf{A}^T \mathbf{F}^{-1} \mathbf{A} + \mathbf{E})^{-1} \nabla_d \mathbf{c} \mathbf{K}. \end{aligned} \quad (3)$$

If we differentiate once again Eq. (2) wrt the demand pattern variable, we get:

$$\frac{\partial^2 h_i}{\partial \alpha_m \partial \alpha_n} = \sum_{k=1}^{n_j} \sum_{j=1}^{n_j} \frac{\partial^2 h_i}{\partial d_j \partial d_k} \frac{\partial d_j}{\partial \alpha_m} \frac{\partial d_k}{\partial \alpha_n} = \sum_{k=1}^{n_j} \sum_{j=1}^{n_j} \frac{\partial^2 h_i}{\partial d_j \partial d_k} K_{jm} K_{kn}$$

Which can be used to calculate the 2nd-order sensitivities. Using the same scheme that in [1] we derive more economical 2nd-order sensitivity wrt d formulae.

Let $\mathbf{x} \in \mathbb{R}^{n_p}$ and $\mathbf{y} \in \mathbb{R}^{n_j}$ be defined by:

$$\begin{aligned} x_p &= -\frac{\partial^2 \xi_p}{\partial q_p^2} \frac{\partial q_p}{\partial \alpha_m} \frac{\partial q_p}{\partial \alpha_n} \\ y_i &= \frac{\partial^2 c_i}{\partial h_i^2} \frac{\partial h_i}{\partial \alpha_m} \frac{\partial h_i}{\partial \alpha_n} + \frac{E_{ii}}{d_i} \left(K_{in} \frac{\partial h_i}{\partial \alpha_m} + K_{im} \frac{\partial h_i}{\partial \alpha_n} \right). \end{aligned}$$

Then the vector

$$\mathbf{h}_{xy} = \left(\frac{\partial^2 h_i}{\partial \alpha_m \partial \alpha_n} \right)_{i=1, \dots, n_j}$$

is given by:

$$\mathbf{h}_{xy} = -(\mathbf{A}^T \mathbf{F}^{-1} \mathbf{A} + \mathbf{E})^{-1} (\mathbf{A}^T \mathbf{F}^{-1} \mathbf{x} + \mathbf{y})$$

Finally we get:

$$\begin{aligned} \mathbf{h}_{xy}^{\text{key}} &= -\mathbf{S}_h(\mathbf{A}^T \mathbf{F}^{-1} \mathbf{A} + \mathbf{E})^{-1}(\mathbf{A}^T \mathbf{F}^{-1} \mathbf{x} + \mathbf{y}), \\ \mathbf{q}_{xy}^{\text{key}} &= \mathbf{S}_q \mathbf{F}^{-1}(\mathbf{A} \mathbf{h}_{xy} + \mathbf{x}). \end{aligned} \quad (4)$$

DIFFERENTIATING ONE HEAD VARIABLE WITH ONE DEMAND COMPONENT

This is a particular case of the preceding section, in which we make the choice:

$$\mathbf{S}_h = \mathbf{e}_j^T, \mathbf{K} = \mathbf{e}_i, K_{in} = \delta_{in} \text{ and } K_{im} = \delta_{im}.$$

Then Eqs (3) and (4) are used.

RESULTS AND DISCUSSION

Firstly, we studied how the change of one head of interest can be predicted with regards to change of demand using formulae from previous sections. For the medium-scale network depicted in Figure 1 (six loops, one tank, 11 junction nodes and 17 pipes), we have selected a node that is distant from the tank (node 7), and have calculated the heads for 21 different demands at a separate location (node 4).

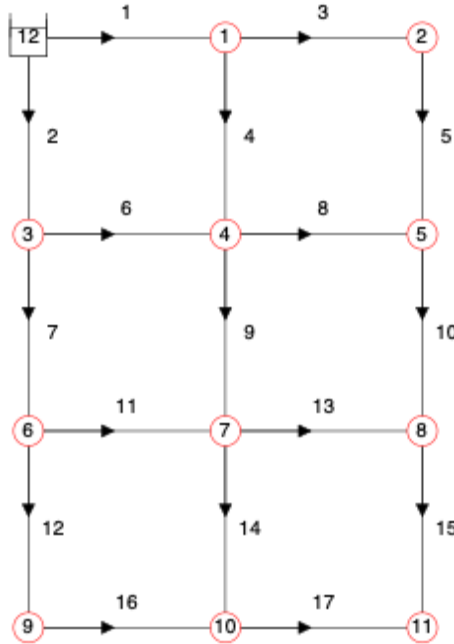


Figure 1. Medium network example with 6 loops, 1 tank, 11 junction nodes and 17 pipes.

The result is shown in Figure 2. The circles represent h_7 from a demand-driven simulation made by our MATLAB code. The demand at node 4 in L/s is chosen to vary within the range $[0, 200]$. With the first-order sensitivity calculated for $d_4 = 90$ L/s and the corresponding head h_7 we built the first-order Taylor polynomial approximation (TPA1), which is the tangent line (blue curve) and is always above the black curve with one intersection at the design point. The maximum error in prediction is approximately 1.36 mH₂O, indicating that TPA1 is not a reliable approximation away from the design point. With the first- and second-order sensitivities at the same design point, we calculate the second-order Taylor approximation (TPA2), represented by the the red curve. It is noteworthy that this curve is much closer to the black curve, with a maximum prediction error reduced to 0.06 mH₂O. This illustrates that the second-order

polynomial expansion provides a reliable approximation for the head, and modifying the head or demand location leads to equivalent conclusions for this network.

It is expected that the same conclusion will apply to every WDN, provided that there is no change in configuration or physical law. For example, if the pressure head falls below the minimum pressure threshold, consumption will decrease, and the quadratic demand-driven approximation will become ineffective. To restore accurate prediction, a second-order Taylor expansion approximation must then be performed under PDM conditions at the design point.

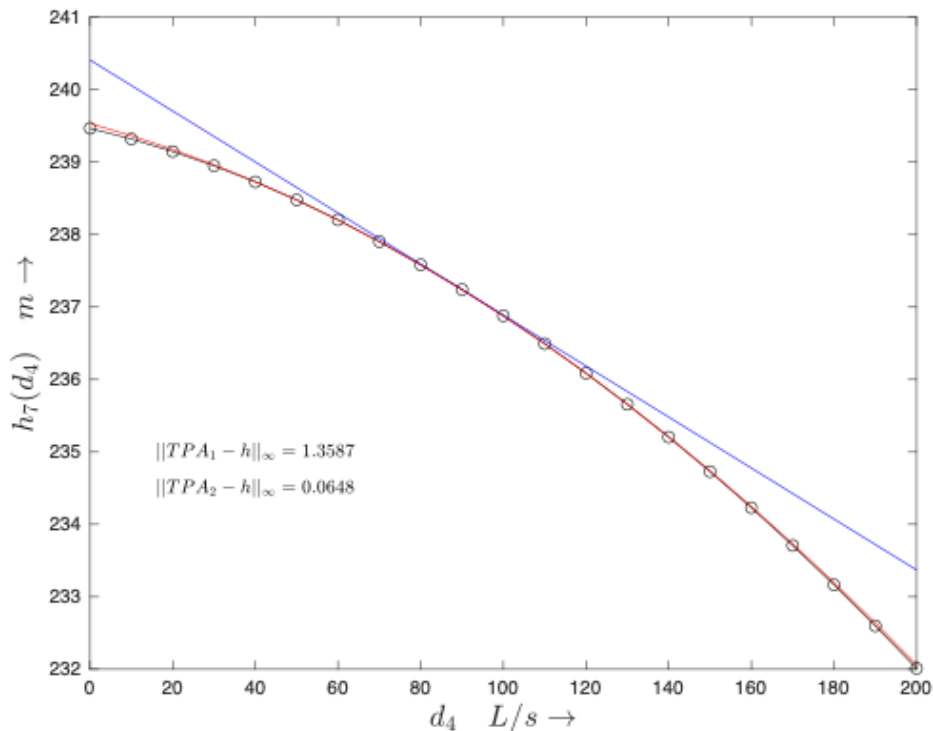


Figure 2. Example of head dependency with the demand in a looped system; The black curve is calculated using the demand-driven MATLAB code developed by the authors. The circles represent the points where the curve is the exact solution; The blue curve represents the tangent line at the selected point (90, $h(90)$), while the red curve shows the second-order Taylor polynomial expansion at that same point.

CONCLUSIONS

This research study involved developing a method and explicit formulae for calculating the first- and second-order sensitivities of hydraulic state variables, such as head and flow rates, when a limited number of parameters (e.g. demand patterns or a single demand of interest) vary. This allows sensitivities to be calculated at a design point and used to create second-order Taylor polynomial approximations, even for large networks.

Second-order Taylor polynomial approximations have been found to be very accurate over a large region around the design point.

Another possible application that will be demonstrated at the conference is improved demand pattern calibration. This anticipates and enables real-time calibration of large networks using a limited number of patterns.

Funding: This research was funded by the French National Research Agency, grant number ANR-22-CE39-0010.

REFERENCES

- [1] O. Piller, S. Elhay, J. W. Deuerlein, and A. R. Simpson, "First- and Second-Order Sensitivities of Steady-State Solutions to Water Distribution Systems," *Engineering Proceedings*, vol. 69, no. 1, 2024.
- [2] P. H. Gleik, "Water and terrorism", *Water Policy*, 8, 481-503, 2006.
- [3] E. Abraham, E.J.M. Blokker, and I. Stoianov. "Network Analysis, Control Valve Placement and Optimal Control of Flow Velocity for Self-Cleaning Water Distribution Systems," *Procedia Eng.*, 186, 576–583, 2017.
- [4] D.B. Steffelbauer, J. Deuerlein, D. Gilbert, E. Abraham, and O. Piller, "Pressure-Leak Duality for Leak Detection and Localization in Water Distribution Systems," *J. Water Res. Plan. Manag.*, 148, 04021106, 2022.
- [5] S. Elhay, J. Deuerlein, O. Piller, and A. Simpson, "Graph Partitioning in the Analysis of Pressure Dependent Water Distribution Systems," *J. Water Res. Plan. Manag.*, 144, 1–13, 2018.
- [6] K. Diao, L. Berardi, D. B. Laucelli, B. Ulanicki, and O. Giustolisi, "Topological and hydraulic metrics-based search space reduction for optimal re-sizing of water distribution networks," *J. of Hydroinformatics*, vol. 24, no. 3, pp. 610-621, 2022, doi: 10.2166/hydro.2022.158.
- [7] O. Piller, S. Elhay, J. Deuerlein, and A. Simpson, "Local Sensitivity of Pressure-Driven Modeling and Demand-Driven Modeling Steady-State Solutions to Variations in Parameters," *J Water Res. Plan. Manag.*, vol. 143, no. 2, p. 12, 2017.

APPENDIX

Table 1. Notations

n_p	: number of links ;
n_j	: number of junction nodes ;
\mathbf{A}	: $n_p \times n_j$ arc-node incidence matrix reduced to junction nodes ;
\mathbf{A}^T	: $n_j \times n_p$ transpose matrix of \mathbf{A} ;
\mathbf{F}	: diagonal matrix with q -derivatives of $\xi_i(q_i)$, the head loss function ;
\mathbf{E}	: diagonal matrix with h -derivatives of $c_j(d_j, h_j)$, the pressure-outflow relationship ;
\mathbf{K}	: $n_j \times n_a$ junction node-demand pattern matrix ;
\mathbf{e}_m	: standard basis vector, with components all zero, except one that equals 1 at m ;
\mathbf{q}	: $(q_1, q_2, \dots, q_{n_p})^T$, q_i link flow rates;
\mathbf{h}	: $(h_1, h_2, \dots, h_{n_j})^T$, h_j nodal heads ;
\mathbf{d}	: $(d_1, d_2, \dots, d_{n_j})^T$, d_j nodal demands ;
$\boldsymbol{\alpha}$: $(\alpha_1, \alpha_2, \dots, \alpha_{n_a})^T$, α_k demand patterns ;
