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EXPONENTIAL ERGODICITY OF MEAN-FIELD LANGEVIN DYNAMICS BY SYNCHRONOUS COUPLING *

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ABSTRACT

As an example of the *nonlinear Fokker-Planck equation*, the *mean field Langevin dynamics* recently attracts attention due to its connection to (noisy) *gradient descent* on infinitely wide *neural networks* in the mean field regime, and hence the *convergence property* of the dynamics is of great theoretical interest. In this paper, In the continuity of [1, 2], we are interested by the long-time behavior and uniform in time propagation of chaos by synchronous coupling for mean-field Langevin dynamics (over- and under-damped).

Keywords and phrases: ergodicity · mean-field · Langevin dynamics · synchronous coupling · U-statistics · propagation of chaos · polynomial interaction · (kinetic) Fokker-Planck equation · McKean-Vlasov equation · functional inequalities · Lyapunov functional · gradient flow · convergence to equilibrium · (hypo)coercivity

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1 Introduction

The present work studies both the quantitative long-time behavior of the first order (over-damped) and second order (under-damped) mean field Langevin dynamics and their uniform-in-time propagations of chaos properties, under a functional not necessary convex in the flat interpolation sense.

In this paper, we consider the first order McKean-Vlasov equation

$$dX_t = \sigma dB_t - \mathcal{D}_m H(\mu_t, X_t) dt; \quad (1.1)$$

and the second order McKean-Vlasov equation

$$\begin{cases} dP_t = V_t dt; \\ dV_t = \sigma dB_t - A(V_t) dt - B(P_t) dt - \mathcal{D}_m H(\mathbb{P}_{P_t}, P_t) dt; \end{cases} \quad (1.2)$$

where $\mathcal{D}_m H$ denotes the intrinsic derivative of the functional H (a mapping from $\mathcal{D}_p(\mathbb{R}^d) \times \mathbb{R}^d$ into \mathbb{R}^d and will be defined in Eq. (2.3)), $\sigma > 0$, $\mu_t := \mathbb{P}_{X_t}$, A a friction field (velocity field) and B an external confinement field. Without loss of generality, in the remainder of the paper, we will take $\sigma = \sqrt{2}$.

Let $n \geq 1$ be an integer and let $\mathbf{x} := (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ be an n -tuple of coordinates in \mathbb{R}^d . We denote by $\mu_{\mathbf{x}}$ the empirical measure formed with the n coordinates x_1, \dots, x_n , that is,

$$\mu_{\mathbf{x}} := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}. \quad (1.3)$$

We consider the first order mean field particle system

$$\forall i \in \{1, \dots, n\}, \quad dX_t^{(i)} = \sigma dB_t^{(i)} - \mathcal{D}_m H(\mu_{\mathbf{X}_t}, X_t^{(i)}) dt; \quad (1.4)$$

and the second order mean field particle system

$$\begin{cases} dP_t^{(i)} = V_t^{(i)} dt; \\ dV_t^{(i)} = \sigma dB_t^{(i)} - A(V_t^{(i)}) dt - B(P_t^{(i)}) dt - \mathcal{D}_m H(\mu_{\mathbf{P}_t}, P_t^{(i)}) dt; \end{cases} \quad (1.5)$$

where $B^{(1)}, \dots, B^{(n)}$ are i.i.d. standard Brownian motions in d -dimensions.

For $\mu \in \mathcal{D}(\mathbb{R}^d)$, we consider the over-damped Langevin dynamics:

$$dX_t^\mu = \sqrt{2} dB_t^\mu - \mathcal{D}_m H(\mu, X_t^\mu) dt. \quad (1.6)$$

By explicit computations, we can find that the Gibbs measure:

$$\Phi(\mu)(dx) := \frac{e^{-\frac{\delta H}{\delta m}(\mu, x)}}{\int_{\mathbb{R}^d} e^{-\frac{\delta H}{\delta m}(\mu, y)} dy} dx \quad (1.7)$$

is invariant to the dynamics ($\frac{\delta H}{\delta m}$ is called *flat derivative* of H and is a continuous mapping from $\mathcal{P}_p(\mathbb{R}^d) \times \mathbb{R}^d$ into \mathbb{R} and will be defined in Eq. (2.8)). By explicit computations, we can find that the set of invariant probability distributions of the mean field Langevin dynamics (over-damped) is defined by

$$\{\mu_\infty, \quad \Phi(\mu_\infty) = \mu_\infty\}. \tag{1.8}$$

By explicit computations, we can find that the n -particle Gibbs measure $\mu^{(n)}$ for the energy H , defined by

$$\mu^{(n)}(d\mathbf{x}) := \frac{e^{-nH(\mu_{\mathbf{x}})}}{\int_{\mathbb{R}^{nd}} e^{-nH(\mu_{\mathbf{y}})} d\mathbf{y}} d\mathbf{x}, \tag{1.9}$$

is invariant to the over-damped dynamics.

In the kinetic case, in general, if the velocity and external confinement fields do not derive from potential gradients, the invariant probabilities are not explicit.

We give in this section a short review of the recent progresses in the long-time behavior and the uniform-in-time propagation of chaos property of McKean–Vlasov dynamics, with an emphasis on (over and under damped) mean field Langevin dynamics ones. We refer readers to [3, 4] for a more comprehensive review of propagation of chaos.

As for the long-time behavior, the uniform in time propagation of chaos and the convergence rates, many works have been devoted to the subject and different approaches have been introduced: coupling methods, functional inequalities...

Coupling approaches. Let us briefly describe the coupling method (the major approach of this paper). The basic idea is that an upper bound on the Wasserstein distance between two probability distributions is given by the construction of any pair of random variables distributed respectively according to those. The goal is thus to construct simultaneously two solutions that have a trend to get closer with time. Each solution being driven by a Brownian motion, a coupling of two solutions then follows from a coupling of the Brownian motions. The nature of the coupling therefore depends on the choice of Brownian motions. The coupling is said to be synchronous if the Brownian motions are the same: it is a coupling for which the Brownian noise cancels out in the infinitesimal evolution of the difference. In that case the contraction of a distance between the processes can only be induced by the drift component, as in [5, 2]. Such a contraction holds under the strong convexity conditions.

Nevertheless, in more general cases, the calculation of the evolution of the difference shows that there is still some deterministic contraction on a subspace: see, among others, [6, 7, 8]. We can therefore use a synchronous coupling in the vicinity of this subspace. Outside of this subspace, it is necessary to make use of the noise to get the processes closer together, at least in the direction orthogonal to this space. In order to maximize the variance of this noise, we then use a so-called reflection coupling, which consists of taking opposing Brownian motions in the direction of space given by the difference process, and synchronous in the orthogonal direction.

The granular media and the Vlasov-Fokker-Planck dynamics have attracted a lot of attention, usually named McKean-Vlasov diffusions, these past twenty-four years, by means of a stochastic interpretation and synchronous couplings as in [5, 2] or the recent by reflection couplings as in [7] enabling to get rid of convexity conditions, but limited to small interactions. Remark however that small interactions are natural to get uniform in time propagation of chaos as for large interactions the non linear limit equation may have several stationary measures (see [9] for example).

Concerning the mean-field Langevin dynamics, propagation of chaos in finite time for the first and second orders McKean-Vlasov dynamics given by Eq. (1.1) and Eq. (1.2) is relatively easy to show using *synchronous coupling*: for example, regarding *polynomial interaction*, see [1, Theorem 5.1]. The bound obtained by this method, however, generally tends to infinity when the time interval extends to infinity. Besides, the dynamics may possess *multiple invariant measures*, so uniform in time convergence can not be expected without additional assumptions, or a more general definition of convergence itself (e.g. *convergence modulo symmetries*). By assuming the functional convexity of the energy, uniform in time propagation of chaos for the mean-field Langevin dynamics and the kinetic mean-field Langevin dynamics have been proven in [10, 11]. In [8], the authors consider a flat energy which could be non-convex but is essentially with small (nonlinear) dependence, and aim at obtaining an exponential contraction result. The authors mainly borrow the tools developed in Eberle, Guillin and Zimmer [12, 6], where they initiate the reflection-synchronous coupling technique, further validate it in the study of the Langevin dynamics with a general non-convex potential, and make the point that the technique offers significant flexibility for additional development. But note that [6] is not concerned with mean-field

interaction and the rate found there is dimension dependent. In [8], the authors design a new metric involving a quadratic form ([8, Section 4.4.2] to obtain the contraction when the coupled particles are far away, and as a result obtain a dimension-free convergence rate. The construction of the quadratic form shares some flavor with the argument in Bolley, Guillin and Malrieu [2]. Notably, their construction helps to capture the optimal rate in the area of interest ([8, Remark 4.16]), so may be more intrinsic. Notice that most of the articles concerning the ergodicity of underdamped Langevin dynamics obtain the convergence rates depending on the dimension, and in particular very few allow both non-convex potential and the mean-field interaction. In [2], the result holds in the noncompact case with small Lipschitz interaction, and can be seen as a first attempt to deal with more general case. Moreover it shows existence and uniqueness of the equilibrium measure, and in particular does not use its explicit expression (which is unknown in our broader situation). It is also not only a result on the convergence to equilibrium, but also a stability result of all solutions. Let us finally note that it is based on the natural stochastic interpretation and a simple coupling argument, and does not need any hypoelliptic regularity property of the solutions.

Functional approaches. Another approach to long-time behavior and uniform-in-time propagation of chaos is the functional one, and this is also the major approach of [10, 11, 13, 14]. In this situation in order to study the long-time behaviors and propagation of chaos properties, we construct appropriate (Lyapunov) functionals and investigate the change of their values along the dynamics. For example, the relative entropy is used as the functional in many works. In fact, for example, if $\mu^{(n)}$ verifies a ρ_n -log-Sobolev inequality :

$$\rho_n \mathbf{H}[\cdot | \mu^{(n)}] \leq 2\mathbf{I}[\cdot | \mu^{(n)}], \quad (1.10)$$

then for all initial value $\mu_0^{(n)}$ of the flow,

$$\mathbf{H}[\mu_t^{(n)} | \mu^{(n)}] \leq e^{-2\rho_n t} \mathbf{H}[\mu_0^{(n)} | \mu^{(n)}] \quad (1.11)$$

where \mathbf{H} is the *relative entropy* defined by

$$\mathbf{H}[v | \mu] = \begin{cases} \mathbb{E}_v[\log \frac{dv}{d\mu}] =: \mathbf{Ent}_\mu[\frac{dv}{d\mu}] & \text{if } v \ll \mu, \\ +\infty & \text{otherwise.} \end{cases} \quad (1.12)$$

and \mathbf{I} is the *Fisher information* defined by

$$\mathbf{I}[v | \mu] = \int \left\| \nabla \sqrt{\frac{dv}{d\mu}} \right\|^2 d\mu = \frac{1}{4} \int \left\| \nabla \log \frac{dv}{d\mu} \right\|^2 dv = \frac{1}{4} \int \left\| \mathcal{D}_m \mathbf{H}[\cdot | \mu](v, y) \right\|^2 v(dy) \quad (1.13)$$

if $v \ll \mu$ and $\sqrt{\frac{dv}{d\mu}} \in \mathbf{H}_\mu^1$, and $\mathbf{I}[v | \mu] = +\infty$ otherwise, where \mathbf{H}_μ^1 is the domain of the Dirichlet form

$$\mathcal{E}_\mu : g \longmapsto \int \|\nabla g\|^2 d\mu. \quad (1.14)$$

We refer to Section 5.2 of the Bakry-Gentil-Ledoux monograph [15] for details. Thus the existence of a log-Sobolev constant independent from n implies that the rate of *exponential convergence in entropy* is independent of the number of particles. The interest of such a universal constant lies in the study of the *long-time behavior of the mean field limit* (for example, [1]). In [16], assuming a *log-Sobolev inequality* for $\Phi(\mu)$ uniform in μ , a *Poincaré inequality* independent of n for the conditional marginals of $\mu^{(n)}$ and the *convexity in flat interpolation sense* of \mathbf{H} , a log-Sobolev inequality for $\mu^{(n)}$ independent of n is proved. Time-uniform log-Sobolev inequalities satisfied by solutions of semi-linear mean-field equations have recently appeared to be a key tool to obtain time-uniform propagation of chaos estimates.

In this paper, the flat energies considered are not necessarily convex. Let the flat energy regularized by relative entropy:

$$H_0 + \frac{\sigma^2}{2} \mathbf{H}[\cdot | \alpha] \quad (1.15)$$

where α is the *Gibbs distribution* associated with a *confinement potential* V and energy H_0 is convex in functional sense:

$$\forall (t, \mu, \nu) \in [0, 1] \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d), \quad H_0((1-t)\mu + t\nu) \leq (1-t)H_0(\mu) + tH_0(\nu). \quad (1.16)$$

In [17], The drift associated with the Hamiltonian of the mean-field Langevin dynamics given by $\mu \mapsto H_0(\mu) + \int V d\mu$ corresponds to the *gradient descent* in L^2 -Wasserstein space of the *free energy*. Since the free energy is strictly convex, there is at most one minimizer μ_∞ of the energy that satisfies

$$\frac{\delta H_0}{\delta m}(\mu_\infty, \cdot) + \frac{\sigma^2}{2} \ln \frac{d\mu_\infty}{d\alpha} = C_{\mu_\infty} \iff \mu_\infty(dx) = \frac{1}{Z_{\mu_\infty}} e^{-\frac{2}{\sigma^2} \frac{\delta H_0}{\delta m}(\mu_\infty, \cdot)} \alpha(dx). \quad (1.17)$$

Contributions. In the continuity of [1, 2], we are interested in problems of contraction and uniform in time propagation of chaos by synchronous coupling for mean-field Langevin dynamics (over- and under-damped).

(i) **Over-damped dynamics**

- ▷ First, as a generalization of the proof of [1, Proposition 4.1], we show that the invariant measure map is a strict contraction of a metric space (Theorem 3.1): existence of at most one equilibrium on this space.
- ▷ Second, by synchronous coupling, we show that there exists a unique equilibrium and that we have exponential convergence towards it (Theorem 3.2).
- ▷ Finally, we show a result of propagation of chaos L^2 with a bound which can be chosen uniform in time (Theorem 3.3).

(ii) **under-damped dynamics**

- ▷ First, as a generalization of [2, Theorem 1], we show that there exists a unique kinetic equilibrium and that we have exponential convergence towards it (Theorem 3.4).
- ▷ Finally, as a generalization of [2, Theorem 2], we show a result of uniform in time propagation of chaos L^2 (Theorem 3.5).

Organization of paper. Let us finish this introduction by the plan of the paper. The paper is mainly divided into five sections. In Section 2, we give the settings and assumptions of the paper. The main theorems are stated in Section 3 and examples are given in Section 4. We discuss the connection with partial differential equations in Section 5. Finally, Section 6 focuses on the proofs of the main results of the paper.

2 Settings and assumptions

We give an informal preview of our settings and assumptions in this section.

2.1 Settings

Let $d \geq 1$ be an integer. In this paper, we consider mean field energy functionals H that are defined on the space of probability measures on \mathbb{R}^d . More precisely, such functionals are mappings ($p \geq 0$)

$$H: \mathcal{P}_p(\mathbb{R}^d) \longrightarrow \mathbb{R} \tag{2.1}$$

where $\mathcal{P}_p(\mathbb{R}^d)$ denotes the set of probability measures of finite p -moment equipped with its weak topology. We denote \mathcal{W}_p the Wasserstein p -metric defined by

$$\forall (\mu, \nu) \in \mathcal{P}_p(\mathbb{R}^d) \times \mathcal{P}_p(\mathbb{R}^d), \quad \mathcal{W}_p(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} (\mathbb{E}[\|X - Y\|^p])^{\min(\frac{1}{p}, 1)}. \tag{2.2}$$

We denote by $\mathcal{D}_m H$ and $\mathcal{D}_m^2 H$ the first and second orders intrinsic derivatives of H : the first order intrinsic derivative is a mapping from $\mathcal{P}_p(\mathbb{R}^d) \times \mathbb{R}^d$ into \mathbb{R}^d ; and the second order intrinsic derivative is a mapping from $\mathcal{P}_p(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^{d \times d}$ and will be defined by

$$\mathcal{D}_m H(\mu, x) := \nabla_x \frac{\delta H}{\delta m}(\mu, x); \tag{2.3}$$

$$[\mathcal{D}_m H]_{1, \infty} := \sup_{\mu} [\mathcal{D}_m H(\mu, \cdot)]_1; \tag{2.4}$$

$$\mathcal{D}_m^2 H(\mu, x, y) := \nabla_{x, y}^2 \frac{\delta^2 H}{\delta m^2}(\mu, x, y); \tag{2.5}$$

$$\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} := \sup_{\mu, x, y} \|\mathcal{D}_m^2 H(\mu, x, y)\|_2; \tag{2.6}$$

where $\frac{\delta H}{\delta m}$ is called (first) *flat derivative* of H and is a continuous mapping from $\mathcal{P}_p(\mathbb{R}^d) \times \mathbb{R}^d$ into \mathbb{R}^d satisfying the two following properties :

- ▷ For all $K \subset \mathcal{P}_p(\mathbb{R}^d)$ compact,

$$\exists C_K \geq 0 \quad \forall x \in \mathbb{R}^d, \quad \sup_{\mu \in K} \left| \frac{\delta H}{\delta m}(\mu, x) \right| \leq C_K (1 + \|x\|^p); \tag{2.7}$$

▷

$$H(\mu_1) - H(\mu_0) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta H}{\delta m}(t\mu_1 + (1-t)\mu_0, x)(\mu_1 - \mu_0)(dx)dt \quad (2.8)$$

$$\iff \frac{d}{dt}H(t\mu_1 + (1-t)\mu_0) = \int_{\mathbb{R}^d} \frac{\delta H}{\delta m}(t\mu_1 + (1-t)\mu_0, x)(\mu_1 - \mu_0)(dx). \quad (2.9)$$

2.2 Assumptions

We impose the following assumptions on the energy H.

Assumptions of over-damped dynamics

Assumption 2.1. The functional H admits first and second order flat derivatives

$$\frac{\delta H}{\delta m} : \mathcal{P}_p(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow \mathbb{R}; \quad (2.10)$$

$$\frac{\delta^2 H}{\delta m^2} : \mathcal{P}_p(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}; \quad (2.11)$$

that are jointly continuous and are \mathcal{C}^2 in the spatial variables. we have the following:

(i) $(\mu, x, y) \mapsto \|\mathcal{D}_m^2 H(\mu, x, y)\|_2$ is bounded ($\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} < +\infty$);

(ii) $(\mu, x) \mapsto \lambda_{\nabla \mathcal{D}_m H(\mu, x)}$ is strictly bounded from below by $\|\mathcal{D}_m^2 H\|_{\text{op}, \infty}$.

Assumption 2.2. Assume that the functional H admits first and second order flat derivatives that are jointly continuous and are \mathcal{C}^2 in the spatial variables. Suppose that

$$\sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \left\| \nabla \frac{\delta H}{\delta m}(\mu, 0) \right\|_2 < +\infty \quad (2.12)$$

and there exists $\lambda > 0$ such that

(i) for all $(\mu, x, y) \in \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\left\langle \nabla \frac{\delta H}{\delta m}(\mu, x) - \nabla \frac{\delta H}{\delta m}(\mu, y), x - y \right\rangle \geq \lambda \|x - y\|_2^2; \quad (2.13)$$

(ii) the map $(\mu, x, y) \mapsto \|\mathcal{D}_m^2 H(\mu, x, y)\|_2$ is bounded ($\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} < +\infty$) and $\lambda > \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}$.

Assumptions of under-damped dynamics

Assumption 2.3. Assume that

▷ $[A]_1 < +\infty$, $B : p \mapsto \lambda_B p + D(p)$ with $\lambda_B \geq 0$ and $[D]_1 < +\infty$.

▷ There exists $\lambda_A \geq 0$ such that for all $(v, w) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\langle A(v) - A(w), v - w \rangle \geq \lambda_A |v - w|^2. \quad (2.14)$$

▷ $[\mathcal{D}_m H]_{1, \infty} < +\infty$ and $\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} < +\infty$.

3 Main theorems

3.1 First order case

Theorem 3.1 (Contraction of Φ). *If the energy functional H satisfies Assumption 2.1, then*

$$\forall p \geq 0, \quad \Phi : \mathcal{P}_p(\mathbb{R}^d) \longrightarrow \mathcal{P}_p(\mathbb{R}^d), \quad (3.1)$$

and for

$$\underline{\lambda} := \inf_{\mu, x} \lambda_{\nabla \mathcal{D}_m H(\mu, x)} \quad (> \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}), \quad (3.2)$$

for all $p \geq 1$, Φ defines a strict contraction of $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_1)$:

$$\forall (\mu, \nu) \in \mathcal{P}_p(\mathbb{R}^d) \times \mathcal{P}_p(\mathbb{R}^d), \quad \mathcal{W}_1(\Phi(\mu), \Phi(\nu)) \leq \frac{\|\mathcal{D}_m^2 H\|_{\text{op}, \infty}}{\lambda} \mathcal{W}_1(\mu, \nu). \quad (3.3)$$

In this case, Φ admits at most one fixed point on $\mathcal{P}_p(\mathbb{R}^d)$: if $p = 1$, there is a unique fixed point.

Proof. See Section 6.1. □

Theorem 3.2 (\mathcal{W}_2 -Exponential contraction by coupling). *If the energy functional H satisfies Assumption 2.2, for all $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ solutions of the mean field Langevin equation Eq. (1.1) with initial conditions $\mu_0^X \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mu_0^Y \in \mathcal{P}_2(\mathbb{R}^d)$, we have*

- ▷ for all $t \geq 0$, $\mu_t^X \in \mathcal{P}_2(\mathbb{R}^d)$;
- ▷ for all $t \geq 0$,

$$\mathcal{W}_2(\mu_t^X, \mu_t^Y) \leq e^{(\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - \lambda)t} \mathcal{W}_2(\mu_0^X, \mu_0^Y). \quad (3.4)$$

In particular, there exists a unique invariant measure $\mu_\infty \in \mathcal{P}_2(\mathbb{R}^d)$ and

$$\forall t \geq 0, \quad \mathcal{W}_2(\mu_t^X, \mu_\infty) \leq e^{(\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - \lambda)t} \mathcal{W}_2(\mu_0^X, \mu_\infty). \quad (3.5)$$

Proof. See Section 6.1. □

Theorem 3.3 (Uniform in time propagation of chaos for mean field Langevin dynamics). *If the energy functional H satisfies Assumption 2.2, for $(\tilde{X}^{(n)})_{n \geq 1}$ a sequence of independent copies of the mean field Langevin process $(X_t)_{t \geq 0}$ synchronously coupled to particles of the over-damped mean field Langevin system, we have*

$$\forall T \geq 0 \quad \forall t \in [0, T], \quad \mathbb{E}[\|X_t^{(1)} - \tilde{X}_t^{(1)}\|^2] \leq \begin{cases} \frac{\alpha_1(T)}{\beta_1} (e^{\beta_1 t} - 1) & \text{if } \beta_1 \neq 0; \\ \alpha_1(T) t & \text{if } \beta_1 = 0. \end{cases} \quad (3.6)$$

$$\alpha_1(T) := \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} C(d) \left(\sup_{t \in [0, T]} \langle \mu_t^X, \|\cdot\|_2^2 \rangle \right) \delta_d(n); \quad (3.7)$$

$$\delta_d(n) := \begin{cases} \frac{1}{\sqrt{n}} & \text{if } d < 4; \\ \frac{\ln(n+1)}{\sqrt{n}} & \text{if } d = 4; \\ \frac{1}{n^{\frac{d}{2}}} & \text{if } d > 4; \end{cases} \quad (3.8)$$

$$\beta_1 := 3 \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - 2\lambda; \quad (3.9)$$

$$\alpha := 2d + 2 \sup_{\mu} \left\| \nabla \frac{\delta H}{\delta m}(\mu, 0) \right\|_2; \quad (3.10)$$

$$\beta := -2 \left(\lambda - \sup_{\mu} \left\| \nabla \frac{\delta H}{\delta m}(\mu, 0) \right\|_2 \right). \quad (3.11)$$

Moreover, this upper bound is uniform in time if, and only if, $\beta < 0$ and $\beta_1 < 0$:

$$\sup_{t \geq 0} \mathbb{E}[\|X_t^{(1)} - \tilde{X}_t^{(1)}\|^2] \leq \frac{\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} C(d)}{2\lambda - 3 \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}} \left(\langle \mu_0^X, \|\cdot\|_2^2 \rangle - \frac{\alpha}{\beta} \right) \delta_d(n). \quad (3.12)$$

Proof. See Section 6.1. □

3.2 Kinetic case

Theorem 3.4 (Kinetic contraction by coupling). *If the fields A and B and the energy functional H satisfy Assumption 2.3, for all positive $[A]_1$, λ_A and λ_B , there exists a positive constant c such that, if*

$$0 \leq \min([D]_1, [\mathcal{D}_m H]_{1, \infty}, \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}) \leq \max([D]_1, [\mathcal{D}_m H]_{1, \infty}, \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}) < c, \quad (3.13)$$

then there exist positive constants C and C' such that, for all $(P_t, V_t)_{t \geq 0}$ and $(\bar{P}_t, \bar{V}_t)_{t \geq 0}$ two solutions of the kinetic mean-field Langevin equation Eq. (1.2) with initial conditions $\mathbb{P}_{(P_0, V_0)} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathbb{P}_{(\bar{P}_0, \bar{V}_0)} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\forall t \geq 0, \quad \mathcal{W}_2(\mathbb{P}_{(P_t, V_t)}, \mathbb{P}_{(\bar{P}_t, \bar{V}_t)}) \leq C' e^{-Ct} \mathcal{W}_2(\mathbb{P}_{(P_0, V_0)}, \mathbb{P}_{(\bar{P}_0, \bar{V}_0)}). \quad (3.14)$$

Moreover, there exists a unique stationary solution ξ_∞ and

$$\forall t \geq 0, \quad \mathcal{W}_2(\mathbb{P}_{(P_t, V_t)}, \xi_\infty) \leq C' e^{-Ct} \mathcal{W}_2(\mathbb{P}_{(P_0, V_0)}, \xi_\infty). \quad (3.15)$$

Proof. See Section 6.2. □

Theorem 3.5 (Uniform in time propagation of chaos for kinetic mean field Langevin dynamics). *Let $(\bar{P}^{(n)}, \bar{V}^{(n)})_{n \geq 1}$ a sequence of independent copies of the kinetic mean field Langevin process $(P_t, V_t)_{t \geq 0}$ synchronously coupled to particles of the under-damped mean field Langevin system with initial datum $(P_0^{(i)}, V_0^{(i)})$ for $i \in \{1, \dots, n\}$ be n independent \mathbb{R}^{2d} -valued random variables with law $f_0 \in \mathcal{P}_2(\mathbb{R}^{2d})$. If the fields A and B and the energy functional H satisfy Assumption 2.3, for all positive $[A]_1$, λ_A and λ_B , there exists a positive constant c such that, if*

$$0 \leq \min([D]_1, [\mathcal{D}_m H]_{1, \infty}, \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}) \leq \max([D]_1, [\mathcal{D}_m H]_{1, \infty}, \|\mathcal{D}_m^2 H\|_{\text{op}, \infty}) < c, \quad (3.16)$$

then there exists a positive constant C , independent of n , such that for $i \in \{1, \dots, n\}$

$$\sup_{t \geq 0} \mathbb{E} \left[|P_t^{(i)} - \bar{P}_t^{(i)}|^2 + |V_t^{(i)} - \bar{V}_t^{(i)}|^2 \right] \leq C \begin{cases} \frac{1}{\sqrt{n}} & \text{if } d < 4; \\ \frac{\ln(n+1)}{\sqrt{n}} & \text{if } d = 4; \\ \frac{1}{n^{\frac{d}{2}}} & \text{if } d > 4. \end{cases} \quad (3.17)$$

Here the constant C depends only on the coefficients of the equation and the second moment of f_0 .

Proof. See Section 6.2. □

3.3 Comments on drift and assumptions

In Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5, we can replace the intrinsic derivative by a more general drift

$$b: \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d) \longrightarrow \mathbb{R}^d. \quad (3.18)$$

In the first order case, we can replace Assumption 2.2 by the following conditions on the new drift b

$$\sup_{\mu} b(0, \mu) < +\infty; \quad (3.19)$$

$$\exists \lambda > \beta \geq 0 \quad \forall (x, y, \mu, \nu), \quad \langle b(x, \mu) - b(y, \nu), x - y \rangle \geq \lambda |x - y|^2 - \beta |x - y| \mathcal{W}_p(\mu, \nu). \quad (3.20)$$

As for the proofs, it is enough to mimic those with the intrinsic derivative (Section 6.1) which is why we do them without loss of generality with the intrinsic derivative.

In kinetic case, we can replace Assumption 2.3 by the same conditions on the velocity field A and the external confinement field B and the following Lipschitz conditions on the new drift b

$$\exists \gamma_1, \gamma_2 \geq 0, \quad \forall (x, y, \mu, \nu), \quad |b(x, \mu) - b(y, \mu)| \leq \gamma_1 |x - y|; \quad (3.21)$$

$$|b(x, \mu) - b(x, \nu)| \leq \gamma_2 \mathcal{W}_p(\mu, \nu). \quad (3.22)$$

As in the previous case, we carry out the proofs with the intrinsic derivative without loss of generality.

4 Examples

In this section, we give examples of flat energies for which our results are satisfied.

4.1 Two-body interaction: granular media flat energy

In this case, we consider the following flat energy

$$H_{VW} : \mu \longmapsto \int V(x)\mu(dx) + \frac{1}{2} \int W(x-y)\mu(dx)\mu(dy) \quad (4.1)$$

with $V \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ called confinement potential and $W \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ called interaction potential such that for all $x \in \mathbb{R}$, $W(-x) = W(x)$.

Proposition 4.1 (Two-body interaction and Assumption 2.1). *If $\|\nabla^2 W\|_{\text{op},\infty} < +\infty$ and*

$$\inf_{(\mu,x)} \lambda_{\nabla^2 V(x) + \nabla^2 W * \mu(x)} > \|\nabla^2 W\|_{\text{op},\infty}, \quad (4.2)$$

H_{VW} satisfies Assumption 2.1.

Proof. See Section 6.3. □

Remark 4.1 (About one-dimensional example). In this case, the contraction conditions become

$$\begin{aligned} & \|W''\|_{\infty} < +\infty; \\ \forall (\mu, x), \quad & V''(x) + \int W''(x-y)\mu(dy) > \|W''\|_{\infty}. \end{aligned}$$

We have

$$V''(x) + \int W''(x-y)\mu(dy) \geq V''(x) - \|W''\|_{\infty}.$$

In particular, if for all $x \in \mathbb{R}$, $V''(x) > 2\|W''\|_{\infty}$, we deduce that

$$\forall (\mu, x), \quad V''(x) + \int W''(x-y)\mu(dy) > \|W''\|_{\infty}.$$

For example, we can choose the following polynomial confinement potentials:

▷ $V(x) := ax^2 + bx + c$ with a, b and c constants such that $a > \|W''\|_{\infty}$ and $(b, c) \in \mathbb{R}^2$.

▷ $V(x) := ax^4 + bx^3 + cx^2 + dx + e$ with a, b, c, d and e constants such that $a > 0$, $c - \frac{3b^2}{8a} > \|W''\|_{\infty}$ and $(d, e) \in \mathbb{R}^2$.

Proposition 4.2 (Two-body interaction and Assumption 2.2). *If $\|\nabla W\|_{2,+\infty} < +\infty$, $\|\nabla^2 W\|_{\text{op},\infty} < +\infty$ and there exists $\lambda > 0$ such that for all (μ, x, y) ,*

$$\left\langle \nabla V(x) - \nabla V(y) + \nabla W * \mu(x) - \nabla W * \mu(y), x - y \right\rangle \geq \lambda \|x - y\|_2^2; \quad (4.3)$$

$$\lambda > \|\nabla^2 W\|_{\text{op},\infty}; \quad (4.4)$$

H_{VW} satisfies Assumption 2.2.

Proof. See Section 6.3. □

Remark 4.2. If for all (x, y) ,

$$\left\langle \nabla V(x) - \nabla V(y), x - y \right\rangle \geq \lambda_V \|x - y\|_2^2; \quad (4.5)$$

$$\left\langle \nabla W(x) - \nabla W(y), x - y \right\rangle \geq 0; \quad (4.6)$$

we have

$$\begin{aligned} \left\langle \nabla V(x) - \nabla V(y) + \nabla W * \mu(x) - \nabla W * \mu(y), x - y \right\rangle &= \left\langle \nabla V(x) - \nabla V(y), x - y \right\rangle \\ &+ \int \left\langle \nabla W(x-z) - \nabla W(y-z), x - z - (y - z) \right\rangle \mu(dz) \\ &\geq \lambda_V \|x - y\|_2^2. \end{aligned} \quad (4.7)$$

4.2 Many-body interaction: polynomial flat energy

In this case, we consider the following flat energy

$$H_{\text{VW}}^N : \mu \longmapsto \int V d\mu + \sum_{k=2}^N \int W^{(k)} d\mu^{\otimes k}, \quad (4.8)$$

where $\forall k \in \{2, \dots, N\}$, $W^{(k)}$ is a *symmetric interaction potential* between k particles and N represents the number of such potentials.

Proposition 4.3 (Many-body interaction and Assumption 2.1). *If for all $k \in \{2, \dots, N\}$, $\|\nabla_{x_1, x_2}^2 W^{(k)}\|_{\text{op}, \infty} < +\infty$ and*

$$\inf_{(\mu, x)} \lambda_{\nabla^2 V(x) + \sum_{k=2}^N k \nabla_{x_1}^2 W^{(k)} * \mu^{\otimes k-1}(x)} > \sum_{k=2}^N k(k-1) \|\nabla_{x_1, x_2}^2 W^{(k)}\|_{\text{op}, \infty}, \quad (4.9)$$

H_{VW}^N satisfies Assumption 2.1.

Proof. See Section 6.3. □

Proposition 4.4 (Many-body interaction and Assumption 2.2). *If for all $k \in \{2, \dots, N\}$, $\|\nabla_{x_1} W^{(k)}(0, \cdot)\|_{2, +\infty} < +\infty$, $\|\nabla_{x_1, x_2}^2 W^{(k)}\|_{\text{op}, \infty} < +\infty$ and there exists $\lambda > 0$ such that for all (μ, x, y) ,*

$$\left\langle \nabla V(x) - \nabla V(y) + \sum_{k=2}^N k \left(\nabla_{x_1} W^{(k)} * \mu^{\otimes k-1}(x) - \nabla_{x_1} W^{(k)} * \mu^{\otimes k-1}(y) \right), x - y \right\rangle \geq \lambda \|x - y\|_2^2; \quad (4.10)$$

$$\lambda > \sum_{k=2}^N k(k-1) \|\nabla_{x_1, x_2}^2 W^{(k)}\|_{\text{op}, \infty}; \quad (4.11)$$

H_{VW}^N satisfies Assumption 2.2.

Proof. See Section 6.3. □

Remark 4.3. Note that the *granular-media flat energy* is a particular case of the polynomial energy:

$$H_{\text{VW}}(\mu) = \int V(x) \mu(dx) + \int W^{(2)}(x, y) \mu(dx) \mu(dy); \quad (4.12)$$

$$W^{(2)}(x, y) := \frac{1}{2} W(x - y). \quad (4.13)$$

We deduce from this that Proposition 4.3 and Proposition 4.4 generalize Proposition 4.1 and Proposition 4.2.

4.3 Internal flat energy

In this case, we consider the following flat energy

$$H_{\Psi} : \mu \longmapsto \Psi(\langle \mu, W \rangle) \quad (4.14)$$

with $\Psi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ and $W \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$.

Proposition 4.5 (Internal flat energy and Assumption 2.1). *If $\sup_{\mu} |\Psi''(\langle \mu, W \rangle)| < +\infty$, $\|\nabla^2(W \otimes W)\|_{\text{op}, \infty} < +\infty$ and*

$$\inf_{(\mu, x)} \lambda_{\Psi'(\langle \mu, W \rangle) \nabla^2 W(x)} > \|\nabla^2(W \otimes W)\|_{\text{op}, \infty} \sup_{\mu} |\Psi''(\langle \mu, W \rangle)|, \quad (4.15)$$

H_{Ψ} satisfies Assumption 2.1.

Proof. See Section 6.3. □

Proposition 4.6 (Internal flat energy and Assumption 2.2). *If $\sup_{\mu} |\Psi'(\langle \mu, W \rangle)| < +\infty$, $\sup_{\mu} |\Psi''(\langle \mu, W \rangle)| < +\infty$, $\|\nabla^2(W \otimes W)\|_{\text{op}, \infty} < +\infty$ and there exists $\lambda > 0$ such that for all (μ, x, y) ,*

$$\Psi'(\langle \mu, W \rangle) \left\langle \nabla W(x) - \nabla W(y), x - y \right\rangle \geq \lambda \|x - y\|_2^2; \quad (4.16)$$

$$\lambda > \|\nabla^2(W \otimes W)\|_{\text{op}, \infty} \sup_{\mu} |\Psi''(\langle \mu, W \rangle)|; \quad (4.17)$$

H_{Ψ} satisfies Assumption 2.2.

Proof. See Section 6.3. □

5 Connection with partial differential equations

The mean field Langevin dynamics corresponds to nonlinear Fokker-Planck equation defined on $[0, +\infty[\times\mathbb{R}^d$ for the flow of probability measures $(\mu_t)_{t \geq 0}$:

$$\frac{d}{dt}\mu_t = \Delta\mu_t + \nabla \cdot (\mu_t \mathcal{D}_m H(\mu_t, \cdot)). \quad (5.1)$$

In the kinetic case, we have

$$\frac{d}{dt}f_t + v \cdot \nabla_p f_t - \mathcal{D}_m H(f_t, p) \cdot \nabla_v f_t = \Delta_v f_t + \nabla_v \cdot \left((A(v) + B(p)) f_t \right). \quad (5.2)$$

The particle system corresponds to a Liouville or Fokker-Planck equation defined on $[0, +\infty[\times\mathbb{R}^{nd}$ for the flow of probability measures defined by

$$\forall t \geq 0, \quad \mu_t^{(n)} := \mathbb{P}_{\mathbf{X}_t}; \quad (5.3)$$

$$H_n(\mathbf{x}) := nH(\mu_{\mathbf{x}}); \quad (5.4)$$

$$\frac{d}{dt}\mu_t^{(n)} = \Delta\mu_t^{(n)} + \nabla \cdot (\mu_t^{(n)} \nabla H_n). \quad (5.5)$$

In the kinetic case, we have the kinetic Fokker-Planck equation :

$$\frac{d}{dt}f_t^{(n)} + \mathbf{v} \cdot \nabla_{\mathbf{p}} f_t^{(n)} - \nabla H_n(\mathbf{p}) \cdot \nabla_{\mathbf{v}} f_t^{(n)} = \Delta_{\mathbf{v}} f_t^{(n)} + \sum_{i=1}^n \nabla_{v_i} \cdot \left((A(v_i) + B(p_i)) f_t^{(n)} \right). \quad (5.6)$$

Theorem 5.1 (Characterization of invariant distributions). *Assume that for all $\mu \in \mathcal{P}(\mathbb{R}^d)$, $x \in \mathbb{R}^d \mapsto \frac{\delta H}{\delta m}(\mu, x)$ is differentiable and*

$$Z_\mu := \int_{\mathbb{R}^d} e^{-\frac{\delta H}{\delta m}(\mu, x)} dx < +\infty. \quad (5.7)$$

The following assertions are equivalent:

(i) μ_∞ is a Maxwellian of the McKean-Vlasov PDE:

$$\mu_\infty \in \{v, \quad \Delta v + \nabla \cdot (v \mathcal{D}_m H(v, \cdot)) = 0\} \quad (5.8)$$

(ii) μ_∞ is a critical point for the free energy:

$$\mu_\infty \in \left\{ v, \quad \mathcal{D}_m H(v, \cdot) + \nabla \ln \frac{dv}{dx} = 0 \right\}. \quad (5.9)$$

(iii) μ_∞ is a fixed point of Φ :

$$\mu_\infty \in \{v, \quad \Phi(v) = v\}. \quad (5.10)$$

Proof. See Section 6.3. □

In the kinetic case, we recall that if the velocity and external confinement fields do not derive from potential gradients, the invariant probabilities are not explicit : in this paper, we will use [2, Lemma 10].

6 Proof of main theorems

6.1 Proof of main results of the first order case

Proof of Theorem 3.1. We first check that $\Phi(\mathcal{P}_p(\mathbb{R}^d)) \subset \mathcal{P}_p(\mathbb{R}^d)$. In Assumption 2.1, the uniform lower bound of the smallest eigenvalue of the Hessian of the first flat derivative implies that

$$\exists c_1^H > 0 \quad \exists c_2^H \in \mathbb{R} \quad \forall x \in \mathbb{R}^d \quad \forall \mu \in \mathcal{P}_p(\mathbb{R}^d), \quad \frac{\delta H}{\delta m}(\mu, x) \geq c_1^H \|x\|_2^2 + c_2^H. \quad (6.1)$$

We deduce that

$$\begin{aligned} \langle \Phi(\mu), \|\cdot\|^p \rangle &= \frac{1}{Z_\mu} \int_{\mathbb{R}^d} \|x\|^p e^{-\frac{\delta H}{\delta m}(\mu, x)} dx \\ &\leq \frac{e^{-c_2^H}}{Z_\mu} \int_{\mathbb{R}^d} \|x\|^p e^{-c_1^H \|x\|^2} dx, \end{aligned} \quad (6.2)$$

and this last integral is finite since

$$\exists M^H > 0 \quad \forall x \in \mathbb{R}^d, \quad \|x\|^p e^{-c_1^H \|x\|^2} \leq M^H e^{-\frac{c_1^H}{2} \|x\|^2}. \quad (6.3)$$

Now, let's prove the contraction of Φ . Let $v_0, v_1 \in \mathcal{D}_p(\mathbb{R}^d)$ and set $v_t = (1-t)v_0 + tv_1$, $t \in [0, 1]$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a 1-Lipschitz smooth function. From the very definition of Φ ,

$$\langle \Phi(v_t), f \rangle = \frac{1}{Z_{v_t}} \int_{\mathbb{R}^d} f(x) e^{-\frac{\delta H}{\delta m}(v_t, x)} dx, \quad (6.4)$$

so that setting

$$g_t(x) := -\frac{d}{dt} \frac{\delta H}{\delta m}(v_t, x), \quad (6.5)$$

we get

$$\begin{aligned} \frac{d}{dt} \langle \Phi(v_t), f \rangle &= -\frac{\partial_t Z_{v_t}}{Z_{v_t}} \langle \Phi(v_t), f \rangle + \langle \Phi(v_t), f g_t \rangle \\ &= \langle \Phi(v_t), f g_t \rangle - \langle \Phi(v_t), f \rangle \langle \Phi(v_t), g_t \rangle \\ &= \text{Cov}_{\Phi(v_t)}(f, g_t). \end{aligned} \quad (6.6)$$

As the functional H admits first and second-order flat derivatives, we have

$$g_t(x) := -\int_{\mathbb{R}^d} \frac{\delta^2 H}{\delta m^2}(v_t, x, y) (v_1 - v_0)(dy). \quad (6.7)$$

For a probability μ , let \mathcal{L}_μ be the operator defined on \mathcal{C}^2 -functions by

$$\mathcal{L}_\mu f = \Delta f - \mathcal{D}_m H(\mu, \cdot) \cdot \nabla f. \quad (6.8)$$

Denoting by φ_t be the solution of the Poisson equation $f - \Phi(v_t)(f) = \mathcal{L}_{v_t} \varphi_t$ and using that \mathcal{L}_{v_t} is self-adjoint in $L^2(\Phi(v_t))$, we get

$$\text{Cov}_{\Phi(v_t)}(f, g_t) = \langle \varphi_t, \mathcal{L}_{v_t} g_t \rangle_{L^2(\Phi(v_t))} = -\langle \nabla \varphi_t, \nabla g_t \rangle_{L^2(\Phi(v_t))}.$$

Note that for the second equality, we used the fact that for some \mathcal{C}^2 -functions f and g , $\mathcal{L}_{v_t}(f \cdot g) = f \mathcal{L}_{v_t} g + g \mathcal{L}_{v_t} f + 2\nabla f \cdot \nabla g$. With the help of Cauchy-Schwarz inequality, this leads to

$$\begin{aligned} |\text{Cov}_{\Phi(v_t)}(f, g_t)| &\leq \|\nabla \varphi_t\|_{L^2(\Phi(v_t))} \|\nabla g_t\|_{L^2(\Phi(v_t))} \\ &\leq \|\nabla \varphi_t\|_\infty \|\nabla g_t\|_\infty. \end{aligned} \quad (6.9)$$

Control of $\|\nabla \varphi_t\|_\infty$. Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function with bounded derivative. Let φ be a unique solution of the Poisson equation $h - \langle \Phi(\mu), h \rangle = \mathcal{L}_\mu \varphi$. It is well-known that $\varphi(x) = \int_0^{+\infty} P_t^\mu h(x) - \langle \Phi(\mu), h \rangle dt$ where $(P_t^\mu)_{t \geq 0}$ denotes the semi-group associated with \mathcal{L}_μ so that under adequate derivation conditions (which will be satisfied in our setting),

$$\nabla \varphi(x) = \int_0^{+\infty} \nabla P_t^\mu h(x) dt.$$

Indeed, it is sufficient to verify the conditions of application of the Lebesgue dominated convergence theorem for the derivation under the integral sign, which is done below.

We have

$$\nabla P_t^\mu h(x) = \mathbb{E}[\nabla h(X_t^{\mu, x}) \partial_x X_t^{\mu, x}]$$

where $(X_t^{\mu, x})$ denotes the solution starting from x of the SDE associated with \mathcal{L}_μ and $(\partial_x X_t^\mu)$ its first variation process solution to

$$dY_t = -\nabla \mathcal{D}_m H(\mu, X_t^{\mu, x}) Y_t dt$$

with $Y_0 = I_d$. Under the assumption

$$\underline{\lambda} := \inf_{\mu, x} \lambda_{\nabla \mathcal{D}_m H(\mu, x)} > 0, \quad (6.10)$$

one easily deduces from a Gronwall argument that for any $z \in \mathbb{R}^d$,

$$|Y_t z|^2 \leq e^{-2\lambda t} |z|^2$$

which implies that

$$|\nabla P_t^\mu h(x)| \leq [h]_1 \|\partial_x X_t^{\mu, x}\|_{\text{op}} \leq [h]_1 e^{-\lambda t}$$

where $[h]_1$ denotes the Lipschitz constant of h . Then,

$$\|\nabla\varphi\|_\infty \leq [h]_1 \underline{\lambda}^{-1}.$$

In particular, we have

$$\|\nabla\varphi_t\|_\infty \leq [f]_1 \underline{\lambda}^{-1} = \frac{1}{\underline{\lambda}}. \quad (6.11)$$

Control of $\|\nabla g_t\|_\infty$. Let us finally focus on ∇g_t . First, one checks that

$$\nabla g_t(x) := - \int_{\mathbb{R}^d} \nabla_x \frac{\delta^2 H}{\delta m^2}(v_t, x, y)(v_1 - v_0)(dy). \quad (6.12)$$

Now, denoting by (X_{v_0}, X_{v_1}) an optimal coupling of v_0 and v_1 for the 1-Wasserstein distance, one obtains

$$\begin{aligned} \left| \nabla_x \left(\int_{\mathbb{R}^d} \frac{\delta^2 H}{\delta m^2}(v_t, x, y)(v_1 - v_0)(dy) \right) \right| &\leq |\mathbb{E}[\nabla_x \frac{\delta^2 H}{\delta m^2}(v_t, x, X_{v_1}) - \nabla_x \frac{\delta^2 H}{\delta m^2}(v_t, x, X_{v_0})]| \\ &\leq \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \mathcal{W}_1(v_0, v_1). \end{aligned} \quad (6.13)$$

Hence,

$$\|\nabla g_t\|_\infty \leq \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \mathcal{W}_1(v_0, v_1),$$

and by (6.9), we get for any smooth 1-Lipschitz function

$$|\langle \Phi(v_1), f \rangle - \langle \Phi(v_0), f \rangle| \leq \int_0^1 \left| \frac{d}{dt} \langle \Phi(v_t), f \rangle \right| dt \leq \frac{\|\mathcal{D}_m^2 H\|_{\text{op}, \infty}}{\underline{\lambda}} \mathcal{W}_1(v_0, v_1).$$

Since

$$\frac{\|\mathcal{D}_m^2 H\|_{\text{op}, \infty}}{\underline{\lambda}} < 1, \quad (6.14)$$

by a density argument and the Kantorovitch-Rubinstein duality relation, it follows that Φ is a contraction on $(\mathcal{P}_p(\mathbb{R}^d), \mathcal{W}_1)$.

Concerning the existence of at most one fixed point, we show it by the absurd: assuming that we have more than one, we would contradict the strict contraction since we would have

$$\frac{\|\mathcal{D}_m^2 H\|_{\text{op}, \infty}}{\underline{\lambda}} \geq 1. \quad (6.15)$$

For $p = 1$, existence follows from Picard's fixed point theorem since Φ is a strict contraction of $(\mathcal{P}_1(\mathbb{R}^d), \mathcal{W}_1)$ which is a complete metric space. \square

Proof of Theorem 3.2. Concerning the moments, we have

$$\begin{aligned} \left\langle \frac{d}{dt} \mu_t^X, \|\cdot\|_2^2 \right\rangle &= \left\langle \mu_t^X, \mathcal{L}_{\mu_t^X} \|\cdot\|_2^2 \right\rangle \\ &= \int \left(2d - 2 \nabla \frac{\delta H}{\delta m}(\mu_t^X, x) \cdot x \right) \mu_t^X(dx) \\ &\leq 2d - 2 \left[\left(\lambda - \left\| \nabla \frac{\delta H}{\delta m}(\mu_t^X, 0) \right\|_2 \right) \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle - \left\| \nabla \frac{\delta H}{\delta m}(\mu_t^X, 0) \right\|_2 \right] \\ &\leq 2d + 2 \left\| \nabla \frac{\delta H}{\delta m}(\mu_t^X, 0) \right\|_2 - 2 \left(\lambda - \left\| \nabla \frac{\delta H}{\delta m}(\mu_t^X, 0) \right\|_2 \right) \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle. \end{aligned} \quad (6.16)$$

We deduce that there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\left\langle \frac{d}{dt} \mu_t^X, \|\cdot\|_2^2 \right\rangle \leq \alpha + \beta \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle. \quad (6.17)$$

By Gronwall's lemma, we have

$$\forall t \geq 0, \quad \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle \leq \begin{cases} \left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle e^{\beta t} + \frac{\alpha}{\beta} (e^{\beta t} - 1) & \text{if } \beta \neq 0; \\ \left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle + \alpha t & \text{if } \beta = 0. \end{cases} \quad (6.18)$$

As for the exponential contraction, to do this, we consider the coupling given by

$$\begin{cases} dX_t &= -\nabla \frac{\delta H}{\delta m}(\mu_t^X, X_t) dt + \sqrt{2} dB_t \\ dY_t &= -\nabla \frac{\delta H}{\delta m}(\mu_t^Y, Y_t) dt + \sqrt{2} dB_t. \end{cases} \quad (6.19)$$

We have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|X_t - Y_t\|_2^2 &= \left\langle \frac{d}{dt} (X_t - Y_t), X_t - Y_t \right\rangle \\ &= -\left\langle \nabla \frac{\delta H}{\delta m}(\mu_t^X, X_t) - \nabla \frac{\delta H}{\delta m}(\mu_t^Y, Y_t), X_t - Y_t \right\rangle \\ &= -\left\langle \nabla \frac{\delta H}{\delta m}(\mu_t^Y, X_t) - \nabla \frac{\delta H}{\delta m}(\mu_t^Y, Y_t), X_t - Y_t \right\rangle \\ &\quad - \left\langle \nabla \frac{\delta H}{\delta m}(\mu_t^X, X_t) - \nabla \frac{\delta H}{\delta m}(\mu_t^Y, X_t), X_t - Y_t \right\rangle. \end{aligned} \quad (6.20)$$

On the one hand, according to (i), we have

$$-\left\langle \nabla \frac{\delta H}{\delta m}(\mu_t^Y, X_t) - \nabla \frac{\delta H}{\delta m}(\mu_t^Y, Y_t), X_t - Y_t \right\rangle \leq -\lambda \|X_t - Y_t\|_2^2. \quad (6.21)$$

On the other hand, by definition of the flat derivative and according to (ii), we have

$$\begin{aligned} &-\left\langle \nabla \frac{\delta H}{\delta m}(\mu_t^X, X_t) - \nabla \frac{\delta H}{\delta m}(\mu_t^Y, X_t), X_t - Y_t \right\rangle \\ &= -\int_0^1 \int \left\langle \nabla_x \frac{\delta^2 H}{\delta m^2}(s\mu_t^X + (1-s)\mu_t^Y, X_t, y), X_t - Y_t \right\rangle (\mu_t^X - \mu_t^Y)(dy) ds \\ &\leq \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \|X_t - Y_t\|_2 \mathcal{W}_1(\mu_t^X, \mu_t^Y) \\ &\leq \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \|X_t - Y_t\|_2 \mathbb{E}[\|X_t - Y_t\|_2]. \end{aligned} \quad (6.22)$$

We deduce that

$$\frac{1}{2} \frac{d}{dt} \|X_t - Y_t\|_2^2 \leq -\lambda \|X_t - Y_t\|_2^2 + \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \|X_t - Y_t\|_2 \mathbb{E}[\|X_t - Y_t\|_2]; \quad (6.23)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathbb{E}[\|X_t - Y_t\|_2^2] &\leq -\lambda \mathbb{E}[\|X_t - Y_t\|_2^2] + \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \left(\mathbb{E}[\|X_t - Y_t\|_2] \right)^2 \\ &\leq (\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - \lambda) \mathbb{E}[\|X_t - Y_t\|_2^2] \quad \text{Jensen's inequality.} \end{aligned} \quad (6.24)$$

According to Gronwall's lemma, we have

$$\forall t \geq 0, \quad \mathbb{E}[\|X_t - Y_t\|_2^2] \leq \mathbb{E}[\|X_0 - Y_0\|_2^2] e^{2(\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - \lambda)t}. \quad (6.25)$$

We deduce that

$$\forall t \geq 0, \quad \mathcal{W}_2^2(\mu_t^X, \mu_t^Y) \leq \mathbb{E}[\|X_t - Y_t\|_2^2] \leq \mathbb{E}[\|X_0 - Y_0\|_2^2] e^{2(\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - \lambda)t}. \quad (6.26)$$

We conclude that

$$\forall t \geq 0, \quad \mathcal{W}_2(\mu_t^X, \mu_t^Y) \leq e^{(\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - \lambda)t} \mathcal{W}_2(\mu_0^X, \mu_0^Y). \quad (6.27)$$

□

Proof of Theorem 3.3. Consider $(\tilde{X}^{(n)})_{n \geq 1}$ a sequence of independent copies of the mean field Langevin process $(X_t)_{t \geq 0}$. By synchronous coupling of the particle system with these copies, we have

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^n \|X_t^{(i)} - \tilde{X}_t^{(i)}\|^2 &= -2 \sum_{i=1}^n \left\langle \nabla \frac{\delta H}{\delta m}(\mu_{X_t}, X_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_t^X, \tilde{X}_t^{(i)}), X_t^{(i)} - \tilde{X}_t^{(i)} \right\rangle \\ &\leq -2\lambda \sum_{i=1}^n \|X_t^{(i)} - \tilde{X}_t^{(i)}\|^2 - 2 \sum_{i=1}^n \left\langle \nabla \frac{\delta H}{\delta m}(\mu_{X_t}, \tilde{X}_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_t^X, \tilde{X}_t^{(i)}), X_t^{(i)} - \tilde{X}_t^{(i)} \right\rangle \\ &\leq -2\lambda \sum_{i=1}^n \|X_t^{(i)} - \tilde{X}_t^{(i)}\|^2 + 2\|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \mathcal{W}_1(\mu_{X_t}, \mu_t^X) \sum_{i=1}^n \|X_t^{(i)} - \tilde{X}_t^{(i)}\|. \end{aligned} \quad (6.28)$$

On the other hand, we have

$$\mathcal{W}_1(\mu_{\mathbf{X}_t}, \mu_t^X) \leq \mathcal{W}_2(\mu_{\mathbf{X}_t}, \mu_t^X); \quad (6.29)$$

$$\mathcal{W}_2(\mu_{\mathbf{X}_t}, \mu_t^X) \leq \mathcal{W}_2(\mu_{\mathbf{X}_t}, \mu_{\tilde{\mathbf{X}}_t}) + \mathcal{W}_2(\mu_{\tilde{\mathbf{X}}_t}, \mu_t^X). \quad (6.30)$$

By Young's inequality, we have

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^n \|\mathbf{X}_t^{(i)} - \tilde{\mathbf{X}}_t^{(i)}\|^2 &\leq 2(\|\mathcal{D}_m^2 \mathbf{H}\|_{\mathbf{op}, \infty} - \lambda) \sum_{i=1}^n \|\mathbf{X}_t^{(i)} - \tilde{\mathbf{X}}_t^{(i)}\|^2 \\ &\quad + n\|\mathcal{D}_m^2 \mathbf{H}\|_{\mathbf{op}, \infty} \left(\mathcal{W}_2^2(\mu_{\mathbf{X}_t}, \mu_{\tilde{\mathbf{X}}_t}) + \mathcal{W}_2^2(\mu_{\tilde{\mathbf{X}}_t}, \mu_t^X) \right). \end{aligned} \quad (6.31)$$

We have

$$\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{X}_t}, \mu_{\tilde{\mathbf{X}}_t})] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\mathbf{X}_t^{(i)} - \tilde{\mathbf{X}}_t^{(i)}\|^2]. \quad (6.32)$$

By symmetry, we have

$$\forall j \in \{1, \dots, n\}, \quad \mathbb{E}[\|\mathbf{X}_t^{(j)} - \tilde{\mathbf{X}}_t^{(j)}\|^2] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\|\mathbf{X}_t^{(i)} - \tilde{\mathbf{X}}_t^{(i)}\|^2]. \quad (6.33)$$

Setting $\Upsilon_t := \mathbb{E}[\|\mathbf{X}_t^{(1)} - \tilde{\mathbf{X}}_t^{(1)}\|^2]$, we have

$$\frac{d}{dt} \Upsilon_t \leq (3\|\mathcal{D}_m^2 \mathbf{H}\|_{\mathbf{op}, \infty} - 2\lambda)\Upsilon_t + \|\mathcal{D}_m^2 \mathbf{H}\|_{\mathbf{op}, \infty} \mathbb{E}[\mathcal{W}_2^2(\mu_{\tilde{\mathbf{X}}_t}, \mu_t^X)]. \quad (6.34)$$

By Eq. (6.18), we have

$$\forall t \geq 0, \quad \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle \leq \begin{cases} \left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle e^{\beta t} + \frac{\alpha}{\beta} (e^{\beta t} - 1) & \text{if } \beta \neq 0; \\ \left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle + \alpha t & \text{if } \beta = 0. \end{cases} \quad (6.35)$$

$$\alpha := 2d + 2 \sup_{\mu} \left\| \nabla \frac{\delta \mathbf{H}}{\delta m}(\mu, 0) \right\|_2; \quad (6.36)$$

$$\beta := -2 \left(\lambda - \sup_{\mu} \left\| \nabla \frac{\delta \mathbf{H}}{\delta m}(\mu, 0) \right\|_2 \right). \quad (6.37)$$

If $\beta < 0$, we have

$$\sup_{t \geq 0} \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle \leq \left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle - \frac{\alpha}{\beta}. \quad (6.38)$$

By the result of Fournier and Guillin ([18, Theorem 1]), we have

$$\mathbb{E}[\mathcal{W}_2^2(\mu_{\tilde{\mathbf{X}}_t}, \mu_t^X)] \leq C(d) \left(\left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle - \frac{\alpha}{\beta} \right) \delta_d(n); \quad (6.39)$$

$$\delta_d(n) := \begin{cases} \frac{1}{\sqrt{n}} & \text{if } d < 4; \\ \frac{\ln(n+1)}{\sqrt{n}} & \text{if } d = 4; \\ \frac{1}{n^{\frac{d}{2}}} & \text{if } d > 4. \end{cases} \quad (6.40)$$

By Gronwall's lemma, we have ($\Upsilon_0 = 0$)

$$\forall t \geq 0, \quad \Upsilon_t \leq \begin{cases} \frac{\alpha_1}{\beta_1} (e^{\beta_1 t} - 1) & \text{if } \beta_1 \neq 0; \\ \alpha_1 t & \text{if } \beta_1 = 0. \end{cases} \quad (6.41)$$

$$\alpha_1 := \|\mathcal{D}_m^2 \mathbf{H}\|_{\mathbf{op}, \infty} C(d) \left(\left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle - \frac{\alpha}{\beta} \right) \delta_d(n); \quad (6.42)$$

$$\beta_1 := 3\|\mathcal{D}_m^2 \mathbf{H}\|_{\mathbf{op}, \infty} - 2\lambda. \quad (6.43)$$

In this case, the bound is uniform in time if, and only if, $\beta_1 < 0$:

$$\sup_{t \geq 0} \Upsilon_t \leq -\frac{\alpha_1}{\beta_1}. \quad (6.44)$$

If $\beta = 0$, we have:

$$\sup_{t \in [0, T]} \left\langle \mu_t^X, \|\cdot\|_2^2 \right\rangle \leq \left\langle \mu_0^X, \|\cdot\|_2^2 \right\rangle + \alpha T. \quad (6.45)$$

We deduce that

$$\forall t \in [0, T], \quad \Upsilon_t \leq \begin{cases} \frac{\alpha_2}{\beta_1} (e^{\beta_1 t} - 1) & \text{if } \beta_1 \neq 0; \\ \alpha_2 t & \text{if } \beta_1 = 0. \end{cases} \quad (6.46)$$

$$\alpha_2 := \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} C(d) \left(\langle \mu_0^X, \|\cdot\|_2^2 \rangle + \alpha T \right) \delta_d(n); \quad (6.47)$$

$$\beta_1 := 3 \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - 2\lambda. \quad (6.48)$$

In this case, since the moments are only bounded in finite time horizon, the propagation of chaos cannot be uniform in time.

If $\beta > 0$, we have

$$\sup_{t \in [0, T]} \langle \mu_t^X, \|\cdot\|_2^2 \rangle \leq \langle \mu_0^X, \|\cdot\|_2^2 \rangle e^{\beta T} + \frac{\alpha}{\beta} (e^{\beta T} - 1). \quad (6.49)$$

We deduce that

$$\forall t \in [0, T], \quad \Upsilon_t \leq \begin{cases} \frac{\alpha_3}{\beta_1} (e^{\beta_1 t} - 1) & \text{if } \beta_1 \neq 0; \\ \alpha_3 t & \text{if } \beta_1 = 0. \end{cases} \quad (6.50)$$

$$\alpha_3 := \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} C(d) \left(\langle \mu_0^X, \|\cdot\|_2^2 \rangle e^{\beta T} + \frac{\alpha}{\beta} (e^{\beta T} - 1) \right) \delta_d(n); \quad (6.51)$$

$$\beta_1 := 3 \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} - 2\lambda. \quad (6.52)$$

In this case, since the moments are only bounded in finite time horizon, the propagation of chaos cannot be uniform in time. \square

6.2 Proof of main results of the kinetic case

Proof of Theorem 3.4. Let $(P_t, V_t)_{t \geq 0}$ and $(\bar{P}_t, \bar{V}_t)_{t \geq 0}$ two solutions of

$$\begin{cases} dP_t = V_t dt; \\ dV_t = \sqrt{2} dB_t - A(V_t) dt - B(P_t) dt - \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, P_t) dt; \end{cases} \quad (6.53)$$

such that $\mathbb{P}_{(P_0, V_0)} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathbb{P}_{(\bar{P}_0, \bar{V}_0)} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$. Then, by difference, $(p_t, v_t) := (P_t - \bar{P}_t, V_t - \bar{V}_t)$ evolves according to

$$\begin{cases} dp_t = v_t dt; \\ dv_t = -\left(A(V_t) - A(\bar{V}_t) + \lambda_B p_t + D(P_t) - D(\bar{P}_t) + \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right) dt. \end{cases} \quad (6.54)$$

Then, if a and b are positive constants to be chosen later on,

$$\begin{aligned} \frac{d}{dt} \left(a|p_t|^2 + 2p_t \cdot v_t + b|v_t|^2 \right) &= 2ap_t \cdot v_t + 2|v_t|^2 \\ &\quad - 2p_t \cdot \left(A(V_t) - A(\bar{V}_t) + \lambda_B p_t + D(P_t) - D(\bar{P}_t) \right) \\ &\quad - 2bv_t \cdot \left(A(V_t) - A(\bar{V}_t) + \lambda_B p_t + D(P_t) - D(\bar{P}_t) \right) \\ &\quad - 2(p_t + bv_t) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right). \end{aligned} \quad (6.55)$$

By the Cauchy-Schwarz inequality and assumptions on A and D , the first four terms are bounded by above by

$$2(a - \lambda_B b)p_t \cdot v_t + 2([A]_1 + [D]_1 b)|p_t| \cdot |v_t| - 2(\lambda_A b - 1)|v_t|^2 - 2(\lambda_B - [D]_1)|p_t|^2. \quad (6.56)$$

As for the last remaining term, we have

$$\begin{aligned} -2(p_t + bv_t) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right) &= -2(p_t + bv_t) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, \bar{P}_t) \right) \\ &\quad - 2(p_t + bv_t) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mathbb{P}_{P_t}, \bar{P}_t) - \nabla \frac{\delta H}{\delta m}(\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right). \end{aligned} \quad (6.57)$$

Then, we have

$$-2p_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) \right) \leq 2[\mathcal{D}_m H]_{1,\infty} |p_t|^2; \quad (6.58)$$

$$-2p_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right) \leq 2\|\mathcal{D}_m^2 H\|_{\text{op},\infty} |p_t| \mathbb{E}[|p_t|]; \quad (6.59)$$

$$-2bv_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) \right) \leq 2b[\mathcal{D}_m H]_{1,\infty} |p_t| \cdot |v_t|; \quad (6.60)$$

$$-2bv_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right) \leq 2b\|\mathcal{D}_m^2 H\|_{\text{op},\infty} |v_t| \mathbb{E}[|p_t|]. \quad (6.61)$$

By the identity $2ab \leq a^2 + b^2$, we have

$$-2bv_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, P_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) \right) \leq b[\mathcal{D}_m H]_{1,\infty} (|p_t|^2 + |v_t|^2). \quad (6.62)$$

By Jensen's inequality and Young's inequality, we have

$$-2\mathbb{E} \left[p_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right) \right] \leq 2\|\mathcal{D}_m^2 H\|_{\text{op},\infty} \mathbb{E}[|p_t|^2]; \quad (6.63)$$

$$-2b\mathbb{E} \left[v_t \cdot \left(\nabla \frac{\delta H}{\delta m} (\mathbb{P}_{P_t}, \bar{P}_t) - \nabla \frac{\delta H}{\delta m} (\mathbb{P}_{\bar{P}_t}, \bar{P}_t) \right) \right] \leq b\|\mathcal{D}_m^2 H\|_{\text{op},\infty} \mathbb{E}[|p_t|^2 + |v_t|^2]. \quad (6.64)$$

Collecting all terms leads to the bound

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[a|p_t|^2 + 2p_t \cdot v_t + b|v_t|^2 \right] &\leq 2 \left(a - \lambda_B b \right) \mathbb{E}[p_t \cdot v_t] + 2 \left([A]_1 + [D]_1 b \right) \mathbb{E}[|p_t| \cdot |v_t|] \\ &\quad - 2 \left(\lambda_B - [D]_1 - ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right. \\ &\quad \left. - \frac{b}{2} ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right) \mathbb{E}[|p_t|^2] \\ &\quad - 2 \left(\lambda_A b - 1 - \frac{b}{2} ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right) \mathbb{E}[|v_t|^2]. \end{aligned} \quad (6.65)$$

For all $a \geq \lambda_B b$, by the Cauchy-Schwarz inequality and the Young's inequality ($2ab \leq a^2 + b^2$), we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[a|p_t|^2 + 2p_t \cdot v_t + b|v_t|^2 \right] &\leq \left(a - \lambda_B b + [A]_1 + [D]_1 b \right. \\ &\quad \left. - 2 \left(\lambda_B - [D]_1 - ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right. \right. \\ &\quad \left. \left. - \frac{b}{2} ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right) \right) \mathbb{E}[|p_t|^2] \\ &\quad + \left(a - \lambda_B b + [A]_1 + [D]_1 b \right. \\ &\quad \left. - 2 \left(\lambda_A b - 1 - \frac{b}{2} ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right) \right) \mathbb{E}[|v_t|^2]. \end{aligned} \quad (6.66)$$

For all $b > \frac{1}{\sqrt{\lambda_B}}$, $Q_{\lambda_B b, b} : (p, v) \in \mathbb{R}^{2d} \mapsto \lambda_B b |p|^2 + 2p \cdot v + b|v|^2$ is a positive quadratic form on \mathbb{R}^{2d} . Then we look for a and b such that

$$\begin{cases} a - \lambda_B b + [A]_1 + [D]_1 b - 2 \left(\lambda_B - [D]_1 - ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) - \frac{b}{2} ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right) < 0; \\ a - \lambda_B b + [A]_1 + [D]_1 b - 2 \left(\lambda_A b - 1 - \frac{b}{2} ([\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}) \right) < 0; \\ a \geq \lambda_B b. \end{cases} \quad (6.67)$$

in such a way that

$$\exists C > 0 \quad \forall t \geq 0, \quad \frac{d}{dt} \mathbb{E} \left[Q_{a,b} \left((p_t, v_t) \right) \right] \leq -C \mathbb{E}[|p_t|^2 + |v_t|^2]. \quad (6.68)$$

Let us set $\gamma := [\mathcal{D}_m H]_{1,\infty} + \|\mathcal{D}_m^2 H\|_{\text{op},\infty}$. If $a = \lambda_B b$, b satisfies

$$\begin{cases} b < \frac{2\lambda_B - 2[D]_1 - [A]_1 - 2\gamma}{[D]_1 + \gamma}; \\ ([D]_1 + \gamma - 2\lambda_A) b < -[A]_1 - 2. \end{cases} \quad (6.69)$$

Such b exists given that

$$2\lambda_B - 2[D]_1 - [A]_1 - 2\gamma > 0; \quad (6.70)$$

$$[D]_1 + \gamma - 2\lambda_A < 0; \quad (6.71)$$

$$\frac{-[A]_1 - 2}{[D]_1 + \gamma - 2\lambda_A} < \frac{2\lambda_B - 2[D]_1 - [A]_1 - 2\gamma}{[D]_1 + \gamma}. \quad (6.72)$$

In fact, for every positive ε , we have

$$\frac{d}{dt} \mathbb{E} \left[\lambda_B b |p_t|^2 + 2p_t \cdot v_t + b |v_t|^2 \right] \leq - \left(2\lambda_B - 2\eta - \varepsilon - \eta b \right) \mathbb{E} [|p_t|^2] - \left((2\lambda_A - \eta) b - 2 - \frac{[A]_1^2}{\varepsilon} \right) \mathbb{E} [|v_t|^2] \quad (6.73)$$

where $\eta := \gamma + [D]_1$. For us to have

$$\begin{cases} 2\lambda_B - 2\eta - \varepsilon - \eta b > 0; \\ (2\lambda_A - \eta) b - 2 - \frac{[A]_1^2}{\varepsilon} > 0; \end{cases} \quad (6.74)$$

it is necessary that $\eta < 2\lambda_A$, which is assumed in the sequel. Then, for instance for $\varepsilon = \lambda_B$, the conditions are equivalent to

$$\frac{2 + \frac{[A]_1^2}{\lambda_B}}{2\lambda_A - \eta} < b < \frac{\lambda_B - 2\eta}{\eta} \quad \text{and} \quad \eta < 2\lambda_A. \quad (6.75)$$

We look for η such that

$$\frac{2 + \frac{[A]_1^2}{\lambda_B}}{2\lambda_A - \eta} < \frac{\lambda_B - 2\eta}{\eta}, \quad (6.76)$$

that is,

$$2\eta^2 - \eta \left(2 + \frac{[A]_1^2}{\lambda_B} + \lambda_B + 4\lambda_A \right) + 2\lambda_A \lambda_B > 0. \quad (6.77)$$

This polynomial takes negative values at $\eta = 2\lambda_A$, so it is positive on an interval $[0, \eta_0[$ for some $\eta_0 < 2\lambda_A$. We further notice that

$$\eta_0 < \frac{\lambda_B \sqrt{\lambda_B}}{1 + 2\sqrt{\lambda_B}}, \quad (6.78)$$

so that there exists b with all the above conditions for any $\eta \in [0, \eta_0[$.

Hence there exists a constant η_0 , depending only on $[A]_1$, λ_A and λ_B , such that, if $\gamma + [D]_1 < \eta_0$, then there exist a positive quadratic form $Q_{\lambda_B b, b}$ on \mathbb{R}^{2d} and a constant C depending only on $[A]_1$, λ_A , λ_B , γ and $[D]_1$ such that

$$\forall t \geq 0, \quad \frac{d}{dt} \mathbb{E} \left[Q_{\lambda_B b, b} \left((p_t, v_t) \right) \right] \leq -C \mathbb{E} [|p_t|^2 + |v_t|^2]. \quad (6.79)$$

In turn, since $\sqrt{Q_{\lambda_B b, b}}$ and $(p, v) \mapsto \sqrt{|p|^2 + |v|^2}$ are equivalent on \mathbb{R}^{2d} , this is bounded by $-2C \mathbb{E} [Q_{\lambda_B b, b}((p_t, v_t))]$ for a new constant $C > 0$, so that,

$$\forall t \geq 0, \quad \mathbb{E} \left[Q_{\lambda_B b, b} \left((P_t, V_t) - (\bar{P}_t, \bar{V}_t) \right) \right] \leq e^{-2Ct} \mathbb{E} \left[Q_{\lambda_B b, b} \left((P_0, V_0) - (\bar{P}_0, \bar{V}_0) \right) \right] \quad (6.80)$$

by integration. We deduce by equivalence of norms on the space of configurations and the existence of an optimal coupling that

$$\forall t \geq 0, \quad \mathcal{W}_2 \left(\mathbb{P}_{(P_t, V_t)}, \mathbb{P}_{(\bar{P}_t, \bar{V}_t)} \right) \leq C' e^{-Ct} \mathcal{W}_2 \left(\mathbb{P}_{(P_0, V_0)}, \mathbb{P}_{(\bar{P}_0, \bar{V}_0)} \right). \quad (6.81)$$

We conclude that there exists a unique equilibrium ([2, Lemma 10]) and that we contract exponentially towards the latter. \square

Proof of Theorem 3.5. The time-uniform propagation of chaos requires a time-uniform bound of the second moment of $\mathbb{P}_{(P_t, V_t)}$. Let $(f_t)_{t \geq 0}$ be a solution to Eq. (5.2) with initial datum $f_0 \in \mathcal{D}_2(\mathbb{R}^d \times \mathbb{R}^d)$ and let a and b be positive numbers to be chosen later on. Note that for all test function φ ,

$$\begin{aligned} & \frac{d}{dt} \int \varphi(p, v) f_t(dp dv) \\ &= \int \left(\Delta_v \varphi(p, v) + v \cdot \nabla_p \varphi(p, v) - \left(A(v) + B(p) + \mathcal{D}_m H(f_t, p) \right) \cdot \nabla_v \varphi(p, v) \right) f_t(dp dv). \end{aligned} \quad (6.82)$$

Then

$$\begin{aligned} & \frac{d}{dt} \int (a|p|^2 + 2p \cdot v + b|v|^2) f_t(dp dv) \\ &= 2bd + 2 \int (v \cdot (ap + v) - (p + bv) \cdot (A(v) + B(p) + \mathcal{D}_m H(f_t, p))) f_t(dp dv) \end{aligned} \quad (6.83)$$

where, by Young's inequality and assumption on A, B and H,

$$-2p \cdot A(v) = -2p \cdot (A(v) - A(0)) - 2p \cdot A(0) \quad (6.84)$$

$$\leq 2[A]_1 |p| \cdot |v| - 2p \cdot A(0);$$

$$-2p \cdot B(p) = -2p \cdot (\lambda_B p + D(p) - D(0) + D(0)) \quad (6.85)$$

$$\leq -(2\lambda_B - 2[D]_1) |p|^2 - 2p \cdot D(0);$$

$$-2bv \cdot A(v) = -2bv \cdot (A(v) - A(0) + A(0)) \quad (6.86)$$

$$\leq -2b\lambda_A |v|^2 - 2bv \cdot A(0);$$

$$-2bv \cdot B(p) = -2bv \cdot (\lambda_B p + D(p) - D(0) + D(0)) \quad (6.87)$$

$$\leq -2b\lambda_B p \cdot v + 2b[D]_1 |p| \cdot |v| - 2bv \cdot D(0);$$

$$-2p \cdot \mathcal{D}_m H(f_t, p) = -2p \cdot (\mathcal{D}_m H(f_t, p) - \mathcal{D}_m H(f_t, 0) + \mathcal{D}_m H(f_t, 0)) \quad (6.88)$$

$$\leq 2[\mathcal{D}_m H]_{1,\infty} |p|^2 - 2p \cdot \mathcal{D}_m H(f_t, 0);$$

$$-2bv \cdot \mathcal{D}_m H(f_t, p) = -2bv \cdot (\mathcal{D}_m H(f_t, p) - \mathcal{D}_m H(f_t, 0) + \mathcal{D}_m H(f_t, 0)) \quad (6.89)$$

$$\leq 2b[\mathcal{D}_m H]_{1,\infty} |p| \cdot |v| - 2bv \cdot \mathcal{D}_m H(f_t, 0)$$

$$\leq b[\mathcal{D}_m H]_{1,\infty} (|p|^2 + |v|^2) - 2bv \cdot \mathcal{D}_m H(f_t, 0).$$

Collecting all terms and using Young's inequality we obtain, with $a = \lambda_B b$ and $\eta = \gamma + [D]_1$,

$$\begin{aligned} \frac{d}{dt} \int (\lambda_B b |p|^2 + 2p \cdot v + b|v|^2) f_t(dp dv) &\leq 2bd + (2[A]_1 + 2[D]_1 b) \int |p| \cdot |v| f_t(dp dv) \\ &+ (b\gamma + 2\gamma - 2\lambda_B + 2[D]_1) \int |p|^2 f_t(dp dv) \\ &+ (2 + \gamma b - 2\lambda_A b) \int |v|^2 f_t(dp dv) \\ &- 2(A(0) + D(0)) \cdot \left(\int p f_t(dp dv) + b \int v f_t(dp dv) \right) \\ &\leq 2bd + \left(\frac{2}{\varepsilon} + \frac{b^2 \varepsilon}{2[A]_1^2} \right) |A(0) + D(0)|^2 \\ &- (2\lambda_B - 2\eta - \varepsilon - \eta b) \int |p|^2 f_t(dp dv) \\ &- \left((2\lambda_A - \eta)b - 2 - \frac{4[A]_1^2}{\varepsilon} \right) \int |v|^2 f_t(dp dv) \end{aligned} \quad (6.90)$$

for all positive ε . Now, as in the proof of Theorem 3.4, with $[A]_1$ replaced by $2[A]_1$, we get the existence of a positive constant η_0 , depending only on $[A]_1$, λ_A and λ_B , such that for all $0 \leq \gamma + [D]_1 < \eta_0$, there exist b (and ε) such that $Q_{\lambda_B b, b} : (p, v) \mapsto \lambda_B b |p|^2 + 2p \cdot v + b|v|^2$ be a positive quadratic form on \mathbb{R}^{2d} and such that

$$\begin{aligned} \frac{d}{dt} \int Q_{\lambda_B b, b}(p, v) f_t(dp dv) &\leq C_1 - C_2 \int (|p|^2 + |v|^2) f_t(dp dv) \\ &\leq C_1 - C_3 \int Q_{\lambda_B b, b}(p, v) f_t(dp dv) \end{aligned} \quad (6.91)$$

for positive constants C_i . It follows that

$$\sup_{t \geq 0} \int Q_{\lambda_B b, b}(p, v) f_t(dp dv) < +\infty \quad (6.92)$$

if initially

$$\int Q_{\lambda_B b, b}(p, v) f_0(dp dv) < +\infty, \quad (6.93)$$

that is,

$$\sup_{t \geq 0} \int (|p|^2 + |v|^2) f_t(dp dv) < +\infty \quad (6.94)$$

if initially f_0 belongs to $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$.

We now turn to the proof of Theorem 3.5. For each $i \in \{1, \dots, n\}$, the law f_t of $(\bar{P}_t^{(i)}, \bar{V}_t^{(i)})$ is the solution to Eq. (5.2) with f_0 as initial datum and the processes $((\bar{P}_t^{(i)}, \bar{V}_t^{(i)}))_{t \geq 0}$ and $((P_t^{(i)}, V_t^{(i)}))_{t \geq 0}$ are driven by the same Brownian motion. In particular the differences $p_t^{(i)} := P_t^{(i)} - \bar{P}_t^{(i)}$ and $v_t^{(i)} := V_t^{(i)} - \bar{V}_t^{(i)}$ evolve according to

$$\begin{cases} dp_t^{(i)} &= v_t^{(i)} dt; \\ dv_t^{(i)} &= -\left(A(V_t^{(i)}) - A(\bar{V}_t^{(i)}) + \lambda_B p_t^{(i)} + D(P_t^{(i)}) - D(\bar{P}_t^{(i)}) + \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, P_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)})\right) dt; \\ (p_0^{(i)}, v_0^{(i)}) &= (0, 0). \end{cases} \quad (6.95)$$

As in the proof of Theorem 3.4, by the Young inequality and assumptions on A and D, for all positive ε , we have

$$\begin{aligned} \frac{d}{dt} \left(\lambda_B b |p_t^{(i)}|^2 + 2p_t^{(i)} \cdot v_t^{(i)} + b |v_t^{(i)}|^2 \right) &\leq \left(\frac{\varepsilon}{2} - 2\lambda_B + 2[D]_1 + [D]_1 b \right) |p_t^{(i)}|^2 \\ &\quad + \left(2 + \frac{2[A]_1^2}{\varepsilon} - 2\lambda_A b + [D]_1 b \right) |v_t^{(i)}|^2 \\ &\quad - 2 \left(p_t^{(i)} + b v_t^{(i)} \right) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mu_{P_t}, P_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) \right). \end{aligned} \quad (6.96)$$

On the other hand, we have

$$\begin{aligned} &-2 \left(p_t^{(i)} + b v_t^{(i)} \right) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mu_{P_t}, P_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) \right) \\ &= -2 \left(p_t^{(i)} + b v_t^{(i)} \right) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mu_{P_t}, P_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) \right) \\ &\quad - 2 \left(p_t^{(i)} + b v_t^{(i)} \right) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) \right). \end{aligned} \quad (6.97)$$

By the Cauchy-Schwarz inequality, the Young inequality and assumptions on H, we have

$$\begin{aligned} &-2 \left(p_t^{(i)} + b v_t^{(i)} \right) \cdot \left(\nabla \frac{\delta H}{\delta m}(\mu_{P_t}, P_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) \right) \\ &\leq \left(2[\mathcal{D}_m H]_{1, \infty} + b[\mathcal{D}_m H]_{1, \infty} \right) |p_t^{(i)}|^2 + b[\mathcal{D}_m H]_{1, \infty} |v_t^{(i)}|^2 \\ &\quad + 2 \left(|p_t^{(i)}| + b |v_t^{(i)}| \right) \left\| \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) - \nabla \frac{\delta H}{\delta m}(\mu_{P_t}, \bar{P}_t^{(i)}) \right\| \\ &\leq \left(2[\mathcal{D}_m H]_{1, \infty} + b[\mathcal{D}_m H]_{1, \infty} \right) |p_t^{(i)}|^2 + b[\mathcal{D}_m H]_{1, \infty} |v_t^{(i)}|^2 \\ &\quad + 2 \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \left(|p_t^{(i)}| + b |v_t^{(i)}| \right) \mathcal{W}_1(\mu_{P_t}, \mathbb{P}_{P_t}). \end{aligned} \quad (6.98)$$

By the triangle inequality, we have

$$\mathcal{W}_1(\mu_{P_t}, \mathbb{P}_{P_t}) \leq \mathcal{W}_1(\mu_{P_t}, \mu_{\bar{P}_t}) + \mathcal{W}_1(\mu_{\bar{P}_t}, \mathbb{P}_{P_t}). \quad (6.99)$$

By the Kantorovitch-Rubinstein duality relation, we have

$$\mathcal{W}_1(\mu_{P_t}, \mu_{\bar{P}_t}) \leq \frac{1}{n} \sum_{j=1}^n |p_t^{(j)}|. \quad (6.100)$$

Moreover, by symmetry, we have

$$\mathbb{E}[\mathcal{W}_1(\mu_{P_t}, \mu_{\bar{P}_t})] \leq \mathbb{E}[|p_t^{(i)}|]. \quad (6.101)$$

From the Cauchy-Schwarz, Jensen and Young inequalities, we deduce that

$$2 \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \left(|p_t^{(i)}| + b |v_t^{(i)}| \right) \mathcal{W}_1(\mu_{P_t}, \mathbb{P}_{P_t}) \quad (6.102)$$

$$\leq \|\mathcal{D}_m^2 H\|_{\text{op}, \infty} \left(|p_t^{(i)}|^2 + b |v_t^{(i)}|^2 + (b+1) \mathcal{W}_1(\mu_{P_t}, \mathbb{P}_{P_t})^2 \right);$$

$$\mathbb{E} \left[2 \left(|p_t^{(i)}| + b |v_t^{(i)}| \right) \mathcal{W}_1(\mu_{P_t}, \mathbb{P}_{P_t}) \right] \quad (6.103)$$

$$\leq (2b+3) \mathbb{E}[|p_t^{(i)}|^2] + b \mathbb{E}[|v_t^{(i)}|^2] + (2b+2) \mathbb{E}[\mathcal{W}_2(\mu_{\bar{P}_t}, \mathbb{P}_{P_t})^2].$$

By the result of Fournier and Guillin ([18, Theorem 1]), we have

$$\sup_{t \geq 0} \mathbb{E}[\mathcal{W}_2(\mu_{\mathbb{P}_t}, \mathbb{P}_{\mathbb{P}_t})^2] \leq C(d) \left(\sup_{t \geq 0} \int (|p|^2 + |v|^2) f_t(dp dv) \right) \delta_d(n). \quad (6.104)$$

Collecting all terms, it follows that there exists a positive constant c such that for all γ and $[D]_1$ in $[0, c)$, there exists a constant M such that

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[\left(\lambda_B b |p_t^{(i)}|^2 + 2p_t^{(i)} \cdot v_t^{(i)} + b |v_t^{(i)}|^2 \right) \right] &\leq - \left(2\lambda_B - 2\eta - \varepsilon - \eta b \right) \mathbb{E}[|p_t^{(i)}|^2] \\ &\quad - \left((2\lambda_A - \eta)b - 2 - \frac{4[A]_1^2}{\varepsilon} \right) \mathbb{E}[|v_t^{(i)}|^2] \\ &\quad + M \left(\frac{2}{\varepsilon} + \frac{b^2 \varepsilon}{2[A]_1^2} \right) \delta_d(n) \end{aligned} \quad (6.105)$$

for all positive t , b and ε , where $\eta = \gamma + [D]_1$.

Now, as in the proof of Theorem 3.4, with $[A]_1$ replaced by $2[A]_1$, we get the existence of a positive constant η_0 , depending only on $[A]_1$, λ_A and λ_B , such that for all $0 \leq \gamma + [D]_1 < \eta_0$, there exist b (and ε) such that $Q_{\lambda_B b, b} : (p, v) \mapsto \lambda_B b |p|^2 + 2p \cdot v + b |v|^2$ be a positive quadratic form on \mathbb{R}^{2d} and such that

$$\frac{d}{dt} \mathbb{E}[Q_{\lambda_B b, b}(p_t^{(i)}, v_t^{(i)})] \leq -C_1 \mathbb{E}[|p_t^{(i)}|^2 + |v_t^{(i)}|^2] + C_2 \delta_d(n) \quad (6.106)$$

for all $t \geq 0$ and for positive constants C_1 and C_2 , also depending on f_0 through its second moment, but not on n . In turn this is bounded by

$$-C_3 \mathbb{E}[Q_{\lambda_B b, b}(p_t^{(i)}, v_t^{(i)})] + C_2 \delta_d(n), \quad (6.107)$$

so that

$$\sup_{t \geq 0} \mathbb{E}[Q_{\lambda_B b, b}(p_t^{(i)}, v_t^{(i)})] \leq C_4 \delta_d(n) \quad (6.108)$$

and finally

$$\sup_{t \geq 0} \mathbb{E}[|p_t^{(i)}|^2 + |v_t^{(i)}|^2] \leq C \delta_d(n) \quad (6.109)$$

where the constant C depends on the parameters of the equation and on the second moment of f_0 , but not on n . \square

6.3 Proof related to the examples

Proof of Theorem 5.1. To do this, simply show the two equivalents (i) \iff (ii) and (ii) \iff (iii).

(i) \iff (ii). Let μ_∞ be such that

$$\Delta \mu_\infty + \nabla \cdot (\mu_\infty \mathcal{D}_m H(\mu_\infty, \cdot)) = 0. \quad (6.110)$$

For any test function φ , we have

$$\langle \Delta \mu_\infty + \nabla \cdot (\mu_\infty \mathcal{D}_m H(\mu_\infty, \cdot)), \varphi \rangle = \int \Delta \varphi d\mu_\infty - \int \mathcal{D}_m H(\mu_\infty, \cdot) \cdot \nabla \varphi d\mu_\infty. \quad (6.111)$$

Since the Laplacian is a self-adjoint operator, we have

$$\begin{aligned} \int \Delta \varphi d\mu_\infty - \int \mathcal{D}_m H(\mu_\infty, \cdot) \cdot \nabla \varphi d\mu_\infty &= - \int \left(\nabla \frac{d\mu_\infty}{dx} \nabla \varphi + \frac{d\mu_\infty}{dx} \mathcal{D}_m H(\mu_\infty, \cdot) \cdot \nabla \varphi \right) dx \\ &= - \int \nabla \left(\ln \frac{d\mu_\infty}{dx} + \frac{\delta H}{\delta m}(\mu_\infty, \cdot) \right) \cdot \nabla \varphi d\mu_\infty. \end{aligned} \quad (6.112)$$

We deduce that

$$(i) \iff \forall \varphi, \int \nabla \left(\ln \frac{d\mu_\infty}{dx} + \frac{\delta H}{\delta m}(\mu_\infty, \cdot) \right) \cdot \nabla \varphi d\mu_\infty \iff \nabla \left(\ln \frac{d\mu_\infty}{dx} + \frac{\delta H}{\delta m}(\mu_\infty, \cdot) \right) = 0, \quad (6.113)$$

which proves the expected equivalence.

(ii) \iff (iii). As for this last equivalence, we have

$$\begin{aligned} \forall (y, z) \in \mathbb{R}^d \times \mathbb{R}^d, \quad & \left| \ln \frac{d\mu_\infty}{dx}(y) + \frac{\delta H}{\delta m}(\mu_\infty, y) - \ln \frac{d\mu_\infty}{dx}(z) - \frac{\delta H}{\delta m}(\mu_\infty, z) \right| \\ & \leq \left\| \nabla \left(\ln \frac{d\mu_\infty}{dx} + \frac{\delta H}{\delta m}(\mu_\infty, \cdot) \right) \right\|_\infty \|y - z\|. \end{aligned} \quad (6.114)$$

We deduce that

$$(ii) \iff \ln \frac{d\mu_\infty}{dx} + \frac{\delta H}{\delta m}(\mu_\infty, \cdot) = C_{\mu_\infty} \iff (iii). \quad (6.115)$$

□

Proof of Proposition 4.1. We have

$$\mathcal{D}_m H_{VW}(\mu, x) = \nabla V(x) + \int \nabla W(x - y) \mu(dy); \quad (6.116)$$

$$\mathcal{D}_m^2 H_{VW}(\mu, x, y) = -\nabla^2 W(x - y). \quad (6.117)$$

□

Proof of Proposition 4.2. We have

$$\mathcal{D}_m H_{VW}(\mu, x) = \nabla V(x) + \int \nabla W(x - y) \mu(dy); \quad (6.118)$$

$$\mathcal{D}_m^2 H_{VW}(\mu, x, y) = -\nabla^2 W(x - y). \quad (6.119)$$

□

Proof of Proposition 4.3. The intrinsic derivatives associated with this functional are given by

$$\mathcal{D}_m H_{VW}^N(\mu, x) \quad (6.120)$$

$$= \nabla V(x)$$

$$+ \sum_{k=2}^N \sum_{j=1}^k \int \nabla_{x_j} W^{(k)}(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_k) \mu^{\otimes k-1}(dx_1, \dots, dx_{j-1}, dx_{j+1}, \dots, dx_k)$$

$$= \nabla V(x) + \sum_{k=2}^N k \int \nabla_{x_1} W^{(k)}(x, y) \mu^{\otimes k-1}(dy)$$

$$\mathcal{D}_m^2 H_{VW}^N(\mu, x, y) = \sum_{k=2}^N k(k-1) \int \nabla_{x_1, x_2}^2 W^{(k)}(x, y, z) \mu^{\otimes k-2}(dz). \quad (6.121)$$

□

Proof of Proposition 4.4. The intrinsic derivatives associated with this functional are given by

$$\mathcal{D}_m H_{VW}^N(\mu, x) = \nabla V(x) \quad (6.122)$$

$$+ \sum_{k=2}^N \sum_{j=1}^k \int \nabla_{x_j} W^{(k)}(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_k) \mu^{\otimes k-1}(dx_1, \dots, dx_{j-1}, dx_{j+1}, \dots, dx_k)$$

$$= \nabla V(x) + \sum_{k=2}^N k \int \nabla_{x_1} W^{(k)}(x, y) \mu^{\otimes k-1}(dy)$$

$$\mathcal{D}_m^2 H_{VW}^N(\mu, x, y) = \sum_{k=2}^N k(k-1) \int \nabla_{x_1, x_2}^2 W^{(k)}(x, y, z) \mu^{\otimes k-2}(dz). \quad (6.123)$$

□

Proof of Proposition 4.5. We have

$$\mathcal{D}_m H_\Psi(\mu, x) = \Psi'(\langle \mu, W \rangle) \nabla W(x); \quad (6.124)$$

$$\mathcal{D}_m^2 H_{VW}(\mu, x, y) = \Psi''(\langle \mu, W \rangle) \nabla W(x) \otimes \nabla W(y). \quad (6.125)$$

□

Proof of Proposition 4.6. We have

$$\mathcal{D}_m H_\Psi(\mu, x) = \Psi'(\langle \mu, W \rangle) \nabla W(x); \quad (6.126)$$

$$\mathcal{D}_m^2 H_{VW}(\mu, x, y) = \Psi''(\langle \mu, W \rangle) \nabla W(x) \otimes \nabla W(y). \quad (6.127)$$

□

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