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Morse Sequences on Stacks and Flooding Sequences

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Abstract. This paper builds upon the framework of *Morse sequences*, a simple and effective approach to discrete Morse theory. A Morse sequence on a simplicial complex consists of a sequence of nested subcomplexes generated by expansions and fillings—two operations originally introduced by Whitehead. Expansions preserve homotopy, while fillings introduce critical simplexes that capture essential topological features. We extend the notion of Morse sequences to *stacks*, which are monotonic functions defined on simplicial complexes, and define *Morse sequences on stacks* as those whose expansions preserve the homotopy of all sublevel sets. This extension leads to a generalization of the fundamental collapse theorem to weighted simplicial complexes. Within this framework, we focus on a refined class of sequences called *flooding sequences*, which exhibit an ordering behavior similar to that of classical watershed algorithms. Although not every Morse sequence on a stack is a flooding sequence, we show that the gradient vector field associated with any Morse sequence can be recovered through a flooding sequence. Finally, we present algorithmic schemes for computing flooding sequences using cosimplicial complexes.

Keywords: Discrete Morse theory · Expansions and collapses · Fillings and perforations · Simplicial complex.

1 Introduction

This work is set within the framework of *Morse sequences* [5] offering a simple approach to discrete Morse theory [13]. If L is a subcomplex of a simplicial complex K , a Morse sequence from L to K is a sequence $\vec{W} = \langle L = K_0, \dots, K_k = K \rangle$ of nested simplicial complexes such that, for each $i \in [1, k]$, K_i is either an *expansion* or a *filling* of K_{i-1} . These two elementary operations were introduced in [23]: expansions (the inverse of collapses) preserve homotopy, while fillings introduce critical simplices. The set of all expansions of the sequence induces a gradient vector field between L and K , while the set of all critical simplexes captures fundamental homological properties of the pair (L, K) .

In this paper, we extend these notions by exploring Morse sequences constructed on stacks defined on simplicial complexes (Section 3). A *stack* on a simplicial complex K is a monotonic function F assigning a value $F(\nu)$ to each

simplex $\nu \in K$. A *Morse sequence on F* is a Morse sequence on K in which the expansions preserve the homotopy of all sublevel sets of F . This naturally generalizes the fundamental collapse theorem of discrete Morse theory to weighted simplicial complexes.

We introduce and study a refined class of Morse sequences on stacks, called *flooding sequences* (Section 4). These sequences satisfy an ordering condition analogous to those found in some classical watershed algorithms. A Morse sequence on a stack is not necessarily a flooding sequence. Nevertheless, we prove that the gradient vector field of an arbitrary Morse sequence on a stack F can be obtained by computing a flooding sequence on F .

Thanks to the notion of a *cosimplicial complex* (a complex which is the difference of two simplicial complexes), we derive simple schemes for computing flooding sequences (Section 5 and 6). In particular, we consider a *maximal increasing scheme* (building \vec{W} from L to K by prioritizing expansions), and a *minimal decreasing scheme* (building \vec{W} from K to L by prioritizing collapses).

These methods follow a propagation paradigm aimed at minimizing the number of critical simplexes. We show that they yield efficient algorithms for computing flooding sequences.

2 Basic definitions

2.1 Simplicial and cosimplicial complexes

In this section, we introduce *cosimplicial complexes*, which generalize the classical notion of simplicial complexes. In the context of combinatorial dynamics, these structures are also referred to as *multivectors* [11]. In this paper, we employ cosimplicial complexes to construct the sequences of simplicial complexes that form Morse sequences.

By a *simplex*, we mean a finite non-empty set. The *dimension* of a simplex ν , written $\dim(\nu)$, is the number of its elements minus one.

Definition 1. *Let S be a finite set of simplexes. The set S is a cosimplicial complex if, for any $\sigma, \tau \in S$, we have $\nu \in S$ whenever $\sigma \subseteq \nu \subseteq \tau$.*

A simplex of a cosimplicial complex S is a *face of S* . A face of S is a *facet of S* if it is maximal for inclusion, and a *cofacet of S* if it is minimal for inclusion.

A finite set K of simplexes is a *simplicial complex* if, for each $\tau \in K$, we have $\sigma \in K$ whenever $\sigma \subseteq \tau$ and $\sigma \neq \emptyset$. Thus, each simplicial complex is also a cosimplicial complex.

Let K be a simplicial complex and let $S \subseteq K$. We say that S is *closed for K* if S is a simplicial complex. We say that S is *open for K* if, for each $\sigma \in S$, we have $\tau \in S$ whenever $\sigma \subseteq \tau$ and $\tau \in K$. Therefore, a set $S \subseteq K$ is closed for K if and only if $K \setminus S$ is open for K . The family \mathcal{O}_K composed of all sets which are open for K satisfies the axioms of a topology on K . So we obtain the structure of a finite topological space, see [1], [2]. We easily get the following.

Proposition 1. *Let K be a simplicial complex and let $S \subseteq K$. The set S is a cosimplicial complex if and only if S is the intersection of an open set for K and a closed set for K .*

In general topology, a subset which satisfies the condition of Proposition 1 is said to be *locally closed*, see [9], Ch.1, §3, no3.

Let R and T be two subsets of K . We have $S = R \cap T$ if and only if $S = R \setminus R'$, with $R' = K \setminus T$. Thus, by Proposition 1 and by the properties of the sets which are open or closed for K , we obtain the following.

Proposition 2. *Let K be a simplicial complex and let $S \subseteq K$.*

- *S is a cosimplicial complex iff S is the difference of two open sets for K .*
- *S is a cosimplicial complex iff S is the difference of two closed sets for K .*

Since a closed set for K is a simplicial complex in its own right, we derive:

Proposition 3. *A set S is a cosimplicial complex if and only if S is the difference of two simplicial complexes.*

2.2 An ambient space

In [17], an *open simplicial complex* is defined precisely as the difference of two simplicial complexes. Consequently, by Proposition 3, cosimplicial complexes and open simplicial complexes are equivalent notions. In this paper, we will take advantage of the local characterization of Definition 1 for deriving some operators which act directly on a cosimplicial complex.

Let S be a finite set of simplexes. Let \overline{S} be the set of simplexes such that $\sigma \in \overline{S}$ if and only if there exists $\tau \in S$ with $\sigma \subseteq \tau$. The set \overline{S} is the *closure* of S . In the sequel of this paper, we write $\underline{S} = \overline{S} \setminus S$.

We have $S \subseteq \overline{S}$ and we observe that \overline{S} is a simplicial complex, this complex is the smallest simplicial complex that contains S . It follows that S is a simplicial complex if and only if $\overline{S} = S$. Also, the set \underline{S} is the smallest set such that $S \cup \underline{S}$ is a simplicial complex.

By the previous remarks, the simplicial complex \overline{S} may be considered as “an ambient topological space for S ”. As a direct consequence of the definition of a cosimplicial complex we obtain:

Proposition 4. *Let S be a finite set of simplexes.*

- 1) *The set S is a cosimplicial complex if and only if S is open for \overline{S} .*
- 2) *The set S is a cosimplicial complex if and only if \underline{S} is a simplicial complex.*

If S is a cosimplicial complex, then the sets $L = \underline{S}$ and $K = \overline{S}$ are two simplicial complexes such that $L \subseteq K$ and $S = K \setminus L$. The following proposition generalizes this situation.

Proposition 5. *Let (L, K) be a pair of simplicial complexes such that $L \subseteq K$. The cosimplicial complex $S = K \setminus L$ is such that $L \cup \overline{S} = K$ and $L \cap \overline{S} = \underline{S}$.*

Let S be a cosimplicial complex and let $\nu \in S$. We write:

- $\partial(\nu, S) = \{\eta \in S \mid \eta \subseteq \nu \text{ and } \dim(\eta) = \dim(\nu) - 1\}$, and
 - $\delta(\nu, S) = \{\mu \in S \mid \nu \subseteq \mu \text{ and } \dim(\mu) = \dim(\nu) + 1\}$;
- $\partial(\nu, S)$ and $\delta(\nu, S)$ are, respectively, the *boundary* and the *coboundary* of ν in S . We observe that:
- the face ν is a facet of S if and only if $\delta(\nu, S) = \emptyset$,
 - the face ν is a cofacet of S if and only if $\partial(\nu, S) = \emptyset$.

2.3 Stacks and filtrations

With the notion of a stack, we introduce some weights on a cosimplicial or a simplicial complex. A stack is just a monotone function that assigns values to the simplices of the complex.

Let F be a map from a cosimplicial complex S to \mathbb{Z} . We say that F is a *stack* on S if we have $F(\sigma) \leq F(\tau)$ whenever $\sigma, \tau \in S$ and $\sigma \subseteq \tau$.

Let F be a map from a cosimplicial complex S to \mathbb{Z} . For any $\lambda \in \mathbb{Z}$, we write:

$$F_\lambda = \{\nu \in S \mid F(\nu) \leq \lambda\} \text{ and } F[\lambda] = \{\nu \in S \mid F(\nu) = \lambda\},$$

F_λ and $F[\lambda]$ are, respectively, the *cut* and the *section* of F at level λ .

If F is a stack on a cosimplicial complex S , then the indexed family $(F_\lambda)_{\lambda \in \mathbb{Z}}$ is a *filtration* on S . That is we have $F_\lambda \subseteq F_{\lambda'}$ whenever $\lambda \leq \lambda'$. Also if F is a stack on a simplicial complex, then any cut of F is a simplicial complex. Furthermore:

Proposition 6. *If F is a stack on a cosimplicial complex, then any cut of F and any section of F is a cosimplicial complex.*

3 Morse sequences on stacks

We recall the definitions of simplicial collapses and simplicial expansions [23].

Let K be a simplicial complex and let $\sigma, \tau \in K$. The couple (σ, τ) is a *free pair* for K , if τ is the only face of K that contains σ . If (σ, τ) is a free pair for K , then the simplicial complex $L = K \setminus \{\sigma, \tau\}$ is an *elementary collapse* of K , and K is an *elementary expansion* of L . We say that K *collapses onto* L , or that L *expands onto* K , if there exists a sequence $\langle K = K_0, \dots, K_k = L \rangle$, such that K_i is an elementary collapse of K_{i-1} , $i \in [1, k]$.

We recall also the definitions of perforations and fillings [23].

Let K, L be simplicial complexes. If $\nu \in K$ is a facet of K and if $L = K \setminus \{\nu\}$, we say that L is an *elementary perforation* of K , and that K is an *elementary filling* of L .

We now introduce the notion of a ‘‘Morse sequence’’ by simply considering expansions and fillings of a simplicial complex [5].

Definition 2. *Let $L \subseteq K$ be two simplicial complexes. A Morse sequence (from L to K) is a sequence $\vec{W} = \langle L = K_0, \dots, K_k = K \rangle$ of simplicial complexes such that, for each $i \in [1, k]$, K_i is either an elementary expansion or an elementary*

filling of K_{i-1} . If $L = \emptyset$, we say that \vec{W} is a Morse sequence on K . Thus, the sequence $\vec{W} = \langle K \rangle$ is trivially a Morse sequence on K .

Note that a Morse sequence $\vec{W} = \langle K_0, \dots, K_k \rangle$ is a filtration, that is, for each $i \in [0, k-1]$, we have $K_i \subseteq K_{i+1}$; see [22,12].

If $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ is a Morse sequence on K , with $k \geq 1$, we also note that K_1 is necessarily a filling of \emptyset .

Let $\vec{W} = \langle L = K_0, \dots, K_k = K \rangle$ be a Morse sequence from L to K . If $k \geq 1$, we write $\diamond\vec{W}$ for the sequence $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$ such that, for each $i \in [1, k]$:

- If K_i is an elementary filling of K_{i-1} , then κ_i is the simplex such that $K_i = K_{i-1} \cup \{\kappa_i\}$; we say that K_i and the face κ_i are *critical for \vec{W}* .
- If K_i is an elementary expansion of K_{i-1} , then κ_i is the free pair (σ, τ) such that $K_i = K_{i-1} \cup \{\sigma, \tau\}$; we say that K_i , the pair κ_i , and the faces σ, τ , are *regular for \vec{W}* .

If $k = 0$, we write $\diamond\vec{W} = \langle \rangle$. That is, $\diamond\vec{W}$ is the empty sequence.

We say that $\diamond\vec{W}$ is a *simplex-wise (Morse) sequence (from L to K)*. Note that $\diamond\vec{W}$ is a sequence of faces and pairs.

Definition 3. Let \vec{W} be a Morse sequence. The gradient vector field of \vec{W} is the set composed of all regular pairs for \vec{W} . We say that two Morse sequences \vec{V} and \vec{W} from L to K are equivalent if they have the same gradient vector field.

Let us consider the simplicial complex K depicted Fig. 1 (a). For convenience, we will describe a simplex by a concatenation of its vertices.

Let $\vec{W} = \langle \emptyset = K_0, \dots, K_7 = K \rangle$ be the sequence such that $K_1 = \{a\}$, $K_2 = K_1 \cup \{b, ab\}$, $K_3 = K_2 \cup \{c, bc\}$, $K_4 = K_3 \cup \{d, cd\}$, $K_5 = K_4 \cup \{e, ed\}$, $K_6 = K_5 \cup \{be\}$, $K_7 = K_6 \cup \{bd, bde\}$. It can be seen that \vec{W} is a Morse sequence on K . The corresponding simplex-wise sequence $\diamond\vec{W}$ is such that $\diamond\vec{W} = \langle a, (b, ab), (c, bc), (d, cd), (e, ed), be, (bd, bde) \rangle$. In Fig. 1 (b) the gradient vector field of \vec{W} is given by arrows, the two critical faces a and be of \vec{W} are in red.

Now, we extend the notion of a Morse sequence for an arbitrary stack F .

In the sequel of this paper, L and K will denote simplicial complexes.

Let F be a stack on K and let L be a subcomplex of K . Let $\kappa = (\sigma, \tau)$ be a free pair for L . If $F(\sigma) = F(\tau)$, then we say that $L' = L \setminus \{\sigma, \tau\}$ is an *(elementary) F -collapse of L* and L is an *(elementary) F -expansion of L'* . We write $F(\kappa) = F(\sigma) = F(\tau)$.

Definition 4. Let \vec{W} be a Morse sequence from L to K and let F be a stack on K . We say that \vec{W} is a (Morse) sequence on F or an F -sequence (from L to K), if we have $F(\sigma) = F(\tau)$ whenever (σ, τ) is a regular pair for \vec{W} .

An example of a stack F on a complex K is given Fig. 1 (c). Let us consider again the Morse sequence \vec{W} and its gradient vector field given in (b).

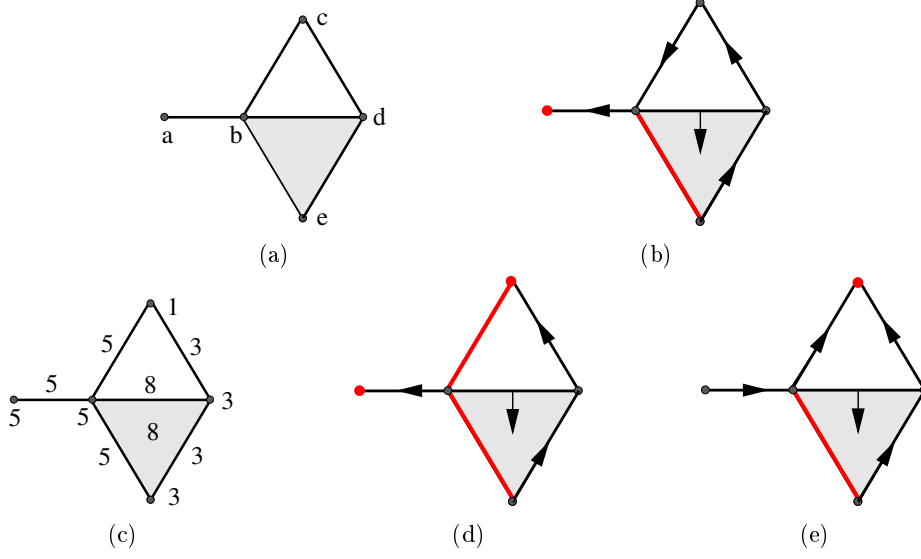


Fig. 1: (a) A simplicial complex K . (b) The gradient vector field of a Morse sequence \vec{W} on K . (c) A stack F on K . (d) The gradient vector field of a Morse sequence \vec{W}' on F . (e) The gradient vector field of a flooding sequence \vec{W}'' on F . See text for the description of the sequences \vec{W} , \vec{W}' , and \vec{W}'' .

We see that \vec{W} does not satisfy the condition of Def. 4 for the stack F . Thus, the sequence \vec{W} is not an F -sequence. Now let us consider the sequence $\diamond\vec{W}' = \langle a, (b, ab), c, bc, (d, cd), (e, ed), be, (bd, bde) \rangle$. This sequence is a simplex-wise Morse sequence on K . Furthermore the corresponding sequence \vec{W}' is an F -sequence on K . The gradient vector field of \vec{W}' is depicted in (d), the four critical faces a , c , bc , and be of \vec{W}' are in red.

Let F be a stack on K and L be a subcomplex of K . If $\lambda \in \mathbb{Z}$, the set $F_\lambda \cap L$ is the cut of L at level λ which is induced by F . Note that $F_\lambda \cap K = F_\lambda$.

Let (σ, τ) be a free pair for L such that $F(\sigma) = F(\tau)$. Then:

- For each $\lambda < F(\sigma)$, we have $\sigma \notin F_\lambda \cap L$ and $\tau \notin F_\lambda \cap L$,
- For each $\lambda \geq F(\sigma)$, one can verify that the pair (σ, τ) is a free pair for $F_\lambda \cap L$.

Now, let $\vec{W} = \langle L = K_0, \dots, K_k = K \rangle$ be an F -sequence. Suppose K_i is an elementary expansion of K_{i-1} . Then, by the above observation, this elementary expansion is a (possibly trivial) expansion for all cuts of K_{i-1} .

By induction, we have:

Theorem 1. *Let F be a stack on K and $\vec{W} = \langle L = K_0, \dots, K_k = K \rangle$ be an F -sequence. Let K_i and K_j , $j > i$, be two consecutive critical complexes for \vec{W} .*

That is, K_{i+1}, \dots, K_{j-1} are regular complexes for \vec{W} . Then, for each $\lambda \in \mathbb{Z}$, the complex $F_\lambda \cap K_{j-1}$ collapses onto $F_\lambda \cap K_i$.

This last property may be seen as an extension to stacks of a fundamental theorem, called *the collapse theorem*, which makes the link between the basic definitions of discrete Morse theory and discrete homotopy (See Theorem 3.3 of [13] and Theorem 4.27 of [22]).

Remark 1. Let $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ be an F -sequence. Then \vec{W} induces a “double filtration”: the indexed family $(K_i)_{i \in [0, k]}$ is a filtration where each K_i induces the filtration $(F_\lambda \cap K_i)_{\lambda \in \mathbb{Z}}$. Also, \vec{W} induces the sequence $\langle F_0, \dots, F_k \rangle$ where each F_i is the stack on K_i which is the restriction of F to K_i .

If S is a set, we write $\mathbb{1}_S$ for the constant function $S \rightarrow \mathbb{Z}$ such that, for each $x \in S$, we have $\mathbb{1}_S(x) = 1$. We observe that a sequence \vec{W} is a Morse sequence from L to K if and only if \vec{W} is an F -sequence from L to K for $F = \mathbb{1}_K$.

It is well known that discrete Morse theory can be formulated from the perspective of gradient vector fields [18]. In the remainder of this section, we present a natural extension of this classical notion to stacks. We then emphasize that a Morse sequence is in no way a restrictive notion with respect to this point of view. That is, any gradient vector field on a stack F is the gradient vector field of some Morse sequence on F .

Let F be a stack on K and let V be a set of pairs (σ, τ) , with $\sigma, \tau \in K$, such that $\sigma \in \partial\tau$ and $F(\sigma) = F(\tau)$. We say that V is a *(discrete) vector field on F* if each simplex of K is in at most one pair of V . A simplex $\nu \in K$ is said to be *critical for V* if it does not belong to any pair in V .

Let V be a vector field on F . A *(p -)gradient path in V (from σ_0 to σ_k)* is a sequence $\pi = \langle \sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_k \rangle$, with $k \geq 0$, composed of faces such that, for all $i \in [0, k-1]$, $\dim(\sigma_i) = \dim(\sigma_k) = p$, (σ_i, τ_i) is in V , $\sigma_{i+1} \subset \tau_i$, and $\sigma_{i+1} \neq \sigma_i$. This sequence π is said to be *trivial* if $k = 0$, that is, if $\pi = \langle \sigma_0 \rangle$; otherwise, if $k \geq 1$, we say that π is *non-trivial*. Also, the sequence π is *closed* if $\sigma_0 = \sigma_k$. We say that a vector field V on F is a *gradient vector field* if V is *acyclic*, that is if V contains no non-trivial closed p -gradient path.

Note that, if F is the constant map $\mathbb{1}_K$, then the above definitions correspond exactly to the classical ones. The proof of the following result is a direct extension of the proof given in [5] (Theorem 51).

Theorem 2. *Let F be a stack on K . A vector field V on F is a gradient vector field if and only if V is the gradient vector field of a Morse sequence on F .*

4 Flooding sequences

We introduce the following refinement of an F -sequence. A flooding sequence corresponds to a process similar to the one often used for computing the watershed transform, which is a key tool for image segmentation in Mathematical Morphology [7], [8].

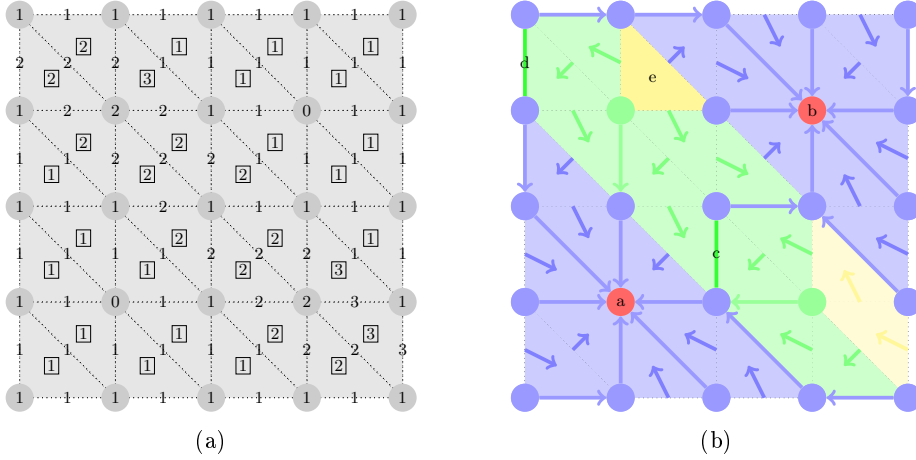


Fig. 2: A flooding sequence. (a) a simplicial stack F on a triangulation K of a square. (b) A flooding sequence on F .

Definition 5. Let F be a stack on K and let $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ be an F -sequence. We say that \vec{W} is a flooding sequence (on F), if we have $F(\sigma) \leq F(\tau)$ whenever $\sigma \in K_i$, $\tau \in K_j \setminus K_i$, and $i < j$.

In the example of Fig. 1, it can be seen that the sequence \vec{W}' is not a flooding sequence on F . The first element of \vec{W}' is the face a . In a flooding sequence, the face c should appear before a since $F(a) = 5$ and $F(c) = 1$. Now, let us consider the sequence $\diamond \vec{W}'' = \langle c, (d, cd), (e, ed), (b, bc), (a, ab), be, (bd, bde) \rangle$. The corresponding sequence \vec{W}'' is a flooding sequence on F . The gradient vector field of \vec{W}'' is given in (e), the two critical faces c and be of \vec{W}'' are in red.

Figure 2 illustrates an example of a flooding sequence on a less simple complex. This sequence \vec{W} can begin from two possible points, both at level 0, as shown in Figure 2.a. If \vec{W} begins with \mathbf{a} , then the second element of \vec{W} must be \mathbf{b} , both of them are critical for \vec{W} . Then the sequence must continue with a subsequence composed of all simplexes at level 1. This subsequence is illustrated in blue, it turns out that it is composed solely of regular elements. The next subsequence composed of all simplexes at level 2 is illustrated in green, it contains two critical simplexes \mathbf{c} and \mathbf{d} . The last subsequence composed of all simplexes at level 3 is illustrated in yellow, it contains one critical simplex \mathbf{e} . See also Figure 1 in [6] which gives, on the same complex, two examples of F -sequences that are not flooding sequences.

Let F be a stack on K and let L be a subcomplex of K . We say that L is a subcomplex for F if, for all $\sigma \in K$ and all $\tau \in L$, we have $\sigma \in L$ whenever $F(\sigma) < F(\tau)$. The following provides two characterizations of flooding sequences.

Proposition 7. *Let F be a stack on K and let $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ be an F -sequence.*

- 1) *The sequence \vec{W} is a flooding sequence on F if and only if, for each $i \in [0, k]$, the complex K_i is a subcomplex for F .*
- 2) *The sequence \vec{W} is a flooding sequence on F if and only if, for each $\lambda \in \mathbb{Z}$, there exists $i \in [0, k]$ such that $F_\lambda = K_i$.*

The next lemma will allow us to transform F -sequences into flooding sequences by means of an exchange property.

Lemma 1. *Let F be a stack on K , and let $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_i, \kappa_{i+1}, \dots, \kappa_k \rangle$, with $k \geq 2$, be a simplex-wise F -sequence. We consider the sequence $\diamond\vec{W}' = \langle \kappa'_1, \dots, \kappa'_i, \kappa'_{i+1}, \dots, \kappa'_k \rangle$, such that $\kappa'_i = \kappa_{i+1}$, $\kappa'_{i+1} = \kappa_i$, and $\kappa'_j = \kappa_j$ for all $j \neq i, j \neq i+1$. If $F(\kappa_i) > F(\kappa_{i+1})$, with $1 \leq i \leq k-1$, then $\diamond\vec{W}'$ is a simplex-wise F -sequence.*

Proof. We write $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ for the Morse sequence corresponding to $\diamond\vec{W}$, and $\vec{W}' = \langle \emptyset = K'_0, \dots, K'_k = K \rangle$ for the one corresponding to $\diamond\vec{W}'$. We consider the different cases for κ_i and κ_{i+1} :

1. Suppose κ_i is a critical simplex; that is, K_i a filling of K_{i-1} .
 - (a) Suppose κ_{i+1} is a critical simplex; that is K_{i+1} is a filling of K_i .
 - If $\kappa_i \notin \partial(\kappa_{i+1})$, then we see that $K'_i = K_{i-1} \cup \{\kappa_{i+1}\}$ is a filling of K_{i-1} , and $K_{i+1} = K'_i \cup \{\kappa_i\}$ is a filling of K'_i . Thus $\diamond\vec{W}'$ is a simplex-wise F -sequence.
 - If $\kappa_i \in \partial(\kappa_{i+1})$, then \vec{W}' is not a filtration. But, in this case, we have $F(\kappa_i) \leq F(\kappa_{i+1})$ since F is a stack on K .
 - (b) Suppose $\kappa_{i+1} = (\sigma, \tau)$; that is K_{i+1} is an expansion of K_i . The pair (σ, τ) is a free pair for F , we have $F(\kappa_{i+1}) = F(\sigma) = F(\tau)$.
 - If $\kappa_i \notin \partial(\tau)$, then $K'_i = K_{i-1} \cup \{\sigma, \tau\}$ is an expansion of K_{i-1} , and $K_{i+1} = K'_i \cup \{\kappa_i\}$ is a filling of K'_i . Thus $\diamond\vec{W}'$ is a simplex-wise F -sequence.
 - If $\kappa_i \in \partial(\tau)$, we have $F(\kappa_i) \leq F(\tau) = F(\kappa_{i+1})$ since F is a stack on K .
2. Suppose $\kappa_i = (\sigma, \tau)$; that is K_i is an expansion of K_{i-1} .
 - (a) Suppose κ_{i+1} is a critical simplex; that is K_{i+1} is a filling of K_i . We observe that we cannot have $\tau \in \partial(\kappa_{i+1})$, otherwise there would exist a face $\nu \in K_i$, with $\nu \in \partial(\kappa_{i+1})$, $\nu \neq \tau$, such that σ is a proper face of ν .
 - If $\sigma \notin \partial(\kappa_{i+1})$, then $\diamond\vec{W}'$ is a simplex-wise F -sequence.
 - If $\sigma \in \partial(\kappa_{i+1})$, then $F(\kappa_i) = F(\sigma) \leq F(\kappa_{i+1})$ since F is a stack on K .
 - (b) Suppose $\kappa_{i+1} = (\sigma', \tau')$; that is K_{i+1} is an expansion of K_i .
 - If $\sigma \notin \partial(\tau')$ and $\tau \notin \partial(\tau')$, then $\diamond\vec{W}'$ is a simplex-wise F -sequence.
 - If $\sigma \in \partial(\tau')$ or $\tau \in \partial(\tau')$, we have $F(\kappa_i) \leq F(\tau') = F(\kappa_{i+1})$ since F is a stack on K . \square

An F -sequence is not necessary a flooding sequence. Nevertheless we have the following result which indicates that the gradient vector field of an arbitrary Morse sequence on F may be obtained by computing a flooding sequence on F .

Theorem 3. *Let F be a stack on K , and let \vec{W} be an F -sequence from \emptyset to K . Then, there exists a flooding sequence on F which is equivalent to \vec{W} .*

Proof. If \vec{W} is not a flooding sequence, then there exist two elements κ_i and κ_{i+1} in $\diamond\vec{W}$ such that $F(\kappa_i) > F(\kappa_{i+1})$. By Lemma 1, we can swap these elements. Note that this swap corresponds exactly to the swap of the bubble sorting algorithm [10]. By repeating this operation, we obtain a sorted sequence \vec{V} which is equivalent to \vec{W} , and which is a flooding sequence. \square

Remark 2. By Lemma 1, a flooding sequence may be obtained from an arbitrary F -sequence from \emptyset to K with a sorting algorithm. In order to effectively obtain a Morse sequence, it can be seen that it is sufficient for the sorting algorithm to be *stable*. That is, equal elements must appear in the same order in the sorted sequence as they do in the original F -sequence [10].

We conclude this section by illustrating Theorem 3 with the sequence \vec{W}' given Fig. 1. We have $\diamond\vec{W}' = \langle a, (b, ab), c, bc, (d, cd), (e, ed), be, (bd, bde) \rangle$. By iteratively applying Lemma 1, we see that we can derive the simplex-wise sequence $\diamond\vec{W}''' = \langle c, (d, cd), (e, ed), a, (b, ab), bc, be, (bd, bde) \rangle$: the corresponding sequence \vec{W}''' is an F -sequence which is equivalent to \vec{W}' . Furthermore, \vec{W}''' is a flooding sequence on F .

5 Flooding sequences and cosimplicial complexes

Let F be a stack on K and let $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ be an F -sequence. Let $\diamond\vec{W} = \langle \kappa_1, \dots, \kappa_k \rangle$. For each $\lambda \in \mathbb{Z}$ we write $\diamond\vec{W}[\lambda]$ for the subsequence of $\diamond\vec{W}$ composed of all κ_i such that $F(\kappa_i) = \lambda$. Recall that a subsequence is a *substring* if it is a sequence of contiguous elements of the original sequence. If the sequence \vec{W} is a flooding sequence then, for each $\lambda \in \mathbb{Z}$, the sequence $\diamond\vec{W}[\lambda]$ is a substring of $\diamond\vec{W}$ which is a simplex-wise Morse sequence from $F_{\lambda-1}$ to F_λ . The set $S = F_\lambda \setminus F_{\lambda-1}$ is a cosimplicial complex. From Proposition 5, we obtain:

Proposition 8. *Let (L, K) be a pair of simplicial complexes such that $L \subseteq K$, and let $S = K \setminus L$. A sequence \vec{W} is a Morse sequence from L to K if and only if \vec{W} is a Morse sequence from \underline{S} to \overline{S} .*

Let S be a cosimplicial complex. If \vec{V} is a Morse sequence from \underline{S} to \overline{S} , then we say that \vec{V} and $\diamond\vec{V}$ are *Morse sequences on S* .

If F is a stack on K , recall that $F[\lambda] = \{\nu \in K \mid F(\nu) = \lambda\}$; we have $F[\lambda] = F_\lambda \setminus F_{\lambda-1}$. We derive the following result from the previous observations.

Proposition 9. *Let F be a stack on K and let $\vec{W} = \langle \emptyset = K_0, \dots, K_k = K \rangle$ be an F -sequence. Let $[0, H]$ be the range of F . The sequence \vec{W} is a flooding sequence if and only if $\diamond\vec{W}$ is a concatenation of the form $\diamond\vec{W} = \diamond\vec{W}[0] \dots \diamond\vec{W}[\lambda] \dots \diamond\vec{W}[\lambda] \dots \diamond\vec{W}[H]$ where, for each $\lambda \in [0, H]$, $\diamond\vec{W}[\lambda]$ is a Morse sequence on $F[\lambda]$.*

Definition 6. Let S be a cosimplicial complex and let $\sigma, \tau, \nu \in S$, with $\sigma \in \partial\tau$.

- If $\delta(\sigma, S) = \{\tau\}$, we say that $S' = S \setminus \{\sigma, \tau\}$ is a reduction of S ,
- If $\delta(\nu, S) = \emptyset$, we say that $S' = S \setminus \{\nu\}$ is a perforation of S ,
- If $\partial(\tau, S) = \{\sigma\}$, we say that $S' = S \setminus \{\sigma, \tau\}$ is a coreduction of S ,
- If $\partial(\nu, S) = \emptyset$, we say that $S' = S \setminus \{\nu\}$ is a coperforation of S .

In each of the above four cases, we say that S' is a contraction of S .

Reductions and coreductions have been introduced in [19] in the context of S -complexes. In our context, the following property is crucial for an efficient computation of Morse sequences.

Proposition 10. Let S be a cosimplicial complex and let $\sigma, \tau, \nu \in S$.

- $\bar{S} \setminus \{\sigma, \tau\}$ is a collapse of \bar{S} iff $S \setminus \{\sigma, \tau\}$ is a reduction of S .
- $\bar{S} \setminus \{\nu\}$ is a perforation of \bar{S} iff $S \setminus \{\nu\}$ is a perforation of S .
- $\underline{S} \cup \{\sigma, \tau\}$ is an expansion of \underline{S} iff $S \setminus \{\sigma, \tau\}$ is a coreduction of S .
- $\underline{S} \cup \{\nu\}$ is a filling of \underline{S} iff $S \setminus \{\nu\}$ is a coperforation of S .

We note that a contraction of a cosimplicial complex is also a cosimplicial complex. Thus, the following definition makes sense.

Let S be a cosimplicial complex. A *Morse cosequence on S* is a sequence $\vec{U} = \langle S = S_0, \dots, S_k = \emptyset \rangle$ such that, for each $i \in [1, k]$, S_i is a contraction of S_{i-1} . The sequence is increasing (resp. decreasing) if each S_i is either a coreduction or a coperforation of S_{i-1} (resp. a reduction or a perforation of S_{i-1}).

Let $L \subseteq K$ be two simplicial complexes. Let $\vec{W} = \langle L = K_0, \dots, K_k = K \rangle$ be a sequence of nested complexes. From Propositions 8 and 10, we obtain:
 Let $\vec{U} = \langle S_0, \dots, S_k \rangle$ with $S_i = K \setminus K_i$. Then \vec{W} is a Morse sequence if and only if \vec{U} is an increasing Morse cosequence on $K \setminus L$.
 Let $\vec{U} = \langle S_0, \dots, S_k \rangle$ with $S_i = K_{k-i} \setminus L$. Then \vec{W} is a Morse sequence if and only if \vec{U} is a decreasing Morse cosequence on $K \setminus L$.

6 Computing a flooding sequence

We have seen that a Morse sequence \vec{W} from L to K may be obtained by computing a Morse sequence $\diamond\vec{V}$ on the set $S = K \setminus L$. The sequence $\diamond\vec{V}$ can be constructed from the left to the right through iterative coreductions and coperforations of the set S . Alternatively, $\diamond\vec{V}$ can be constructed from the right to the left through iterative reductions and perforations of S .

Scheme 1 allows us to compute a Morse sequence $\diamond\vec{V}$ on an arbitrary cosimplicial complex S by considering the two previous constructions. At each step of the scheme, the set S' is a contraction of S .

We consider three cases of Scheme 1:

1. If $\diamond\vec{V}$ is constructed using only coreductions and coperforations, then the scheme is an *increasing scheme*. We say that $\diamond\vec{V}$ is a *maximal sequence* if we make a coperforation only when no coreduction can be made.

Scheme 1: Sequence(S)

Data: A cosimplicial complex S
Result: **Sequence(S)**, which is a simplex-wise Morse sequence on S .

```

1  $\vec{L} := \langle \rangle$ ;  $\vec{R} := \langle \rangle$ ;
2 while  $S \neq \emptyset$  do
3   Extract  $\kappa$  such that:
4   - either  $\kappa = (\sigma, \tau)$ , with  $\sigma \subset \tau$ , and  $S' = S \setminus \{\sigma, \tau\}$  is a reduction or a
     coreduction of  $S$ ,
5   - or  $\kappa = \nu$  and  $S' = S \setminus \{\nu\}$  is a perforation or a coperforation of  $S$ .
6   if  $S'$  is a coreduction or a coperforation of  $S$  then  $\vec{L} := \vec{L} \cdot \kappa$ ;
7   if  $S'$  is a reduction or a perforation of  $S$  then  $\vec{R} := \kappa \cdot \vec{R}$ ;
8    $S := S'$ 
9  $\diamond \vec{V} := \vec{L} \cdot \vec{R}$ ;
10 return  $\diamond \vec{V}$ 

```

2. If $\diamond \vec{V}$ is constructed using only reductions and perforations, then the scheme is a *decreasing scheme*. We say that $\diamond \vec{V}$ is a *minimal sequence* if we make a perforation only when no reduction can be made.
3. Otherwise we say that the scheme is an *interleaved scheme*. We say that $\diamond \vec{V}$ is a *min/max sequence* if we make a perforation or a coperforation only when no reduction and no coreduction can be made.

Let us illustrate these three schemes with the complex K of Figure 1 (a).

1. The Morse sequence \vec{W} corresponding to Figure 1 (b) is such that:

$$\diamond \vec{W} = \langle a, (b, ab), (c, bc), (d, cd), (e, ed), be, (bd, bde) \rangle.$$

This sequence is a maximal sequence. It can be built from the left to the right with Scheme 1. Under this ordering, no coreduction can be made when the two coperforations a and be are done.

2. Let \vec{W}''' be the Morse sequence such that:

$$\diamond \vec{W}''' = \langle b, (c, bc), (d, cd), (e, ed), be, (a, ab), (bd, bde) \rangle.$$

This sequence is a minimal sequence. It can be built from the right to the left with Scheme 1. Under this ordering, no reduction can be made when the two perforations be and b are done. Note that \vec{W} is not a minimal sequence and \vec{W}''' is not a maximal sequence.

3. For a min/max sequence, at the second step of Scheme 1, we must have either $\vec{L} = \langle \rangle$ and $\vec{R} = \langle (a, ab), (bd, bde) \rangle$ or $\vec{L} = \langle \rangle$ and $\vec{R} = \langle (bd, bde), (a, ab) \rangle$. At the third step, we can choose to make the coperforation b . In this case, we must make three coreductions at the following steps. Thus, at line 9 of Scheme 1, we may obtain:

$$\vec{L} = \langle b, (c, bc), (d, cd), (e, ed), be \rangle \text{ and } \vec{R} = \langle (a, ab), (bd, bde) \rangle.$$

We retrieve the sequence \vec{W}''' after the concatenation of \vec{L} and \vec{R} .

The purpose of maximal, minimal, and min/max sequences is to try to minimize the number of critical simplexes. This problem is, in general, NP-hard [16]. Therefore, these sequences do not, in general, yield optimal results. Note also that different numbers of critical simplexes may be obtained by these three types of sequences, see Figure 11 of [14] and Figure 4 of [4] for examples which illustrate this fact.

The computational models of increasing and decreasing schemes are classical models for extracting gradient vector fields, see [3], [19], [15], [14]. Also the interleaved model is introduced in [14]. A distinctive feature of our approach is the use of these schemes to compute Morse sequences. A Morse sequence on a complex K not only defines a gradient vector field on K , but also provides a specific structure on K .

With Proposition 9, we derive Scheme 2 which is a direct extension of Scheme 1 for computing flooding sequences. Given a stack F on K , we consider the different sections $F[\lambda]$ of F at level λ . For each λ , a Morse sequence $\diamond\vec{W}[\lambda]$ on $F[\lambda]$ is computed thanks to Scheme 1. Then, the sequences $\diamond\vec{W}[\lambda]$ are simply concatenated in order to obtain a flooding sequence on F .

Scheme 2: Flood(F)

Data: A simplicial complex K and a stack F on K ; the range of F is $[0, H]$.

We set $F[\lambda] := \{\nu \in K \mid F(\nu) = \lambda\}$.

Result: **Flood**(F), which is a simplex-wise flooding sequence on F .

```

1 for  $\lambda \in [0, H]$  do in parallel
2    $\diamond\vec{W}[\lambda] := \mathbf{Sequence}(F[\lambda])$ 
3  $\diamond\vec{W} := \langle \rangle$ ;  $\lambda := 0$ ;
4 while  $\lambda \leq H$  do
5    $\diamond\vec{W} := \diamond\vec{W} \cdot \diamond\vec{W}[\lambda]$ ;
6    $\lambda := \lambda + 1$ 
7 return  $\diamond\vec{W}$ 

```

With the three above instances of Scheme 1, we obtain three instances of Scheme 2 which allow to compute maximal, minimal, and min/max flooding sequences.

Let us consider again the stack F given Figure 1 (c). We have:

$$F[1] = \{c\}, F[3] = \{d, e, de, cd\}, F[5] = \{a, b, ab, bc, be\}, F[8] = \{bd, bde\}.$$

With Scheme 1, we may obtain the following maximal sequences:

- **Sequence**($F[1]$) = $\langle c \rangle$,
- **Sequence**($F[3]$) = $\langle (d, cd), (e, ed) \rangle$,
- **Sequence**($F[5]$) = $\langle (b, bc), (a, ab), be \rangle$,
- **Sequence**($F[8]$) = $\langle (bd, bde) \rangle$.

Thus, with Scheme 2, we may obtain precisely the flooding sequence $\diamond\vec{W}'''$ which is illustrated Figure 1 (e). Observe that, in this simple case, we obtain an *optimal*

F-sequence, that is, an *F*-sequence with the smallest number of critical faces. Observe also that, by Theorem 3, an optimal *F*-sequence may always be obtained by considering solely flooding sequences.

With Scheme 1, it can be seen that a maximal, minimal, or min/max sequence $\diamond \vec{W}[\lambda]$ of Scheme 2 may be obtained with a time complexity $\mathcal{O}(dN_\lambda)$, where d is the dimension of the complex K , and N_λ is the number of simplexes of $F[\lambda]$, see [6]. Therefore, since the sets $F[\lambda]$ are disjoint, all the sequences $\diamond \vec{W}[\lambda]$ can be computed with a time complexity $\mathcal{O}(dN)$, where N is the number of simplexes of K . Now, if each sequence is stored as a list, and if the range of F is $[0, H]$, it can be seen that the concatenations may be done with complexity $\mathcal{O}(H)$. Since H is bounded by N , we obtain $\mathcal{O}(dN)$ for computing a flooding sequence with a sequential algorithm.

Also the computations of all sequences may be done in parallel. In the best case, that is, if the values of the simplexes are equally distributed, we achieve $\mathcal{O}(\frac{dN}{H} + H)$ for computing the flooding sequence.

7 Conclusion

This work extends the framework of Morse sequences to stacks on simplicial complexes, introducing a flexible and expressive model for discrete Morse theory. By incorporating stack-based constraints, we define flooding sequences—a refined version of Morse sequences that respect monotonic ordering, reflecting processes commonly observed in watershed algorithms.

We have shown that the gradient vector field arising from any Morse sequence on a stack can be recovered by computing an equivalent flooding sequence, underscoring the generality of the flooding model. We also proposed general construction schemes, based on cosimplicial complexes, that provide efficient methods for computing flooding sequences. It should be noted that all the notions introduced in the paper may be directly adapted to more general complexes, for example to cubical complexes.

Importantly, a Morse sequence encodes richer structural information about a complex than a gradient field alone. In particular, the Morse complex of an object can be easily derived via the reference map introduced in [5], requiring only a linear scan of the sequence.

It is also worth emphasizing that a stack is not necessarily an injective function—different faces in the complex may share the same value. This contrasts with some approaches that require distinct values for each face [14,20]. Allowing arbitrary stacks makes the model more suitable for real-world data, where such constraints are typically not satisfied [21].

Overall, the Morse sequence framework opens promising directions for future research, including extensions to persistent homology as well as applications in image segmentation and topological data analysis.

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