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MONOPOLISTIC BEHAVIORS AND GENERAL
EQUILIBRIUM :
A generalization of Nikaido's work

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Abstract

This paper exhibits alternative conditions that ensure the existence of a general equilibrium with price-making firms in pure as well as in mixed strategies. The firms as in Nikaido's early work are considered as having a perfect knowledge of their demand. The existence of an objective demand is proved in a more general setting than the one used by Nikaido. Moreover to weaken the assumptions which guarantee the existence of an objective demand, I introduce a continuous random demand and prove its generic existence

Keywords

Monopolistic pricing, general equilibrium, objective demand, random selections

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1 Introduction

The purpose of this paper is to study, along the line of Oscar Lange and Hukukaine Nikaido, a wide class of general equilibrium situations in the context of a private ownership economy in which a group of firms acts as price-makers. The introduction of non-competitive behaviors however modifies the decentralization mechanism at work in an economy. Following **Lange's** [14] seminal work, one notices that in these models “*monopolistic supply is always equal to the demand [...] A monopolistic (or more generally a non competitive) market is in equilibrium when the quantity sold is such that it maximizes the profit of the monopolist*”. In other words, an equilibrium with imperfect competition coincides to a situation in which the non competitive agents choose, by profit maximization, an optimal price and quantity strategy among the market clearing prices and production levels¹. To go further, following **Nikaido** [21], “*the economy is in equilibrium if and only if, monopolists' maximizations of their profit are realized consistently and simultaneously*”. If one agrees with this last definition, an equilibrium with imperfect competition simply reduces to a Nash equilibrium in which any deviation from the equilibrium price and quantity choices must be consistent with market clearing. The main problem with this concept is therefore to construct, in a preliminary step, the set of all “*consistent*” deviations available to each non-competitive agent. This market clearing requirement leads in fact to two alternative approaches : either the quantities are chosen strategically and the prices are determined by the market forces or the prices are set by the price-makers and the quantities that clear the markets are produced by these agents². The former approach was essentially developed, in the last years, in the spirit of **Gabszewicz-Vial's** concept of Cournot-Walras [9] (see also **Mas-Colell** [16] **Roberts** [10], **Allen** [3] **Gary-Bobo** [10]...). But the later one only gives rise to few works (see **Negishi** [20], **Marschak-Selten** [15], **Nikaido** [21] and **Stahn** [25] [27] [28]).

The principal motivation of this paper is to exhibit some existence conditions of a general equilibrium with price-making firms. More precisely, my objective is to extend **Nikaido's** contribution [21] by introducing on the one hand more general productions abilities and on the other hand a random objective demand curve. But, in order to reach

¹In Oscar Lange's approach, one implicitly assumes that the choices of the non-competitive agents are compatible with market clearing at least at equilibrium. For an approach using rationing see **Benassy** [4].

²At that point, and in order to allowed such a methodological choice, I implicitly assume the existence of agents who act as price-takers. If such agents are not involved, only pure game theory arguments can be exploited in the line of the literature dealing with strategic market games (see for instance **Dubey - Mas-Colell - Shubik** [8]).

this goal, two questions must be answered : first what is the suitable demand notion which is perceived by the non-competitive agents and, secondly given this demand curve, do the behaviors of price-makers lead to a Nash equilibrium ?

To answer the first question, remember that “*monopolists’ maximizations of their profit must be consistent*”. As a matter of consequence, the price-makers are, in this paper, considered as having a perfect knowledge of their demand (the firms know their “*objective demand curve*”). In this context, they are said to behave rationally in particular as a result of their perfect awareness of their profit opportunities in opposition to a situation in which these agents only have access to a subjective perception of their “demand curve” and, therefore misestimate their potential profits. Consequently, this first problem simply reduces to the construction of a function which associates to any price structure a vector of market clearing production levels in a way to take into account all the interdependence between the agents.

The existence of such an objective demand curve was first proved by **Nikaido [21]** for a class of economies characterized by a Leontief production system. My first proposition, simply extends his work to a more general setting which covers a more general class of production sets (i.e. the technology of each monopolist is characterized by a standard cost function). However, as in Nikaido’s early proof, the conditions which ensure the existence of a “*deterministic objective demand curve*” issue from examining, for any price vector, the existence and the uniqueness of a vector of production levels which simultaneously clears every markets. So, if one attempts to weaken these conditions, one has to deal with a situation in which several monopolistic market clearing production levels may exist at the current prices. To overcome this indeterminacy, I introduce a randomization which may be thought of as representing the inherent ex-ante uncertainty in the minds of the producers as to which quantities clear the markets. To be more precise, I introduce a “*random objective demand curve*” defined by means of a continuous random selection in the set of all possible market clearing production levels and prove its generic existence. This therefore shows that the suggestions made by **Allen [3]** for models with quantity strategies also apply to models with price strategies. However, instead of working in a universe of the parameterized excess demand functions, my result will be obtained by perturbing the cost functions of at least two monopolists.

Yet, obtaining a well-defined objective demand curve is not sufficient to ensure the existence of a equilibrium with price-making firms. Along the lines of **Lange [14]** and **Nikaido [21]**, this is only a preliminary step in which one constructs, for each monopolist, the set of his potential deviations from a Nash equilibrium price vector. This existence

problem is addressed in a last part of this paper. Precise definitions of an equilibrium are stated in pure as well as and mixed strategies, and some sufficient conditions are introduced in order to prove four existence results associated alternatively to a deterministic and a random objective demand curve. The most interesting outcome of this discussion is that the existence of an equilibrium with price-making behaviors can be obtained with only few assumptions as long as one accepts mixed strategies as a proxy of a Nash equilibrium with incomplete information and a random demand curve as representing some common ex-ante uncertainty on the market clearing production levels.

This paper will be organized as follows : in section 2, I describe the model, section 3 is devoted to the extension of **Nikaido**'s objective demand concept, in section 4, I analyze the existence of a continuous random selection, section 5 is devoted to some remarks on the existence of a monopolistic equilibrium in pure as well as in mixed strategies, and to the presentation of a simple example and finally section 6 concludes this paper. The proofs are developed in an appendix.

2 The model

In the lines laid out by **Nikaido** [21], let me consider an economy with a set of ℓ goods indexed by $h = 1, \dots, \ell$ and an additional one, labor, which serves both as a “numéraire” and as a non-produced input. The price of good h in terms of this “numéraire” is p_h . The vector of these prices is denoted by $p \in \mathbb{R}_{++}^\ell$ and let $p_{-h} = (p_1, \dots, p_{h-1}, p_{h+1}, \dots, p_\ell) \in \mathbb{R}_{++}^{\ell-1}$. The price of each commodity is set by a monopolist and the labor market works competitively. The agents of this economy are firms and consumers.

For simplicity, I assume that there are ℓ firms³, indexed by $j = 1, \dots, \ell$. Firm j produces the commodity j and sets the price p_j . However to produce the amount of output which balances market j , producer j has to purchase some inputs on the other markets especially on the labor market. As a buyer, he acts as a pure competitor. To characterize these choices, one could, of course, associate a production set to firm j . However, because each firm only produces one good and behaves competitively on his input markets, a standard duality argument allows me to simply characterize each firm by a cost function $c^j(p_{-j}, y_j)$ that takes as arguments $p_{-j} \in \mathbb{R}_{++}^{\ell-1}$, the prices of the inputs,

³This assumption can easily be relaxed by introducing for each firm a market portfolio, that is set of markets on which this firm acts as price-maker. As long as these portfolios describe a partition of the set of all markets, the monopolistic competition assumption is satisfied (see **Stahn** [25])

and $y_j \in \mathbb{R}_+$ the level of monopolistic production. Moreover, I assume as usually that :

- (i) $c^j(p_{-j}, y_j)$ is a C^∞ function,
- (ii) $\forall h = 1, \dots, j-1, j+1, \dots, \ell, \quad \partial_{p_h} c^j(p_{-j}, y_j) \geq 0$,
- (iii) $c^j(p_{-j}, y_j)$ is concave in p_{-j} .

By Shephard's lemma, the conditional competitive demands for produced inputs are given by $\hat{x}_h^j(p_{-j}, y_j) = \partial_{p_h} c^j(p_{-j}, y_j)$ for all $h = 1, \dots, j-1, j+1, \dots, \ell$ and the demand for labor, due to the price normalization rule, is defined by $\lambda_j(p_{-j}, y_j) = c^j(p_{-j}, y_j) - p_{-j} \cdot \partial_{p_{-j}} c^j(p_{-j}, y_j)$. To avoid any aggregation problem, let me redefine the demand for produced inputs in the following way :

$$\forall h = 1, \dots, \ell \quad x_h^j(p_{-j}, y_j) = \begin{cases} \hat{x}_h^j(p_{-j}, y_j) & \text{if } h \neq j \\ 0 & \text{if } h = j \end{cases}$$

Let me also denote by $y = (y_1, \dots, y_\ell)$ the overall vector of monopolistic productions and by $\pi_j(p, y_j) = p_j \cdot y_j - c^j(p_{-j}, y_j)$ the indirect profit

In this economy there are also m consumers, indexed by i who act as pure competitor. Each one, say consumer i , purchases produced goods, supplies labor and earns profits with respect to his profit shares $\theta_{ij} \in [0, 1]$ $j = 1, \dots, \ell$. Given $\pi = (\pi_1, \dots, \pi_\ell)$, the profits realized by the firms, the behavior of consumer i , under standard "smoothness" assumptions on his preferences \preceq_i , can be described by two C^∞ functions $\hat{z}_i(p, \pi)$ and $\hat{\lambda}_i(p, \pi)$ which associate to the prices and the profit levels his demand for produced goods and his labor supply. Taking this behavior for granted, it is however important to notice that the factual choice of this demand is only worked out when the profits are known, or in other words, when the prices are set by the monopolists and when the required amount of goods are produced. From an "ex ante" point of view, the behavior of each consumer is therefore contingent to the different profit levels. Moreover, one also notices that these benefits are related to the price choices and to the production levels. In other words, because "ex ante" $\pi = \pi(p, y) = (\pi_j(p, y_j))_{j=1}^\ell$, the functions describing consumer i 's behavior can be written as $z_i(p, y) = \hat{z}_i(p, \pi(p, y))$ and $\lambda_i(p, y) = \hat{\lambda}_i(p, \pi(p, y))$. Now let me denote by $z(p, y)$, the aggregated demand for produced goods, and $\lambda(p, y)$, the aggregated labor supply. These two functions are of course C^∞ . Moreover, because the consumers perceive no endowment in produced goods, let me assume that $z(p, y) \gg 0$. Finally, by local non satiation, Walras law must be true, thus one verifies that $p \cdot z(p, y) = \lambda(p, y) + p \cdot y - \sum_{j=1}^\ell c^j(p_{-j}, y_j)$.

As a matter of consequence the overall demand to the monopolistic sector of the economy is given by :

$$d(p, y) = z(p, y) + \sum_{j=1}^{\ell} x^j(p_{-j}, y_j)$$

3 Objective demand in the spirit of Nikaido

Following the lines laid out by **Nikaido** [21], an objective perception of the demand must simultaneously takes into account the interdependence between the economic agents on every commodity market. Otherwise, the price-makers would not be aware of their real profit opportunities. This general equilibrium point of view however involves several difficulties because these connections are totally circular. To briefly illustrate this argument, let me, for instance, assume that the excess demand can be chosen as an objective demand curve. Acting in this way, the price-makers set the prices without taking into account the effect of the profit distribution on their own demand. In other words, for an optimal price choice, the consumers' effective demand certainly departs from the one predicted by the excess demand because the profits of the other firms are misanticipated. As a matter of consequence, each monopolist sustains a non-anticipated change in his market clearing production level as well as in his competitive input demands. To overcome this difficulty, it is always possible to assume that these profits are correctly anticipated at least at equilibrium⁴. But these profits levels are determined by the strategical choices of the price-setters. Thus adjusting parametrically the profits levels is inconsistent with profit maximization because the elasticity of the demand with respect to the prices is not correctly anticipated. But, on the other hand, taking into account this fact leads to a circular interdependence between demand and supply.

After having called attention to the importance of these circularities, it becomes essential to build an objective demand curve that overcomes these difficulties. To construct this function, one notes that the demand side on each monopolistic market is fully captured by the excess demand $d(p, y)$. With regards the supply side, one knows, following **Lange's** tradition [14], that a feasible level of monopolistic production $y = (y_1, \dots, y_\ell)$ must be market clearing. Moreover, because of Walras Law, if the commodity markets clear, the same must be true for the labor market. All the interdependence among the economic agents are therefore perfectly summarized in the following market clearing condition $y = d(p, y)$. An objective demand curve which takes into account all these circularities,

⁴This approach is developed in **Marchak-Selten's** paper [15]. In order to obtain a consistent allocation process result, they parametrically adjust the profits at equilibrium..

can therefore be viewed as a function which associates to each price $p \in \mathbb{R}_{++}^\ell$ a vector of monopolistic effective sales $y \in \mathbb{R}_+^\ell$ such that the preceding equation is satisfied. Indeed :

Definition 1

A deterministic objective demand curve is a function $\phi : \mathbb{R}_{++}^\ell \longrightarrow \mathbb{R}^\ell$ such that :

- (i) $\forall p \in \mathbb{R}_{++}^\ell$, $y = \phi(p)$ is the unique solution to $y = d(p, y)$
- (ii) for any sequence $(p_n) \longrightarrow p_0$, one verifies that $\phi(p_n) \longrightarrow \phi(p_0)$

Having this definition in mind, the construction of a objective demand curve is mathematically equivalent to verify, for each price vector p , the existence and the uniqueness of solution in y to the equation $y = d(p, y)$. The existence part of this result can be proved by means of a simple “boundary condition” on the excess supply when the production levels go to infinite. But, as usually, uniqueness requires more. Thus, to build an objective demand curve in **Nikaido’s** lines [21], let me assume that :

Assumption 1 (A1)

- (i) Let $(y_n)_{n \in \mathbf{N}}$ such that $\|y_n\| \rightarrow +\infty$, let $I_{+\infty} = \left\{ h = 1, \dots, \ell \mid (y_h)_n \xrightarrow{n \rightarrow \infty} +\infty \right\}$ then $\forall p \in \mathbb{R}_{++}^\ell$, $\exists h \in I_{+\infty}$, $\exists N \in \mathbf{N}$, $\forall n > N$, $(y_h)_n - d_h(p, y_n) > 0$
- (ii) $\forall p \in \mathbb{R}_{++}^\ell$, $\partial_y z(p, y) \geq 0$, $\partial_y \lambda(p, y) \leq 0$, and $\partial_{y_j} x^j(p, y_j) \geq 0$, $\partial_{y_j} \lambda^j(p, y_j) > 0$ for all $j = 1, \dots, \ell$.

Assumption A1(i) ensures the existence of market clearing production levels for each price $p \in \mathbb{R}_{++}^\ell$. This result is obtained by assuming that high production levels cannot be absorbed by the competitive demand. To be more precise, if the production of several goods goes to infinite, A1(i) asserts that an excess supply appears on at least one of these markets. This assumption is therefore not very restrictive. But concerning uniqueness, stronger restrictions are needed. Assumption A1(ii) requires that the demand functions for intermediate (including labor) and final goods must be increasing with any monopolistic production level while the labor supply must be non-increasing with the same arguments. With regards the demand for factors, this assumption seems quite natural and its interpretation is straightforward. Concerning the consumers, the signification of this assumption is more confusing. To have a better understanding of this one, let me identify an increase in at least one of the production levels to an increase in the real wealth of the economy. In this case, this hypothesis means that there is no inferior good at the aggregated level and that a substitution happens between labor and profit incomes.

These assumptions are however very closed to those introduced by **Nikaido** (see [21] pp. 47-50)⁵. From this point of view, my approach in this section simply extends his existence result of an objective demand to a more general class of economies. The construction of the proof is however quite different. Instead of having recourse to Brouwer's fixed point theorem for existence and to Gale-Nikaido's result for uniqueness, some arguments dealing with degree theory are required in order to prove that :

Proposition 1

Under A1, there exists a continuous objective demand curve $\phi(p)$ with the property that for all $p \in \mathbb{R}_{++}^\ell$, $y = \phi(p)$ is the unique vector of monopolistic production levels that simultaneously clears the commodity markets.

This extension of **Nikaido's** work [21] is nevertheless not really satisfying. In fact, the assumptions which are required for uniqueness (A1(ii)) induce some strong restrictions on the shape of the demand curve for final and intermediate goods. However, without A1 (ii), only few can be said about the cardinality of set of the market clearing production levels. To overcome this major difficulty, one could perhaps try to pick up, for each $p \in \mathbb{R}_{++}^\ell$, a unique vector in the correspondence $\mathcal{Y}(p) = \{y \in \mathbb{R}_+^\ell \mid y = d(p, y)\}$ which associates to each price the set of the market clearing production levels. Yet nothing ensures that this selection would be a continuous one and accordingly that the price-maker strategies lead to a Nash equilibrium in pure as well as in mixed strategies⁶. At that point of the discussion, two alternatives are available. On the one hand, one could arbitrary assume the continuity of this selection, but such a strong restriction on the structure of the objective demand correspondence \mathcal{Y} is difficult to accept. On the other hand, one could try to convexify this correspondence by randomization and to choose a continuous selection from the randomized correspondence.

4 Objective demand : a random selection approach

In fact the non-existence of a continuous selection in the objective demand correspondence is essentially due to the jumps in the cardinality of $\mathcal{Y}(p)$. This point can be easily illustrated

⁵In fact, **Nikaido** [21] only introduces some restrictions on the aggregate demand and on the labor supply because the assumptions on the demand for intermediate goods are trivially satisfied within a Leontief system.

⁶For instance, **Dierker** and **Grodal** [7] have constructed an example, in a Cournot-Walras framework, of an economy in which no equilibrium in mixed strategies exists due to lack of continuity in the selection chosen in the Walras correspondence.

in a two dimensional example. If one assumes for instance that \mathcal{Y} contains a “fold” (see figure 1), every deterministic selection in \mathcal{Y} is at least discontinuous at p_1 or at p_2 .

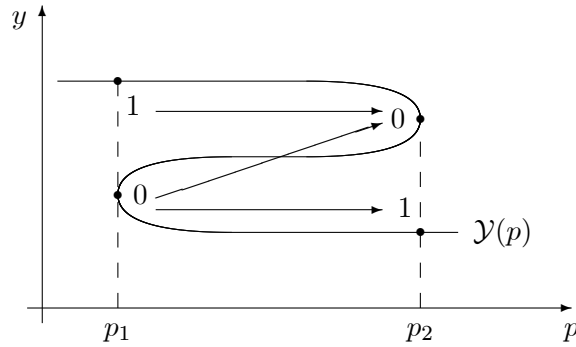


Figure 1: the case of a “fold”

But in the neighborhood of a “fold, it is impossible to associate to each price vector a description of the market clearing production plans. Hence every selection which encounter a “fold” is ineluctably discontinuous.. However, if one allows random selections, this problem disappears. In fact, if one has in mind that this selection represents the inherent uncertainty about the market clearing production levels, one can assume that each producer assigns a zero probability to a production vector on a fold because any small mistake in the price observation can lead to drastic errors in appraising the market clearing production plans. The use of a random selection also permits to reduce “smoothly” the probability of occurrence of production vectors nearby a fold and by this way restores the continuity of the expected demand. In the preceding example (figure 1), this selection can be constructed in the following way : one puts probability 1 on the unique solutions before p_1 and after p_2 and between these two prices one continuously decreases the probability from 1 to 0 on the upper part of the fold, symmetrically increases the probability from 0 to 1 on the lower part and in the middle one takes the complement to 1.

Random selections are therefore useful. This method was first introduced in imperfect competition models by **Allen** [1][2][3]. She proves the generic existence of a random selection in the Walras correspondence or equivalently the generic existence of a random inverse objective demand curve in a Cournot-Walras framework (i.e. in a world in which quantities are set in a first step by some oligopolists and in which one reaches in the second step a Walrasian equilibrium in an exchange economy conditionally to these strategic

choices). My objective, in this paper, is to verify that a similar result can be obtained in models with price strategies. More precisely, I prove the generic existence of a continuous random objective demand curve defined from the price set, \mathbb{R}_{++}^ℓ , into $\mathcal{M}(\mathbb{R}_+^\ell)$ the set of all probability measures constructed on $\mathcal{B}(\mathbb{R}_+^\ell)$ the borelian subsets of the monopolist production levels. Indeed :

Definition 2

A continuous random objective demand curve is a map $\phi : \mathbb{R}_{++}^\ell \longrightarrow \mathcal{M}(\mathbb{R}_+^\ell)$ which associates to each price a probability measure over the production levels with the property that :

- (i) $\forall p \in \mathbb{R}_{++}^\ell, \quad \phi(p)[\mathcal{Y}(p)] = 1$
- (ii) if $(p_n) \longrightarrow p_0$ then $\phi(p_n) \longrightarrow \phi(p_0)$ for the topology of weak convergence

It however turns out that a randomization is not sufficient to guarantee the existence of a continuous selection. To illustrate this fact let me again consider a two dimensional example and let me assume that \mathcal{Y} contains a “cliff” at p_0 (see figure 2)

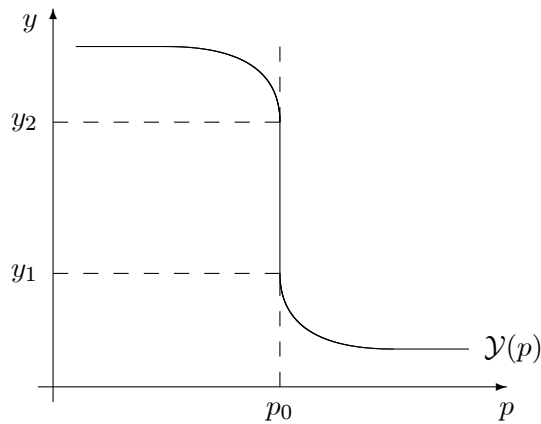


Figure 2: the case of a “cliff”

One notices that for every p closed to p_0 but $p \neq p_0$, the random demand necessarily puts measure 1 on the unique $y \in \mathcal{Y}(p)$ and at p_0 , one can select any probability measure with support $[y_1, y_2]$. But in any case, one cannot pick up a distribution at p_0 that ensures the continuity of the random selection for the topology of weak convergence. In this figure, one however notices that the singularities for which a selection does not exist

are very peculiar : they disappears after a small perturbation of the economy. This low dimensional example leads therefore to the following intuitions : first local uniqueness of the components of $\mathcal{Y}(p)$ is a prerequisite to the existence of a random demand curve and secondly this property seems to be true for a generic set of economies.

To obtain this generic argument, I simply introduce some perturbations on the data of the economy, more precisely on the cost functions of at least two producers. At that point, my approach departs from **Allen's** one [1] because in her Cournot-Walras framework instead of perturbing the data, she works in the set of all C^∞ functions from $\mathbb{R}_{++}^\ell \times \mathbb{R}_+^\ell$ into \mathbb{R}^ℓ and identifies this one to the set of all excess demand functions. In this respect, my approach is closer to the one of **Mas-Colell** and **Nachbar** [17]. In a Cournot-Walras approach, they show the generic existence of a continuous random inverse demand curve by simply perturbing the preferences of at least one consumer. To obtain this result they first show the generic finiteness of the number of critical equilibrium in an exchange economy by perturbing the preferences and secondly construct for these economies a random selection following **Allen's** suggestions [2].

My approach however departs from their paper in several respects : first I do not work with the Walras correspondence but with $\mathcal{Y}(p) = \{y \in \mathbb{R}_+^\ell \mid y - d(p, y) = 0\}$ the correspondence which associates to each price p the set of all market clearing production levels. Secondly the perturbations are introduced on the cost functions instead of the utility functions. This is due to the fact that the competitive input demand of each monopolist is a component of the excess demand. By Hotelling's lemma, a perturbation of the cost functions has an immediate consequence on the excess demand $d(p, y)$. So let me denote by \mathcal{C}^j the set of all cost functions for agent j . By perturbing the cost functions of agent $\ell - 1$ and ℓ , one obtains the following result :

Proposition 2

As long as A1(i) (the boundary behavior) is true, there exists a residual set $C \subset \mathcal{C}^{\ell-1} \times \mathcal{C}^\ell$ of cost functions for agent $\ell - 1$ and agent ℓ such that $\forall p \in \mathbb{R}^\ell$, $\mathcal{Y}(p)$ is at most countable.

Within this generic class of economies, one is now able to construct a random objective demand curve. The basic intuition goes as follows : one first constructs the correspondence \mathcal{S} which associates to each price a set of probability mixtures over \mathcal{Y} and applies a continuous selection argument. But in order to use **Michael's** selection theorem [19], one has to make sure that $\mathcal{S}(p)$ is non-empty, closed, convex and that \mathcal{S} is lower-hemi-continuous. The three first properties are achieved by considering probability mixtures over the non-empty set $\mathcal{Y}(p)$, but nothing ensures that \mathcal{S} is lower-hemi-continuity. In order to have

an intuition of this problem, let me come back to the example given in figure 1. If one chooses a price p with the property that $\mathcal{Y}(p)$ contains a non-regular solution that leads to a reversal of the orientation of the equilibrium manifold, then the correspondence which associates to any p the probability mixtures over $\mathcal{Y}(p)$ cannot be lower-hemi-continuous. For instance at p_1 there exists no increasing sequence $(p_n) \rightarrow p_1$ such that a sequence of probability distribution over $\mathcal{Y}(p_n)$ converges to $(1/2, 1/2)$. (see figure 3)

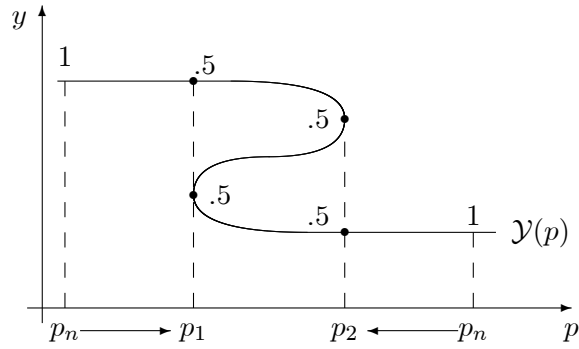


Figure 3: the problem of lower-hemi-continuity

A random selection however exists in this example. But this one has a peculiar feature : it associates measure 0 to each non-regular solution of $y = d(p, y)$ which leads to a reversal of the orientation. In other words, as long as one excludes from \mathcal{Y} the subset of these peculiar points, one can assert that this new correspondence \mathcal{Y}' is lower-hemi-continuous. Moreover taking the probability mixtures over a lower-hemi-continuous correspondence leads to a correspondence with the same property (see **Hildendrand [12]**). It appears in addition fairly reasonable to exclude these points. In fact, having in mind that a random selection represents the inherent uncertainty about the market clearing production levels, the producers surely assign a zero probability to configurations which are not structurally stable because, at these points, a small mistake in the price observation can lead to drastic errors in appraising the market clearing production plan. More technically, I will consider the following sub-correspondence :

$$\mathcal{Y}'(p) = \left\{ y_0 \in \mathcal{Y}(p) \mid \exists \varepsilon > 0, \forall y \in \partial B_\varepsilon(y_0), y \neq d(p, y) \text{ and } \deg \left(\frac{y - d(p, y)}{\|y - d(p, y)\|} \right) \neq 0 \right\}$$

where $B_\varepsilon(y_0)$ is a ball with center y_0 and radius ε . In this case one can prove that :

Proposition 3

Under A1(i) and for all $(c^{\ell-1}, c^\ell) \in C$, $\mathcal{Y}'(p)$ is a non-empty, lower-hemi-continuous correspondence

In order to obtain a continuous random objective demand curve, one simply has to randomize the correspondence \mathcal{Y}' and to apply Michael's selection theorem.

Proposition 4

Under A1(i) (the boundary condition), there exists a continuous random objective demand curve $\phi(p)$ for a generic class of economies.

However the existence of either a deterministic or a random objective demand concept is no longer sufficient to ensure the existence of a monopolistic equilibrium in pure or in mixed strategies. Let me now address this problem.

5 The existence of a monopolistic equilibrium

5.1 The equilibrium in pure strategies

Having in mind the definitions of an objective demand, it is now easy to describe the price making behaviors of the firms. In pure strategies, the optimal price choice of firm j is simply obtained by profits' maximization given the price quoted by the other firms. Moreover, as a consequence of the introduction of an objective demand concept, the Nash equilibrium price vector of this game induces a consistent allocation : that is a commodity reallocation in which the competitive behaviors are optimal at the current equilibrium prices (i.e. either utility maximizing or cost minimizing) and in which the market clearing production level of each monopolist is correctly prescribed by the objective demand curve.

To be more precise let me denote by π_j^d the function

$$\pi_j^d(p_j, p_{-j}) = p_j \cdot \phi_j(p_j, p_{-j}) - c^j(p_{-j}, \phi_j(p_j, p_{-j}))$$

with $\phi_j(p_j, p_{-j})$ is the j^{th} component of the deterministic objective demand curve. A monopolistic equilibrium associated to a deterministic objective demand curve is therefore well-defined if there exists a price vector \hat{p} with the property that :

$$\forall j = 1, \dots, \ell, \quad \forall p_j \in P_j \quad \pi_j^d(\hat{p}_j, \hat{p}_{-j}) \geq \pi_j^d(p_j, \hat{p}_{-j})$$

A similar definition can be stated for a monopolistic equilibrium associated to a random objective demand curve. In fact, if $\phi_j(p_j, p_{-j})$ denotes the marginal distribution of $\phi(p_j, p_{-j})$ over $y_j \in \mathbb{R}_+$, one defines the function π_j^r by :

$$\pi_j^r(p_j, p_{-j}) = \int_{y_j \in \mathbb{R}_+} (p_j \cdot y_j - c^j(p_{-j}, y_j)) d\phi_j(p_j, p_{-j})$$

and an equilibrium \hat{p} exists if and only if .

$$\forall j = 1, \dots, \ell, \quad \forall p_j \in P_j \quad \pi_j^r(\hat{p}_j, \hat{p}_{-j}) \geq \pi_j^r(p_j, \hat{p}_{-j})$$

Let me now briefly address the questions related to the existence of these two kinds of equilibrium. In the two cases, the problem reduces to the existence of a Nash equilibrium in pure strategies. Concerning these games, one knows that an equilibrium exists as soon as the payoff function of each player is continuous and quasi-concave on the relevant compact convex strategy set (see **Debreu** [6]). But here neither quasi-concavity nor compactness of the strategy sets are satisfied.

One is therefore constrained to assume arbitrarily that the payoff functions satisfy this quasi-concavity requirement. This assumption is surely restrictive but also wholly standard in the area of imperfect competition in general equilibrium. The general setting of these models is not able to provide enough structure in order to obtain well-defined reaction functions⁷. In fact, as long as one works with an objective demand curve (i.e. a function derived from a competitive excess demand function), it is hopeless to try to deduce these properties from some restrictions on individual behaviors : it is well-known that the competitive excess demand only satisfies few properties.

In opposition to a Cournot-Walras approach, one also notices that the strategy sets are here essentially non-compact. To be more precise, one can hope that the set of price strategies is at least bounded from below. In fact, one knows that labor is required in order to produce and that its price is set to one. So let me denote by \underline{p}_j this lower bound. But this lower bound may perhaps be zero⁸. An assumption is therefore required in order to make sure that the best response of each firm never goes either to this lower bound or

⁷Of course, such kind of assumptions are not only needed in models with price strategies (see for instance **Benassy** [4] or **Nikaido** [21]) but they are also required in models with quantity strategies (see for instance **Gabszewicz-Vial** [9] or **Mas-Colell** [16]).

⁸In fact, this lower bound can take different values. For instance, if the production sets are linear in labor, it is clear that this lower bound can be identified to the amount of labor required to produce one unit of good. But if one considers a production function for which labor can be substituted by other inputs this bound may be zero.

to infinite. In other word, one has to assume that the strategies closed to the boundary are strictly dominated by an other strategy⁹.

More formally, I assume that :

Assumption 2 (A2)

(i) Let p_j^0 be either \underline{p}_j or $+\infty$. The strategies closed to the boundary are strictly dominated iff $\forall j = 1, \dots, \ell, \exists \tilde{p}_j \in]\underline{p}_j, +\infty[$ with the property that :

$$\forall p_{-j} \in \prod_{\substack{k=1 \\ k \neq j}}^{\ell} [\underline{p}_k, +\infty[, \quad \lim_{p_j \rightarrow p_j^0} \pi_j^d(p_j, p_{-j}) < \pi_j^d(\tilde{p}_j, p_{-j})$$

(i)' Let p_j^0 be either \underline{p}_j or $+\infty$. $\forall j = 1, \dots, \ell, \exists \tilde{p}_j \in]\underline{p}_j, +\infty[$ such that :

$$\forall p_{-j} \in \prod_{\substack{k=1 \\ k \neq j}}^{\ell} [\underline{p}_k, +\infty[, \quad \lim_{p_j \rightarrow p_j^0} \pi_j^r(p_j, p_{-j}) < \pi_j^r(\tilde{p}_j, p_{-j})$$

(ii) $\forall j = 1, \dots, \ell, \pi_j^d(\bullet, p_{-j})$ is quasi-concave in p_j

(ii)' $\forall j = 1, \dots, \ell, \pi_j^r(\bullet, p_{-j})$ is quasi-concave in p_j

Of course, this assumption immediately induces the following proposition

Proposition 5

(i) Under A1(i)(ii) and A2(i)(ii) there exists a monopolistic equilibrium in pure strategies which extends *Nikaido's work*

(ii) Under A1(i) and A2(i)'(ii)' there exists a monopolistic equilibrium in pure strategies associated to a random objective demand curve

If one has a look at this existence conditions, one notes, as planed that the restrictions which ensure the existence of a monopolistic equilibrium associated to a random objective demand curve are weaker then the one needed in the natural extension of **Nikaido's** work [21]. This is due to the fact that the existence of a random objective demand curve can be obtained only with a boundary condition (A1(i)). The conditions that ensure the existence of a Nash equilibrium in pure strategies are however the same : dominant strategies closed to the boundary and quasi-concavity. But this last assumption induces again a strong

⁹In order to bound the strategy sets, one could also, as **Benassy** [4], introduce reservation prices by bounding the marginal rates of substitution between commodities and labor.

restriction on the shape of the profit functions. Moreover, it is well-known that it is rather difficult to base this assumption on restrictions on the data of the economy¹⁰. Hence, in order to avoid this assumption let me introduce mixed strategies.

5.2 The equilibrium in mixed strategies

In this case, the strategy set of each monopolist is no more given by $\left[\underline{p}_j, +\infty\right]$ the set of all prices, but is defined by the set of all probability measures $\mathcal{M}\left(\left[\underline{p}_j, +\infty\right]\right)$ over the price set. So, let me denote by $\mu_j \in \mathcal{M}\left(\left[\underline{p}_j, +\infty\right]\right)$ a mixed strategy for monopolist j , let me define by $\mu = (\mu_j)_{j=1}^\ell$ a vector of mixed strategies and let $\mu_{-j} = (\mu_1, \dots, \mu_{j-1}, \mu_{j+1}, \dots, \mu_\ell)$. The payoff functions of the players are of course the same as in the preceding subsection. There are given either by $\pi_j^d(p_j, p_{-j})$ or by $\pi_j^r(p_j, p_{-j})$.

Moreover, the allocation process, due to the use of mixed strategies, is given, in this economy, by a set of probability measures over quantities. To be more precise, these distributions are induced by the equilibrium distribution over prices and the functions that characterize the competitive behaviors. But, due to the use of an objective demand curve, one knows that for any price in the support of the equilibrium distribution, the induced allocation is feasible and is either utility maximizing or cost minimizing. Hence, as in the preceding subsection, market clearing conditions do not matter in the definition of a monopolistic equilibrium as long as the pricing strategies are computed by means of an objective demand.

Thus, a monopolistic equilibrium with mixed strategies which extends Nikaido's work exists if one can define $\hat{\mu} \in \mathcal{M}\left(\prod_{k=1}^\ell \left[\underline{p}_k, +\infty\right]\right)$ such that: $\forall j = 1, \dots, \ell, \forall \mu_j \in \mathcal{M}\left(\left[\underline{p}_j, +\infty\right]\right)$

$$\int_{(p_j, p_{-j}) \in \prod_{k=1}^\ell \left[\underline{p}_k, +\infty\right]} \pi_j^d(p_j, p_{-j}) d\hat{\mu}_j \times d\hat{\mu}_{-j} \geq \int_{(p_j, p_{-j}) \in \prod_{k=1}^\ell \left[\underline{p}_k, +\infty\right]} \pi_j^d(p_j, p_{-j}) d\mu_j \times d\hat{\mu}_{-j}$$

And of course a monopolistic equilibrium associated to a random objective demand curve is well-defined, if one can construct $\hat{\mu} \in \mathcal{M}\left(\prod_{k=1}^\ell \left[\underline{p}_k, +\infty\right]\right)$ such that $\forall j = 1, \dots, \ell$,

¹⁰This quasi-concavity assumption is surely not satisfied for a large class of economies. Counter-examples are not exhibited, for the moment, within a monopolistic competition framework. But in the dual Cournot-Walras approach, several examples exist (see for instance **Roberts** and **Sonnenschein** [22])

$$\forall \mu_j \in \mathcal{M} \left(\left[\underline{p}_j, +\infty \right] \right)$$

$$\int_{(p_j, p_{-j}) \in \prod_{k=1}^{\ell} [\underline{p}_k, +\infty[} \pi_j^r(p_j, p_{-j}) d\hat{\mu}_j \times d\hat{\mu}_{-j} \geq \int_{(p_j, p_{-j}) \in \prod_{k=1}^{\ell} [\underline{p}_k, +\infty[} \pi_j^r(p_j, p_{-j}) d\mu_j \times d\hat{\mu}_{-j}$$

With regards the existence of these two kinds of equilibrium, by construction, one knows that no quasi-concavity restriction is required. However, it is again important to notice that the supports of the price strategies are not compact. To be more precise, one knows by **Glicksberg**'s result [12], that games with continuous payoffs and non-empty compact strategy sets admit a mixed strategy equilibrium. But, if one takes a sequence of such games converging to a game with unbounded strategy sets, one has to make sure that a sequence of equilibrium converges. Intuitively speaking, one has to make sure that “no mass escapes to infinite or to \underline{p}_j along this sequence”. But, the boundary condition, introduced in A2(i), prevents this even because, this assumption ensures that the support of any best response mixed strategy never “meets the boundary”. Hence

Proposition 6

(i) Under A1(i)(ii) and A2(i) there exists a monopolistic equilibrium in mixed strategies which extends Nikaido's work

(ii) Under A1(i) and A2(i)' there exists a monopolistic equilibrium in mixed strategies associated to a random objective demand curve

5.3 A simple example

In order to illustrate these existence results, I propose in this subsection a simple example in which there exists at least a mixed strategy equilibrium associated to a deterministic objective demand curve. This example illustrates therefore a case in which assumptions A1(i), A1(ii) and A2(i) apply.

So let me consider an economy composed by one consumer and two firms. Two produced goods are available. The consumer buys these commodities in quantity x_1 and x_2 , and supplies an amount λ of labor. He also earns the profits of the two firms. His utility is a C.E.S. function given by $u(x_1, x_2, \lambda) = (x_1^\rho + x_2^\rho + (\Lambda_0 - \lambda)^\rho)^{1/\rho}$. Λ_0 is maximum amount of labor that this agent is able to supply. I assume that the elasticity of substitution is greater than 1 or in other words that $\rho \in]0, 1[$ and, for simplicity, I set $\Lambda_0 = 1$. Firm i ($i = 1, 2$) produces good i by means of labor. The production functions are linear. Let α_i be the required amount of labor in order to produce one unit of commodity i .

Having in mind the price of labor is normalized to 1. The cost function is defined by $c_i(p_{-i}, y_i) = \alpha_i \cdot y_i$.

With this assumption, one immediately notices that the computation of an objective demand is very simple. In fact the market clearing conditions on the commodity markets lead to the following system of equations :

$$\begin{cases} y_1 = h_1(p_1, p_2) \cdot (1 + (p_1 - \alpha_1) \cdot y_1 + (p_2 - \alpha_2) \cdot y_2) \\ y_2 = h_2(p_1, p_2) \cdot (1 + (p_1 - \alpha_1) \cdot y_1 + (p_2 - \alpha_2) \cdot y_2) \end{cases}$$

with $h_1(p_1, p_2) = \frac{p_1^{r-1}}{p_1^r + p_2^r + 1}$, $h_2(p_1, p_2) = \frac{p_2^{r-1}}{p_1^r + p_2^r + 1}$ and $r = \frac{\rho}{\rho - 1}$
This system is equivalent to the following linear system

$$\begin{cases} (1 - h_1(p_1, p_2) \cdot (p_1 - \alpha_1)) \cdot y_1 - (h_1(p_1, p_2) \cdot (p_2 - \alpha_2)) \cdot y_2 = h_1(p_1, p_2) \\ -(h_2(p_1, p_2) \cdot (p_1 - \alpha_1)) \cdot y_1 + (1 - h_2(p_1, p_2) \cdot (p_2 - \alpha_2)) \cdot y_2 = h_2(p_1, p_2) \end{cases}$$

whose determinant is given by :

$$\begin{aligned} \det(p_1, p_2) &= 1 - h_1(p_1, p_2) \cdot (p_1 - \alpha_1) - h_2(p_1, p_2) \cdot (p_2 - \alpha_2) \\ &= \frac{1 + \alpha_1 \cdot p_1^{r-1} + \alpha_2 \cdot p_2^{r-1}}{p_1^r + p_2^r + 1} > 0 \end{aligned}$$

It follows that the objective demand curve is given by :

$$\forall i = 1, 2 \quad y_i(p_1, p_2) = \frac{p_i^{r-1}}{1 + \alpha_1 \cdot p_1^{r-1} + \alpha_2 \cdot p_2^{r-1}}$$

and the profit function of the two firms are defined by :

$$\forall i = 1, 2 \quad \pi_i(p_1, p_2) = \frac{(p_i - \alpha_i) \cdot p_i^{r-1}}{1 + \alpha_1 \cdot p_1^{r-1} + \alpha_2 \cdot p_2^{r-1}}$$

One immediately notices that the set of price strategies is for each agent bounded below by α_i . So let me note by P_i $i = 1, 2$ the set $[\alpha_i, +\infty[$.

Because $\rho \in]0, 1[$ or equivalently $r < 0$, the reader also remarks that

$$\forall i = 1, 2 \quad \forall p_i \in P_i \quad \partial_{p_j} \pi_i(p_1, p_2) = (1 - r) \cdot \frac{\alpha_j \cdot (p_i - \alpha_i) \cdot p_i^{r-1} \cdot p_j^{r-2}}{(1 + \alpha_1 \cdot p_1^{r-1} + \alpha_2 \cdot p_2^{r-1})^2} \geq 0$$

It follows that $\forall i = 1, 2, \quad \forall p_i \in P_i$

$$\sup_{p_j \in P_j} \pi_i(p_1, p_2) = \frac{(p_i - \alpha_i) \cdot p_i^{r-1}}{1 + \alpha_i \cdot p_i^{r-1}} \quad \text{and} \quad \inf_{p_j \in P_j} \pi_i(p_1, p_2) = \frac{(p_i - \alpha_i) \cdot p_i^{r-1}}{1 + \alpha_i \cdot p_i^{r-1} + \alpha_j^r}$$

Because $r < 0$, one can assert that

$$\forall i = 1, 2, \quad \lim_{p_i \rightarrow \alpha_i} \left(\sup_{p_j \in [\alpha_j, +\infty[} \pi_i(p_1, p_2) \right) = \lim_{p_i \rightarrow +\infty} \left(\sup_{p_j \in [\alpha_j, +\infty[} \pi_i(p_1, p_2) \right) = 0$$

One also notices that

$$\forall \tilde{p}_i > \alpha_i \quad \inf_{p_j \in P_j} \pi_i(p_1, p_2) > 0$$

In other words, if p_i^0 is either α_i or $+\infty$, one can assert that $\forall i = 1, 2, \exists \tilde{p}_i \in]\alpha_i, +\infty[$ with the property that :

$$\forall p_j \in [\alpha_j, +\infty[, \quad \lim_{p_i \rightarrow p_i^0} \pi_i(p_i, p_j) < \pi_i(\tilde{p}_i, p_j)$$

Because the strategies “closed to the boundary” are strictly dominated, there exists, in this example, at least an equilibrium in mixed strategies.

6 Conclusion

The principal motivation of this paper was to investigate the circumstances under which there exists a monopolistic general equilibrium in the spirit of **Lange** [14] or **Nikaido** [21]. In this framework, an equilibrium coincides to a situation in which the imperfect competitors choose by profit maximization an optimal price and quantity strategy among the market clearing price and quantity levels. Having in mind that the economies studied in the paper encompass a competitive sector, it became manifest that an approach build on price-making behaviors is the symmetric of the standard Cournot-Walras approach, i.e. an approach in which quantities are set and prices are determined by the market equilibrium conditions. Taking this analogy for granted, my approach of this problem was also splitted into two steps. In a first one, I studied the set of all market clearing prices and quantities in order to construct, a objective demand curve. Using methods closed to those developed in a Cournot-Walras approach, I was able to exhibit several conditions that guarantee the existence of either a deterministic or a random objective demand curve. These results allowed me, in a second step, to investigate the circumstances under which an equilibrium with price-making behaviors exists in pure as well as in mixed strategies.

The results of my investigation with respect to these two steps are the follows. As in Cournot-Walras approach, I first verified that the introduction of a random objective demand curve based on a continuous selection in the set of all market clearing production levels weakens the restrictions required for the existence of an objective demand curve. In fact, its generic existence was obtained by means of an intuitive boundary condition, (i.e. if several production levels go to infinite then at least of one of these markets an excess supply appears) as opposed to the existence of a deterministic objective demand curve the proof of which required restrictions on the shape of the excess demand. In the second step of this study, I analyzed the existence of a monopolistic equilibrium in pure and mixed strategies associated to these two objective demand concepts. The discussion of these four existence results seemed to reinforce the idea that mixed strategies are more appropriated to the study of a monopolistic equilibrium. As long as the introduction of mixed strategies is thought as a proxy of a game with incomplete information, their use seems less costly than the quasi-concavity assumption. In this case the existence of a Nash equilibrium simply requires a dominant strategy argument closed to boundary. As a matter of consequence, if one accepts a random demand concept as representing the some common ex-ante uncertainty on the market clearing production level and one agrees with mixed strategies as representing a proxy of a Nash equilibrium with incomplete information, the main result of this paper is that the existence of a monopolistic equilibrium requires two boundary conditions .

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Appendix

A Proof of proposition 1

In order to prove this proposition let me define $f : \mathbb{R}_{++}^\ell \times \mathbb{R}_+^\ell \longrightarrow \mathbb{R}^\ell$ by $f(p, y) = y - d(p, y)$. In this case one remarks that :

Claim 1 $\forall p \in \mathbb{R}_{++}^\ell$, $\partial_y f(p, y)$ is a positive diagonal dominant matrix.

To obtain this result let me check that

$$\forall h = 1, \dots, \ell \quad \sum_{\substack{k=1 \\ k \neq h}}^{\ell} p_k \cdot \left| \partial_{y_h} z_k(p, y) + \sum_{j=1}^{\ell} \partial_{y_h} x_k^j(p_{-h}, y_h) \right| < p_h \cdot (1 - \partial_{y_h} z_h(p, y))$$

or equivalently, because of assumption A1 (ii), that

$$p \cdot \left(\partial_y z(p, y) + \sum_{j=1}^{\ell} \partial_y x^j(p_{-j}, y_j) \right) \ll p$$

Now note that by Walras Law

$$p \cdot z(p, y) = \lambda(p, y) + p \cdot y - \sum_{j=1}^{\ell} c^j(p_{-j}, y_j)$$

By differentiating this expression with respect to y , one obtains

$$p \cdot \partial_y z(p, y) = \partial_y \lambda(p, y) + p - \sum_{j=1}^{\ell} (p \cdot \partial_y x^j(p_{-j}, y_j) + \partial_y \lambda^j(p_{-j}, y_j))$$

or equivalently that:

$$p \cdot \left(\partial_y z(p, y) + \sum_{j=1}^{\ell} \partial_y x^j(p_{-j}, y_j) \right) = p - \sum_{j=1}^{\ell} \partial_y \lambda^j(p_{-j}, y_{-j}) + \partial_y \lambda(p, y)$$

By assumption A1(ii), one satisfies $\partial_y \lambda(p, y) - \sum_{j=1}^{\ell} \partial_y \lambda^j(p_{-j}, y_{-j}) \ll 0$. Hence

$$p \cdot \left(\partial_y z(p, y) + \sum_{j=1}^{\ell} \partial_y x^j(p_{-j}, y_j) \right) \ll p$$

Claim 2 $\forall \bar{p} \in \mathbb{R}_{++}^\ell$, $\deg(f(\bar{p}, y)) = 1$

Because the degree is invariant by homotopy let me verify that there exists a map $g :$

$\mathbb{R}_+^\ell \rightarrow \mathbb{R}^\ell$ with $\deg(g) = 1$ such that the map $H : \mathbb{R}_{++}^\ell \times [0, 1] \rightarrow \mathbb{R}^\ell$ defined by $H(y, \lambda) = \lambda \cdot f(\bar{p}, y) + (1 - \lambda) \cdot g(y)$ for $\lambda \in [0, 1]$ satisfies $H^{-1}(0) \subset \mathbb{R}_{++}^\ell \times [0, 1]$ and $H^{-1}(0)$ compact.

Let me choose $g(y) = y - q$ with $q \in \mathbb{R}_{++}^\ell$, clearly $\deg(g) = 1$. By continuity of g and f , $H^{-1}(0)$ is closed. Thus, one only needs to verify that this set is also compact. To obtain this result, consider a sequence (y_n, λ_n) such that $\forall n \in \mathbb{N}$, $(y_n, \lambda_n) \in H^{-1}(0)$. Clearly $\lambda_n \rightarrow \lambda_0 \in [0, 1]$, but assume that $y_n \rightarrow \partial\mathbb{R}_{++}^\ell$ and define the sets $I_0 = \left\{ h = 1, \dots, \ell \mid (y_h)_n \xrightarrow{n \rightarrow \infty} 0 \right\}$ and $I_\infty = \left\{ h = 1, \dots, \ell \mid (y_h)_n \xrightarrow{n \rightarrow \infty} +\infty \right\}$.

Suppose first that $I_0 \neq \emptyset$ and note that $\forall \bar{p} \in \mathbb{R}_{++}^\ell$ and $\forall y \in \mathbb{R}_+^\ell$, one verifies that $z_h(\bar{p}, y) > 0$ and $x_h^j(\bar{p}_{-j}, y_j) \geq 0$ for $j = 1, \dots, \ell$. As a matter of consequence, one obtains that $\forall h \in I_0$, $\lim_{y_h \rightarrow 0} f_h(\bar{p}, y) = \lim_{y_h \rightarrow 0} y_h - z_h(\bar{p}, y) - \sum_{j=1}^\ell x_h^j(\bar{p}_{-j}, y_j) < 0$. Hence, by construction of $H(y, \lambda)$, one can assert that $\forall h \in I_0, \exists N \in \mathbb{N}$ such that $\forall n > N$, $H_h(y_n, \lambda_n) < 0$ which contradicts the fact that $\forall n \in \mathbb{N}$, $(y_n, \lambda_n) \in H^{-1}(0)$.

Secondly assume that $I_\infty \neq \emptyset$. By assumption A1(i), $\exists h \in I_\infty, \exists N \in \mathbb{N}$ such that $\forall n > N$, $f_h(\bar{p}, y_n) > 0$. By continuity of $H(y, \lambda)$, one also verifies that $\forall n > N$, $H_h(y_n, \lambda_n) > 0$ which again contradicts the fact that $\forall n \in \mathbb{N}, (y_n, \lambda_n) \in H^{-1}(0)$.

Claim 3 *A C^∞ objective demand curve exists*

Let me define $\mathcal{Y}(p) = \{y \in \mathbb{R}_+^\ell \mid f(p, y) = 0\}$. Because of claim 2, one knows that $\forall p \in \mathbb{R}_{++}^\ell$, $\mathcal{Y}(p) \neq \emptyset$. Now remember that a strictly positive diagonal dominant matrix has a strictly positive determinant (see **Mc Kenzie [13]**). Hence, by claim 1, $\forall p \in \mathbb{R}_{++}^\ell, f \pitchfork \{0\}$. It follows that each $y \in \mathcal{Y}(p)$ is locally unique. Moreover, by the boundary condition A1(i) and by continuity of f , $\mathcal{Y}(p)$ is for each p compact and so finite. Thus, there exists open disjoint neighborhoods $V_i, i = 1, \dots, \#\mathcal{Y}(p)$ around each solution such that $f_i(p, y)$, the restriction of $f(p, y)$ on V_i , is invertible. By **Hirsch [18]**, one knows that $\sum_{i=1}^{\#\mathcal{Y}(p)} \deg(f_i(p, y)) = \deg(f(p, y))$. But $\forall p \in \mathbb{R}_{++}^\ell$ and $\forall y \in \mathbb{R}_+^\ell$, $\det(\partial_y f(p, y)) > 0$, thus $\deg(f_i(p, y)) = 1$ for $i = 1 \dots, \#\mathcal{Y}(p)$. Because $\deg(f(p, y)) = 1$, it follows that $\forall p \in \mathbb{R}_{++}^\ell, \#\mathcal{Y}(p) = 1$. Finally because f is a C^∞ function the same is true for $\mathcal{Y}(p)$.

B Proof of proposition 2

In order to prove this proposition, let me first introduce some definitions. Let $Y = \{Y_1, \dots, Y_{\ell+1}\}$ be a collection of disjoint closed balls in \mathbb{R}_+^ℓ with rational radius and center with rational coordinates. There exists a countable number of such Y . Denote by \mathcal{Y} the

set of all Y . Moreover let (P_n) be a sequence of compact subset converging to \mathbb{R}_{++}^ℓ such that $P_{n-1} \subset P_n$. Let $f(p, y) = y - d(p, y)$ be the excess supply and define F such that:

$$F : P_n \times Y_1 \times \dots \times Y_{\ell+1} \longrightarrow \mathbb{R}^{\ell(\ell+1)}$$

$$(p, y_1, \dots, y_{\ell+1}) \longmapsto \begin{pmatrix} f(p, y_1) \\ \vdots \\ f(p, y_{\ell+1}) \end{pmatrix}$$

Step 1 *Specification of the problem*

For any $\bar{p} \in \mathbb{R}_{++}^\ell$, if $f(\bar{p}, y) = 0$ admit $\ell+1$ or more solutions such that $\text{rank}(\partial_y f(\bar{p}, y)) \leq \ell - 1$, then $F \not\cap \{0\}$

Suppose that there exists $\hat{y}_1, \dots, \hat{y}_{\ell+1}$ such that $\forall i = 1, \dots, \ell + 1$, $f(\bar{p}, \hat{y}_i) = 0$ and $\text{rank}(\partial_y f(\bar{p}, \hat{y}_i)) \leq \ell - 1$. Choose a $Y \in \mathcal{Y}$ such that $\forall i = 1, \dots, \ell + 1$, $\hat{y}_i \in Y_i$ and choose a P_n such that $\bar{p} \in P_n$. Now remark that ∂F is a $[\ell(\ell + 1), \ell + \ell(\ell + 1)]$ matrix defined by $\partial F = [\partial_p F \mid I_{\ell+1} \otimes \partial_y f]$ and evaluate this matrix at $(\bar{p}, \hat{y}_1, \dots, \hat{y}_{\ell+1})$. Because $\text{rank}(\partial_y f(\bar{p}, \hat{y}_i)) \leq \ell - 1$, note that $\text{rank}(\partial F|_{(\bar{p}, \hat{y}_1, \dots, \hat{y}_{\ell+1})}) \leq \ell + (\ell + 1)(\ell - 1) = \ell(\ell + 1) - 1 < \ell(\ell + 1)$. Thus $F \not\cap \{0\}$

Step 2 : *Introduction of a perturbation*

For all $P_n \times Y_1 \times \dots \times Y_{\ell+1}$, there exists a residual set $C_{n,Y} \subset \mathcal{C}_{\ell-1} \times \mathcal{C}_\ell$ such that $F \cap \{0\}$ ($\mathcal{C}_{\ell-1}$ and \mathcal{C}_ℓ are the sets of all cost functions of, respectively, firm $\ell - 1$ and firm ℓ).

Openness follows from continuity of the function determinant and compactness of $P_n \times Y_1 \times \dots \times Y_{\ell+1}$. So I only need to worry about density, that is, about approaching any couple of cost functions $(c^{\ell-1}, c^\ell)$ by an appropriate one (i.e. satisfying $F \cap \{0\}$). In order to achieve this result, let me first introduce some notations. Define by Y' a family of sets such that $\forall i = 1, \dots, \ell + 1$, $Y_i \subset \text{int}(Y'_i)$ and denote by $P'_n \in \mathbb{R}_{++}^\ell$ a set such that $P_n \subset \text{int}(P'_n)$. Let π^j (π^{-j} resp.) be the projection which takes the j -th (all but the j -th resp.) component of any $x \in \mathbb{R}^\ell$ and construct for $j = \ell - 1, \ell$, and $i = 1, \dots, \ell + 1$, a set of $2(\ell + 1)$ C^∞ maps $\alpha_i^j : \mathbb{R}_{++}^\ell \times \mathbb{R}_+^\ell \rightarrow [0, 1]$ such that :

$$\alpha_i^j(p_{-j}, y_j) = \begin{cases} 1 & \text{if } (p_{-j}, y_j) \in \pi^{-j}(P_n) \times \pi^j(Y_i) \\ 0 & \text{if } (p_{-j}, y_j) \notin \pi^{-j}(P'_n) \times \pi^j(Y'_i) \end{cases}$$

Now let me introduce the following perturbed cost functions

$$c_X^\ell(p_{-\ell}, y_\ell) = c^\ell(p_{-\ell}, y_\ell) + \epsilon \sum_{i=1}^{\ell+1} \alpha_i^\ell(p_{-\ell}, y_\ell) \times p_{-\ell} \cdot x_{-\ell}^i$$

$$c_X^{(\ell-1)}(p_{-(\ell-1)}, y_{\ell-1}) = c^{(\ell-1)}(p_{-(\ell-1)}, y_{(\ell-1)}) + \epsilon \sum_{i=1}^{\ell+1} \alpha_i^{(\ell-1)}(p_{-(\ell-1)}, y_{(\ell-1)}) \times p_{\ell} \cdot x_{\ell}^i$$

with $X = (x_{-\ell}^i, x_{\ell}^i)_{i=1}^{\ell+1} \in \mathbb{R}_{++}^{\ell(\ell+1)}$ and $\epsilon \in \mathbb{R}_+$

It is a simple exercise of duality to verify that for ϵ sufficiently small c_X^{ℓ} and $c_X^{(\ell-1)}$ satisfies the standard properties of a cost function. Moreover by taking a ϵ arbitrarily closed to 0, $(c_X^{\ell}, c_X^{(\ell-1)})$ is arbitrarily closed to $(c^{\ell}, c^{(\ell-1)})$. Thus if one verifies that for any $\epsilon > 0$, there exists a pertubation X such that $F_X \pitchfork \{0\}$ the claim is proved.

In order to obtain this result define

$$F_X : P \times Y_1 \times \dots \times Y_{\ell+1} \times \mathbb{R}_{++}^{\ell(\ell+1)} \longrightarrow \mathbb{R}^{\ell(\ell+1)}$$

$$(p, y_1, \dots, y_{\ell+1}, x_1, \dots, x_{\ell+1}) \longmapsto \begin{pmatrix} f_X(p, y_1, x_1) \\ \vdots \\ f_X(p, y_{\ell+1}, x_{\ell+1}) \end{pmatrix}$$

with $f_X(p, y_i, x_i)$ the excess supply associated to the pertubated economy.

Now let me compute the matrix $\partial F_X = [\partial_p F_X \mid I_{\ell+1} \otimes \partial_y f_X \mid \partial_{(x_1, \dots, x_{\ell+1})} F_X]$ of dimension $[\ell(\ell+1), \ell+2\ell(\ell+1)]$. But remember that $\forall i = 1, \dots, \ell+1, \forall y \in Y_i, \forall p \in P_n, c_X^{\ell}(p_{-\ell}, y_{\ell}) = c^{\ell}(p_{-\ell}, y_{\ell}) + \epsilon p_{-\ell} \cdot x_{-\ell}^i$ and $c_X^{(\ell-1)}(p_{-(\ell-1)}, y_{(\ell-1)}) = c^{(\ell-1)}(p_{-(\ell-1)}, y_{(\ell-1)}) + \epsilon p_{\ell} \cdot x_{\ell}^i$. Hence, by Shepard's Lemma,

$$x^{\ell}(p_{-\ell}, y_{\ell}, x^i) + x^{(\ell-1)}(p_{-(\ell-1)}, y_{(\ell-1)}, x^i) = x^{\ell}(p_{-\ell}, y_{\ell}) + x^{(\ell-1)}(p_{-(\ell-1)}, y_{(\ell-1)}) + \epsilon x_i$$

Thus $\forall i = 1, \dots, \ell+1, \partial_{x_i} f(p, y_i, x_i) = I_{\ell}$. As a matter of consequence, $\partial_{(x_1, \dots, x_{\ell+1})} F_X = [I_{\ell+1} \otimes I_{\ell}]$ which is of rank $\ell(\ell+1)$ hence $\text{rank}(\partial F_X) = \ell(\ell+1)$. Thus by the generic transversality theorem, one verifies that for almost all $x \in \mathbb{R}_{++}^{\ell(\ell+1)}$, $F_X \pitchfork \{0\}$.

Step 3 : The main result

Now construct, for each $n \in \mathbb{N}$ and each $Y \subset \mathcal{Y}$, $C_{n,Y}$, the residual set of cost functions for which F restricted on $P_n \times Y_1 \times \dots \times Y_{\ell+1}$ is transversal to $\{0\}$ and compute $\mathcal{C} = \bigcap_{n \in \mathbb{N}} \bigcap_{Y \in \mathcal{Y}} C_{n,Y}$. By Baire's theorem, one knows that the intersection of a countable number of residual sets is also a residual set. Thus by step 1, one can assert that there exists a residual class of cost functions such that the number of critical solution of $f(p_0, y) = 0$ is less then ℓ . It follows, by local uniqueness of the regular solutions of $\mathcal{Y}(p)$ that $\mathcal{Y}(p)$ is at most countable.

C Proof of proposition 3

Claim 1 : $\mathcal{Y}'(p)$ is non-empty

Because $\mathcal{Y}(p)$ is compact and at most countable, it can be covered by finitely many disjoint open balls B_s $s = 1, \dots, S$ such that their closure are nonintersecting. Now define $g_s : \partial B_s \rightarrow S^{\ell-1}$ by $y \mapsto (f(p, y)/\|f(p, y)\|)$ for a given $p \in \mathbb{R}_{++}^\ell$. By **Hirsch [13]** (p 136), one knows that $\deg(g_s) = \text{ind}_y f$. Moreover because $\sum_{f(y)=0} \text{ind}_y = 1$ (by proposition 1), it follows that $\sum_{s=1}^S \deg(g_s) = 1$. Thus there exists, for any p , at least one s_0 such that $\deg\left(f(p, y)/\|f(p, y)\| |_{\partial B_{s_0}}\right) = 1$. If B_{s_0} contains only one point of $\mathcal{Y}(p)$, the result is established. If not cover again $B_{s_0} \cap \mathcal{Y}(p)$ with disjoint open balls and apply the same argument. Repeate this operation as many times as necessary by choising balls $B_{s_0}^n$ with radius strictly decreasing and converging to zero. At least at infinite $\bigcap_{n \in \mathbb{N}} \bar{B}_{s_0}^n$ consists of a single point which belongs to $\mathcal{Y}'(p)$.

Claim 2 : $\mathcal{Y}'(p)$ is lower-hemi-continuous

Take any sequence $p_n \rightarrow p_0$ and let $y_0 \in \mathcal{Y}'(p_0)$. Because $y_0 \in \mathcal{Y}'(p_0)$, there exists an open ball B_ϵ with center y_0 and radius less then ϵ such that one verifies $\deg(f(p_0, y)/\|f(p_0, y)\| |_{\partial B_\epsilon}) \neq 0$. Now remark that, by continuity, for n large enough say n_1 , $f(p_{n_1}, y)$ is homotopic to $f(p_0, y)$ on ∂B_ϵ because $f(p_n, y) \rightarrow f(p_0, y)$. Thus $\deg(f(p_{n_1}, y)/\|f(p_{n_1}, y)\| |_{\partial B_\epsilon}) \neq 0$ and , by the boundary theorem (see **Hirsch [13]** p136), there exists a $y_1 \in B_\epsilon$ such that $f(p_{n_1}, y_1) = 0$. Moreover because y_0 is locally unique one verifies that for any open ball $B_{\epsilon/m}$ with center y_0 and radius ϵ/m , $\deg\left(f(p_0, y)/\|f(p_0, y)\| |_{\partial B_{\epsilon/m}}\right) \neq 0$. Applying the preceding argument to each m , there exists a sequence of $y_m \in \mathcal{Y}'(p_{n_m})$ and $y_m \in B_{\epsilon_m}$. Thus as $m \rightarrow \infty$, p_{n_m} can be choosen such that $p_{n_m} \rightarrow p_0$ and of course $y_m \rightarrow y_0$

D Proof of proposition 4

Define the correspondence $s : \mathbb{R}_{++}^\ell \rightarrow \mathcal{M}(\mathbb{R}_{++}^\ell)$ by

$$s(p) = \left\{ \mu \in \mathcal{M}(\mathbb{R}_{++}^\ell) \mid \mu[y] = 1 \text{ alternatively for each } y \in \mathcal{Y}'(p) \right\}$$

and the correspondence $\mathcal{S} : \mathbb{R}_{++}^\ell \rightarrow \mathcal{M}(\mathbb{R}_{++}^\ell)$ given by $\mathcal{S}(p) = \text{cov}(s(p))$

Clearly $s(p)$ is non-empty because $\mathcal{Y}'(p)$ has the same properties. Now let me verify that s is lower hemi continuous. In other word, let me take a sequence p_n of prices such that

$p_n \rightarrow p_0$ and let choose $\mu_0 \in s(p_0)$. By construction μ_0 puts measure 1 on a $y_0 \in \mathcal{Y}'(p_0)$. But remember that \mathcal{Y}' is lower hemi continuous. It follows that there exists a sequence $y_n \rightarrow y_0$ such that $\forall n \in \mathbb{N}, y_n \in \mathcal{Y}'(p_n)$. By the definition of s , one is able to construct a sequence $\mu_n \in s(p_n)$ with the property that $\forall n \in \mathbb{N}, \mu_n[y_n] = 1$. This sequence converges of course to μ_0 for the weak topology. It follows that s is lower hemi continuous.

Let me now move to the properties of \mathcal{S} . It is immediate that this correspondence as the convex hull of s ; is non-empty and convex value. Moreover because $\mathcal{M}(\mathbb{R}_{++}^\ell)$ is a Hausdorff topological vector space, one knows that the convex hull of a lower hemi continuous correspondence is also lower hemi continuous (see **Hildendrand [12]** proposition 10 p 28).

In order to construct a continuous random objective demand curve ϕ , it remains to apply **Michael's** selection theorem [19] to the correspondence \mathcal{S} in order to extract the continuous selection ϕ .

E Proof of proposition 5

Point (i)

Let $(\alpha_j^n)_{j=1 \dots \ell}^{n \in \mathbb{N}}$ and $(\beta_j^n)_{j=1 \dots \ell}^{n \in \mathbb{N}}$ be $2 \cdot \ell$ sequences of real numbers which satisfy $\forall j = 1, \dots, \ell$ $\alpha_j^0 < \beta_j^0, \alpha_j^{n+1} < \alpha_j^n, \beta_j^{n+1} > \beta_j^n, \alpha_j^n \rightarrow \underline{p}_j$ and $\beta_j^n \rightarrow +\infty$ and let $K_j^n = [\alpha_j^n, \beta_j^n]$. Now let me construct the following sequence of truncated games $\Gamma^n = (K_j^n, \pi_j^n)_{j=1}^\ell$ in which $\pi_j^{d,n} : \prod_{j=1}^\ell K_j^n \rightarrow \mathbb{R}$ is the restriction of $\pi_j^d(p_j, p_{-j})$ to $\prod_{j=1}^\ell K_j^n$.

By assumption A1(ii), one knows that $\pi_j^{d,n}(\bullet, p_{-j})$ is quasi-concave in p_j for all $j = 1, \dots, \ell$. Moreover because $\phi_j(p)$ and $c^j(p_{-j}, y_j)$ are continuous, by a standard argument (see **Debreu [6]**), there exists an equilibrium in pure strategies for each Γ^n . One is therefore able to construct $p^n = ((p_j)_{j=1}^\ell)^n \in \prod_{j=1}^\ell K_j^n$ a sequence of prices with the property that p^n is, for each $n \in \mathbb{N}$, a Nash equilibrium of Γ^n

If this sequence admits a sub-sequence which converges to $p^0 \in \prod_{j=1}^\ell [\underline{p}_j, +\infty[$, then, by continuity of the payoff functions, p^0 is an equilibrium of the non-truncated game. So let me verify that there exists at least a subsequence (p^{n_k}) such that $(p^{n_k}) \rightarrow p^0$. Assume that $p^n \rightarrow p^0 \in \partial \left(\prod_{j=1}^\ell [\underline{p}_j, +\infty[\right)$ and take a j with the property that $p_j^n \rightarrow p_j^0$ with $p_j^0 = \underline{p}_j$ or $p_j^0 = +\infty$. By assumption A(i), one knows that $\exists N \in \mathbb{N}$, such that $\forall n > N$, and $\forall p_{-j} \in A$, $\pi^j(p_j^n, p_{-j}) < \pi^j(\tilde{p}_j, p_{-j})$. This is in particular true for $p_{-j} = p_{-j}^j$. If one chooses N' large enough (i.e. $N' \geq N$ and $\tilde{p}_j \in K_j^{N'}$), one notices that $\forall n > N$, p_j^n cannot be a best response of agent j which contradicts the fact that (p_n) is for all n a sequence of Nash equilibria.

Point (ii)

One simply repeats the same argument with $\pi_j^r(p_j, p_{-j})$ and notices that this function is continuous because the random objective demand curve is continuous for the weak topology.

F Proof of proposition 6**Point (i)**

As in the proof of proposition 5, let me construct the same sequence of truncated games $\Gamma^n = \left(K_j^n, \pi_j^{d,n} \right)_{j=1}^\ell$ where $\pi_j^{d,n} : \prod_{j=1}^\ell K_j^n \rightarrow \mathbb{R}$ is the restriction of $\pi_j^d(p_j, p_{-j})$ to $\prod_{j=1}^\ell K_j^n$. By **Glicksberg's** result [12] one knows that $\Gamma^n = \left(K_j^n, \pi_j^{d,n} \right)_{j=1}^\ell$ admits a mixed strategy equilibrium $\hat{\mu}^n = (\hat{\mu}_j^n)_{j=1}^\ell$ because each K_j^n is non-empty compact set and $\pi_j^{d,n}$ is continuous.

Let me now verify that $\exists N \in \mathbf{N}$ with the property that $\forall n > N$ the support $\text{supp}(\hat{\mu}_j^n)$ belongs to K_j^N . To verify this statement, let me assume the contrary. In other words, let me assume that there exists a j and a sequence p_j^n such that $\forall n \in \mathbf{N}$, $p_j^n \in \text{supp}(\hat{\mu}_j^n)$ and $p_j^n \rightarrow p_j^0$ with either $p_j^0 = \underline{p}_j$ or $p_j^0 = +\infty$. Because $\hat{\mu}_j^n$ is a best response against $\hat{\mu}_{-j}^n$, if $p_j^n \in \text{supp}(\hat{\mu}_j^n)$, one knows that

$$\int \pi_j^d(p_j, p_{-j}) d(\hat{\mu}_{-j}^n \times \hat{\mu}_j^n) = \int \pi_j^d(p_j, p_{-j}) d(\hat{\mu}_{-j}^n \times \delta_{p_j^n})$$

where δ_{p_j} is a Dirac measure. But if $p_j^n \rightarrow p_j^0$, one can assert by the dominant strategy condition A2(i) that :

$$\exists \tilde{p}_j \in]\underline{p}_j, +\infty[, \quad \exists N' \in \mathbf{N}, \quad \forall n > N', \quad \pi_j^d(\tilde{p}_j, p_{-j}) > \pi_j^d(p_j^n, p_{-j})$$

or in other words that :

$$\forall n > N', \quad \int \pi_j^d(\tilde{p}_j, p_{-j}) d(\hat{\mu}_{-j}^n \times \delta_{\tilde{p}_j}) > \int \pi_j^d(p_j^n, p_{-j}) d(\hat{\mu}_{-j}^n \times \delta_{p_j^n})$$

Because $K_n^j \subset K_{n+1}^j$ and $K_n^j \rightarrow]\underline{p}_j, +\infty[$, let me choose N large enough in order to ensure that $N \geq N'$ and that $\tilde{p}_j \in K_j^N$. It follows $\forall n > N$, $\delta_{\tilde{p}_j} \in \mathcal{M}(K_j^N)$ which contradicts the fact that $\forall n > N$, $\hat{\mu}_j^n$ is a best response.

Because of this result, one can consider that the sequence $\hat{\mu}^n = (\hat{\mu}_j^n)_{j=1}^\ell$ of mixed-strategy equilibrium is sequence of measures which belongs to $\mathcal{M}(K_j^N)$. Because the

set of probability measures over a compact set K_j^N is compact under the topology of weak convergence, one can assert by passing to a subsequence if necessary that each $\hat{\mu}_j^n$ converges to $\hat{\mu}_j$. and let $\hat{\mu} = (\hat{\mu}_j)_{j=1}^\ell$. With the same argument, one can assert that $\forall j = 1 \dots \ell, \forall n > N, \int \pi_j^{d,n}(p_j, p_{-j}) d\hat{\mu}^n = \int \pi_j^{d,N}(p_j, p_{-j}) d\hat{\mu}^n$ and because $\pi_j^{d,N}$ is a continuous function restricted on a compact set, it follows that, at least for a subsequence, $\int \pi_j^{d,n}(p_j, p_{-j}) d\hat{\mu}^n \rightarrow \int \pi_j^d(p_j, p_{-j}) d\hat{\mu}$. It remains to verify that $\hat{\mu} = (\hat{\mu}_j)_{j=1}^\ell$ is an equilibrium. By construction of K_j^N one knows that any pure strategy outside this set cannot be a best response against $\hat{\mu}_{-j}$. Moreover $\forall \mu_j \in \mathcal{M}(K_j^N)$,

$$\int \pi_j^d(p_j, p_{-j}) d\hat{\mu} \geq \int \pi_j^{d,N}(p_j, p_{-j}) d\hat{\mu}^N \geq \int \pi_j^{d,N}(p_j, p_{-j}) d(\hat{\mu}_{-j}^N \times \mu_j)$$

Point (ii)

One simply repeats the same argument with $\pi_j^r(p_j, p_{-j})$ and notices that this function is continuous because the random objective demand curve is continuous for the weak topology.