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► To cite this version:

Stephan Tomaszewski, Marcus Nührenbörger. Race among the number line: Functional thinking among primary school students. Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14), Free University of Bozen-Bolzano; ERME, Feb 2025, Bozen-Bolzano, Italy. <hal-05199174>

HAL Id: hal-05199174

<https://hal.science/hal-05199174v1>

Submitted on 4 Aug 2025

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Race among the number line: Functional thinking among primary school students

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In an international context, algebraic thinking is an established subject area in mathematics lessons – even at primary school level. In German primary schools, however, the topic is often addressed unconsciously (Akinwunmi & Steinweg, 2024), which means that a wide range of potentials for stimulating substantive learning processes remain untapped. In the following, a case study will be used to illustrate the thematization of functional thinking in primary school. From an epistemological perspective, it is reconstructed to what extent primary school students demonstrate conceptual understanding with respect to the four function aspects in their engagement with the learning environment ‘race among the number line’. Results show that learners as young as fourth grade can find access to functionally characterized situations and can develop and express ideas.

Keywords: Algebraic thinking, functional thinking, functions, primary school, function aspects.

Introduction

“The true mathematical wealth is created by the perspective of function” (Freudenthal, 1983, p. 511). Freudenthal’s quote shows how important functional thinking is as part of algebraic thinking in mathematics. However, functional thinking is not only important from a mathematical point of view, but is crucial in society itself.

Although the importance of functional thinking is undisputed, it is far less clear what exactly functional thinking means. Smith (2008) describes functional thinking as a type of

[...] representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances. (Smith, 2008, p. 143)

Functional thinking is therefore not limited to dealing with functions, but describes a broader way of thinking. In primary school in particular, it is of course not about a symbolic examination of functions, but about thinking in *functional relationships*.

Functional relationships inevitably shift the focus of thinking away from individual solutions to the relationships themselves. The 'solution' of activities to functional relationships usually involves the description of this very relationship in the form of an equation, a term, a verbal rule. (Steinweg, 2013, p. 199; translated by author)

In the following, it is shown how fourth-graders can focus on such *functional relationships* using a suitable learning environment and how they express their intuitive approaches. To do so requires considerations on functions, notions of functions, and the epistemology of mathematical concepts.

Theoretical Background

A function defines a mathematical relationship (relation) between the elements of two sets (X , Y) (Rudin, 1964). An element of the domain (the independent value) ($x \in X$) is uniquely assigned to an element of the codomain (the dependent value) ($y \in Y$), such that $f: x \mapsto y$. A relation is a function if it is *well-defined* and *total on its domain*. Total on its domain: $\forall x \in X \exists y \in Y : (x, y) \in f$; Well-defined: $\forall x \in X, \forall y, z \in Y : (x, y) \in f \wedge (x, z) \in f \Rightarrow y = z$.

The function types of the linear function, the quadratic function and the exponential function are relevant for this study:

A function f with $f(x) = ax + b$ where $a, b \in \mathbb{R}$ is called a *linear function*.

A function f with $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ is called a *quadratic function*.

A function f with $f(x) = a^x$ where $a \in \mathbb{R}_{>0} \setminus \{1\}$ is called an *exponential function*.

Understanding functions, however, is not limited to knowing their definitions and formal notations; rather, it involves the active engagement with functions — functional thinking — which constitutes the conceptual core of the concept of function (Warren & Cooper, 2005). Functional thinking is thus not tied to a formal concept of function and can already occur in primary school (ibid.), by "focusing on specific patterns that provide initial access to functions and their structures" (Akinwunmi & Steinweg, 2024, p. 306; translated by author).

Characteristic of this functional thinking are the so-called *function aspects*. In the international context, the four aspects of *input-output*, *covariation*, *correspondence*, and *mathematical object* have become established (Pittalis et al., 2020).

The *input-output* aspect of function is understood as a straightforward computation according to a predefined rule, which does not require the learner to understand the causal relationship between the input and the output. It is often perceived by students merely as a calculative operation, where the insertion of one variable (the input) leads directly to another variable (the output) (Pittalis et al., 2024).

The *covariational* aspect involves analyzing, manipulating, and understanding the relationships between at least two simultaneously changing quantities. This aspect emphasizes the conceptualization of how two or more quantities vary together, aiming to coordinate the changes in the values of these variables (ibid.).

The *correspondence* aspect, frequently employed to formally introduce functions at school, highlights understanding the structural conceptualization of functions. It focuses on grasping the relationship between independent and dependent variables (ibid.).

The *mathematical object* aspect views a function as a mathematical entity that can be depicted through various representations, including arrow chains, tables, graphs, formulas, and verbal descriptions. Each representation offers a unique perspective on the same mathematical object (ibid.).

According to Sfard (1991), there is a gradual shift from a process-oriented perspective, which emphasizes the input-output aspect, to a more structural viewpoint focusing on the object aspect. However, it is important to note that these different perspectives are often intertwined, posing challenges for teaching and learning (Pittalis et al., 2024). In primary school, students

will primarily engage with functions through input-output, covariation, and correspondence approaches.

Keeping these four function-aspects in mind, the research interest for the following case study is: *To what extent do primary school students demonstrate conceptual understanding with respect to the four function-aspects in their engagement with the learning environment 'Race among the number line'?*

Setting & Methodology

The learning environment 'Race among the number line' (Akinwunmi & Lang, 2017) allows learners in primary school to playfully gather initial experiences in handling functions. The basic task is as follows (Fig. 1):

A race is played on the number line with four game pieces. Each game piece moves forward according to specific rules. The game piece that first reaches the number 1000 wins.

- The red game piece is allowed to move 50 steps in each round.
- The blue game piece moves 10 steps in the first round. In each subsequent round, it moves 10 steps more than in the previous round.
- The green game piece moves one step in the first round. In each subsequent round, it moves twice as many steps as in the previous round.
- The yellow game piece moves 100 steps in the first round. In each subsequent round, it moves 10 steps less than in the previous round.

Figure 1: Basic task and specific movement rules

The focus of the task lies on the different movement rules for the game pieces, or the underlying concept of slope. Particularly, the function aspects of *covariation* and *correspondence* are relevant in this context.

The mappings between the number of rounds and the position on the number line represent different types of functions for each game piece (Table 1):

Table 1: Overview of the game pieces and the underlying functions

Red game piece	linear function	$f(x) = 50 \cdot x ; x \in \mathbb{N}$
Blue game piece	quadratic function	$g(x) = 5 \cdot x^2 + 5 \cdot x ; x \in \mathbb{N}$
Green game piece	exponential function	$h(x) = 2^x - 1 ; x \in \mathbb{N}_{>0}$
Yellow game piece	quadratic function	$i(x) = -5 \cdot x^2 + 105 \cdot x ; x \in \mathbb{N}$

The learning environment was conducted with three pairs of learners from a fourth-grade class (ages 9-10 years) at a German primary school, chosen by their teacher. As part of the survey, which lasted 3 times 60 minutes, the pairs were taken out of their regular lessons. The following will examine the work of the learner pair J. and T. They were described by the teacher as “rather high-performing”. The actual survey addressed various follow-up tasks. The analysis only considers individual snapshots.

The sequences were videotaped and transcribed. For analysis, the *epistemological triangle* according to Steinbring (2005) was used. This involves reconstructing the understanding of mathematical *concepts* (e.g., linear functions) through the learners negotiation between specific *signs* (e.g., representations, statements) and *reference contexts* (conceptual reference points of the learners). New mathematical knowledge is ultimately understood as an extension of old knowledge through new, rich relationships, in that possible meanings are transferred from a relatively familiar, or partially familiar *reference context* to a new, as yet meaningless system of *signs* (ibid.). Content knowledge is constructed in interaction. Following this idea, negotiation processes and conceptions of the learners can be reconstructed by understanding expressions, representations, or notations as signs to be interpreted. For instance, conceptions of covariation can be elaborated through the interpretations made by the learners such as when a learner says, *it rises faster and faster*, indicating their perception of the rate of change, or *if this one speeds up, then the other value also rises*, demonstrating their understanding of the relationship between variables in a dynamic system. In this sense, the students processes of constructing new mathematical knowledge in the interaction are reconstructed with the help of the epistemological triangle.

Analysis & Selected Results

After the task was presented to the learners, they were initially asked to write down their first prediction about which game piece would win the race. Both agreed that the red game piece would win (J.: "I believe the red game piece will win because it runs 50 steps in every round."; translated by author) (T.: "I think the red game piece will go further because the yellow game piece runs 10 steps less each time."; translated by author).

Since human thinking and most everyday situations are based on linear processes, exponential growth is a phenomenon that is unfamiliar, especially for primary school students. Consequently, initial guesses are predictably as expected. The difference between repeated addition and repeated multiplication is likely not perceived by J. and T., and certainly not considered decisive. This is evident from the fact that at no point (before the start of the game) was there a discussion about whether the green game piece could win. The small number of "one step in the first round" seems too insignificant compared to the fifty steps of the red game piece. Instead, considerations are made about whether it is better to start with a high number and slow down (yellow game piece), or to run with a consistent stride length (red game piece).

If we consider the red set piece, the following epistemological triangle can be modeled (Fig. 2):

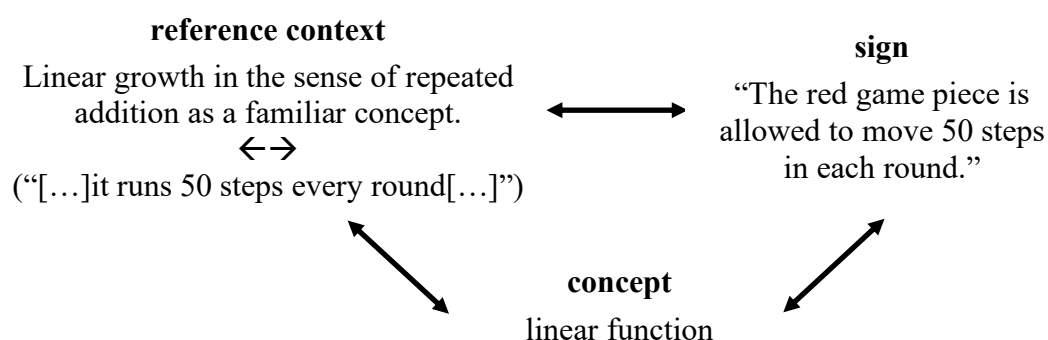


Figure 2: Mediation between specific rule and conceptual reference point for the red game piece

The specified moving pattern for the red game piece (“The red game piece is allowed to move 50 steps in each round.”) serves here as the sign that the learners need to interpret. The direct reference to the number of steps per round of the game piece (“[...]it runs 50 steps every round[...]”) suggests that J. and T. relate the number of steps to the number of rounds. The emphasis on the phrase *every round* indicates that they recognize the constancy of the slope and identify it as important. Even as their mathematical justifications remain at a more narrative level (see Krummheuer, 2013), they demonstrate conceptual considerations regarding the *correspondence* and, to some extent, the *covariational* aspect in terms of continued addition.

After the learners had noted down their predictions, they were asked to check which game piece would win. J. and T. decided to note the positions of the game pieces step by step for each round and to present them in a table (Fig. 3):

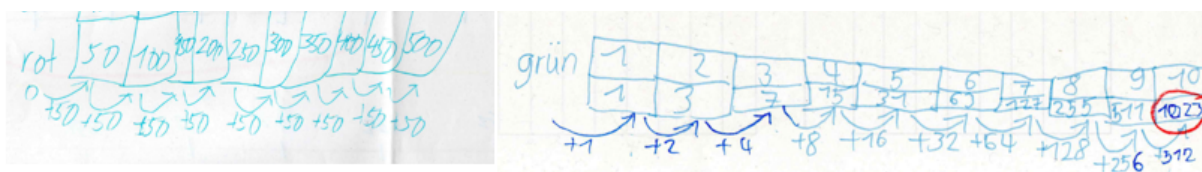


Figure 3: Tabular representation of the game pieces and their position (left: red; right: green)

This leads to the following situation, while J. calculates the position that the green game piece will reach in the 6th round:

- J. [...] and 31 plus 32 are (...) are 63 (*notes result*)
 Int. Mmh.
 J. (...) And doubling thirty-two is (...) well, if I think about it, I really don't know if, if (...) green will make it but (...) well, it could happen that it, it (unintelligible) (...) it will probably happen that, that green takes over a hundred steps.

This passage clearly shows that although J. initially approaches the problem purely arithmetically by calculating the concrete numerical values (“[...]and 31 plus 32 are (...) are 63”), he simultaneously detaches himself from these concrete values (“[...]well, if I think about it[...]”). He is already considering the increase in subsequent rounds without having specifically calculated it (“[...]it will probably happen[...]”) and realizes that it grows much faster than initially suspected. This is a clear indication of thinking in *covariation*. After calculating a few more positions of the green figure, they confirm that the green game piece will in fact reach 1000 first. The green game piece not behaving according to their expectations leads to a strong urge to justify. The interviewer asks:

- Int. Why does the green game piece win now, how can that be?
 J. Because (...) eh, because it keeps increasing, if you're already at a hundred#
 T. # it always doubles. And (...) uh and then, you are, you are here (*points to the table*), then you are, then it keeps increasing and if#
 J. # you must not underestimate the smallest one!
 Int. (*laughs*) you must not underestimate the smallest one?
 J. One must not underestimate that if the smallest one, if one starts running very slowly, and then, and then does, then does, then suddenly goes into turbo mode (*laughs*).
 Int. What do you mean by he suddenly goes into turbo mode?
 J. Because, um (...) from one, you think that, you think you're not walking that far, but it doubles. From two to four, from four to eight, from eight to sixteen.
 T. And then suddenly the steps increase very quickly.

In their explanations, they repeatedly refer to "doubling" and "increasing" (*verb*), but the verbalization makes it clear how difficult this is, since the learners naturally lack the algebraic terminology like *slope* or *increase (noun)*.

At the same time, the choice of words ("[...]it keeps increasing[...]") reveals that the focus is on the change in values (*covariation*), not on the specific values itself. Again they use narrative phrases like "you must not underestimate the smallest one!" or "[...]suddenly goes into turbo mode[...]" to bridge the gap between their available linguistic means and the conceptual ideas they wish to express.

Those phrases, as well as the emphasis on the word "doubling" in their statements, suggest that they see more than just the movement rule of the game piece in this term. They likely see the doubling (or the continued multiplication by two) as the reason for the rapidly increasing growth. This assumption is supported by phrases such as "then the steps increase very quickly", or by listing examples ("From two to four, from four to eight, from eight to sixteen") that simultaneously have a generalizing character. It is apparent that the steeper slope of repeated multiplication compared to repeated addition is the conceptual core of their argument (function aspect of *covariation*).

If we consider the green set piece, the following epistemological triangle can be modeled (Fig. 4):

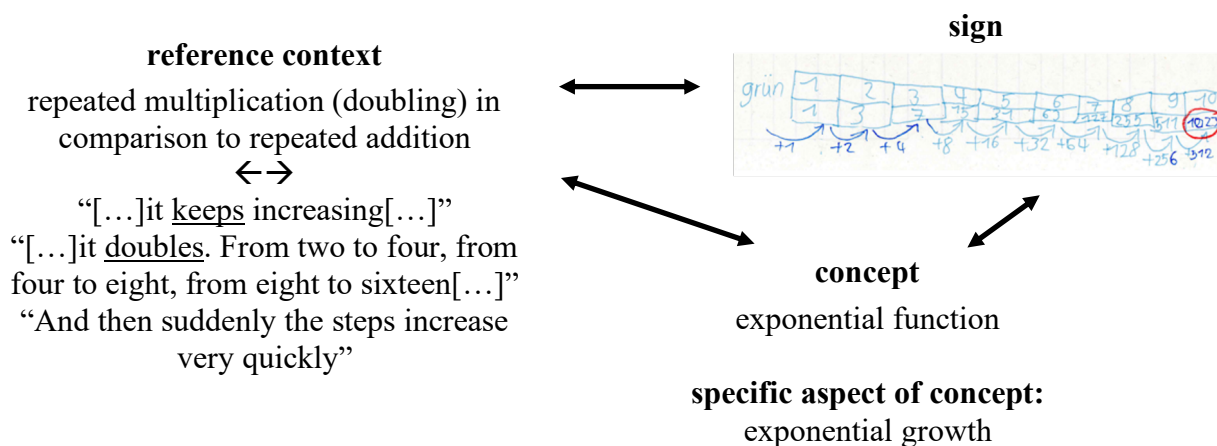


Figure 4: Mediating the irritation by referencing the differences between repeated multiplication and repeated addition

To test these considerations, J. & T. are confronted with a new game piece: this game piece moves 100 steps in the first round. In each subsequent round, it moves 50 steps more than in the previous round ($j(x) = 25 \cdot x^2 + 75 \cdot x$; $x \in \mathbb{N}$). The learners determine some positions of this game piece and quickly recognize that it will reach the 1000 faster than the green game piece. Then the interviewer asks whether this new game piece will also win if the game is played indefinitely, or whether the green figure will catch up again.

- J. But it is now questionable (...) uh (...) so in ten rounds, the new figure would win, but if you continue running, then green would overtake again.
- Int. Why is that?
[...]
- J. Because, because then (*stands up from the chair*), because when you double, steps are, very, very often, are very, when you then, eventually (...) there

(points towards the table of the green figure) you were already at five hundred twelve, and we had calculated that up to here *(points to round 11 of the green figure)*, there (...) two thousand forty-seven, that would be, that *(points to round 10 of the 'new figure')* is indeed more, but here *(points again at the table of the green figure)* the steps are then doubled, but since the steps there are doubled and there *(points towards the table of the 'new figure')* only, there only fifty is added.

J.'s explanations very aptly underscore the considerations previously worked out: the arithmetic values provide the starting point for further thinking processes that, in the sense of functional thinking, detach from the concrete numbers. In terms of covariation, J. recognizes the difference between repeated addition and repeated multiplication and realizes that in the long run, the latter must always grow faster. His linguistically halting and simultaneously gesturally supported explanations once again show the attempt to express this complex relation with limited linguistic means.

Conclusion and Outlook

By adopting the epistemological approach, insights into students' processing and conceptualization of mathematical functions were gained, confirming the notion that fundamental algebraic concepts, typically introduced later, can be understood by young learners through contextually driven, discursively rich learning environments.

Considering the main research question of this paper "*To what extent do primary school students demonstrate conceptual understanding with respect to the four function aspects in their engagement with the learning environment 'race among the number line'?*", it is evident that fourth graders were able to make conceptually significant experiences with functionally defined situations within the provided learning environment. These experiences particularly allowed the students to express concepts of *covariation* and *input-output*. Although the learners lack the explicit technical vocabulary regarding functions, they are able to express their ideas using the linguistic resources available to them. Similarly, there were moments of generalization observed, indicating a shift from purely arithmetic considerations. The results also emphasize the importance of language (including gestures, objects, drawings, graphs, tables, etc.) for mathematical learning processes (e.g., Moschkovich, 2015), as well as the intertwined use of *mathematical language registers* and *everyday language registers* (e.g., "goes into turbo mode").

For future research, it might be interesting to continue focusing on the approaches and linguistic negotiations of primary school students, as well as to observe student teachers dealing with school-related algebraic tasks. It would be relevant not only to examine which subject-specific concepts the student teachers have developed themselves, but also how they pedagogically assess corresponding tasks for primary school, especially those that provide suitable interfaces for algebraic thinking in primary school.

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