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A Novel Approach to Cost Estimation in Lot Sizing with Uncertain Lead Times ^{*}

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Abstract: The lot sizing problem involves meeting the demand of the clients of a given retailer by placing orders with different suppliers. Decision-making models were developed to minimize the retailer’s total cost while considering their retailers’ different characteristics, such as their capacities, prices, and lead times. However, when integrating uncertainty into the environment, this challenging problem becomes even more complex, and the exact methods developed need extremely long computational time to go over all the possible scenarios. This paper proposed a novel formulation to minimize the operations needed when finding the optimal solution to the lot sizing problem under uncertain lead times.

Keywords: Lot-sizing problem, uncertainty, lead-time, agriculture, short food supply chain.

1. INTRODUCTION AND RELATED PUBLICATIONS

In the field of operational research, the lot-sizing problem is a challenging task that requires the retailer to plan the orders placed with different suppliers in order to satisfy the demands of their clients. The retailer has to take into account the different parameters of the suppliers when making these decisions. In addition to the demand of the clients, the production capacities of their suppliers, the lead time of their deliveries, and their pricing, multiple other factors should be taken into account (Dolgui et al., 2010; Akbalik et al., 2012). In a usual setting, the retailer would receive in advance the demands of their clients. The retailer would then need to inspect the production capacities of their suppliers, their pricing, and the time it takes for the order to be delivered after it has been placed. Usually, the production capacity, the price, and the lead time would vary from one supplier to the other. As the retailer is choosing which supplier to buy from, they would indicate which quantity is to be ordered from each supplier at a given time period. The retailer would then need to manage the inventory level and backlogging level of their products (Brahimi et al., 2017). In addition, there is uncertainty about the different aspects of this environment. Different factors can affect client demand (Rodado et al., 2017), supplier pricing (Andriolo et al., 2015), and delivery time (jun Shao and Ke (2006)). Incorporating this uncertainty into the given problem further increases its complexity (Akartunali and Dautère-Pères (2022)).

As this work is part of the **SMALLDERS** project, the main objective is to assist the different actors of the short agri-food supply chain (Longo et al., 2023; Miranda et al., 2023;

Chabouh et al., 2024; Germanos et al., 2024b; Galaverni et al., 2025). One partner of this work is “Le Mas des Agriculteurs”, a French retailer that supplies fresh produce to institutions such as hospitals, schools, universities, and smaller clients. They buy the products from the smallholders, i.e., small-scale farmers, of the Gard region in France. One challenge that this retailer faces is the small and varying production capacities of the smallholders. Therefore, “Le Mas des Agriculteurs” has to plan months in advance the orders placed to satisfy the demand. Moreover, they need to manage an inventory and preemptively store additional product quantities to ensure the satisfaction of the demand whenever the production capacity of their suppliers drops. Finally, the lead time of the smallholders is not certain. However, by working with the smallholders for multiple years, “Le Mas des Agriculteurs” can estimate the probability distribution of the lead time of each one of them. Through this collaboration, we model “Le Mas des Agriculteurs” as the retailer in the lot sizing problem, and the smallholders as the suppliers. In Germanos et al. (2024a, 2023), we developed different preliminary models to tackle this challenge. However, the lead times in those methods were assumed to be certain. In this work, we present a mathematical model that accounts for lead time uncertainty.

Different models were developed to assist in the retailers in this complex task such as the work of Zhu et al. (2022) where the authors developed a hybrid chance-constrained programming model to address material requirements planning in smart factories by managing uncertainties. Their work combined randomness and fuzziness whenever making lot-size decisions. The work of Ben-Ammar et al. (2024) addressed multi-period replenishment planning with dynamic demand and multiple suppliers by proposing linear models to optimize order quantities,

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minimize total costs, and evaluate the benefits of splitting orders and flexibility, supported by numerical experiments. Ali et al. (2021) developed a multi-objective mixed-integer non-linear model using fuzzy programming to optimize the supply chain for perishable milk products under uncertain conditions, aiming to reduce transportation costs, minimize wastage, and address storage inadequacies. Slama et al. (2021) proposed three optimization methods for the stochastic multi-period disassembly lot-sizing problem, using Monte Carlo simulation and genetic algorithms to minimize total costs in a two-level disassembly system with random lead times, demonstrating their effectiveness in supporting replenishment and disassembly planning. Wang et al. (2020) utilized simulation modeling with response surface methodology to optimize order planning in the construction supply chain, identifying optimal settings for review period, order quantity, and safety stock to minimize costs and maximize service levels. Hu and Hu (2020) combined robust optimization and Sample Average Approximation to tackle lot-sizing and scheduling under demand and overtime cost uncertainties. Stadler and Heinrichs (2024) used Multi-Period Descriptive Sampling to generate a small set of representative scenarios that captured the period-specific and cumulative probability distributions of stochastic variables, addressing the trade-off between model accuracy and computational complexity. Additionally, a fine-tuning heuristic was proposed to enhance the performance of scenario-based models by iteratively adjusting production quantities to meet service level constraints.

This paper proposes a novel mathematical formulation of the lot sizing problem under lead time uncertainty with the objective of minimizing the total cost. In the following parts, Section 2 presents the description of the problem, Section 3 lists the proposed mathematical approach, to finally conclude in Section 4.

2. PROBLEM DESCRIPTION

2.1 Problem statement and assumptions

In this work, we investigate the single-item, multi-period, multi-supplier lot sizing problem with known and dynamic demand under uncertainty of lead time. In this formulation, “Le mas des Agriculteurs” is the retailer, and the smallholders are the suppliers. Each of the suppliers has their own independent capacities and prices, as well as an uncertain lead time with known probability distribution. This environment includes a dynamic unit holding cost, a dynamic unit backloging cost, a unit price that varies between the suppliers. Each smallholder has their own delays in delivery with a known probability distribution. The lead-times are independent, discrete random variables with known and bounded probability distributions.

The list of notations used to model the problem is presented in Table 1. The objective is to meet the demand of the clients while minimizing the ordering, holding, and backloging cost. The following definitions will be needed to calculate the expression of the total cost and its expected value.

Definition 2.1. Let \mathbf{Q} be the matrix grouping the decision variables Q_t^s of the non-null ordered quantities:

Table 1. Notations.

Parameters	
\mathcal{T}	planning time periods
\mathcal{S}	set of smallholders
s	index of smallholders, $s \in \mathcal{S}$
t	index of the period, $t \in \mathcal{T}$
d_t	monthly known demand
p_s	unit selling price of smallholder s
h_t	unit inventory holding cost for time period t
b_t	unit inventory backloging cost for time period t
L_-^s	minimum lead time of supplier s
L_+^s	maximum lead time of supplier s
L_t^s	real lead time of the order Q_t^s placed with supplier s at time period t , ranging between $[L_-^s, L_+^s]$
l_t^s	planned lead time of the order Q_t^s placed with supplier s at time period t
Functions	
$\mathbb{1}_A$	indicator function, equals 1 if event A is true; 0 otherwise
$\mathbb{P}(\cdot)$	Probability value
$\mathbb{E}(\cdot)$	Expected value
$\mathcal{P}(\cdot)$	Power set
$[A]^+$	maximum between A and 0
Variables	
Q_t^s	integer decision variable that gives the total quantity to order from smallholder s at period t
$I_t(\mathbf{Q})$	inventory level at the end of period t
$I_t^+(\mathbf{Q})$	inventory to carry at the end of period t . It is equal to $I_t(\mathbf{Q})$ if $I_t(\mathbf{Q}) \geq 0$ and zero otherwise
$I_t^-(\mathbf{Q})$	backloging level at the end of period t . It is a positive integer, equal to $-I_t(\mathbf{Q})$ if $I_t(\mathbf{Q}) \leq 0$ and zero otherwise

$$\mathbf{Q} := \{Q_t^s \in \mathbb{N}^* : (s, t) \in \mathcal{S} \times \mathcal{T}\}$$

Definition 2.2. Let $\mathcal{N}(\mathbf{Q})$ be the set of pair indices (supplier, time period) that groups the indices (s, t) of the decision variables Q_t^s of the non-null ordered quantities:

$$\mathcal{N}(\mathbf{Q}) := \{(s, t) \in \mathcal{S} \times \mathcal{T} : Q_t^s \in \mathbf{Q}\}$$

Definition 2.3. let $\mathcal{N}_t(\mathbf{Q})$ be the set of pair indices (supplier, time period) grouping the indices (s, τ) of non-null orders that are sure to be delivered before the end of time period t :

$$\mathcal{N}_t(\mathbf{Q}) := \{(s, \tau) \in \mathcal{S} \times \mathcal{T} : \tau + L_+^s \leq t \wedge (s, \tau) \in \mathcal{N}(\mathbf{Q})\}$$

The set $\mathcal{N}_t(\mathbf{Q})$ includes a subset of indices of time periods in which orders were placed at time τ such that $\tau + L_+^s \leq t$. This condition ensures that all the time periods of the orders in this set have been delivered to the retailers, as it accounts for the worst case of lead time delay L_+^s .

Definition 2.4. let $\mathcal{M}_t(\mathbf{Q})$ be the set of pair indices (supplier, period) that groups the indices (s, τ) of orders placed with supplier s that might be delivered before the end of time period t .

$$\mathcal{M}_t(\mathbf{Q}) := \{(s, \tau) \in \mathcal{S} \times \mathcal{T} : L_-^s \leq t - \tau \leq L_+^s - 1 \wedge (s, \tau) \in \mathcal{N}(\mathbf{Q})\}$$

The set $\mathcal{M}_t(\mathbf{Q})$ accounts for all the orders that cause the uncertainty of arrivals on a given period t . This is because t falls within the range defined by the shortest possible lead time L_-^s , and the longest possible one L_+^s for an order placed at time τ .

Remark 2.5. The subsets $\mathcal{N}_t(\mathbf{Q})$ and $\mathcal{M}_t(\mathbf{Q})$ do not overlap:

$$\mathcal{N}_t(\mathbf{Q}) \cap \mathcal{M}_t(\mathbf{Q}) = \emptyset$$

The set $\mathcal{N}_t(\mathbf{Q})$ does not account for any orders that cause uncertainty in the inventory level at t , and filters out τ with the following constraint $\tau \leq t - L_+^s$. Whereas the set $\mathcal{M}_t(\mathbf{Q})$ only accounts for orders that are uncertain to arrive before the end of time period t and filters out τ with the following constraint $L_-^s \leq t - \tau \leq L_+^s - 1$. Therefore, the smallest possible value of τ in the set $\mathcal{M}_t(\mathbf{Q})$ is $\tau = t - L_+^s + 1$, while the largest possible value for τ in $\mathcal{N}_t(\mathbf{Q})$ is $\tau = t - L_+^s$. Hence, these two sets will not overlap.

3. MATHEMATICAL MODEL

3.1 Total cost calculation

The objective is to determine the quantities to order from each supplier at each period to meet client demand and minimize costs. The general expression of the total cost in a deterministic environment is given in equation (1):

$$\text{TC}(\mathbf{Q}) = \sum_{t \in \mathcal{T}} \left(h_t I_t^+(\mathbf{Q}) + b_t I_t^-(\mathbf{Q}) + \sum_{s \in \mathcal{S}} p_s \cdot Q_t^s \right) \quad (1)$$

where

$$I_t^+(\mathbf{Q}) = \left[\sum_{\tau=1}^t A_\tau(\mathbf{Q}) - \sum_{i=\tau}^t d_\tau(\mathbf{Q}) \right]^+ \quad \forall t \in \mathcal{T} \quad (2)$$

$$I_t^-(\mathbf{Q}) = I_t^+(\mathbf{Q}) - \sum_{\tau=1}^t A_\tau(\mathbf{Q}) + \sum_{\tau=1}^t d_\tau \quad \forall t \in \mathcal{T} \quad (3)$$

$$\sum_{\tau=1}^t A_\tau(\mathbf{Q}) = \sum_{\tau=1}^{t-L_-^s} \sum_{s \in \mathcal{S}} Q_\tau^s \quad \forall t \in \mathcal{T} \quad (4)$$

However, given the lead-time uncertainty, the total cost value cannot be accurately computed when deciding which orders to place preemptively. Due to the uncertainty in lead time, the period in which an order is received is uncertain. This uncertainty translates into a variation in the arrival quantities $A_t(\mathbf{Q})$ and the inventory levels $I_t^+(\mathbf{Q})$ and $I_t^-(\mathbf{Q})$ in each period. Therefore, to calculate the total cost under lead time uncertainty, Proposition 3.1 and Proposition 3.2 are introduced to calculate the arrival quantities, the inventory level, and inventory to carry over at time period t , respectively. Whereas Proposition 3.6, Proposition 3.7, and Proposition 3.8 are used to calculate the estimated arrival quantity, the inventory to carry, which allows us to extrapolate the backlogging level at time period t respectively.

Proposition 3.1. *The total received quantities from period $\tau = 1$ till period t under uncertainty is defined as follows:*

$$\sum_{\tau=1}^t A_\tau(\mathbf{Q}) = \sum_{(s,\tau) \in \mathcal{N}_t(\mathbf{Q})} Q_\tau^s + \sum_{(s,\tau) \in \mathcal{M}_t(\mathbf{Q})} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}} \quad (5)$$

Proof. In an uncertain environment, the total received quantity from time $\tau = 1$ till time t depends on the suppliers' lead time. An indicator function filters out quantities that could not be received by the end of time period t to account for this uncertainty. By using equation (4), we obtain:

$$\sum_{\tau=1}^t A_\tau(\mathbf{Q}) = \sum_{\tau=1}^t \sum_{i=1}^{\tau} \sum_{s \in \mathcal{S}} Q_i^s \mathbf{1}_{\{i + L_i^s = t\}} \quad (6)$$

The sum $\sum_{i=1}^{\tau} \sum_{s \in \mathcal{S}} Q_i^s \mathbf{1}_{\{i + L_i^s = t\}}$ in equation (6) contains all the order quantities that were ordered between time period $i = 1$ till τ and were received at time t . this summation can be simplified to the element $\sum_{s \in \mathcal{S}} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}}$. then, expression (6) can be written as follows:

$$\sum_{\tau=1}^t A_\tau(\mathbf{Q}) = \sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}} \quad (7)$$

The elements in the sum $\sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}}$ can be categorized into two groups. The first group consists of orders that are guaranteed to be received by the end of time t . For these orders, even when taking into consideration the longest lead time of L_+^s , the orders would still be delivered by the end of time period t , as the time period where they were ordered τ is more than L_+^s time periods away from t . These orders were placed in the time period τ where $\tau \leq t - L_+^s$. As these elements are sure to be delivered before t , the indicator function is not needed to filter any elements out. The second group comprises the elements that are unsure to be received by the end of time period t . These orders are placed at time period τ and might be received before t , at t , or after t . The possible range of τ is $t - L_+^s + 1 \leq \tau \leq t - L_-^s$. The uncertainty in the reception time of these elements necessitates the indicator function to filter out the elements. The division of $\sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}}$ from equation (7) into these two groups is shown in equation (8):

$$\sum_{\tau=1}^t A_\tau(\mathbf{Q}) = \sum_{\tau=1}^{t-L_+^s} \sum_{s \in \mathcal{S}} Q_\tau^s + \sum_{\tau=t-L_+^s+1}^{t-L_-^s} \sum_{s \in \mathcal{S}} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}} \quad (8)$$

Looking at the first right-hand side elements of equation (8), the time periods τ that will be considered in the first summation $\sum_{\tau=1}^{t-L_+^s} \sum_{s \in \mathcal{S}} Q_\tau^s$ are the same ones that would be found in the set $\mathcal{N}_t(\mathbf{Q})$ as detailed in Definition 2.3. Whereas, the time periods τ that will be considered in the second summation $\sum_{\tau=t-L_+^s+1}^{t-L_-^s} \sum_{s \in \mathcal{S}} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}}$ are the same ones that would be found in the set $\mathcal{M}_t(\mathbf{Q})$ as detailed in Definition 2.4. Therefore, equation (5) can be easily deduced. \square

Proposition 3.2. *The positive inventory level $I_t^+(\mathbf{Q})$ at time period t , is given by the following expression:*

$$I_t^+(\mathbf{Q}) = \left[\sum_{(s,\tau) \in \mathcal{N}_t(\mathbf{Q})} Q_\tau^s + \sum_{(s,\tau) \in \mathcal{M}_t(\mathbf{Q})} Q_\tau^s \mathbf{1}_{\{\tau + L_+^s \leq t\}} - \sum_{\tau=1}^t d_\tau \right]^+ \quad (9)$$

Proof. Taking into account the uncertainty of lead time, the received quantities will vary. Therefore, by replacing $\sum_{\tau=1}^t A_\tau$ in equation (2) with its equivalent shown in equation (5), expression (9) can be easily deduced. \square

By using expressions (1-4), and Propositions 3.1 and 3.2, the general expression of the total cost under uncertainty of lead times is deduced:

$$\text{TC}(\mathbf{Q}) = \sum_{t \in \mathcal{T}} \left(h_t I_t^+(\mathbf{Q}) + b_t I_t^-(\mathbf{Q}) + \sum_{s \in \mathcal{S}} p_s \cdot Q_t^s \right) \quad (10)$$

where

$$I_t^+(\mathbf{Q}) = \left[\sum_{\substack{(s,\tau) \\ \in \mathcal{N}_t(\mathbf{Q})}} Q_\tau^s + \sum_{\substack{(s,\tau) \\ \in \mathcal{M}_t(\mathbf{Q})}} Q_\tau^s \mathbf{1}_{\{\tau + L_\tau^s \leq t\}} \quad \forall t \in \mathcal{T} \right. \\ \left. - \sum_{\tau=1}^t d_\tau \right]^+ \quad (11)$$

$$I_t^-(\mathbf{Q}) = I_t^+(\mathbf{Q}) - \sum_{\tau=1}^t A_\tau(\mathbf{Q}) + \sum_{\tau=1}^t d_\tau \quad \forall t \in \mathcal{T} \quad (12)$$

$$\sum_{\tau=1}^t A_\tau(\mathbf{Q}) = \sum_{(s,\tau) \in \mathcal{N}_t(\mathbf{Q})} Q_\tau^s \quad \forall t \in \mathcal{T} \\ + \sum_{(s,\tau) \in \mathcal{M}_t(\mathbf{Q})} Q_\tau^s \mathbf{1}_{\{\tau + L_\tau^s \leq t\}} \quad (13)$$

As the total cost is a finite random variable with a finite possible values, it is possible to compute its expected value. To evaluate the expected value of $\text{TC}(\mathbf{Q})$, denoted by $\mathbb{E}(\text{TC}(\mathbf{Q}))$, it is important to determine $\mathbb{E}(I_t^+(\mathbf{Q}))$ and $\mathbb{E}(A_t(\mathbf{Q}))$ first.

3.2 Expected total cost calculation

In the field of set theory, a power set $\mathcal{P}(\mathcal{X})$ of a set \mathcal{X} is defined as all the possible subsets of \mathcal{X} , including the empty set $\{\emptyset\}$:

$$\mathcal{P}(\mathcal{X}) = \{A : A \subseteq \mathcal{X}\}$$

Definition 3.3. We define $\mathcal{P}_k(\mathcal{X})$ as a subset of $\mathcal{P}(\mathcal{X})$ that includes all subsets with cardinality equal to k .

$$\mathcal{P}_k(\mathcal{X}) = \{A : A \subseteq \mathcal{X}, |A| = k\}$$

Remark 3.4. The set $\mathcal{P}(\mathcal{X})$ is the union of all possible sets $\mathcal{P}_k(\mathcal{X})$ for $0 \leq k \leq |\mathcal{X}|$

$$\mathcal{P}(\mathcal{X}) = \bigcup_{k=0}^{|\mathcal{X}|} \mathcal{P}_k(\mathcal{X})$$

The power set enumerates all possible combinations of elements in a given set. Therefore, when enumerating the possible orders considered when computing the inventory level, the use of a power set is necessary.

The total cost is a discrete random variable with a finite number of outcomes. The following propositions are introduced to express $\mathbb{E}(A_t(\mathbf{Q}))$, $\mathbb{E}(I_t^+(\mathbf{Q}))$ and $\mathbb{E}(I_t^-(\mathbf{Q}))$.

Proposition 3.5. The expected arrival quantity at time t depends on the probability that all the previously ordered quantities and can be expressed using the formula below:

$$\mathbb{E}(A_t(\mathbf{Q})) = \sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s \mathbb{P}(\tau + L_\tau^s = t) \quad (14)$$

Proof. The expected arrival quantity of supplier s at time t is the following:

$$\mathbb{E}(A_t(\mathbf{Q})) = \mathbb{E}\left(\sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s\right) = \sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s \mathbb{P}(\tau + L_\tau^s = t)$$

This formula would sum up all the ordered quantities from supplier s whose order time period τ , along with their lead

time L_τ^s , would result in the quantities arriving at time t . \square

Proposition 3.6. The total expected arrivals quantity from time $\tau = 1$ till t is defined as follows:

$$\sum_{\tau=1}^t \mathbb{E}(A_\tau(\mathbf{Q})) = \sum_{(s,\tau) \in \mathcal{N}_t(\mathbf{Q})} Q_\tau^s \\ + \sum_{(s,\tau) \in \mathcal{M}_t(\mathbf{Q})} Q_\tau^s \mathbb{P}(\tau + L_\tau^s \leq t) \quad (15)$$

Proof. When summing up all the quantities that have arrived until the end of time period t , and using the Equation (14), the formula would be the following:

$$\sum_{\tau=1}^t \mathbb{E}(A_\tau(\mathbf{Q})) = \sum_{\tau=1}^t \sum_{j=1}^{\tau} \sum_{s \in \mathcal{S}} Q_j^s \mathbb{P}(j + L_j^s = \tau)$$

The middle summation of this equation can be discarded by modifying the probability so that it accounts for elements arriving before the end of period τ in the following manner:

$$\sum_{\tau=1}^t \mathbb{E}(A_\tau(\mathbf{Q})) = \sum_{\tau=1}^t \sum_{s \in \mathcal{S}} Q_\tau^s \mathbb{P}(\tau + L_\tau^s \leq t)$$

Looking at the first right-hand side elements of this equation, the time periods τ that will be considered in the first summation $\sum_{\tau=1}^{t-L_\tau^s} Q_\tau^s$ are the same ones that would be found in the set $\mathcal{N}_t(\mathbf{Q})$ as detailed in Definition 2.3. Whereas, the time periods τ that will be considered in the second summation $\sum_{\tau=t-L_\tau^s+1}^{t-L_\tau^s} Q_\tau^s \mathbf{1}_{\{\tau + L_\tau^s \leq t\}}$ are the same ones that would be found in the set $\mathcal{M}_t(\mathbf{Q})$ as detailed in Definition 2.4. Therefore, Equation (15) can be easily deduced. \square

Proposition 3.7. The expected inventory level to carry at time t can be expressed as the following:

$$\mathbb{E}(I_t^+(\mathbf{Q})) = \sum_{k=0}^{|\mathcal{M}_t(\mathbf{Q})|} \sum_{W \subset \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]} \left[\sum_{(s,\tau) \in \mathcal{N}_\tau(\mathbf{Q})} Q_\tau^s \right. \\ \left. + \sum_{\tau \in W} Q_\tau^s - \sum_{\tau=1}^t d_\tau \right]^+ \\ \times \prod_{(s,\tau) \in \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]} \mathbb{P}(\tau + L_\tau^s \leq t) \\ \times \prod_{\substack{\tau \in \mathcal{M}_t(\mathbf{Q}) \\ (s,\tau) \notin \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]}} \mathbb{P}(\tau + L_\tau^s > t) \quad (16)$$

Proof. The expected inventory level to carry at time t is derived using the inventory balance formula of equation (2):

$$\mathbb{E}(I_t^+(\mathbf{Q})) = \mathbb{E}\left(\left[\sum_{\tau=1}^t A_\tau(\mathbf{Q}) - \sum_{\tau=1}^t d_\tau\right]^+\right)$$

Using the equation (9), we get the following expression:

$$\mathbb{E}(I_t^+(\mathbf{Q})) = \mathbb{E}\left(\left[\sum_{(s,\tau) \in \mathcal{N}_\tau(\mathbf{Q})} Q_\tau^s + \sum_{(s,\tau) \in \mathcal{M}_\tau(\mathbf{Q})} Q_\tau^s \mathbf{1}_{\{\tau + L_\tau^s \leq t\}} \right. \right. \\ \left. \left. - \sum_{\tau=1}^t d_\tau\right]^+\right)$$

For a given time t , the inventory level depends on the quantities received and the quantities delivered from the first time period until that time. The uncertainty of lead times affects the quantities that are received, therefore, the expected inventory level is dependent on that. However, not all the amounts received at a given time t will cause uncertainty in inventory. The orders that are sure to be delivered, i.e., the ones whose indices can be found in $\mathcal{N}_t(\mathbf{Q})$, do not cause variability in the inventory levels. Only the orders whose indices can be found in $\mathcal{M}_t(\mathbf{Q})$ do. Therefore, the probability of an inventory level occurring depends only on that of the elements of $\mathcal{M}_t(\mathbf{Q})$:

$$\mathbb{E}(I_t^+(\mathbf{Q})) = \sum_{k=0}^{|\mathcal{M}_t(\mathbf{Q})|} \sum_{\omega \subset \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]} \left(\left[\sum_{(s,\tau) \in \mathcal{N}_\tau(\mathbf{Q})} Q_\tau^s + \sum_{(s,\tau) \in \omega} Q_\tau^s - \sum_{\tau=1}^t d_\tau \right]^+ \times \mathbb{P}(\omega) \right) \quad \forall t \in \mathcal{T} \quad (17)$$

where

$$\mathbb{P}(\omega) = \prod_{(s,\tau) \in \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]} \mathbb{P}(\tau + L_\tau^s \leq t) \times \prod_{\substack{(s,\tau) \in \mathcal{M}_t(\mathbf{Q}) \\ (s,\tau) \notin \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]}} \mathbb{P}(\tau + L_\tau^s > t) \quad (18)$$

In equation (17), the outer summation $\sum_{k=0}^{|\mathcal{M}_t(\mathbf{Q})|}$ loops through all the possible number of orders that will arrive at time t , i.e., no orders arrive at $k = 0$, only one order arrives at $k = 1$, two orders arrive at $k = 2$. So on and so forth until $k = |\mathcal{M}_t(\mathbf{Q})|$, whereby, in such case, all possible orders that can arrive by the end of time t , do arrive. The summation $\sum_{\omega \subset \mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]}$ loops through all possible sets in $\mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]$. Finally, $\mathbb{P}(\omega)$, expressed in equation (18), is the probability that the set of orders ω arrives. Since ω loops through all subsets of $\mathcal{P}_k[\mathcal{M}_t(\mathbf{Q})]$, and k loops through all possible lengths of $\mathcal{P}[\mathcal{M}_t(\mathbf{Q})]$, the combination of these two sums ensures that all possible values for $\mathcal{M}_t(\mathbf{Q})$ are taken into consideration when computing the expected inventory level $\mathbb{E}(I_t^+(\mathbf{Q}))$.

The value of the probability of an exact scenario of ω happening, $\mathbb{P}(\omega)$, takes into account the items that will be delivered and are represented in ω , and those that will not be delivered, therefore, not present in ω . Therefore, equation (17) could be rewritten as seen in equation (16). \square

Proposition 3.8. *The expected backlog level at time t can be calculated using the following formula:*

$$\mathbb{E}(I_t^-(\mathbf{Q})) = \mathbb{E}(I_t^+(\mathbf{Q})) - \sum_{\tau=1}^t \mathbb{E}(A_\tau(\mathbf{Q})) + \sum_{\tau=1}^t d_\tau \quad \forall t \in \mathcal{T} \quad (19)$$

Proof.

$$\mathbb{E}(I_t^-(\mathbf{Q})) = \mathbb{E}(I_t^+(\mathbf{Q})) - \sum_{\tau=1}^t \mathbb{E}(A_\tau(\mathbf{Q})) + \sum_{\tau=1}^t d_\tau$$

When taking Equation (15) into account, the expected inventory level could be re-written as seen in Equation (19). \square

The total expected total cost equation is expressed as the following:

$$\mathbb{E}(TC) = \sum_{t \in \mathcal{T}} \left(h_t \mathbb{E}(I_t^+(\mathbf{Q})) + b_t \mathbb{E}(I_t^-(\mathbf{Q})) + \sum_{s \in \mathcal{S}} p_s \cdot Q_t^s \right) \quad (20)$$

By using expression (12), it can be rewritten as follows:

$$\mathbb{E}(TC) = \sum_{t \in \mathcal{T}} \left((h_t + b_t) \mathbb{E}(I_t^+(\mathbf{Q})) - b_t \sum_{\tau=1}^t (\mathbb{E}(A_\tau(\mathbf{Q})) - d_\tau) + \sum_{s \in \mathcal{S}} p_s \cdot Q_t^s \right) \quad (21)$$

where $\mathbb{E}(I_t^+(\mathbf{Q}))$ and $\sum_{\tau=1}^t \mathbb{E}(A_\tau(\mathbf{Q}))$ are given in proposition 3.5 and 3.7 respectively.

4. CONCLUSION

As part of the SMALLDERS project, this work aims to help retailers plan their orders from their suppliers to meet the demands of their customers. The model takes into account the purchasing cost, inventory holding cost, backlog cost, and uncertain lead times with known probability distributions. To do so, we propose a novel method to model the expected cost of the lot-sizing problem under lead-time uncertainty. This approach employs different sets of orders for each time step to account for the order that causes uncertainty at the inventory level. The first set is the $\mathcal{M}_t(\mathbf{Q})$ set that contains the indices of orders that are unsure to arrive by the end of time t . The second set is the set $\mathcal{N}_t(\mathbf{Q})$ that includes the indices of the orders that are certain to arrive by the end of time t . After defining these sets, the proposed approach performs calculations using power sets, considers all possible arrival times of the ordered quantities, and computes the inventory levels used in the total cost calculations. This approach takes into account all possible arrival times for a given ordered quantity and thus presents a way to compute the expected total cost.

However, this approach oversimplifies the real-life scenario. Further modifications to the method are necessary to account for the different capacities and constraints present that complicate the retailers' decision-making process. Furthermore, this model overlooks additional uncertainties in the system. For example, it does not account for possible uncertainty in the client's demand, the supplier's unit pricing, or the cost of transportation and storage. The model also plans for a limited number of time periods as defined by \mathcal{T} . Integrating a rolling horizon approach like the one seen in Germanos et al. (2024a) would add to the usefulness of the model. Finally, even without integrating the different capacities, constraints, and additional uncertainties, given a sufficiently large problem instance with a large number of possible suppliers and a long planning period, the model would suffer from long computational times. Integrating a well-adapted optimization approach such as heuristic would help shorten the runtime of the model without significant degradation of performance.

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