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# Hyperquaternionic Unitary Symplectic Groups: A Unifying Tool for Physics

Patrick R. Girard, Patrick Clarysse, Romaric Pujol and Philippe Delachartre

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**Abstract.** The mathematical tools of physics, based on group theory, are in permanent evolution. Major covariance groups are the orthogonal, unitary and symplectic groups. These groups are generally expressed in terms of real and complex matrices. Here we shall develop a new representation of the unitary symplectic groups  $USp(n)$  in terms of Clifford algebras constituted by tensor products of quaternion algebras called hyperquaternions. Concise expressions of the generators are obtained and a concrete example  $USp(4)$  is provided. Isomorphic quaternion matrix representations will also be used in the applications. The first application concerns classical mechanics. The Hamiltonian formalism, Poisson brackets and canonical transforms are related to the unitary symplectic groups. The  $1D$  and  $2D$  harmonic oscillators are examined within that framework. The second application concerns quantum mechanics. The Schrödinger and Heisenberg equations are derived in a new hyperquaternionic unitary symplectic way, the complex imaginary  $i$  being replaced by the quaternion  $k$  in phase space. The  $1D$  and  $2D$  quantum harmonic oscillators are treated within that formalism. Allowing a representation of both classical and quantum mechanics, it is hoped that the hyperquaternion algebras might deepen our mathematical comprehension of the foundational principles of physics.

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## 1. Introduction

The mathematical modeling of physics, based on group theory, is in permanent evolution [13]. Main covariance groups are the orthogonal, unitary and symplectic groups. The importance of the symplectic group in classical and quantum mechanics is well-known [3,8,22]. New tools such as Clifford algebras have also been applied to these fields [4,23]. In recent years, a new representation of Clifford algebras in terms of hyperquaternions defined as a tensor product of quaternion algebras (or subalgebra thereof) has been introduced [14,15,17–19] and applied, in particular, to the Poincaré group, the conformal group and to some extent to the unitary symplectic group also called compact symplectic group [15]. Here, we shall develop more extensively, the unitary symplectic group in terms of hyperquaternions together with physical applications. This group is particularly adapted to the formulation of problems in various physics domains and therefore opens up potential applications in both classical and quantum mechanics, as well as electromagnetism, signal and image processing [10,31].

Throughout this paper,  $\mathbb{H}^{\otimes m}$  will denote the tensor product of  $m$  quaternion algebras, i.e.  $\mathbb{H}^{\otimes m} = \mathbb{H} \otimes \mathbb{H} \otimes \dots \otimes \mathbb{H}$  ( $m$  terms) and will be called hyperquaternion algebra [20, 21, 27–29]. A hyperconjugation  $(\mathbb{H}^{\otimes m})^* = (\mathbb{H}_c^{\otimes m})$  where  $c$  is the quaternion conjugate has been defined in [15]. The isomorphism between the hyperquaternion algebras  $\mathbb{H}^{\otimes 2p+1}$ , quaternionic matrices and Clifford algebras with  $p$  integer ( $p \geq 0$ ) has been established in [19]

$$\mathbb{H}^{\otimes 2p+1} \simeq M_{4^p}(\mathbb{H}) \simeq Cl_{4p+2}(2p, 2p+2) \quad (1.1)$$

where  $M_n(\mathbb{H})$  designates a square matrix of order  $n$  over  $\mathbb{H}$ . Thus, a hyperquaternion  $A \in \mathbb{H}^{\otimes 2p+1}$  can be represented by a quaternionic matrix  $Q = f(A)$  with  $Q^* = f(A^*) = Q_c^T$  where  $Q_c^T$  is the matrix transpose quaternion conjugate of  $Q$  [15].

The paper is structured as follows. After the introduction, we give in the second section the standard definition of the unitary symplectic groups in terms of quaternionic and complex matrices. In the third section, we develop a new hyperquaternionic representation of the unitary symplectic groups  $USp(n)$  yielding concise expressions of the generators and examine as concrete example  $USp(4)$ . The expressions of the generators and representation modules are provided both in hyperquaternionic, quaternionic and complex matrix forms together with a `Mathematica` implementation. In the fourth section, a hyperquaternionic representation of classical mechanics is presented. The Poisson brackets are derived within the Hamiltonian formalism. The classical  $1D$  and  $2D$  harmonic oscillators are discussed within that framework. In the fifth section, a hyperquaternionic representation of quantum mechanics is developed. The Schrödinger and Heisenberg equations are introduced in a new unitary symplectic way, the complex imaginary  $i$  being replaced by the quaternionic  $k$  in phase space. As examples, the quantum  $1D$  and  $2D$  harmonic oscillators are presented within that formalism. Allowing a representation of both classical and quantum mechanics, hyperquaternions appear as a unifying tool for physics.

## 2. Background: Unitary, Symplectic and Unitary Symplectic Groups

Here, we first recall the definitions of the unitary groups  $U(n)$  and the symplectic groups  $Sp(2n)$ . Then we proceed to the standard definition of the compact unitary symplectic groups  $USp(n)$  in terms of quaternionic and complex matrices.

### 2.1. Unitary and symplectic groups $U(n)$ , $Sp(2n)$

The unitary group  $U(n)$  is the set of  $n \times n$  unitary matrices, i.e. complex matrices such that  $(\overline{U})^T = U^{-1}$  where  $(\overline{U})^T$  denotes the complex conjugate transpose of  $U$ . The unitary matrices preserve the Hermitian scalar product with  $(x, y) \in \mathbb{C}^n$

$$\langle x, y \rangle = \langle Ux, Uy \rangle = \overline{\langle y, x \rangle}. \quad (2.1)$$

One has,  $|\det U| = 1$ ; if  $\det U = 1$ ,  $U$  is called a special unitary matrix  $SU(n)$ .

The symplectic group  $Sp(2n)$  is the group of  $2n \times 2n$  complex matrices  $S$  which preserve a nondegenerate antisymmetric bilinear form, i.e. a symplectic scalar product denoted  $[x | y]$  with  $(x, y) \in \mathbb{C}^{2n}$

$$[x | y] = [Sx | Sy] = -[y | x]. \quad (2.2)$$

## 2.2. Quaternionic matrix representation of $USp(n)$

The quaternion algebra  $\mathbb{H}$  is constituted by quaternions  $q = a_1 + a_2i + a_3j + a_4k$ . The conjugate  $q_c$  of  $q$  is defined by  $q_c = a_1 - a_2i - a_3j - a_4k$ .

The compact unitary symplectic group  $USp(n)$  is generally defined as the group of invertible quaternion  $n \times n$  matrices preserving the quaternionic scalar product, or, equivalently

$$USp(n) = \{A \in M_n(\mathbb{H}) : AA^* = E_n\}, \quad (2.3)$$

where  $E_n$  is the identity matrix of order  $n$  and  $A^* = (A_c)^T$  is the transpose quaternion conjugate of  $A$  [6,7,9]. The group  $USp(n)$  has  $n(2n+1)$  parameters and satisfies the relation  $USp(n) \cong U(2n) \cap Sp(2n, \mathbb{C})$  where the symbol  $\cong$  indicates an isomorphism. The matrix  $A$  is called normal if  $AA^* = A^*A$ ; hermitian, if  $A^* = A$ ; antihermitian if  $A^* = -A$  and unitary symplectic if  $A^*A = E_n (= A^*A)$  [23, 24, 30]. Important properties are

$$(AB)^* = B^*A^*, \quad (A^*)^{-1} = (A^{-1})^*. \quad (2.4)$$

As first example, consider the unitary symplectic group  $USp(1)$  expressed within the quaternion algebra  $\mathbb{H}$  [9, 23]. Given a quaternion  $q = a_1 + a_2i + a_3j + a_4k$  and its conjugate  $q_c = a_1 - a_2i - a_3j - a_4k$ , one can express  $q$  as a set of two complex numbers  $(\alpha, \beta)$ ,  $q = \alpha + j\beta$  with  $\alpha = a_1 + a_2i$ ,  $\beta = a_3 - a_4i$  and  $zj = j\bar{z}$  where  $\bar{z}$  is the complex conjugate of  $z \in \mathbb{C}$ . The conjugate of  $q$  is given by  $q_c = \bar{\alpha} - j\beta$ . A quaternionic scalar product is defined by

$$\begin{aligned} (q_1 | q_2) &= q_{1c}q_2 = (\bar{\alpha}_1 - j\beta_1)(\alpha_2 + j\beta_2) \\ &= (\bar{\alpha}_1\alpha_2 + \bar{\beta}_1\beta_2) + j(\alpha_1\beta_2 - \alpha_2\beta_1) \\ &= \langle q_1 | q_2 \rangle + j[q_1 | q_2]. \end{aligned} \quad (2.5)$$

It is the sum of a hermitian scalar product  $\langle q_1 | q_2 \rangle = (\bar{\alpha}_1\alpha_2 + \bar{\beta}_1\beta_2)$  and a symplectic scalar product  $[q_1 | q_2] = (\alpha_1\beta_2 - \alpha_2\beta_1)$ . The quaternionic scalar product is invariant, and thus the hermitian and symplectic forms, under the transformations  $a \in \mathbb{H}$  ( $aa_c = 1$ )

$$(aq_1 | aq_2) = q_{1c}a_c a q_2 = q_{1c}q_2. \quad (2.6)$$

Hence, this proves the relation  $USp(1) \cong U(2) \cap Sp(2, \mathbb{C})$ .

The general case  $USp(n)$  is treated similarly as is done in [9]. Consider a quaternion  $n \times n$  matrix and the standard (right) vector space  $\mathbb{H}^n$  (quaternion scalars multiplying on the right) having the basis  $(e_1 \cdots e_n, e_1j \cdots e_nj)$

over  $\mathbb{C}$ . Let  $X^i = x^i + jx^{n+i}$  (and similarly  $Y$ ) be an element of  $\mathbb{H}^n$  with  $(x^i, x^{n+i} \in \mathbb{C})$ . The quaternionic scalar product, denoted  $\langle X | Y \rangle$ , is

$$\begin{aligned} \langle X | Y \rangle &= X^* Y = \sum_{i=1}^n (\overline{x^i - jx^{n+i}}) (y^i + jy^{n+i}) \\ &= \sum_{k=1}^{2n} \overline{x^k} y^k + j \sum_{i=1}^n (x^i y^{n+i} - x^{n+i} y^i) \\ &= \langle X | Y \rangle + j [X | Y]. \end{aligned} \quad (2.7)$$

It is the sum of a hermitian form  $\langle X | Y \rangle = \sum_{k=1}^{2n} \overline{x^k} y^k$  and a symplectic form  $[X | Y] = \sum_{i=1}^n (x^i y^{n+i} - x^{n+i} y^i)$ . The quaternionic scalar product is invariant under the transformation for  $A \in M_n(\mathbb{H})$

$$\langle AX | AY \rangle = (AX)^* AY = X^* (A^* A) Y = X^* Y \quad (2.8)$$

if  $A^* A = E_n$ . Hence, one has the isomorphism  $USp(n) \cong U(2n) \cap Sp(2n, \mathbb{C})$ .

After the quaternionic matrix representation, we shall now give an isomorphic complex matrix representation.

### 2.3. Complex matrix representation of $USp(n)$

Quaternionic matrices can be represented as complex matrices as follows [23, 24, 30]. Rewriting the matrix  $A \in M_n(\mathbb{H})$  as  $A = A_1 + jA_2$  with  $A_i \in M_n(\mathbb{C})$ , one introduces the complex matrix

$$\tilde{A} = T(A) = \begin{bmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{bmatrix} \in M_{2n}(\mathbb{C}). \quad (2.9)$$

The complex representation  $T(A)$  is an isomorphism between the matrix ring  $M_n(\mathbb{H})$  into the matrix ring  $M_{2n}(\mathbb{C})$ . One has

$$T(AB) = T(A)T(B), \quad T(E_n) = E_{2n}, \quad (2.10)$$

$$T(A^{-1}) = [T(A)]^{-1} \quad (2.11)$$

if  $A^{-1}$  exists. Moreover, since  $A^* = A_1^* - jA_2^T$ , one has  $T(A^*) = [T(A)]^*$ . Hence, if a quaternionic matrix  $A \in M_n(\mathbb{H})$  is unitary symplectic then  $T(A) \in M_{2n}(\mathbb{C})$  is also unitary symplectic.

The condition for a complex  $2n \times 2n$  matrix  $S = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix}$  to be of the type  $\begin{bmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{bmatrix}$  is  $JSJ^T = \overline{S}$  with  $J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix}$ . The matrix  $S$  is unitary symplectic if  $SS^* = S^*S = E_{2n}$  with  $S^* = (\overline{S})^T$  which entails the relations (see Appendix A)

$$S^T JS = J \text{ or } SJS^T = J. \quad (2.12)$$

The eigenvalues and eigenvectors of a quaternionic matrix  $A$  are obtained via the relation  $AX = X\lambda$  with  $X \in \mathbb{H}^n, \lambda \in \mathbb{C}$  [23, 24]. Developing  $X$  as  $X = X_1 + jX_2$  with  $X_i \in \mathbb{C}^n$ , one has

$$(A_1 + jA_2)(X_1 + jX_2) = (X_1 + jX_2)\lambda \quad (2.13)$$

leading to the complex matrix equation ( $\lambda \in \mathbb{C}$ )

$$\begin{bmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \lambda. \quad (2.14)$$

The quaternionic eigenvector  $X \in \mathbb{H}^n$  is thus replaced by the complex vector  $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{2n}$ . A quaternionic unitary symplectic matrix  $A$  can be diagonalized via a quaternionic unitary symplectic matrix  $U$  according to  $UAU^* = D$ , the diagonal elements of  $D$  being complex numbers of unit norm [23, 24]. In the complex representation, one obtains with  $D = D_1 + jD_2$  ( $D_2 = 0$ )

$$\tilde{D} = T(D) = \begin{bmatrix} D_1 & -\overline{D_2} \\ D_2 & \overline{D_1} \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & \overline{D} \end{bmatrix}. \quad (2.15)$$

The complex representation yields the trace and the determinant of the matrix  $\tilde{D}$ .

Having presented the quaternionic and complex matrix representations, we shall now discuss the hyperquaternionic Clifford algebra representation.

### 3. Hyperquaternionic Representation of $USp(n)$

In this section, we develop a new hyperquaternionic representation of the unitary symplectic group  $USp(n)$  within the Clifford algebra  $\mathbb{H}^{\otimes 2p+1}$ . The generators of the group have a concise expression of the type  $U = e^{g_i\theta}$  with  $U^* = U^{-1}$  where  $g_i$  can be read off directly from the basis of  $\mathbb{H}^{\otimes 2p+1}$ . The trace and the determinant of the representation are derived. As example, the group  $USp(4)$  is developed in three formalisms: hyperquaternionic, quaternionic and complex one [15].

#### 3.1. General formalism

An element  $A$  of the unitary symplectic group  $USp(n)$  is a hyperquaternion such that  $\{A \in \mathbb{H}^{\otimes 2p+1} : AA^* = 1\}$  with  $n = 4^p$ ,  $p$  integer ( $p \geq 0$ ) and where  $A^*$  is the hyperconjugate of  $A$ . The generators of  $USp(n)$  can be expressed in the concise form  $A = e^{g_i\theta}$  where  $g_i$  is an element of the basis of  $\mathbb{H}^{\otimes 2p+1}$  constituted by an odd number of quaternionic systems  $(i, j, k), (I, J, K), (l, m, n)$  etc.. Thus, the group  $USp(1)$  has three generators  $A \in \{e^{i\theta}, e^{j\theta}, e^{k\theta}\}$ . Similarly, within  $\mathbb{H}^{\otimes 3}$ , the hyperquaternion

$$A = e^{iJl\theta} = \cos \theta + iJl \sin \theta \quad (3.1)$$

satisfies  $A^* = e^{-iJl\theta} = A^{-1}$  and is therefore a generator of  $USp(4)$ . The number of real parameters of  $USp(n)$  with  $n = 4^p$  corresponds to the number of combinations of an odd number of distinct quaternionic systems within a

total of  $2p+1$  systems. This yields  $n(2n+1)$  real parameters. (See appendix B).

The representation module of the hyperquaternion algebra  $\mathbb{H}^{\otimes 2p+1}$  is obtained as follows. Consider a left module  $M$ ,  $u = Ae$  where  $e$  is a primitive idempotent of  $\mathbb{H}^{\otimes 2p} \simeq M_{4^p}(\mathbb{R})$  (such that  $e^2 = e, e^* = e$ ) and  $A$  a general element of  $\mathbb{H}^{\otimes 2p+1}$  [25]. The elements  $u = \sum u_\alpha X^\alpha$  ( $X^\alpha \in \mathbb{H}$ ) with the basis  $u_\alpha$  constitute a (right) vector space over  $\mathbb{H}$ . The dual right module  $v = eA$  has a basis  $v_\alpha$  defined in such a way that  $v_\alpha u_\beta = \delta_{\alpha\beta} e$  [5]. A representation of the algebra  $\mathbb{H}^{\otimes 2p+1}$  is given by  $u' = Au$  ( $A \in \mathbb{H}^{\otimes 2p+1}$ ). The trace of the endomorphism  $u' = Au$  is defined as  $tr_M(A) = \sum_\alpha v_\alpha (Au_\alpha)$  and satisfies the relations [5]

$$tr_M(A + A') = tr_M(A) + tr_M(A'), \quad tr_M(AA') = tr_M(A'A). \quad (3.2)$$

The norm of the element  $A$  relatively to  $M$  is defined by the formula

$$(Au_1) \wedge (Au_2) \wedge \cdots \wedge (Au_n) = N_M(A) \cdot u_1 \wedge u_2 \wedge \cdots \wedge u_n \quad (3.3)$$

where  $\wedge$  stands for the exterior product. The norm  $N_M$  satisfies the relation  $N_M(AA') = N_M(A)N_M(A')$  and is equal to the determinant of the endomorphism [5].

A quaternionic scalar product is defined by

$$(X | Y) = X^*Y = (Ae)^* Be = e(A^*B)e \in \mathbb{H}. \quad (3.4)$$

If  $A^*A = 1$  (for  $A \in \mathbb{H}^{\otimes 2p+1}$ ), one has the conservation of the quaternionic scalar product

$$(AX | AY) = X^*(A^*A)Y = X^*Y. \quad (3.5)$$

The representation module  $M$  is isomorphic to  $\mathbb{H}^n$ . We shall call  $(i, j, k)$  its basis, not to be confused with the first quaternionic system of  $\mathbb{H}^{\otimes 2p+1}$ . Developing the vectors  $X, Y$  of  $\mathbb{H}^n$  on  $\mathbb{C}$ ,  $X^i = x^i + jx^{n+i}, Y^i = y^i + jy^{n+i}$  where the coefficients  $(x^i, x^{n+i}, y^i, y^{n+i})$  are complex numbers, with  $X^{i*} = \overline{x^i} - jx^{n+i}$ , one finds as in the previous section

$$(X | Y) = X^*Y = \langle X | Y \rangle + j[X | Y], \quad (3.6)$$

where  $\langle X | Y \rangle = \sum_{k=1}^{2n} \overline{x^k} y^k$ ,  $[X | Y] = \sum_{i=1}^n (x^i y^{n+i} - x^{n+i} y^i)$ , are respectively a hermitian and a symplectic forms. Hence,  $USp(n) = U(2n) \cap Sp(2n, \mathbb{C})$ .

### 3.2. Example: $USp(4)$ group within the $\mathbb{H}^{\otimes 3}$ algebra

We discuss here the  $\mathbb{H}^{\otimes 3}$  algebra, provide the representation module and the explicit hyperquaternionic expression of the generators of  $USp(4)$  together with a numerical and algebraic software. Then we develop a specific example in hyperquaternionic, quaternionic and complex matrix form.

Consider the Clifford algebra  $\mathbb{H}^{\otimes 3} = (i, j, k) \otimes (I, J, K) \otimes (l, m, n)$ . Any hyperquaternion  $A$  can be expressed as a set of sixteen (real) quaternions

$q_i = a_i + b_i l + c_i m + d_i n = [a_i, b_i, c_i, d_i]$  with respect to the sets  $ijk/IJK/lmn$

$$A = q_1 + Iq_2 + Jq_3 + Kq_4 + iq_5 + iIq_6 + iJq_7 + iKq_8 + jq_9 + jIq_{10} + jJq_{11} + jKq_{12} + kq_{13} + kIq_{14} + kJq_{15} + kKq_{16}. \quad (3.7)$$

The multiplication can be implemented on **Mathematica** using its quaternion product  $q_i * * q_j$  (see Appendix C).

As representation module, consider in  $\mathbb{H}^{\otimes 2}$  the primitive idempotent

$$e = \frac{1}{4} (1 - iI - jK - kJ) \quad (3.8)$$

such that  $e^2 = e, e^* = e$ . The left module  $M, u = Ae$  with  $A \in \mathbb{H}^{\otimes 3}$  is a (right) vector space over  $\mathbb{H}$  (quaternion scalars multiplying on the right),

$$u = Ae = u_1 q^1 + u_2 q^2 + u_3 q^3 + u_4 q^4 \quad (q^i \in \mathbb{H}) \quad (3.9)$$

with the basis vectors  $u_1 = e, u_2 = Ie, u_3 = Je, u_4 = Ke$ . The dual basis is  $v_1 = u_1^* = e, v_2 = -eI, v_3 = -eJ, v_4 = -eK$  with  $v_\alpha u_\beta = \delta_{\alpha\beta} e$ .

Taking two vectors of  $M, X = u_i X^i, Y = u_i Y^i$  and decomposing each of the vectors  $X^i = x^i + jx^{n+i}, Y^i = y^i + jy^{n+i}$  into complex numbers  $(x^\alpha, y^\alpha) \in \mathbb{C}$ , one obtains as above, the quaternionic scalar product

$$(X | Y) = X^* Y = \langle X | Y \rangle + j [X | Y] \quad (3.10)$$

with  $\langle X | Y \rangle = \sum_{k=1}^{2n} \overline{x^k} y^k, [X | Y] = \sum_{i=1}^n (x^i y^{n+i} - x^{n+i} y^i)$  being respectively a hermitian and symplectic forms.

The group  $USp(4)$  is the group which leaves invariant the quaternionic form and has as generators the elements  $A = e^{g_i \theta}$  such that  $A^* = A^{-1}$  with

$$g_i \in \left\{ \begin{array}{l} (l, m, n), (i, j, k), (I, J, K), (l, m, n)iI, \\ (i)jJ, (i)kK, (i)kI, (i)kJ, (i)kK \end{array} \right\} \quad (3.11)$$

where  $()$  stands for  $(l, m, n)$ . The group has  $n(2n+1) = 36$  parameters and  $USp(1), USp(2), USp(3)$  as subgroups [15].

As specific generator of  $USp(4)$  consider the hyperquaternion  $a$

$$a = e^{iIm\theta} = \cos \theta + iIm \sin \theta = 1 [\cos \theta, 0, 0, 0] + iI [0, 0, \sin \theta, 0]. \quad (3.12)$$

The trace of the endomorphism  $u' = au$  is given by  $tr(a) = \Sigma v_\alpha a u_\alpha$  with

$$a u_1 = u_1 (\cos \theta - m \sin \theta), a u_2 = u_2 (\cos \theta + m \sin \theta), \quad (3.13)$$

$$a u_3 = u_3 (\cos \theta + m \sin \theta), a u_4 = u_4 (\cos \theta - m \sin \theta). \quad (3.14)$$

One finds  $tr(a) = (4 \cos \theta) e$ , hence the trace of  $a$  is equal to  $4 \cos \theta$ .

The norm of  $a$  is given by

$$a u_1 \wedge a u_2 \wedge a u_3 \wedge a u_4 = N(a) u_1 \wedge u_2 \wedge u_3 \wedge u_4 \quad (3.15)$$

yielding the result  $N(a) = 1$ , and thus  $det(a) = 1$ .

Concerning the quaternionic matrix representation of  $a = e^{iIm\theta}$ , one might proceed as follows. Since  $\mathbb{H}^{\otimes 2} \simeq M_4(\mathbb{R})$ , any element of  $\mathbb{H}^{\otimes 2}$  can be expressed as a  $4 \times 4$  real matrix  $M$  [15]. Thus

$$M(iI) = M(i \otimes i) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (3.16)$$

Relabeling  $(l, m, n)$  as  $(i, j, k)$ , the quaternionic representation matrix  $A = f(a)$  is given by

$$A = f(a) = \begin{bmatrix} \cos \theta - j \sin \theta & 0 & 0 & 0 \\ 0 & \cos \theta + j \sin \theta & 0 & 0 \\ 0 & 0 & \cos \theta + j \sin \theta & 0 \\ 0 & 0 & 0 & \cos \theta - j \sin \theta \end{bmatrix} \quad (3.17)$$

with  $A^* = A^{-1}$  and  $\text{tr} A = 4 \cos \theta$ ,  $\det A = 1$ .

The complex matrix representation is then obtained by decomposing  $A = A_1 + jA_2$  into two complex  $4 \times 4$  complex matrices (and remembering that  $k = j(-i)$ ), one has

$$A_1 = \begin{bmatrix} \cos \theta & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & \cos \theta \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\sin \theta & 0 & 0 & 0 \\ 0 & \sin \theta & 0 & 0 \\ 0 & 0 & \sin \theta & 0 \\ 0 & 0 & 0 & -\sin \theta \end{bmatrix}. \quad (3.18)$$

The complex matrix representation  $S$  of  $a$  is then

$$S = \tilde{A} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \in M_8(\mathbb{C}). \quad (3.19)$$

One verifies that  $\det S = 1$ ,  $\text{tr}(S) = 8 \cos \theta$ ,  $SJS^T = J$ ,  $S^TJS = J$ . The eigenvalues and eigenvectors of  $S$  are obtained in a standard way. The other generators of the unitary symplectic group are treated similarly.

Next, we shall proceed to applications of the unitary symplectic groups.

## 4. Hyperquaternionic Classical Mechanics

In this section, the Hamiltonian formalism, Poisson brackets and the canonical transforms are related to the unitary symplectic group within the quaternionic matrix representation. As examples, the classical harmonic  $1D$  and  $2D$  oscillators are discussed.

### 4.1. Hamiltonian, Poisson bracket and canonical transforms

The Hamiltonian formalism can be formulated within the quaternionic matrix representation as follows. Consider the quaternionic matrices  $M_n(\mathbb{H})$  and the standard (right) vector space  $\mathbb{H}^n$  having the basis  $(e_1, \dots, e_n, e_1j, \dots, e_nj)$  over  $\mathbb{C}$  where  $(i, j, k)$  denotes the quaternionic system. Since the basis is over

complex numbers, one can also adopt the basis  $(e_1i, \dots, e_ni, e_1j, \dots, e_nj)$  satisfying the following quaternionic scalar products

$$(e_\alpha i \mid e_\beta j) = (e_\alpha i)^* e_\beta j = -i (e_\alpha^* e_\beta) j = -k \delta_{\alpha\beta}, \quad (4.1)$$

$$(e_\alpha i \mid e_\beta i) = (e_\alpha j \mid e_\beta j) = 0 \quad (\alpha \neq \beta). \quad (4.2)$$

These formulas seem to yield much of classical and quantum mechanics. The Hamiltonian formalism is obtained as follows. Let us first consider the 1D case with the Hamiltonian  $H(q, p, t)$  where  $q, p, t$  are real variables (unless otherwise indicated) representing respectively the position, momentum of the particle and  $t$  the time. The Hamiltonian equations read

$$\dot{q} = \frac{\partial H}{\partial p} = \nabla_p H = \{q, H\}_{q,p}, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\nabla_q H = \{p, H\}_{q,p} \quad (4.3)$$

where  $\{f, g\}_{q,p} = \left( \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \right)$  is the Poisson bracket. Writing  $X = iq + jp$ , one has

$$\begin{aligned} \dot{X} &= i\dot{q} + j\dot{p} = i\nabla_p H - j\nabla_q H \\ &= -k(i\nabla_q H + j\nabla_p H) = -k\nabla H = \{X, H\}_{q,p} \end{aligned} \quad (4.4)$$

with  $\nabla H = i\nabla_q H + j\nabla_p H$ . Furthermore,  $\{iq, jp\}_{q,p} = k$  a relation which is of importance both in classical and quantum mechanics. Considering two numerical functions  $f, g$  of  $(q, p)$  one has for the quaternionic scalar product with  $\nabla f = i\nabla_q f + j\nabla_p f$  (similarly for  $\nabla g$ )

$$\begin{aligned} (\nabla f \mid \nabla g) &= (\nabla f)^* \nabla g = (-i\nabla_q f - j\nabla_p f)(i\nabla_q g + j\nabla_p g) \\ &= (\nabla_q f \nabla_q g + \nabla_p f \nabla_p g) - k(\nabla_q f \nabla_p g - \nabla_p f \nabla_q g) \\ &= \{f, g\}_+ - k\{f, g\}_- \end{aligned} \quad (4.5)$$

where  $\{f, g\}_- = \{f, g\}$  is the standard Poisson bracket [1], with

$$\{f, g\}_+ = (\nabla_q f \nabla_q g + \nabla_p f \nabla_p g), \quad (4.6)$$

$$\{f, g\}_- = (\nabla_q f \nabla_p g - \nabla_p f \nabla_q g). \quad (4.7)$$

Since the quaternion scalar product is invariant under the unitary symplectic group, the same holds in particular for the Poisson bracket. Hence, unitary symplectic transforms are canonical transforms which by definition conserve the Poisson bracket.

The  $nD$  case with the Hamiltonian  $H(q_\alpha, p_\alpha, t)$  is treated similarly within the (right) vector space  $\mathbb{H}^n$ . Writing  $X = \sum_{\alpha=1}^n iq_\alpha + jp_\alpha$  where  $q_\alpha, p_\alpha$  are real variables (unless otherwise indicated), one obtains

$$\begin{aligned} \dot{X} &= \sum_{\alpha=1}^n i\dot{q}_\alpha + j\dot{p}_\alpha = \sum_{\alpha=1}^n i\nabla_{p_\alpha} H - j\nabla_{q_\alpha} H \\ &= -k\nabla H = \{X, H\}_{q,p} \end{aligned} \quad (4.8)$$

with  $\{f, g\}_{q,p} = \left( \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha} \right)$  and  $\nabla H = \sum_{\alpha=1}^n i \nabla_{q_\alpha} H + j \nabla_{p_\alpha} H$ . Furthermore, one has the important relations

$$\{iq_\alpha, jp_\beta\}_{q,p} = k\delta_{\alpha\beta}, \quad \{iq_\alpha, iq_\beta\}_{q,p} = 0, \quad \{jp_\alpha, jp_\beta\}_{q,p} = 0. \quad (4.9)$$

For two numerical functions  $f, g$  of  $(q_\alpha, p_\alpha)$ , one has for the quaternionic scalar product

$$(\nabla f | \nabla g) = (\nabla f)^* \nabla g = \{f, g\}_+ - k \{f, g\}_-, \quad (4.10)$$

$$\{f, g\}_+ = \sum_{\alpha=1}^n (\nabla_{q_\alpha} f \nabla_{q_\alpha} g + \nabla_{p_\alpha} f \nabla_{p_\alpha} g), \quad (4.11)$$

$$\{f, g\}_- = \sum_{\alpha=1}^n (\nabla_{q_\alpha} f \nabla_{p_\alpha} g - \nabla_{p_\alpha} f \nabla_{q_\alpha} g). \quad (4.12)$$

Both brackets are invariant under unitary symplectic transforms.

As concrete applications of the hyperquaternionic representation of the unitary symplectic groups, we develop the classical 1D and 2D harmonic oscillators.

#### 4.2. Classical 1D harmonic oscillator

Consider the 1D harmonic oscillator having the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = \frac{\omega}{2} \left( \frac{p^2}{m\omega} + m\omega q^2 \right) = \frac{\omega}{2} (Q^2 + P^2) \quad (4.13)$$

where  $q, p$  are respectively the position and momentum of the particle with  $Q = \sqrt{m\omega}q, P = \frac{p}{\sqrt{m\omega}}$ . Writing  $X = iQ + jP$ , we obtain

$$\nabla H = i \nabla_Q H + j \nabla_P H = \omega (iQ + jP) = \omega X. \quad (4.14)$$

Hence, the equation of motion reads

$$\dot{X} = -k \nabla H = -k\omega X = X(k\omega). \quad (4.15)$$

The solution is  $X = X_0 e^{k\omega t}$  with  $X_0 = iQ_0 + jP_0, Q_0 = \sqrt{m\omega}q_0, P_0 = \frac{p_0}{\sqrt{m\omega}}$ . Developing  $X$ , one finds

$$\begin{aligned} X &= \left( i\sqrt{m\omega}q_0 + j\frac{p_0}{\sqrt{m\omega}} \right) (\cos \omega t + k \sin \omega t) \\ &= i\sqrt{m\omega}q + j\frac{p}{\sqrt{m\omega}}. \end{aligned} \quad (4.16)$$

Finally, one obtains the classical result

$$q = q_0 \cos \omega t + \frac{p_0}{m\omega} \sin \omega t, \quad p = p_0 \cos \omega t - m\omega q_0 \sin \omega t. \quad (4.17)$$

The solution  $X = X_0 e^{k\omega t}$  shows the explicit unitary symplectic transform where  $e^{k\omega t}$  corresponds to a rotation in the phase plane  $(iq, jp)$ .

### 4.3. Classical 2D harmonic oscillator

Consider a 2D harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{1}{2} m \omega^2 (q_1^2 + q_2^2) = \frac{\omega}{2} (P_1^2 + P_2^2 + Q_1^2 + Q_2^2) \quad (4.18)$$

where  $q_i, p_i$  are respectively the components of the position and momentum with  $Q_1 = \sqrt{m\omega}q_1$ ,  $Q_2 = \sqrt{m\omega}q_2$ ,  $P_1 = \frac{p_1}{\sqrt{m\omega}}$ ,  $P_2 = \frac{p_2}{\sqrt{m\omega}}$ . The Poisson brackets are conserved

$$\{iq_\alpha, jq_\beta\}_{q,p} = \{iQ_\alpha, jQ_\beta\}_{q,p} = 0 \quad (4.19)$$

$$\{ip_\alpha, jp_\beta\}_{q,p} = \{iP_\alpha, jP_\beta\}_{q,p} = 0 \quad (4.20)$$

$$\{iq_\alpha, jp_\beta\}_{q,p} = \{iQ_\alpha, jP_\beta\}_{q,p} = k\delta_{\alpha\beta}. \quad (4.21)$$

Hence, the transformation  $(q_\alpha, p_\alpha) \rightarrow (Q_\alpha, P_\alpha)$  is a canonical unitary symplectic transformation yielding the same form of Hamilton's equations, i.e.

$$\dot{Q}_\alpha = \frac{\partial H'}{\partial P_\alpha}, \quad \dot{P}_\alpha = -\frac{\partial H'}{\partial Q_\alpha}, \quad (4.22)$$

with  $H'(Q_\alpha, P_\alpha) = H(q_\alpha, p_\alpha)$ . Writing the quaternion vector  $X$  of  $\mathbb{H}^2$  as

$$X = \sum_{\alpha=1}^n iQ_\alpha + jP_\alpha = \begin{bmatrix} iQ_1 + jP_1 \\ iQ_2 + jP_2 \end{bmatrix}, \quad (4.23)$$

one has

$$\dot{X} = \sum_{\alpha=1}^n i\dot{Q}_\alpha + j\dot{P}_\alpha = \sum_{\alpha=1}^n i\frac{\partial H'}{\partial P_\alpha} + j\left(-\frac{\partial H'}{\partial Q_\alpha}\right) \quad (4.24)$$

$$= -k \sum_{\alpha=1}^n i\frac{\partial H'}{\partial Q_\alpha} + j\frac{\partial H'}{\partial P_\alpha} = -k\nabla H', \quad (4.25)$$

where

$$\nabla H' = \sum_{\alpha=1}^n i\nabla_{Q_\alpha} H' + j\nabla_{P_\alpha} H' = \begin{bmatrix} i\frac{\partial H'}{\partial Q_1} + j\frac{\partial H'}{\partial P_1} \\ i\frac{\partial H'}{\partial Q_2} + j\frac{\partial H'}{\partial P_2} \end{bmatrix} = \omega X. \quad (4.26)$$

Finally, the equation of motion reads

$$\dot{X} = -k\nabla H' = -k\omega X = X(k\omega). \quad (4.27)$$

The solution is  $X = X_0 e^{k\omega t}$  with the initial conditions

$$X_0 = \begin{bmatrix} iQ_{01} + jP_{01} \\ iQ_{02} + jP_{02} \end{bmatrix}. \quad (4.28)$$

The quaternion  $k$  plays the role of a rotation operator in the phase plane  $(iq_\alpha, jp_\alpha)$  or  $(iQ_\alpha, jP_\alpha)$ .

After a hyperquaternionic representation of classical mechanics, we now develop quantum mechanics within the hyperquaternionic unitary symplectic formalism.

## 5. Hyperquaternionic Quantum Mechanics

The importance in quantum mechanics of quaternions, Clifford algebras and symplectic groups has been recognized in [4, 8]. Early efforts to formulate a quaternionic quantum mechanics were made in [2, 11, 12]. Here, we shall proceed with a specific form of Clifford algebras, i.e. hyperquaternions defined as a tensor product of quaternion algebras. Furthermore, we shall use the unitary symplectic groups distinct from the symplectic ones. We derive the Schrödinger and Heisenberg equations in a new hyperquaternionic unitary symplectic way, the complex imaginary  $i$  being replaced by the quaternion  $k$  in phase space. As applications, the  $1D$  and  $2D$  quantum harmonic oscillators are treated within that framework.

### 5.1. Schrödinger and Heisenberg equations

Quantum mechanics is generally presented in terms of a Hilbert space, a hermitian form, unitary operators together with the Schrödinger equation and the Heisenberg representation. The relation between the state vector  $|\psi(t)\rangle$  and the initial vector  $|\psi(0)\rangle$  is assumed to be  $|\psi(t)\rangle = U |\psi(0)\rangle$  where  $U = e^{-\frac{itH}{\hbar}}$  is a unitary operator and  $H$  the Hamiltonian operator which is here supposed to be time independent. Furthermore, one assumes the relation  $[q, p] = i\hbar$ .

Here, we shall proceed in analogy with the approach of classical mechanics described above and present a unitary symplectic formulation of quantum mechanics. Having introduced quaternionic matrices, a quaternionic scalar product and unitary symplectic operators, we postulate the relation  $|\psi(t)\rangle = U |\psi(0)\rangle$  where

$$U = e^{-\frac{ktH}{\hbar}} \quad (k^2 = -1) \quad (5.1)$$

is a unitary symplectic operator and  $H$  time independent. In particular, one has for the time derivative

$$\frac{dU}{dt} = \frac{-kH}{\hbar} U. \quad (5.2)$$

Deriving the above relation  $|\psi(t)\rangle = U |\psi(0)\rangle$  with respect to the time, one obtains

$$\frac{d|\psi(t)\rangle}{dt} = \frac{-kH}{\hbar} |\psi(t)\rangle \quad (5.3)$$

or,

$$\hbar k \frac{d|\psi(t)\rangle}{dt} = H |\psi(t)\rangle \quad (5.4)$$

which is the Schrödinger equation

$$E |\psi(t)\rangle = H |\psi(t)\rangle \quad (5.5)$$

where  $E$  is the energy operator  $E = \hbar k \frac{\partial}{\partial t}$ . The Schrödinger equation is thus introduced in a natural way justifying the appearance of the hypercomplex  $k$ . Furthermore, since the Schrödinger equation is assumed to be valid even if  $H$  is time dependent, we shall consider the relation 5.2 to hold even if  $H$  is time dependent, the operator  $U$  resulting from a succession of products of

infinitesimal unitary symplectic operators. Finally, we postulate the relation  $[q, p] = k\hbar$ .

Denoting  $A_S, A_H$  respectively the Schrödinger and Heisenberg operators, one has for the quaternionic scalar product

$$\langle \psi(t) | A_S | \psi(t) \rangle = \langle \psi(0) | U^* A_S U | \psi(0) \rangle \quad (5.6)$$

where  $U^*$  is the hyperquaternionic conjugate of  $U$  ( $U^*U = 1$ ). The Heisenberg operator is

$$A_H = U^* A_S U \quad (5.7)$$

where  $A_S$  is a constant operator independent of the time. Rewriting the Heisenberg operator definition as  $U A_H = A_S U$  and taking the time derivative, one obtains

$$\frac{dU}{dt} A_H + U \frac{dA_H}{dt} = A_S \frac{dU}{dt}, \quad (5.8)$$

$$\frac{-kH}{\hbar} U A_H + U \frac{dA_H}{dt} = A_S \left( \frac{-kH}{\hbar} \right) U. \quad (5.9)$$

After rearrangement, one has

$$k\hbar U \frac{dA_H}{dt} = -H U A_H + A_S H U \quad (5.10)$$

yielding

$$\begin{aligned} k\hbar \frac{dA_H}{dt} &= -(U^* H U) A_H + U^* A_S H U \\ &= (U^* A_S U) (U^* H U) - (U^* H U) A_H \\ &= A_H H_H - H_H A_H. \end{aligned} \quad (5.11)$$

Finally, one obtains the Heisenberg equation of the operator  $A_H$

$$k\hbar \frac{dA_H}{dt} = [A_H, H_H]. \quad (5.12)$$

If  $A_S$  depends explicitly upon the time, the above relation becomes [26]

$$k\hbar \frac{dA_H}{dt} = [A_H, H_H] + k\hbar \left( \frac{\partial A_S}{\partial t} \right)_H \quad (5.13)$$

with  $\left( \frac{\partial A_S}{\partial t} \right)_H = U^* \left( \frac{\partial A_S}{\partial t} \right) U$ .

Having derived the Schrödinger and Heisenberg equations in the hyperquaternionic unitary symplectic formalism, we now develop the 1D and 2D quantum harmonic oscillators within that framework.

## 5.2. Quantum 1D harmonic oscillator

We consider the quantum harmonic 1D oscillator in the Heisenberg representation. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 \quad (5.14)$$

where  $q, p$  are respectively the position and momentum operators satisfying the relation  $[q, p] = k\hbar$ . The Heisenberg equations yield

$$k\hbar \frac{dq}{dt} = [q, H] = \frac{p}{m} k\hbar, \quad (5.15)$$

$$k\hbar \frac{dp}{dt} = [p, H] = -m\omega^2 q k\hbar \quad (5.16)$$

where we have used the relation  $[A, BC] = [A, B]C + B[A, C]$ .

Introducing the operator

$$a = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega q + kp), \quad (5.17)$$

the Heisenberg equation gives

$$k\hbar \frac{da}{dt} = [a, H] = \hbar\omega a \quad (5.18)$$

or  $\frac{da}{dt} = -k\omega a$  by multiplying both sides on the left by  $k$  with  $k^2 = -1$ . The solution is

$$a = e^{-k\omega t} a_0 = a_0 e^{-k\omega t} \quad (5.19)$$

where  $a_0 = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega q_0 + kp_0)$  is a constant operator commuting with  $e^{-k\omega t}$ .

Similarly, one has with the hyperquaternion conjugate operator

$$a^* = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega q - kp), \quad (5.20)$$

the Heisenberg equation reads

$$k\hbar \frac{da^*}{dt} = [a^*, H] = -\hbar\omega a^* \quad (5.21)$$

or  $\frac{da^*}{dt} = k\omega a^*$  with the solution

$$a^* = e^{k\omega t} a_0^* = a_0^* e^{k\omega t}. \quad (5.22)$$

Introducing the operator  $N = a^*a$ , one finds

$$\begin{aligned} a^*a &= a_0^* e^{k\omega t} e^{-k\omega t} a_0 = a_0^* a_0 \\ &= \frac{1}{\hbar\omega} \left( \frac{m\omega^2 q_0^2}{2} + \frac{p_0^2}{2} \right) - \frac{1}{2} \\ &= \frac{H}{\hbar\omega} - \frac{1}{2}. \end{aligned} \quad (5.23)$$

Similarly, one obtains  $aa^* = \frac{H}{\hbar\omega} + \frac{1}{2}$ . Hence,  $[a, a^*] = 1$  and

$$H = \left( a^*a + \frac{1}{2} \right) \hbar\omega = \left( N + \frac{1}{2} \right) \hbar\omega \quad (5.24)$$

The eigenvalues and eigenvectors of  $H$  can thus be obtained from those of  $N$ . Defining the eigenvalues and eigenvectors of  $N$  by  $N|n\rangle = n|n\rangle$ , one finds in the standard way [26] the relations

$$[N, a^*] = a^*, \quad [N, a] = -a \quad (5.25)$$

and

$$Na^* |n\rangle = (n+1) a^* |n\rangle, \quad Na |n\rangle = (n-1) a |n\rangle \quad (5.26)$$

yielding the relations

$$a^* |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle. \quad (5.27)$$

The eigenvalues of  $H$  are  $E_n = (n + \frac{1}{2}) \hbar\omega$  with  $n \in \mathbb{N}$ .

We obtain, as expected, the standard quantified energy levels. Next we discuss the 2D quantum harmonic oscillator within the same framework.

### 5.3. Quantum 2D harmonic oscillator

Concerning the quantum harmonic 2D oscillator in the Heisenberg representation, the Hamiltonian is

$$H = \left( \frac{p_1^2}{2m} + \frac{1}{2} m\omega^2 q_1^2 \right) + \left( \frac{p_2^2}{2m} + \frac{1}{2} m\omega^2 q_2^2 \right) \quad (5.28)$$

where  $q_\alpha, p_\alpha$  are respectively the position and momentum components operators satisfying the relation

$$[q_\alpha, p_\beta] = k\hbar\delta_{\alpha\beta} \quad (\alpha, \beta = 1, 2). \quad (5.29)$$

The Heisenberg equations read

$$k\hbar \frac{dq_\alpha}{dt} = [q_\alpha, H] = \frac{p_\alpha}{m} k\hbar, \quad (5.30)$$

$$k\hbar \frac{dp_\alpha}{dt} = [p_\alpha, H] = -m\omega^2 q_\alpha k\hbar. \quad (5.31)$$

Introducing the operators

$$a_\alpha = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega q_\alpha + kp_\alpha), \quad (5.32)$$

the Heisenberg equations give

$$k\hbar \frac{da_\alpha}{dt} = [a_\alpha, H] = \hbar\omega a_\alpha. \quad (5.33)$$

The solutions are  $a_\alpha = e^{-k\omega t} a_{0\alpha}$  where  $a_{0\alpha} = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega q_{0\alpha} + kp_{0\alpha})$  are constant operators.

Similarly, the hyperquaternion conjugate operators

$$a_\alpha^* = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega q_\alpha - kp_\alpha), \quad (5.34)$$

satisfy the Heisenberg equations

$$k\hbar \frac{da_\alpha^*}{dt} = [a_\alpha^*, H] = -\hbar\omega a_\alpha^* \quad (5.35)$$

with the solutions  $a^* = e^{k\omega t} a_{0\alpha}^*$ . Furthermore, one has the commutation relations [26]

$$[a_\alpha, a_\beta] = [a_\alpha^*, a_\beta^*] = 0, \quad [a_\alpha, a_\beta^*] = \delta_{\alpha\beta}. \quad (5.36)$$

Introducing the operator  $N_\alpha = a_\alpha^* a_\alpha$ , one has

$$H = (N_1 + N_2 + 1) \hbar\omega. \quad (5.37)$$

The eigenvalues and eigenvectors of  $H$  are obtained from those of  $N_\alpha$ . Defining the eigenvalues and eigenvectors of  $N_\alpha$  by  $N_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle$ , the eigenvalues of the energy are  $(n_1 + n_2 + 1)\hbar\omega$  with  $n_\alpha \in \mathbb{N}$ .

## 6. Conclusion

The paper has given a new representation of the unitary symplectic groups in terms of Clifford algebras constituted by tensor products of quaternions called hyperquaternions yielding concise expressions of the generators. A concrete example  $USp(4)$  has been discussed. A hyperquaternionic formulation of classical and quantum mechanics has been presented. A simple natural derivation of the Schrödinger and Heisenberg equations has been developed in terms of the unitary symplectic groups, the complex imaginary  $i$  being replaced by the quaternion  $k$  in phase space. As applications, the  $1D$  and  $2D$  classical and quantum harmonic oscillators have been discussed in the hyperquaternionic framework. Allowing a representation of both classical and quantum mechanics, it is hoped that the hyperquaternion algebras might constitute a step towards a greater unification of the foundational principles of physics.

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### Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

### Declarations

#### Conflict of interest

The author declares that he has no conflict of interest.

## Appendix A

The first relation  $SJS^T = J$  is derived from  $JSJ^T = \tilde{S}$  and  $SS^* = S^*S = E_{2n}$  with  $S^* = \begin{pmatrix} \tilde{S} \\ \tilde{S} \end{pmatrix}^T$ . Since  $(JSJ^T)^T = S^*$ , one has

$$S(JS^T J^T) = SS^* = E_{2n} \quad (6.1)$$

$$\implies (SJS^T)J^T J = E_{2n}J = J. \quad (6.2)$$

Finally, with  $J^T J = E_{2n}$ , one obtains  $SJS^T = J$ .

The second relation  $(S^T JS) = J$  is obtained as follows. From  $(\tilde{S})^T S = E_{2n}$ , one has  $S^T \tilde{S} = E_{2n}$ ; thus

$$S^T (JSJ^T) = S^T (\tilde{S}) = E_{2n} \quad (6.3)$$

$$\implies (S^T JS) J^T J = E_{2n} J = J. \quad (6.4)$$

Finally,  $(S^T JS) = J$ .

## Appendix B

The number of real parameters  $S$  of  $USp(1)$  with one quaternionic system  $(i, j, k)$  of  $\mathbb{H}^{\otimes 1}$  is  $3C_1^1$  with  $C_n^p = \frac{n!}{p!(n-p)!}$ . For 3 systems of quaternions (i.e.  $\mathbb{H}^{\otimes 3}$ ), one has  $S = 3C_3^1 + 3^3 C_3^3$ . In the general case, within  $\mathbb{H}^{\otimes 2p+1}$  for  $USp(n)$  with  $n = 4^p$  and  $2p + 1$  quaternionic systems, the number of real parameters is given by

$$S = 3C_{2p+1}^1 + (3)^3 C_{2p+1}^3 + \dots + (3)^{2p+1} C_{2p+1}^{2p+1}. \quad (6.5)$$

$S$  can be obtained via the binomial formula

$$(x + a)^n = x^n + C_n^1 a x^{n-1} + C_n^2 a^2 x^{n-2} + C_n^p a^p x^{n-p} + \dots + C_n^n a^n. \quad (6.6)$$

Taking  $x = 1$  and respectively  $a = 3$  and  $-3$ , one obtains

$$(1 + 3)^{2p+1} = 1 + 3C_{2p+1}^1 + (3)^2 C_{2p+1}^2 + (3)^3 C_{2p+1}^3 + \dots + (3)^{2p+1} C_{2p+1}^{2p+1} = 4^{2p+1}, \quad (6.7)$$

$$(1 - 3)^{2p+1} = 1 - 3C_{2p+1}^1 + (-3)^2 C_{2p+1}^2 + (-3)^3 C_{2p+1}^3 + \dots + (-3)^{2p+1} C_{2p+1}^{2p+1} = (-2)^{2p+1}. \quad (6.8)$$

Subtracting the second equation from the first one, yields with  $n = 4^p$

$$S = \frac{1}{2} (4^{2p+1} + 2^{2p+1}) = 4^p (4^p + 1) = n (2n + 1). \quad (6.9)$$

## Appendix C: Mathematica code, $USp(4)$

### C1: Quaternionic matrices

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<< Quaternions

Remove["Global\*"]

(\*quaternion - 4x4 - matrix : {{q1, q2, q3, q4}, {q5, q6, ..., }, { }, { } }\*)

(\*HP : quaternion - matrix - multiplication - product\*)

HP[a\_, b\_] := { (a[[1]]\*\*b[[1]]) + (a[[2]]\*\*b[[5]]) + (a[[3]]\*\*b[[9]]) + (a[[4]]\*\*b[[13]]),  
(a[[1]]\*\*b[[2]]) + (a[[2]]\*\*b[[6]]) + (a[[3]]\*\*b[[10]]) + (a[[4]]\*\*b[[14]]),  
(a[[1]]\*\*b[[3]]) + (a[[2]]\*\*b[[7]]) + (a[[3]]\*\*b[[11]]) + (a[[4]]\*\*b[[15]]),  
(a[[1]]\*\*b[[4]]) + (a[[2]]\*\*b[[8]]) + (a[[3]]\*\*b[[12]]) + (a[[4]]\*\*b[[16]]),  
(a[[5]]\*\*b[[1]]) + (a[[6]]\*\*b[[5]]) + (a[[7]]\*\*b[[9]]) + (a[[8]]\*\*b[[13]]),

$(a[[5]**b[[2]]) + (a[[6]**b[[6]]) + (a[[7]**b[[10]]) + (a[[8]**b[[14]]),$   
 $(a[[5]**b[[3]]) + (a[[6]**b[[7]]) + (a[[7]**b[[11]]) + (a[[8]**b[[15]]),$   
 $(a[[5]**b[[4]]) + (a[[6]**b[[8]]) + (a[[7]**b[[12]]) + (a[[8]**b[[16]]),$   
 $(a[[9]**b[[1]]) + (a[[10]**b[[5]]) + (a[[11]**b[[9]]) + (a[[12]**b[[13]]),$   
 $(a[[9]**b[[2]]) + (a[[10]**b[[6]]) + (a[[11]**b[[10]]) + (a[[12]**b[[14]]),$   
 $(a[[9]**b[[3]]) + (a[[10]**b[[7]]) + (a[[11]**b[[11]]) + (a[[12]**b[[15]]),$   
 $(a[[9]**b[[4]]) + (a[[10]**b[[8]]) + (a[[11]**b[[12]]) + (a[[12]**b[[16]]),$   
 $(a[[13]**b[[1]]) + (a[[14]**b[[5]]) + (a[[15]**b[[9]]) + (a[[16]**b[[13]]),$   
 $(a[[13]**b[[2]]) + (a[[14]**b[[6]]) + (a[[15]**b[[10]]) + (a[[16]**b[[14]]),$   
 $(a[[13]**b[[3]]) + (a[[14]**b[[7]]) + (a[[15]**b[[11]]) + (a[[16]**b[[15]]),$   
 $(a[[13]**b[[4]]) + (a[[14]**b[[8]]) + (a[[15]**b[[12]]) + (a[[16]**b[[16]])\}$   
 (\*TK : matrix – transpose, quaternion – conjugate\*)

TK[a.]:=

{Quaternion[a[[1, 1]], -a[[1, 2]], -a[[1, 3]], -a[[1, 4]],  
 Quaternion[a[[5, 1]], -a[[5, 2]], -a[[5, 3]], -a[[5, 4]],  
 Quaternion[a[[9, 1]], -a[[9, 2]], -a[[9, 3]], -a[[9, 4]],  
 Quaternion[a[[13, 1]], -a[[13, 2]], -a[[13, 3]], -a[[13, 4]],  
 Quaternion[a[[2, 1]], -a[[2, 2]], -a[[2, 3]], -a[[2, 4]],  
 Quaternion[a[[6, 1]], -a[[6, 2]], -a[[6, 3]], -a[[6, 4]],  
 Quaternion[a[[10, 1]], -a[[10, 2]], -a[[10, 3]], -a[[10, 4]],  
 Quaternion[a[[14, 1]], -a[[14, 2]], -a[[14, 3]], -a[[14, 4]],  
 Quaternion[a[[3, 1]], -a[[3, 2]], -a[[3, 3]], -a[[3, 4]],  
 Quaternion[a[[7, 1]], -a[[7, 2]], -a[[7, 3]], -a[[7, 4]],  
 Quaternion[a[[11, 1]], -a[[11, 2]], -a[[11, 3]], -a[[11, 4]],  
 Quaternion[a[[15, 1]], -a[[15, 2]], -a[[15, 3]], -a[[15, 4]],  
 Quaternion[a[[4, 1]], -a[[4, 2]], -a[[4, 3]], -a[[4, 4]],  
 Quaternion[a[[8, 1]], -a[[8, 2]], -a[[8, 3]], -a[[8, 4]],  
 Quaternion[a[[12, 1]], -a[[12, 2]], -a[[12, 3]], -a[[12, 4]],  
 Quaternion[a[[16, 1]], -a[[16, 2]], -a[[16, 3]], -a[[16, 4]]}

X:={Quaternion[0, x1, p1, 0], Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, x2, p2, 0], Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, x3, p3, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0], Quaternion[0, x4, p4, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0]}

A:={Quaternion[Cos[ $\theta/2$ ], 0, -Sin[ $\theta/2$ ], 0], Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0], Quaternion[Cos[ $\theta/2$ ], 0, +Sin[ $\theta/2$ ], 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[Cos[ $\theta/2$ ], 0, +Sin[ $\theta/2$ ], 0], Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0],  
 Quaternion[0, 0, 0, 0], Quaternion[0, 0, 0, 0], Quaternion[Cos[ $\theta/2$ ], 0, -Sin[ $\theta/2$ ], 0]}

w1:=Simplify[HP[TK[X], X]]

w2:=Simplify[HP[A, X]]

w3:=Simplify[HP[TK[w2], w2]]

**C2: Complex matrices**

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<< Quaternions

Remove["Global\*"]

A:={{cos[θ], 0, 0, 0, sin[θ], 0, 0, 0},

{0, cos[θ], 0, 0, 0, -sin[θ], 0, 0}, {0, 0, cos[θ], 0, 0, 0, -sin[θ], 0},

{0, 0, 0, cos[θ], 0, 0, 0, sin[θ]}, {-sin[θ], 0, 0, 0, cos[θ], 0, 0, 0},

{0, sin[θ], 0, 0, 0, cos[θ], 0, 0}, {0, 0, sin[θ], 0, 0, 0, cos[θ], 0},

{0, 0, 0, -sin[θ], 0, 0, 0, cos[θ]}}

JJ:={{0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0},

{0, 0, 0, 0, 0, 0, 0, 1}, {-1, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0},

{0, 0, -1, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0}}

A//MatrixForm

JJ//MatrixForm

w1 = Simplify[Det[A]]

w2:=Simplify[A.JJ.Transpose[A]]//MatrixForm

w3:=Simplify[Transpose[A].JJ.A]//MatrixForm

$$\begin{pmatrix} \cos[\theta] & 0 & 0 & 0 & \sin[\theta] & 0 & 0 & 0 \\ 0 & \cos[\theta] & 0 & 0 & 0 & -\sin[\theta] & 0 & 0 \\ 0 & 0 & \cos[\theta] & 0 & 0 & 0 & -\sin[\theta] & 0 \\ 0 & 0 & 0 & \cos[\theta] & 0 & 0 & 0 & \sin[\theta] \\ -\sin[\theta] & 0 & 0 & 0 & \cos[\theta] & 0 & 0 & 0 \\ 0 & \sin[\theta] & 0 & 0 & 0 & \cos[\theta] & 0 & 0 \\ 0 & 0 & \sin[\theta] & 0 & 0 & 0 & \cos[\theta] & 0 \\ 0 & 0 & 0 & -\sin[\theta] & 0 & 0 & 0 & \cos[\theta] \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\cos[\theta]^2 + \sin[\theta]^2)^4$$

**C3: Hyperquaternion algebra  $\mathbb{H}^{\otimes 3}$**

See: software notebookarchive in Ref. [16].

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