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# An analytical reference compensator for dynamic set-point tracking with qLPV NMPCs

M.M. Morato<sup>1,2</sup>, Tito L.M. Santos<sup>3</sup>

**Abstract**—In the literature, recent works have systematically shown that the use of quasi-Linear Parameter Varying (qLPV) embeddings in the place of nonlinear models can significantly enhance the numerical performances of Nonlinear Model Predictive Control (NMPC) algorithms. However, the corresponding available formulations for the reference tracking problem typically enable offset-free steady-state tracking only for the case of piece-wise constant or fixed set-points. In order to extend these algorithms for the more generic case of time-varying reference signals, we propose an analytical target modification scheme that can be directly integrated to the prior. In particular, the proposed compensator scheme has low numerical complexity, being based on the unconstrained optimisation solution of the NMPC. We also provide an explicit input-state reference trajectory representation and demonstrate that qLPV NMPC schemes coupled to the proposed scheme maintain input-to-state-practical-stability (ISpS) and recursive feasibility. A cart-spring benchmark nonlinear simulation is presented to demonstrate the effectiveness of the proposed target modification block, in comparison to the standard setting.

## I. INTRODUCTION

Model Predictive Control (MPC) is a consolidated control strategy, with several experimental results registered [1]. Nowadays, MPC has been validated for many different applications in both linear and nonlinear (NMPC) settings, cf. [2], [3], [4]. Moreover, since [5], formal theoretical synthesis procedures to establish stability, optimality and feasibility are available - recently extended in [6] by means of dissipativity.

Notwithstanding, the use of (quasi)-Linear Parameter Varying (qLPV) embeddings in NMPC design has recently been shown as an efficient way to achieve (near-)optimal control performances with very low computational cost - when compared to standard NMPC techniques [7], [8], [9]. In short, the qLPV framework consists in parametrising and scheduling linear system models to *exactly* represent nonlinear state transitions, cf. [10], [11]. To this end, a scheduling parameter needs to be computed in each iteration of the corresponding MPC (iteratively [12], recursively [13], [14], etc). The main advantage of qLPV MPC design relies on the resulting optimisation taking the form of Sequential

Quadratic Programs (SQPs), whose real-time resolutions are well-consolidated using standard, off-the-shelf solvers.

In practical engineering applications, the problem of reference tracking is particularly important. However, typical NMPC formulations “*for tracking*” are only able to ensure null tracking error in steady-state for piece-wise constant or fixed set-points. Although there exist formulations for dynamic reference signals [4], [15], including in the qLPV setting [14], these generally require more complex optimisation problems to be solved during the implementation, using, e.g., dynamic targets and hierarchical trajectory planning [16], [17] or reference-generic terminal ingredients [18], tuned offline for possible reference values (which can also be complex to render). **In sum, the main challenge related to the previous approaches is that resulting numerical complexity of the NMPC becomes tougher when dealing with time-varying set-points, in relation to the fixed set-point case. In this paper, we provide a solution with minor numerical burden, which dynamically modifies the reference passed to the NMPC.**

Accordingly, we extend our previous results from [19] to the more generic nonlinear setting using qLPV embeddings. In particular, we propose an analytical reference compensation mechanism with low associated numerical cost that can be directly integrated into qLPV NMPC schemes for tracking. Our **contributions** are the following:

- 1) Based on the unconstrained optimisation, we derive an admissible stabilising tracking control law, which is used to adapt and compensate dynamic set-point signals to standard Tracking qLPV NMPC algorithms (Sec. III). An important novelty is that our method does not require the system to be square. Complementary, we also construct a novel qLPV parametrisation for artificial target variables in the context of LPV tracking schemes, in order to avoid nonlinear target parametrisations (Appendix A).
- 2) Furthermore, we develop a simple explicit parameter-dependent feed-forward reference trajectory that can be used to construct corresponding generic time-varying state and input sequences (Sec. III-C).
- 3) We also demonstrate that our method ensures Input-to-State-Practical-Stability (ISpS) and recursive feasibility of the optimisation (Sec. IV).
- 4) Finally, a nonlinear Cart-Spring benchmark is used to demonstrate, in simulation, the effectiveness of the proposed reference compensator when integrated to Tracking qLPV MPC algorithms (Sec. V).

**Notation.** The index set  $\mathbb{N}_{[a,b]}$  gives  $\{i \in \mathbb{N} \mid a \leq i \leq b\}$ ,

<sup>1</sup>Instituto Nacional de Ciência e Tecnologia em Controle e Automação de Processos de Energia (INCT CAPE), Dept. de Eng. de Automação e Sistemas (EAS), Univ. Fed. de Santa Catarina, Florianópolis-SC, Brazil.

<sup>2</sup>Univ. Grenoble Alpes, CNRS, Grenoble INP<sup>T</sup>, GIPSA-lab, 38000 Grenoble, France. <sup>T</sup>Institute of Engineering Univ. Grenoble Alpes. Contact: marcelo.menezes@ufsc.br. An **extended version** of this paper is available online in the open repository: <https://hal.science/hal-05166133>.

<sup>3</sup>Dept. of Electrical and Computer Eng., Univ. Fed. da Bahia, Rua Aristides Novis, 02, CEP 402-630, Salvador-BA, Brazil. Contact: tlsantos@ufba.br

with  $0 \leq a \leq b$ .  $\ell_{2e}^m$  is the space of sequences with elements in  $\mathbb{R}^m$ . The identity matrix of size  $j$  is denoted as  $I_j$ . The predicted value of a given variable  $v(k)$  at time instant  $k+i$ , computed based on the information available at instant  $k$ , is denoted as  $v(k+i|k)$ . For an  $m$ -sized vector  $v$ ,  $v_j$  denotes its  $j$ -th variable,  $\text{col}(v_1, \dots, v_m) := (v_1^\top, \dots, v_m^\top)^\top$ , and  $\text{diag}(v_1, \dots, v_m)$  denotes the block diagonal matrix with  $v_1, \dots, v_m$  over the diagonal;  $\|\cdot\|_P$  denotes the  $P$ -weighted 2-norm and  $\ominus$  denotes the Pontryagin set difference.

## II. RECAP: QLPV NMPC FOR TRACKING

Before presenting our new method for time-varying reference signals, we revisit the fundamental aspects of qLPV MPC design [8], [14]. In particular, we consider nonlinear systems that are *exactly*<sup>1</sup> described in an LPV format<sup>2</sup>. That is, given some real matrices  $(A(\rho), B(\rho))$  and an initial condition  $x(0) \in \mathbb{R}^{n_x}$ , we use the following discrete-time state-space:

$$\begin{aligned} x(k+1) &= A(\rho(k))x(k) + B(\rho(k))u(k), \\ y(k) &= C(\rho(k))x(k). \end{aligned} \quad (1)$$

Here,  $x \in \ell_{2e}^{n_x}$ ,  $y \in \ell_{2e}^{n_y}$  and  $u \in \ell_{2e}^{n_u}$  stand, respectively, for the states, outputs and inputs, while  $\rho \in \ell_{2e}^{n_\rho}$  represents the scheduling variables, which are bounded and known at each sampling instant  $k$ . Specifically, the embedding in (1) is imposed over an *admissible* region (related to physical constraints of the plant) given by  $x \in \mathcal{X}$  and  $u \in \mathcal{U}$  (which implies in the output admissibility constraint  $y \in \mathcal{Y}$ ), for which the scheduling proxy holds:  $\rho = f_\rho(x, u) \in \mathcal{P}$ , being  $\mathcal{P} \subset \mathbb{R}^{n_\rho}$  a known convex *scheduling set*.

Considering the nonlinear system (1), the application of a corresponding NMPC algorithm for tracking requires a rolling performance criterion  $J_k := J(x(k), U_k, r(k))$ , mapped along a finite *prediction horizon* of  $N_p$  samples, and the operational constraints ( $\mathcal{X}$  and  $\mathcal{U}$ ), considering  $r(k) \in \mathbb{R}^{n_y}$  as a piece-wise constant reference signal. Such an algorithm determines, at sample  $k$ , the sequence of optimal candidate control inputs along the horizon, i.e.  $U_k^* := \text{col}(u^*(k|k), \dots, u^*(k+N_p-1|k))$ . Then, given that at each discrete-time sample  $k$ , only the sampled scheduling variable  $\rho(k)$  and state  $x(k)$  are available, we consider a slightly modified version of Tracking qLPV NMPC from [14], which relies in solving the following optimisation:

$$V_k^* := \arg \min_{U_k, \theta_a} \overbrace{J(x(k), U_k, r(k))}^{J_k}, \text{ s.t. } \forall j \in \mathbb{N}_{[1, N_p]} : \quad (2)$$

$$\left\{ \begin{array}{l} x(k+j|k) = A(\rho(k+j-1|k))x(k+j-1|k) \\ \quad + B(\rho(k+j-1|k))u(k+j-1|k), \\ x(k+j|k) \in (\mathcal{X} \ominus \mathcal{S}(j)), \\ u(k+j-1|k) \in \mathcal{U}, \\ (x(k+N_p|k), \theta_a) \in \Omega_t, \\ (x_a, u_a, y_a) \in M_\theta(\rho(k))\theta_a. \end{array} \right.$$

<sup>1</sup>By exactness, we mean that if the scheduling relation mapped by  $\rho$  is re-introduced in (1), we retrieve a nonlinear discrete-time model.

<sup>2</sup>In many standard references, nonlinear systems given in the format of (1) are said *quasi*-LPV, due to the endogenous parameter-dependency. For brevity, we use only the term LPV.

In this modified formulation, for which  $V_k^* := (U_k^*, \theta_a^*)$ , the steady-state equilibrium for a given scheduling is defined by  $(x_a, u_a, y_a) = M_\theta(\rho(k))\theta_a$  such that the new approach can be applied to non-square systems ( $n_y \neq n_u$ ) which is a significant novelty in comparison to [4] and [14].

The NMPC optimisation in (2) comprises the following elements: (i) a quadratic tracking cost function given by:

$$J_k := \overbrace{\|x(k+N_p|k) - x_a\|_P^2 + \|r(k) - y_a\|_T^2}^{V(x(k+N_p|k))} \quad (3)$$

$$+ \sum_{j=0}^{N_p-1} (\|x(k+j|k) - x_a\|_Q^2 + \|u(k+j|k) - u_a\|_R^2),$$

being  $T$ ,  $Q$ ,  $R$  and  $P$  well-defined positive definite matrices; (ii) box-type state and input admissibility sets, i.e.  $\mathcal{X} := \{x \in \mathbb{R}^{n_x} : -\bar{x}_i \leq x_i \leq \bar{x}_i, \forall i \in \mathbb{N}_{[1, n_x]}\}$  and  $\mathcal{U} := \{u \in \mathbb{R}^{n_u} : -\bar{u}_j \leq u_j \leq \bar{u}_j, \forall j \in \mathbb{N}_{[1, n_u]}\}$ ; (iii) an robustly admissible augmented invariant set for tracking  $\Omega_t$ , (iv) the artificial (state, input, and output) target variables  $x_a$ ,  $u_a$  and  $y_a$ , as proposed in the original ‘‘NMPC for Tracking’’ [4]; (v) an artificial decision variable  $\theta_a \in \mathbb{R}^{n_u}$  related to the target set-point, parametrised by the sampled scheduling variable  $\rho(k)$ ; and (vi) a constraint tightening mechanism, implied by the uncertainty propagation set<sup>3</sup>  $\mathcal{S}(j)$ , and the stabilising terminal feedback gain  $\kappa_t$ . **In the open-loop stable case, it is implied that the state prediction error (a result of the predicted scheduling trajectory) is always bounded, i.e.  $(A(\rho(k+j-1|k)) - A(\rho(k+j-1|k+1)))x + (B(\rho(k+j-1|k) - B(\rho(k+j-1|k+1)))u \in \mathcal{W}$ ,  $\forall x \in \mathcal{X}$ ,  $\forall u \in \mathcal{U}$ , being  $\mathcal{W}$  a known compact set. For unstable plants, this can be established using a hierarchical local LPV feedback law  $u(k) = K_s(\rho)x(k) + v(k)$  s.t.  $A(\rho) + K_s(\rho)B(\rho)$  is stable.**

Some typical additional stabilising assumptions are considered to achieve recursive feasibility and input-to-state stability [5]: (i)  $P > 0$  is the solution for  $(A(\rho) + B(\rho)\kappa_t)^\top P(A(\rho) + B(\rho)\kappa_t) - P \leq -\kappa_t^\top R\kappa_t - Q$ ,  $\forall \rho \in \mathcal{P}$ ; (ii)  $\Omega_t$  ensures the following invariant property  $(x, \theta) \in \Omega_t \Rightarrow ((A(\rho) + B(\rho)\kappa_t)x + BL(\rho)\theta + w, \theta) \in \Omega_t$ , where  $L(\rho) = -\kappa_t M_x(\rho) + M_u(\rho)$ ,  $x \in \mathbb{X} \ominus \mathcal{S}(N_p)$ ,  $\kappa_t x + L(\rho)\theta \in \mathbb{U}$ ,  $M_x(\rho)\theta \in \lambda(\mathbb{X} \ominus \mathcal{S}(N_p))$ ,  $M_u(\rho)\theta \in \lambda\mathbb{U}$ ,  $0 < \lambda < 1$ , where  $\lambda$  is a scalar used to contract the equilibrium constraints,  $\forall w \in \mathcal{W}$ .

Along this paper, as considered in similar works, e.g. [4], [18], [19], [14], we assume that the piece-wise constant reference signal  $r(k)$  can be directly converted into an input-output reference pair  $(x_r(k), u_r(k))$ . This conversion can be handled, e.g., as an inverse problem or by an unknown-input observer scheme, as long as the system in (1) is observable or standard trajectory planning algorithms. We notice that, in the linear time-invariant (LTI) setting, the parametrisation of the artificial target variables w.r.t. the set-point  $r(k)$  is trivial (and can be derived from the null space condition related to the system description).

<sup>3</sup>There are several design procedures to construct the uncertainty propagation sets  $\mathcal{S}(j)$  that ensure robust constraint satisfaction and recursive feasibility of the optimisation. In particular, zonotopes can be directly used for this goal, considering qLPV models, as we demonstrate in [14].

However, up to our best knowledge, NMPC schemes “for Tracking” - such as the qLPV one we propose in [14] - typically require nonlinear functions to map the input-state pair  $(x_r, u_r)$  from its corresponding output target  $r(k)$ , cf. [4], [20]. These functions are not direct and can even be non-existent for particular nonlinearities. Here, as a side contribution, we also formulate an explicit qLPV parameter-dependent target characterisation, which serves to remove such nonlinear relations from the MPC optimisation problem. In particular, we detail how to compute the explicit matrix  $M_\theta(\rho(k))$  during the implementation in Appendix A.

The constraints in (2) account for the *predicted* system dynamics along the future horizon. The main difficulty in qLPV MPC, thus, lies in the unavailability of the future scheduling variable values, i.e.  $\rho(k+j)$ ,  $\forall j \in \mathbb{N}_{[1, N_p-1]}$ , which depend on the future states and inputs. The usual implementation approach relies in using estimates for these variables, e.g. computed using a Taylor expansion [13], and accounting for the corresponding introduced uncertainty<sup>4</sup>.

The most important properties in NMPC design for tracking are *recursive feasibility* of (2), *stability* of the corresponding closed-loop interconnection, and offset-free tracking of  $r(k)$  as long as this signal is reachable/admissible, i.e. given in  $\mathcal{Y}$ . For such, the considered tracking cost  $J_k$  includes the terminal penalty  $V(\cdot)$  that, combined with the terminal set  $\Omega_t$ , ensures closed-loop stability despite the model uncertainties. Moreover, the tightened constraints ensure robust satisfaction. We consider, along the sequel, that the uncertainty propagation sets  $\mathcal{S}(j)$  and the terminal ingredients are given and that the qLPV NMPC in (2), thus, ensures the aforementioned properties. We indicate that, in recent works, the seminal criteria proposed by Mayne and colleagues [5] is adapted to the LPV context [6], [14]. In short,  $P$  and  $\Omega_t$  can be computed using linear matrix inequalities. A simple choice for  $\Omega_t$  can be, for instance,  $\Omega_t = \{(x, \theta) \in \mathbb{R}^{n_x+n_u} : -\bar{\theta} \leq \theta \leq \bar{\theta}, (x - M_x(\rho)\theta)^\top P(x - M_x(\rho)\theta) \leq \alpha, \forall \rho \in \mathcal{P}\}$ , where  $\alpha$  and  $\bar{\theta}$  are defined to impose the admissible conditions. Further details are omitted herein for brevity.

### III. SET-POINT MODIFICATION SCHEME

Considering the qLPV NMPC for Tracking algorithm from [14], given by the optimisation in (2) with appropriate terminal ingredients and uncertainty propagation sets, we provide the main result of this work next. In particular, our method is as follows: we propose an analytical rule that modifies any given bounded time-varying reference  $r_{TV}(k) \in \mathbb{R}^{n_v}$ , converting it into a trackable, admissible signal  $r(k)$  that is passed to the NMPC scheme from [14] - which is designed to track piece-wise constant targets (not time-varying ones).

#### A. Unconstrained NMPC law

To develop our reference compensation method, as illustrated in Fig. 1, requires to generalise the arguments from [19] to

<sup>4</sup>The goal of this paper is not to debate how to construct such scheduling sequences and the corresponding robust sets (for tracking), since these issues have been previously addressed by the Authors in [6], [14].

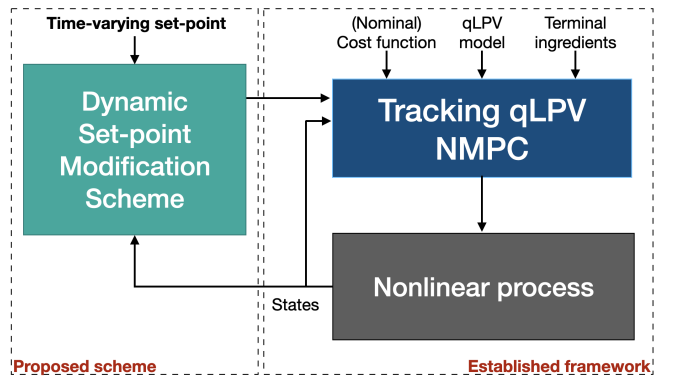


Fig. 1. Proposed scheme and qLPV NMPC.

the qLPV setting. **To this end, we first consider that**, at a given sampling instant  $k$ , during the implementation of the standard Tracking qLPV NMPC in (2), the output reference target  $r(k)$  maps the correlated input-state tracking pair  $(x_r(k), u_r(k))$  - in Sec. III-C we detail how to generate these elements. Moreover, we re-state<sup>5</sup> (2) in a generic quadratic format, i.e. as the optimisation problem  $\mathfrak{P}_{\text{NMPC}}$  - which seeks to determine the decision vector  $v$  that minimises the cost  $J(v, z)$  s.t.  $Fv \in \mathcal{C}$ , being  $z$  known parameters. Then, we assume that the generic cost  $J(v, z)$  is quadratic w.r.t.  $v$ , and that its second-order derivative  $\frac{\partial^2}{\partial v^2} J(v, z)$  is positive (i.e.  $J(v, z)$  is a strictly convex function).

Hence, our argument from [19] is valid: consider that  $v^*$  is the optimal solution related to  $\mathfrak{P}_{\text{NMPC}}$ , under  $Fv^* \in (\mathcal{C} \ominus \delta\mathcal{C})$ , being  $\delta > 1$  the smallest scalar **such that the boundaries of  $\mathcal{C}$  are excluded**. That is, without any constraints active - and thus with artificial variables valued as their respective targets ( $x_a = x_r$  and  $u_a = u_r$ ). Then, it is well-established in the literature that  $\nabla_v J(v^*, z) = 0$  gives the optimal unconstrained solution of  $\mathfrak{P}_{\text{NMPC}}$ . **By exploring this simple result**, we propose to dynamically alter the set-point  $r_{TV}(k)$  to the closest possible target  $r(k)$  which can be tracked by this unconstrained predictive control law. The rationale is to pass to the NMPC an adapted reference value  $r(k)$  to correct the possible tracking deviation from the original target  $r_{TV}(k)$ , considering the input region associated with unconstrained law.

Consider the analytical quadratic expression for  $J(v, z)$ :

$$J(v, z) := \frac{1}{2} v^\top H(P_k) v + z^\top S(P_k) v + f^\top(P_k) v + h_z(z) + c. \quad (4)$$

Then, from the optimality result applied to (4), it is implied that the divergent  $\nabla_v J(v^*, z) = 0$  and, thus, that the corresponding optimal unconstrained solution is given by:

$$v^* = -H^{-1}(P_k) (S^\top(P_k) z + f(P_k)). \quad (5)$$

*Remark 1:* The predicted scheduling trajectory associated to (2),  $P_k = \text{col}(\rho(k), \rho(k+1|k) \cdots, \rho(k+N_p-1|k))$ ,

<sup>5</sup>The Tracking qLPV NMPC in (2) can be expressed in the generic format of  $\mathfrak{P}_{\text{NMPC}}$  with the decision variable  $v := (U_k)$  and the compact parametric term  $z := (x(k), P_k, x_r(k), u_r(k))$ .

can be computed via several different methods, cf. [12], [8], [13]. For brevity, we assume  $P_k$  is at hand at each sampling instant and omit further discussions due to scope.

Next, we translate the optimal unconstrained minimiser of  $\mathfrak{P}_{\text{NMPC}}$  in (5) to the Tracking qLPV NMPC scheme in (2). That is, we rewrite the tracking cost function from (3), with  $x_a = x_r$  and  $u_a = u_r$ , in the form of (4), which yields:

$$J_k = \frac{1}{2}U_k^\top H(P_k)U_k + U_k^\top g_x^\top(P_k)x(k) + U_k^\top g_{x_r}^\top(P_k)x_r(k) + U_k^\top g_{u_r}^\top(P_k)u_r(k) + c. \quad (6)$$

Accordingly, the optimal unconstrained predictive control law related to (2), i.e.  $u^*(k)$  as the first entry of  $U_k^*$ , can be obtained by differentiating (6) w.r.t.  $U_k$ , which leads to:

$$\begin{aligned} u^*(k) &= K(P_k)x(k) + L_x(P_k)x_r(k) \\ &+ L_u(P_k)u_r(k), \\ &= K(P_k)x(k) + L_\theta(P_k)\theta(k), \end{aligned} \quad (7)$$

being  $K(P_k)$  a parameter-dependent feedback gain,  $L_x(P_k)$  a parameter-dependent feed-forward gain related to the state target and  $L_u(P_k)u_r(k)$ , and  $L_\theta(P_k) = L_x(P_k)M_x(\rho) + L_u(P_k)M_u(\rho)$ . These gains are analytically given by:

$$\begin{cases} K(P_k) & := -\overbrace{[1_{n_u}, 0_{n_u, (N_p-1)}]}^{\mathcal{I}} H^{-1}(P_k)g_x(P_k), \\ L_x(P_k) & := -\mathcal{I}H^{-1}(P_k)g_{x_r}(P_k), \\ L_u(P_k) & := -\mathcal{I}H^{-1}(P_k)g_{u_r}(P_k), \end{cases}$$

and the related (sampled) Hessian and gradients are:

$$\begin{cases} H(P_k) & := 2(\mathbf{B}(P_k)^\top \check{Q} \mathbf{B}(P_k) + \check{R}), \\ g_x(P_k) & := 2\mathbf{B}(P_k)^\top \check{Q} \mathbf{A}(P_k), \\ g_{x_r}(P_k) & := -2\mathbf{B}(P_k)^\top \check{Q}, \\ g_{u_r}(P_k) & := -2\check{R}, \end{cases} \quad (9)$$

considering that  $\check{R} = \text{blkdiag}(R, \dots, R)$  and that  $\check{Q} = \text{blkdiag}(Q, \dots, Q, P)$ . The matrix-valued terms in (9) related to (6) implicitly depend on the scheduling trajectory  $P_k$  at each sampling. Such dependency accounts for the parametric matrices  $\mathbf{A}(\hat{P}_k)$  and  $\mathbf{B}(\hat{P}_k)$ , given in (10).

### B. Reference compensation law

The unconstrained predictive control law in (7) is time-varying, given the qLPV nature of the process - yet parametrised by the scheduling term  $\rho(k)$ . As of this, one can directly map the relation from the arguments  $x(k)$  and  $(x_r(k), u_r(k))$  to the law  $u^*(k)$  by constructing the different gains  $[K(P_k) L_x(P_k) L_u(P_k)]$  for the whole scheduling space  $\mathcal{P}$ . We note that the state target  $x_r(k)$  is some given trajectory to be followed, related to  $r(k)$ , while  $u_r(k)$  can be computed analytically according to the result in Sec. III-C.

Then, considering that  $x_r(k)$  and  $u_r(k)$  are at hand, it is implied from (7) that there always exists a reverse output target computation from the unconstrained law, as long as  $\det(L_\theta(\cdot)) \neq 0$ . In practice, given that (7) defines the unconstrained solution of the qLPV NMPC in (2) (which ensures piece-wise constant reference tracking), it is implied that  $\det(L_\theta(\cdot)) \neq 0$  since any admissible target can be

tracked cf. [19, Proposition 1]. Therefore, for a time-varying reference  $r_{\text{TV}}(k)$ , our analytical compensation law is:

$$r(k) = M_y(\rho)L_\theta^{-1}(P_k)(-K(P_k)x_{\text{TV}}(k) + u_{\text{TV}}(k)), \quad (11)$$

being  $u_{\text{TV}}(k)$  and  $x_{\text{TV}}(k)$  in (11) the input-state target pair related to the dynamic set-point  $r_{\text{TV}}(k)$ .

In short, the proposed dynamic set-point modification scheme in (11) only requires to: (i) compute the feedback and feed-forward gains ( $K(P_k)$ ,  $L_x(P_k)$ ,  $L_u(P_k)$  and  $L_\rho(P_k)$ ) for sampled scheduling parameter value  $\rho(k)$ ; and (ii) solve the transformation in (11) to derive a modified reference target  $r(k)$  - which is passed to the qLPV NMPC (2). These operations are numerically inexpensive (simple algebraic operations) and, thus, do not significantly alter the resulting computational complexity of the NMPC algorithm. Notice that, in the absence of active constraints, the NMPC law converges to  $u^*(k) = K(P_k)(x(k) - x_{\text{TV}}(k)) + u_{\text{TV}}(k)$ .

### C. Feed-forward trajectory and input sequence

The proposed reference compensation scheme requires the time-varying state-input target sequences  $(x_{\text{TV}}(k), u_{\text{TV}}(k))$  so that the reference can be adapted via (11). The following remedy<sup>6</sup> can be used to construct the state-input sequences.

*Assumption 1:* Consider that outputs of the qLPV system in (1) should track a given sequence of targets  $R_{\text{TV}} := \{r_{\text{TV}}(k), r_{\text{TV}}(k+1), \dots\}$ , related to a corresponding state sequence  $X_{\text{TV}} := \{x_{\text{TV}}(k), x_{\text{TV}}(k+1), \dots\}$ . Assume that, at each sampling instant  $k$ , the controller has access to the current and successor state targets, i.e.  $x_{\text{TV}}(k)$  and  $x_{\text{TV}}(k+1)$ . Then, the corresponding feed-forward input trajectory  $U_{\text{TV}} := \{u_{\text{TV}}(k), u_{\text{TV}}(k+1), \dots\}$  should be defined such that  $x_{\text{TV}}(k+1) = A(\rho(k))x_{\text{TV}}(k) + B(\rho(k))u_{\text{TV}}(k)$  holds.

Notice that Assumption 1 considers that  $X_{\text{TV}}$  can be directly retrieved from  $R_{\text{TV}}$ , which, in practice, can be established by, e.g., an unknown input observer or an optimisation. Then, at each sample  $k$ , given  $\rho(k)$ , there exists an input target s.t.  $B(\rho(k))u_{\text{TV}}(k) = x_{\text{TV}}(k+1) - A(\rho(k))x_{\text{TV}}(k)$ . Given a stabilising feedback gain  $\kappa_t$ , the control law  $u(k) = \kappa_t(x(k) - x_r(k)) + u_{\text{TV}}(k)$  indeed drives  $e_x(k) = (x(k) - x_{\text{TV}}(k))$  to the origin.

## IV. CERTIFICATES

*Proposition 1:* Consider the qLPV process in (1), with states  $x(k)$  and scheduling parameters  $\rho(k)$  measured each sampling instant  $k$ . Then, qLPV NMPC for Tracking optimisation in (2), based on the adapted reference signal  $r(k)$  according to (11), is recursively feasible.

*Proof:* The recursive feasibility property does not depend on the  $r(k)$  since it has no effect on the constraints. Also, notice that the adaptation from  $r_{\text{TV}}(k)$  to  $r(k)$  in (11) is a mere characterisation of the target signal. Thus, the certificate from [14, Proposition 5.6] is preserved. For brevity, we only revisit the main argument: Consider that (2) (with cost (3) and tracking ingredients, i.e. terminal

<sup>6</sup>In the case of piece-wise constant set-points, the input-state steady-state pair  $(x_r, u_r)$  is constant and can be computed directly from the reference value, i.e., Assumption 1 is not necessary.

$$\overbrace{\begin{bmatrix} A(\rho(k)) \\ A(\rho(k+1))A(\rho(k)) \\ A(\rho(k+2))A(\rho(k+1))A(\rho(k)) \\ \vdots \\ \prod_{j=0}^{N_p-1} A(\rho(k+j)) \end{bmatrix}}^{\mathbf{A}(P_k) :=} \quad \text{and} \quad \overbrace{\begin{bmatrix} B(\rho(k)) & 0 & 0 & \cdots & 0 \\ A(\rho(k+1))B(\rho(k)) & B(\rho(k+1)) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A(\rho(k+N_p-1)) \cdots A(\rho(k+1))B(\rho(k)) & \cdots & \cdots & \cdots & B(\rho(k+N_p-1)) \end{bmatrix}}^{\mathbf{B}(P_k) :=} \quad (10)$$

cost  $V(\cdot)$ , stabilising gain  $\kappa_t$ , uncertainty sets  $\mathcal{S}(j)$  and the robustly admissible invariant set for tracking  $\Omega_t$  ensures robust constraint satisfaction and that  $(U_k^*, \theta_a^*(k))$  is the optimal solution at  $k$ , with  $x^*(k+j|k)$  being the optimal state evolution. Then, the candidates  $\theta_a^c(k+1) = \theta_a^*(k)$  and  $U_{k+1}^c = \text{col}(u^*(k+1|k), \dots, u^*(k+N_p-1|k), \kappa_t x^*(k+N_p|k) + L_\theta(P_k)\theta_a^*(k))$  with  $x^*(k+j|k) \in \mathcal{X} \ominus \mathcal{S}(j) \implies x^c(k+1+j|k+1) \in \mathcal{X} \ominus \mathcal{S}(j)$  define a feasible solution for (2) at instant  $k+1$ , thus establishing recursive feasibility. ■

*Lemma 1:* Consider the qLPV process in (1), with measured  $\rho(k)$  and  $x(k)$  at each sampling instant. Assume that the scheduling parameter deviation  $\delta\rho(k) := (\rho(k+1) - \rho(k))$  is bounded. Then, the predictive control law obtained by solving the qLPV NMPC for Tracking optimisation in (2) based on the reference signal  $r(k)$ , modified w.r.t. to the tracking target  $r_{\text{TV}}(k)$  according to (11) ensures input-to-state-practical-stability (ISpS).

*Proof:* Assume that  $r_{\text{TV}}(k) \in \mathcal{R}$  is bounded, being  $\mathcal{R} \subseteq \mathbb{R}^{n_y}$  a compact set. Then,  $r(k) \in \tilde{\mathcal{R}}$  for all  $k \geq 0$ , being  $\tilde{\mathcal{R}}$  is also a compact set, as a consequence of the modified target transformation, cf. [19, Proposition 1]. Moreover, assume that the qLPV NMPC scheme in (2) is recursively feasible. Accordingly, there exists a feasible region of attraction related to (2), namely  $\mathcal{D}$ , such that  $(x(k), \rho(k), r(k)) \in \mathcal{D}$  implies in  $(x(k+1), \rho(k+1), r(k+1)) \in \mathcal{D}$ . Thus, the following bound related to the artificial output target can be established:

$$\|r(k+1) - y_a^c(k+1)\|_T^2 - \|r(k) - y_a^*(k)\|_T^2 \leq (12) \\ \|r(k+1) - r(k)\|_T^2 < c_r,$$

considering  $c_r$  as a positive finite constant,  $y_a^*(k)$  as the optimal value of  $y_a$  in (2) at instant  $k$  and  $y_a^c(k+1) = y_a^*(k)$  as its candidate at the following instant. Indeed, the boundedness of  $\|r(k+1) - r(k)\|_T^2$  is reasonable for the fact that we consider (bounded) time-varying output tracking set-points. This bound is the main difference from ISpS to the typical ISS condition for constant references. Then, by defining  $V_L(k) = V_L(x(k), \rho(k), r(k)) := J_k - \|r(k+1) - y_a^*(k)\|_T^2$  as a Lyapunov candidate, we find that:

$$V_L(k+1) - V_L(k) \leq -\|x(k) - x_a^*\|_Q^2 + \gamma(\|\delta\rho(k)\|) + c_r, \quad (13)$$

considering that  $\gamma(\cdot)$  is a strictly positive scalar function that passes through the origin. Accordingly, (13) ensures ISpS in the presence of arbitrary, bounded time-varying target  $r_{\text{TV}}(k)$ . Moreover, if the set-point signal is non-admissible, but constant, ISS holds since  $c_r$  decays to zero. This concludes the proof. ■

## V. SIMULATION RESULTS

In order to demonstrate the effectiveness of the proposed reference modification scheme, in the context of qLPV NMPC algorithms “for Tracking”, we consider the cart and spring-damper benchmark system, as originally described in [21]. This nonlinear process is represented here by the following qLPV discrete-time model, operating under the sampling period of  $T_s = 400$  ms:

$$x(k+1) = \begin{bmatrix} 1 & 0.4 \\ -0.1320\rho(k) & 0.56 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} u(k), \\ y(k) = x(k), \\ \rho(k) = \exp(-x_1(k)).$$

This system has state and input constraints associated to its physical characteristics (position, speed and applied force). We consider that admissibility is defined by the following symmetric state and input bounds:  $\bar{x}_1 = \bar{x}_2 = 2$  m and  $\bar{u} = 5$  N. The steady-state parameterisation is simply  $M_u(\rho) = 0.33\rho$ ,  $M_x(\rho) = M_x = \text{col}(1, 0)$ , and  $M_y(\rho) = M_x$ .

Accordingly, we design the qLPV MPC for Tracking from [14], given in (2), by using the following tuning parameters: a prediction horizon of  $N_p = 10$  samples, performances weights  $Q = \mathbb{I}_2$  and  $R = 0.1\mathbb{I}_1$ , and and output tracking cost is  $T = \mathbb{I}_{n_y} 10^3 \|P\|_2$ . In particular, we use the (terminal) stabilising gain  $\kappa_t = \text{col}(-0.8907, -2.2529)^\top$  (required to compute the state-input target characterisation, cf. Appendix A) and the admissible invariant set for tracking  $\Omega_t := \{(x, \theta) \in \mathbb{R}^{n_x+n_u} : (x - x_a)^\top P(x - x_a) \leq \frac{1}{32.46}, |\theta| \leq 1.6435, x_a = M_x\theta\}$  defined by the positive definite matrix

$$P = \begin{bmatrix} 5.6934 & 1.8477 \\ \star & 2.1664 \end{bmatrix}.$$

This robust invariant set for tracking, in practice, imposes that  $|u_a| < M_u(\rho)1.565$  and  $|x_{a,1}| < 1.565$ . Notice that these sets are slightly reduced w.r.t. the original admissible sets - which is necessary to ensure recursive feasibility despite uncertainties, cf. [14].

Then, we compare<sup>7</sup> the tracking performances using the standard qLPV MPC for Tracking from [14], and the performances obtained with the same controller joint with the proposed reference compensation scheme (as illustrated in Fig. 1). In particular, we consider an arbitrarily varying (non-periodical and, in parts, non-admissible) output set-point signal  $r_{\text{TV}}(k)$ . The tracking results are presented in

<sup>7</sup>The results presented in the sequel were obtained using Matlab software and Yalmip toolbox; the numerics are relative to a M1 Pro 16 GB RAM Macintosh computer. QPs were solved using gurobi.

Figures 2 and (state trajectories and applied predictive control signal, respectively), considering the initial condition  $x(0) = \text{col}(-2.3, 1.5)$ . We highlight the main features:

- Both controllers are, as expected [9], real-time capable, in the sense that the average computational time required to solve the optimisation (QP) is of 39.70 ms with the standard controller and 40.80 ms with the standard controller coupled to the proposed dynamic reference modification. Indeed, the 1.10 ms time difference (in average) indicates the residual computational complexity associated to the reference compensation law in (11). As shown in [22], we emphasise that embedding these NMPCs to low-cost microcontrollers is today practical and possible, particularly thanks to the numerical ease that comes with the qLPV formulation.
- Moreover, the main advantage of the proposed scheme is that it considerably enhances the tracking performances. In practice, the standard qLPV NMPC is not able to track the considered time-varying reference with precision, while the proposed compensation scheme makes it follow the set-point very closely - it only avoids tracking when the reference is unreachable (i.e. outside the admissibility set  $\mathcal{Y}$ ). In numerical terms, we highlight that the resulting tracking cost  $J_T(k) = \left( \|x(k) - x_{\text{TV}}(k)\|_Q^2 + \|u(k) - u_{\text{TV}}(k)\|_Q^2 \right)$  is considerably better with the inclusion of the compensation scheme: the aggregated cost  $\sum_k J_T$  is reduced by 33.92%.

## VI. FINAL REMARKS

In this brief note, we discussed how to dynamically adapt time-varying output set-points in order to enhance the tracking performances of standard qLPV NMPC schemes. In particular, the proposed reference modification scheme is analytical, with residual numerical complexity (only algebraic transformations), and is derived from the unconstrained NMPC law. That is, given a time-varying set-point  $r_{\text{TV}}(k)$ , our scheme determines an adapted reference  $r(k)$  that corrects possible tracking deviations considering the input region associated with the unconstrained law. We show, using a cart and spring-damper nonlinear benchmark simulation, that our method indeed enables standard qLPV NMPC algorithms for tracking piece-wise constant signals, as the one in [14], to track arbitrarily time-varying output targets with ease.

**In future works, we plan to further compare the proposed scheme with NMPC scheme for tracking with dynamic terminal set optimisation, e.g. [18] and “oracle” NMPCs, for which the complete future reference signal is known/estimated. We also plan to exploit partitions of the unconstrained qLPV MPC input region to enhance results.**

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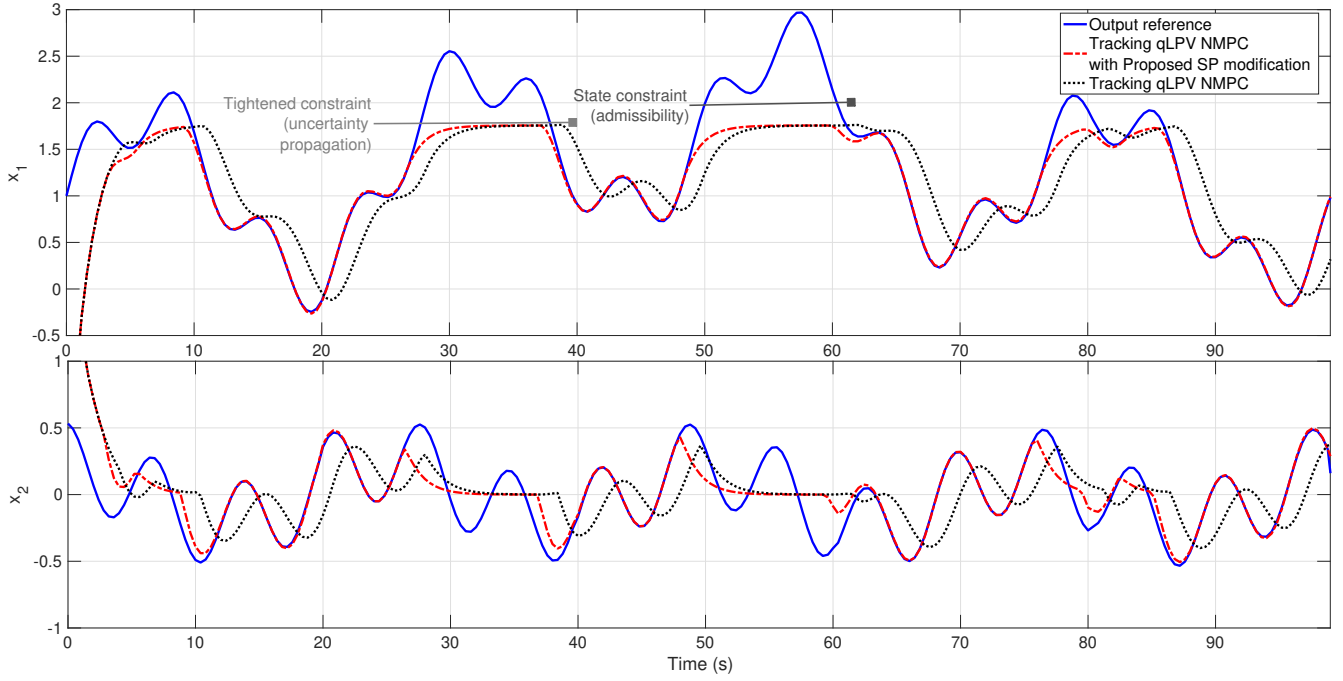


Fig. 2. State trajectories.

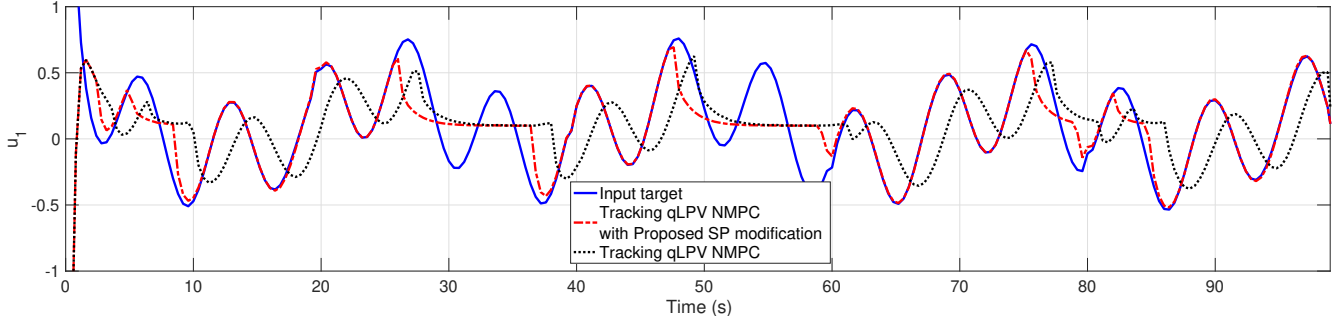


Fig. 3. Input signal.

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## APPENDIX

### A. LPV target parametrisation

Consider the qLPV model in (1) and the corresponding NMPC for Tracking in (2). Next, we analyse the (last constraint) target parametrisation  $(x_r, u_r, r) \in M_\theta(\rho)\theta$ . In particular, we derive the parameter-dependent matrix

$M_\theta(\rho)$ , scheduled by a fixed/sampled value for  $\rho$ , that maps the input selection variable  $\theta$  from the steady-state pair  $(x_r, u_r)$  and output reference  $r$ . For such, consider that  $u(k) = \kappa_t x(k) + \theta(k)$  as a stabilising feedback law that drives the state trajectories towards  $x_r$  and, thus, the output to the reference  $r$ . Accordingly, the corresponding closed-loop dynamics are given by  $x(k+1) = A_\pi(\rho(k))x(k) + B(\rho(k))\theta(k)$ , where  $A_\pi(\rho) := A(\rho) + B(\rho)\kappa_t$ . Then, by imposing  $\lim_{k \rightarrow +\infty} x(k) \rightarrow x_r$  for a fixed  $\rho$ , it follows that  $x_r = M_x(\rho)\theta_r$  and  $\lim_{k \rightarrow +\infty} \theta(k) \rightarrow \theta_r$  with  $M_x(\rho) := (\mathbb{I}_{n_x} - A_\pi(\rho))^{-1} B(\rho)$ . Furthermore,  $\lim_{k \rightarrow +\infty} u(k) \rightarrow u_r$  and  $\lim_{k \rightarrow +\infty} y(k) \rightarrow r$  imply that  $u_r = M_u(\rho)\theta_r$  with  $M_u(\rho) := (\kappa_t M_x(\rho) + \mathbb{I}_{n_x})$  and  $r = M_y(\rho)x_r$  with  $M_y(\rho) := C(\rho)M_x(\rho)$ , respectively. Thus, the target parametrisation is with the following matrix;  $M_\theta(\rho) := \text{col}(M_x(\rho) \ M_u(\rho) \ M_y(\rho))$ .