



HAL
open science

A useful invariant for elastic rods

Florence Bertails-Descoubes, Sébastien Neukirch

► **To cite this version:**

Florence Bertails-Descoubes, Sébastien Neukirch. A useful invariant for elastic rods. ESMC 2025 - 12th European Solid Mechanics Conference, Jul 2025, Lyon, France. <hal-05162593>

HAL Id: hal-05162593

<https://hal.science/hal-05162593v1>

Submitted on 15 Jul 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

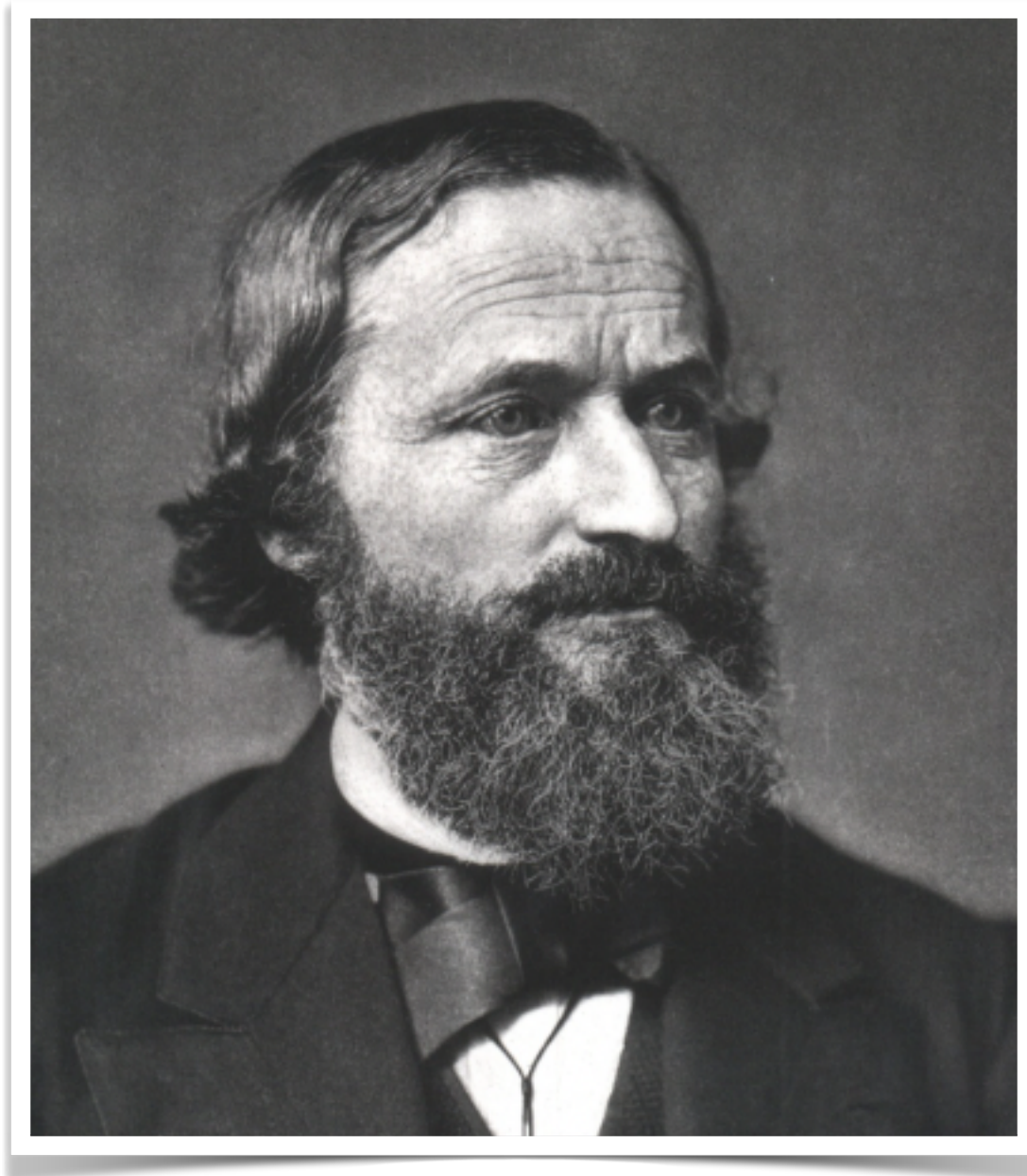


Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

KIRCHHOFF, NOETHER, AND MADDOCKS

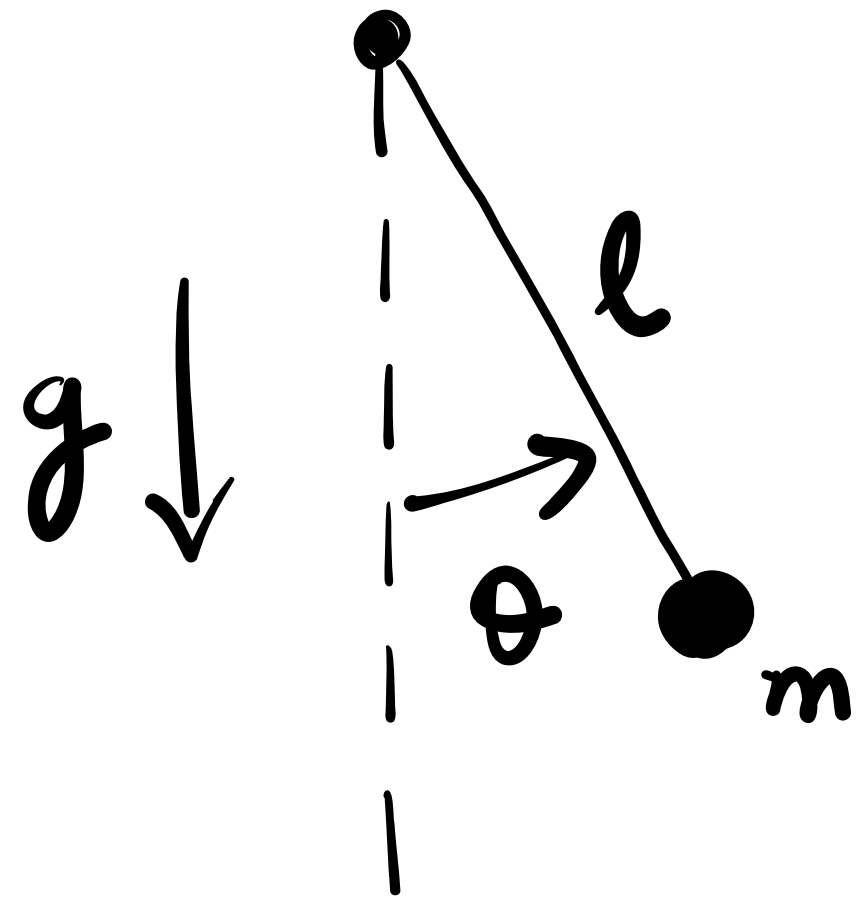
Florence Bertails-Descoubes (INRIA Grenoble, France)

Sébastien Neukirch (CNRS & Sorbonne University, Paris, France)



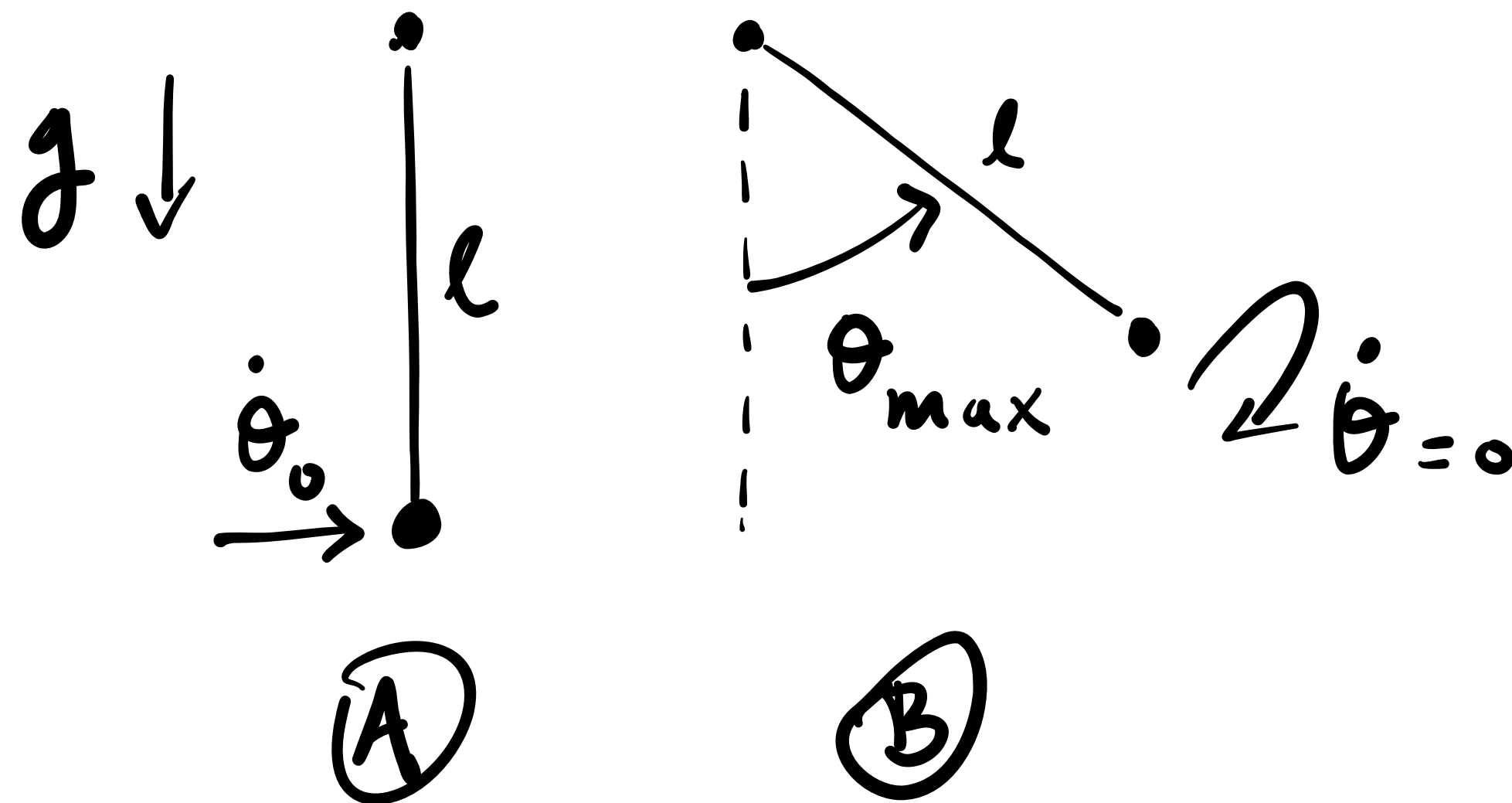
CONSERVATION OF MECHANICAL ENERGY

Dynamics of the nonlinear pendulum



$$\ddot{\theta}(t) = -\frac{g}{\ell} \sin \theta(t)$$

$\theta(t)$ using Jacobi Elliptic functions



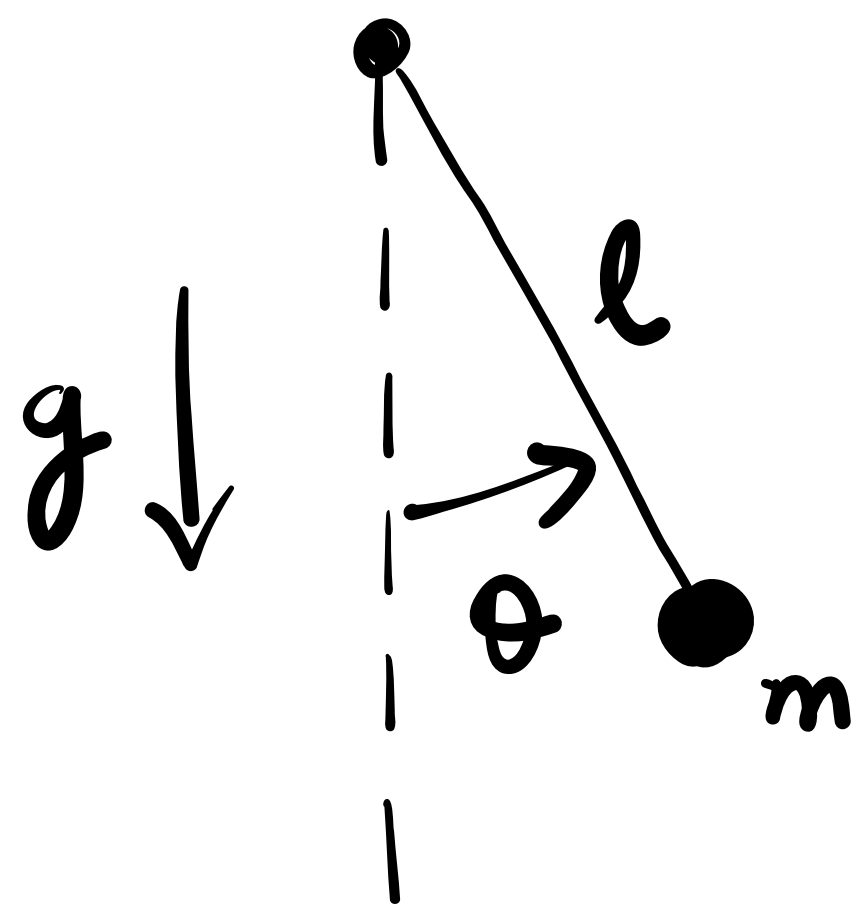
$$H = \frac{1}{2} \dot{\theta}^2 - \frac{g}{\ell} \cos \theta$$

(conserved) mechanical energy, i.e. $\dot{H} = 0$

$$H_A = H_B \Rightarrow \cos \theta_{max} = 1 - \frac{\ell}{2g} \dot{\theta}_0^2$$

KIRCHHOFF ANALOGY

nonlinear pendulum



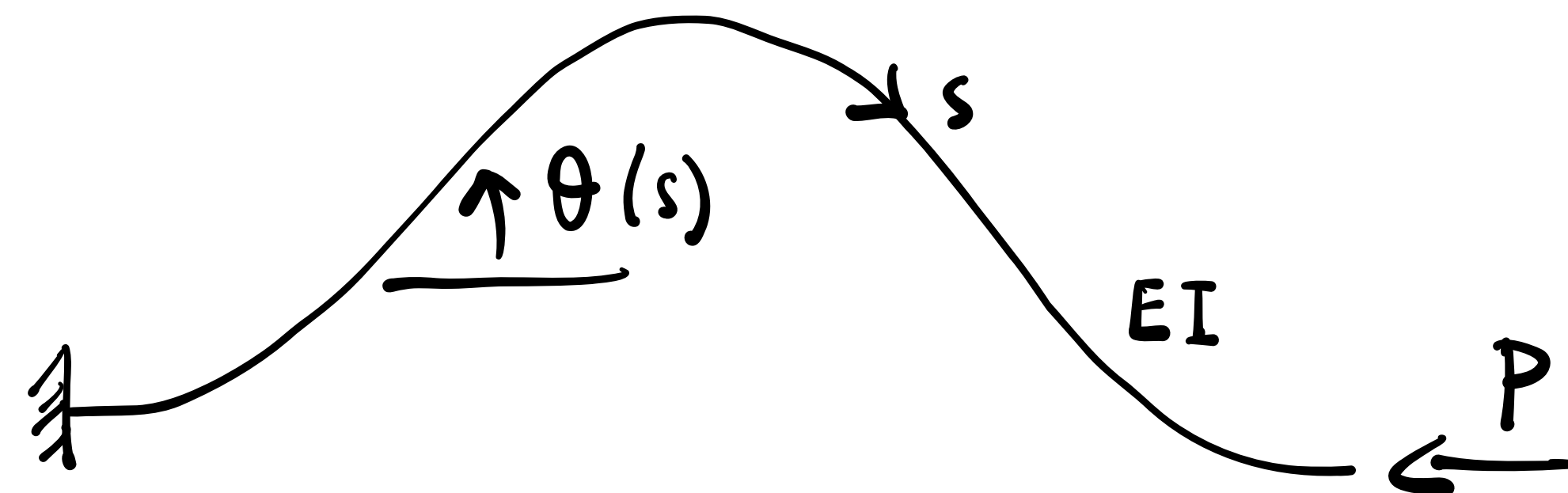
dynamics

$$\ddot{\theta}(t) = -\frac{g}{\ell} \sin \theta(t)$$

mechanical energy

$$H = \frac{1}{2} \dot{\theta}^2 - \frac{g}{\ell} \cos \theta$$

planar elastica



$$\theta''(s) = -\frac{P}{EI} \sin \theta(s)$$

statics

$$H = \frac{1}{2} EI \theta'^2 - P \cos \theta$$

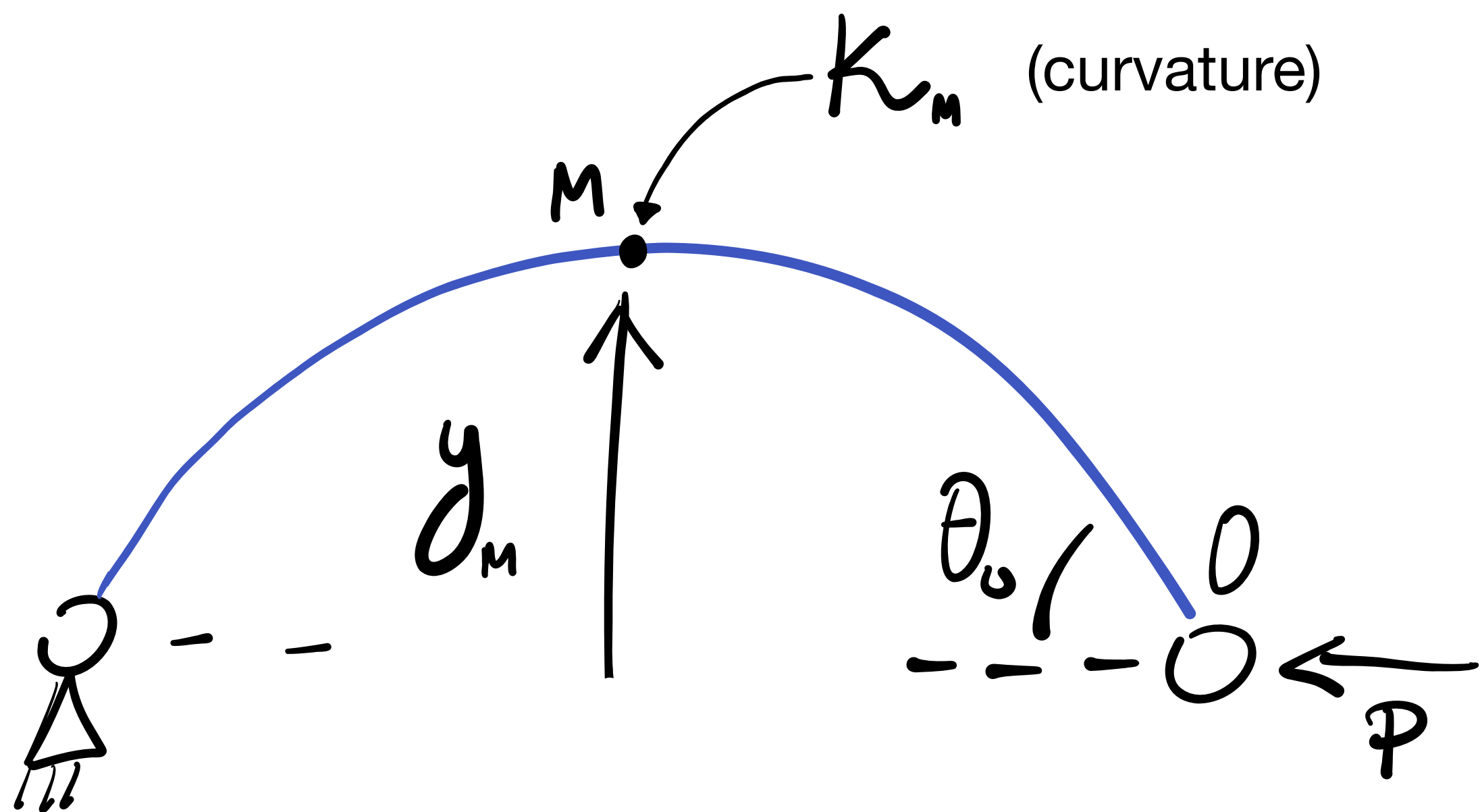
curvature energy tension

Maddocks invariant: $H' = 0$

A.E.H Love 1944 (first occurrence?)
 Kehrbaum & Maddocks 1997
 O'Reilly 2007
 Jung 2011 (discrete version)
 Singh 2021 (pseudo-momentum)

MAKING USE OF THE INVARIANT

pinned-pinned planar elastica



We impose P , we obtain the deflection θ_0

Q: values of y_M and κ_M ?

$$H_0 = -P \cos \theta_0$$

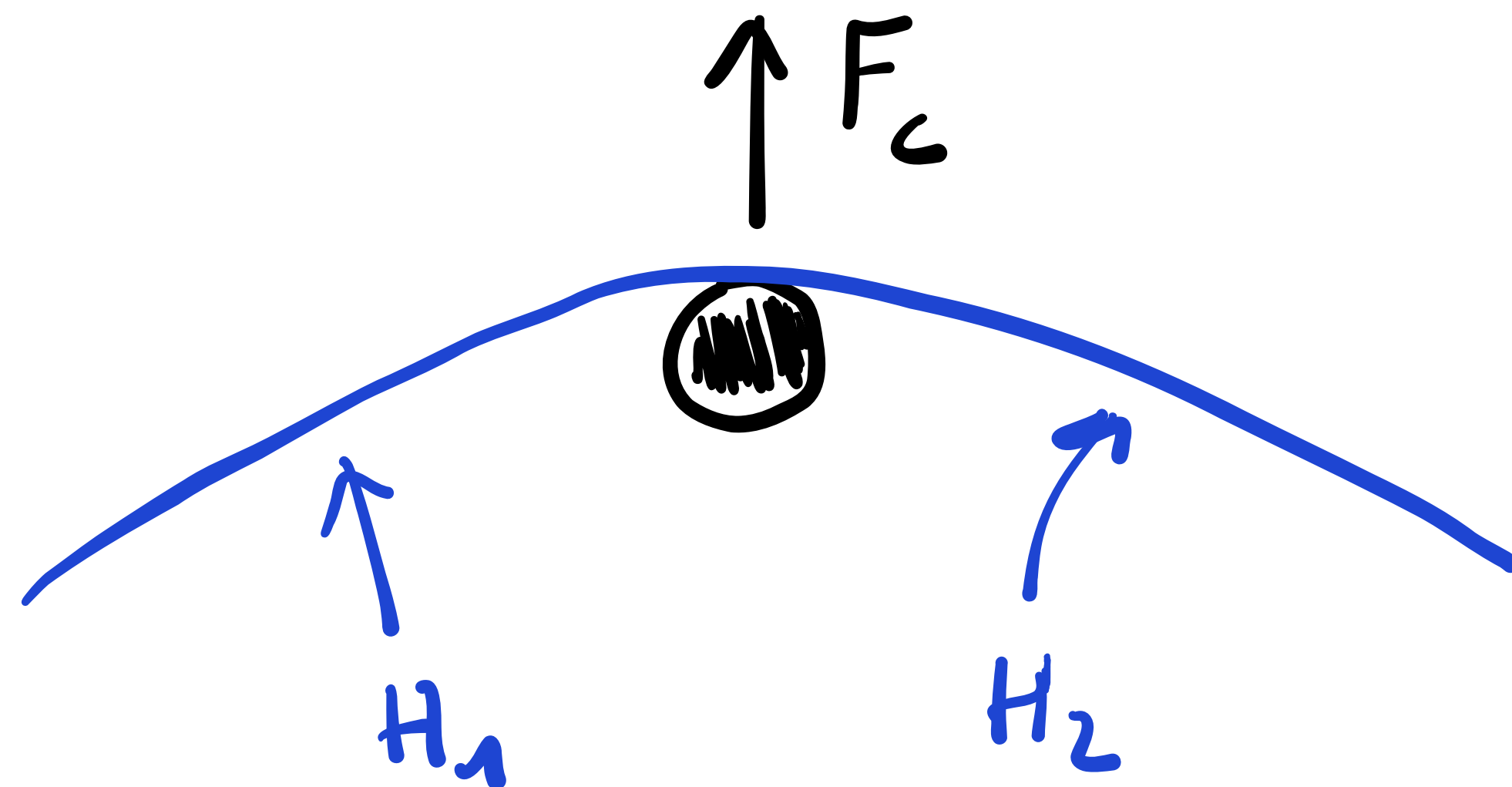
$$H_M = \frac{1}{2} EI \kappa_M^2 - P$$

$$H_0 = H_M \Rightarrow \begin{cases} \kappa_M = \sqrt{\frac{2P}{EI} (1 - \cos \theta_0)} \\ y_M = \sqrt{\frac{2EI}{P} (1 - \cos \theta_0)} \end{cases}$$

=> could be used for validation of numerics

HOW UNIVERSAL IS IT?

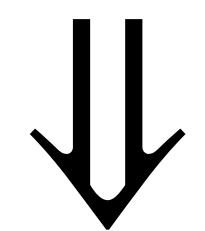
frictionless contact



curvature energy

$$H = \frac{1}{2} EI \theta'^2 + \text{tension}$$

- the curvature does not jump
- the tension does not jump



$$H_1 = H_2$$

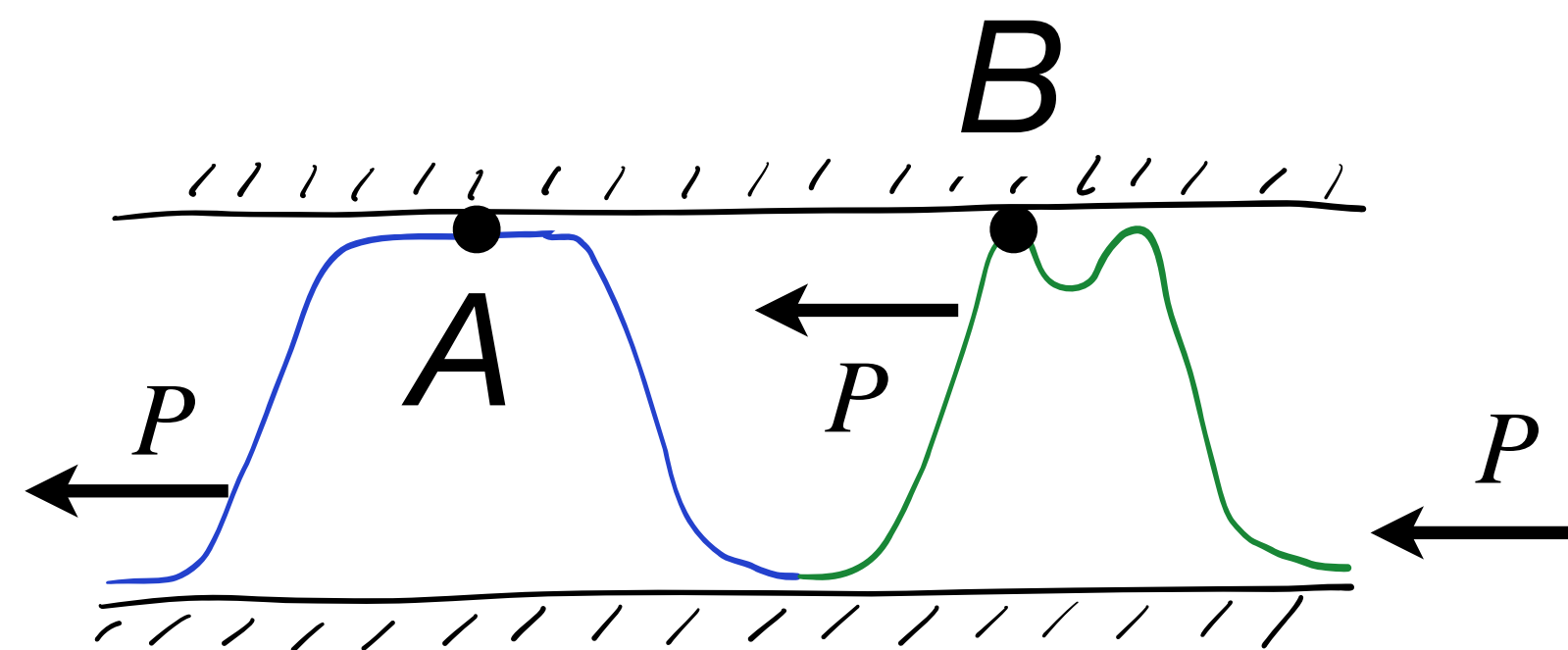
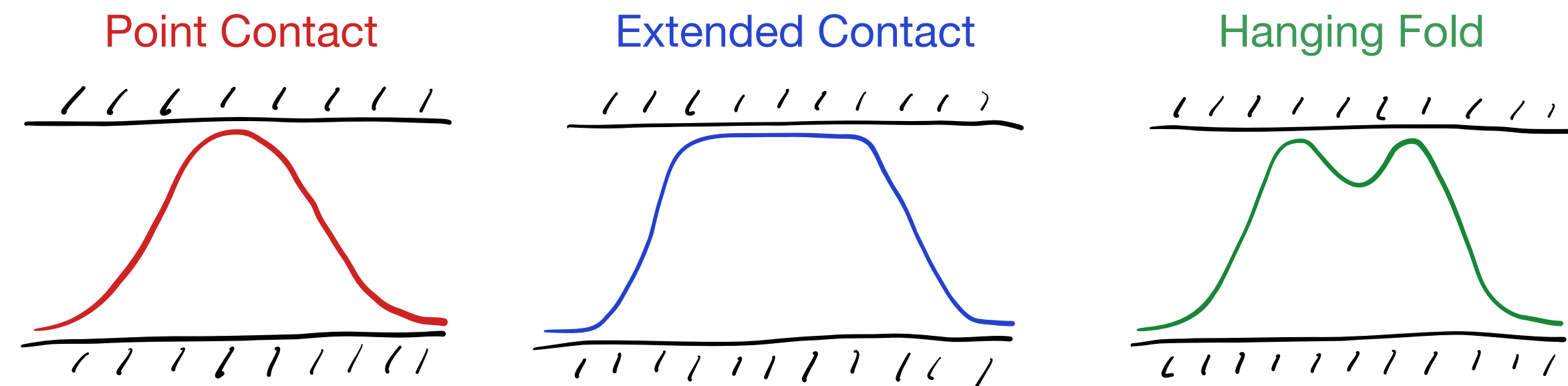
=> works in presence of **contact**

see also
Clauvelin 2009 (knot)
O'Reilly 2017 (jump cond.)
Singh 2022 (capstan)

MAKING USE OF THE INVARIANT (CONTACT)

Euler buckling in cavity

Lubinski 1962, Domokos 1996, Roman 1999, Deboeuf 2024, etc.

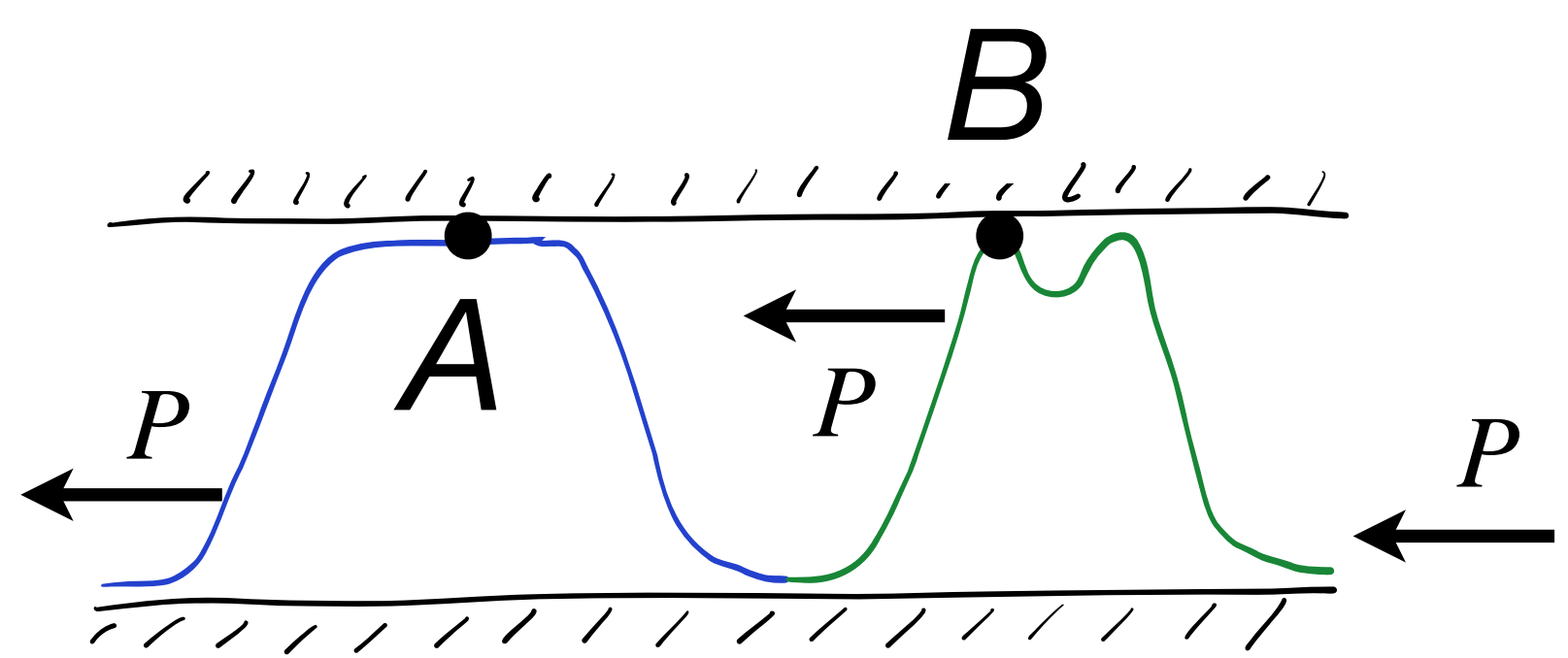
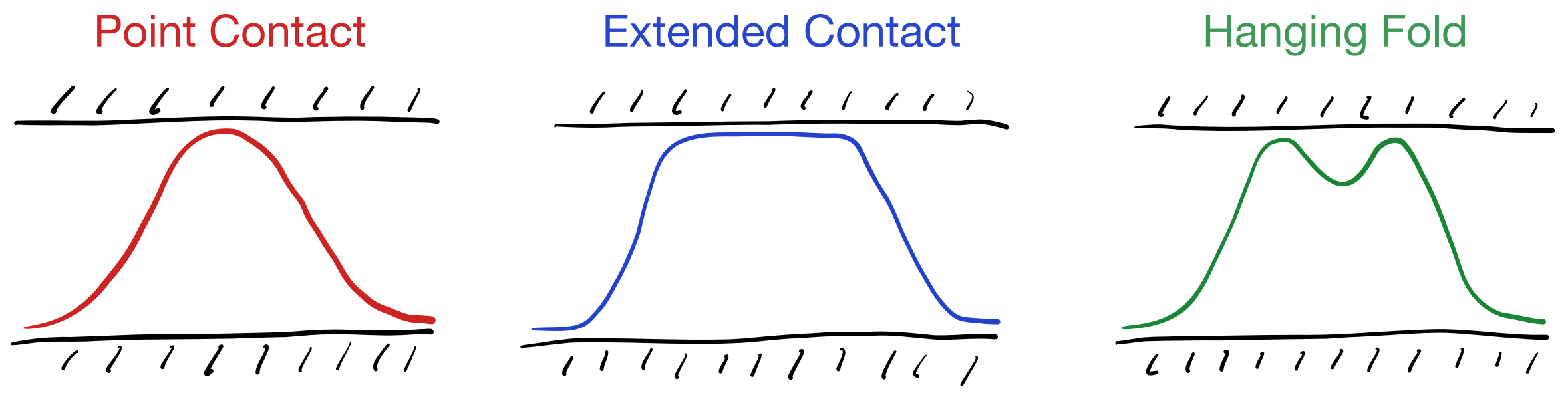


Question: coexistence?

MAKING USE OF THE INVARIANT (CONTACT)

Euler buckling in cavity

Lubinski 1962, Domokos 1996, Roman 1999, Deboeuf 2024, etc.



Question: coexistence?

Answer:

no friction $\Rightarrow P$ uniform

$$H_A = 0 - P$$

$$H_B = \frac{1}{2} EI \kappa_B^2 - P \text{ with } \kappa_B > 0$$

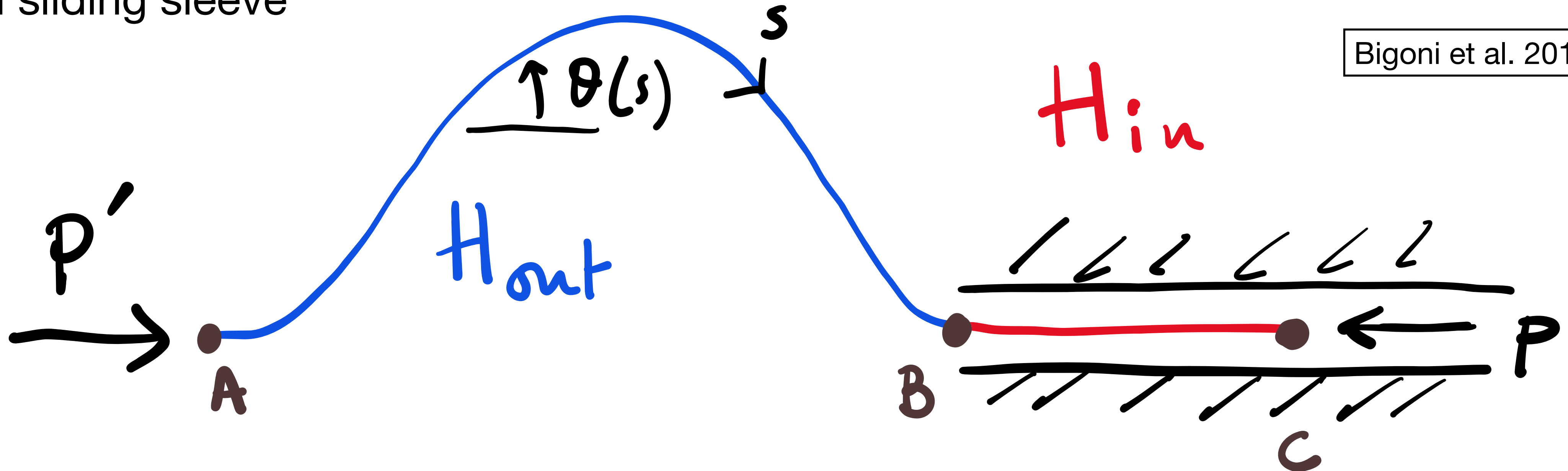
$\Rightarrow H_A = H_B$ impossible

\Rightarrow coexistence is impossible

MAKING USE OF THE INVARIANT (3)

rigid sliding sleeve

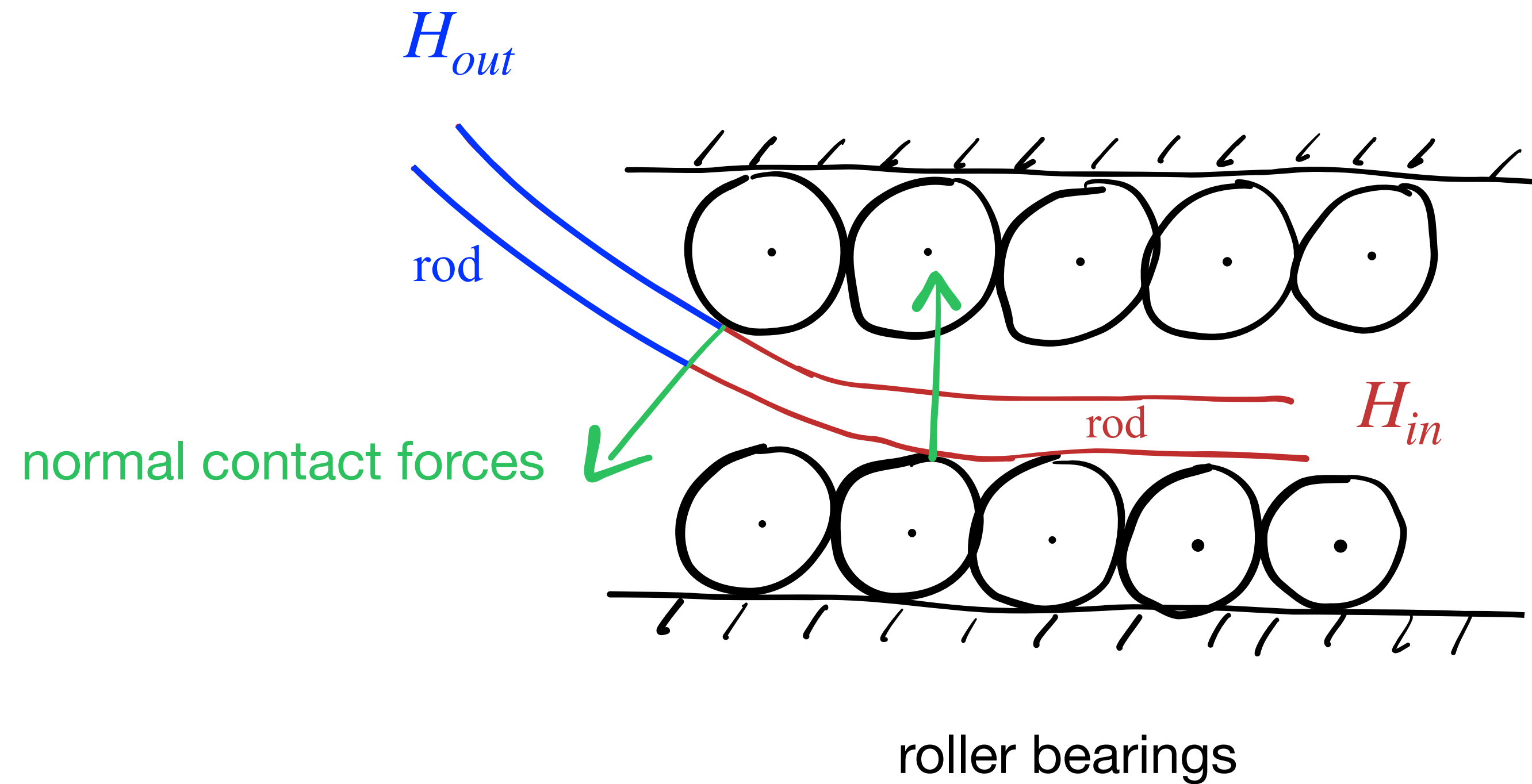
Bigoni et al. 2015



$$\left. \begin{aligned}
 H_{out} = H_A = H_B = \frac{1}{2} EI \kappa_B^2 - P' \\
 H_{in} = H_C = 0 - P
 \end{aligned} \right\} H_{out} = H_{in} \Rightarrow P' = P + \frac{1}{2} EI \kappa_B^2$$

CONSERVATION OF THE INVARIANT AT THE EXIT OF A SLIDING SLEEVE

Bigoni et al. 2015

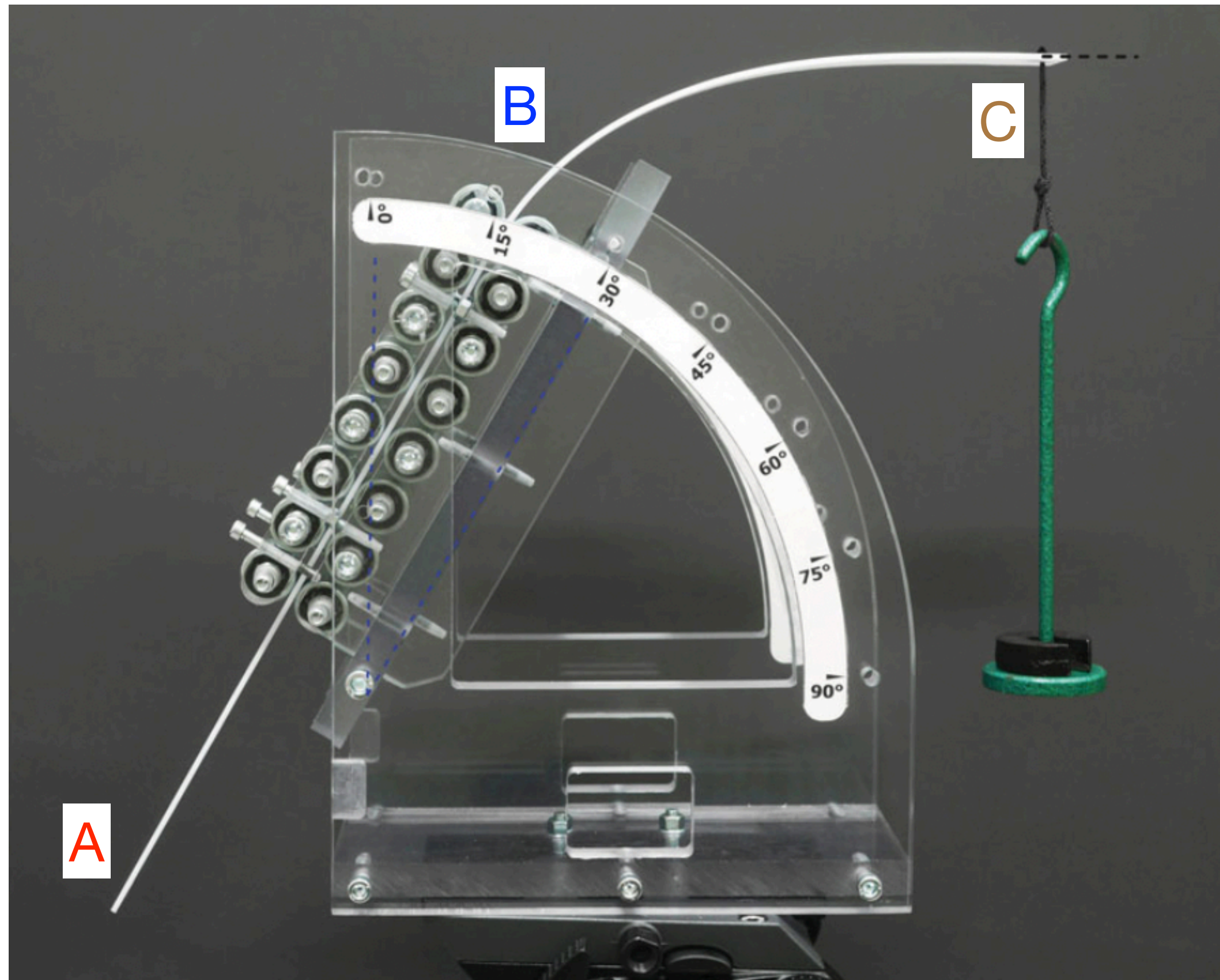


only normal contact forces $\Rightarrow H_{out} = H_{in}$

MAKING USE OF THE INVARIANT (4)

The Elastica Arm Scale

Bosi et al. 2014



$$H_A = 0 + 0$$

$$H_B = \frac{1}{2} EI \kappa^2 + tension = 0$$

=> the beam is under compression at point B

$$H_C = \frac{1}{2} EI \kappa_C^2 - Mg \mathbf{e}_y \cdot \text{tangente}_C = 0$$

bound. cond. => $\kappa_C = 0$

=> $\text{tangente}_C \perp \mathbf{e}_y$

THE INVARIANT: WHERE DOES IT COME FROM?

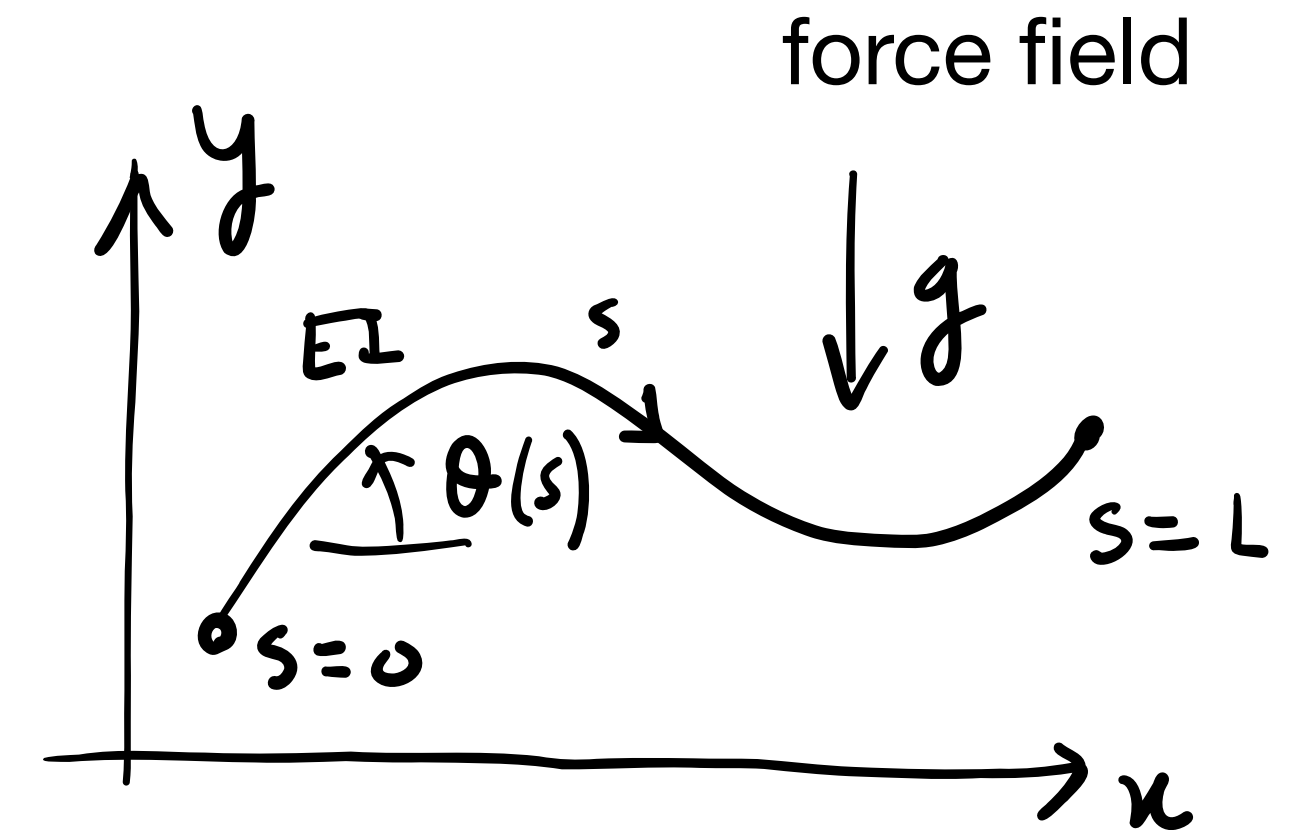
Variational approach $A = \int_0^L \mathcal{L}[\theta(s), \dots] ds$

$$W = \frac{1}{2} EI (\kappa(s) - \hat{\kappa})^2 + \frac{1}{2} EA (e(s) - 1)^2 + \rho A g y(s)$$

curvature

extension

gravity



bound. cond.

$$x(0) = \dots \quad x(L) = \dots$$

$$y(0) = \dots \quad y(L) = \dots$$

$$\theta(0) = \dots \quad \theta(L) = \dots$$

Under the kinematic constraints between $x(s), y(s), \theta(s), \kappa(s), e(s)$

$$\mathcal{L} = W + \text{constraints}$$

$$\mathcal{L} = \frac{1}{2} EI (\kappa(s) - \hat{\kappa})^2 + \frac{1}{2} EA (e(s) - 1)^2 + \rho A g y(s) + \lambda_1 (x' - e \cos \theta) + \lambda_2 (y' - e \sin \theta) + \lambda_3 (\theta' - \kappa)$$

study $\mathcal{L}(\mathbf{q}, \mathbf{q}')$ with $\mathbf{q} = \{x, y, e, \theta, \kappa\} \Rightarrow$ Euler-Lagrange: $\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \mathbf{q}'}$

THE INVARIANT: WHERE DOES IT COME FROM?

force field: $-\nabla V$

$$\frac{\partial \mathcal{L}}{\partial \kappa} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \kappa'} = 0 \Rightarrow \lambda_3 = EI (\kappa(s) - \hat{\kappa})$$

bending moment

$$\frac{\partial \mathcal{L}}{\partial e} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial e'} = 0 \Rightarrow \lambda_1 \cos \theta + \lambda_2 \sin \theta = EA (e(s) - 1)$$

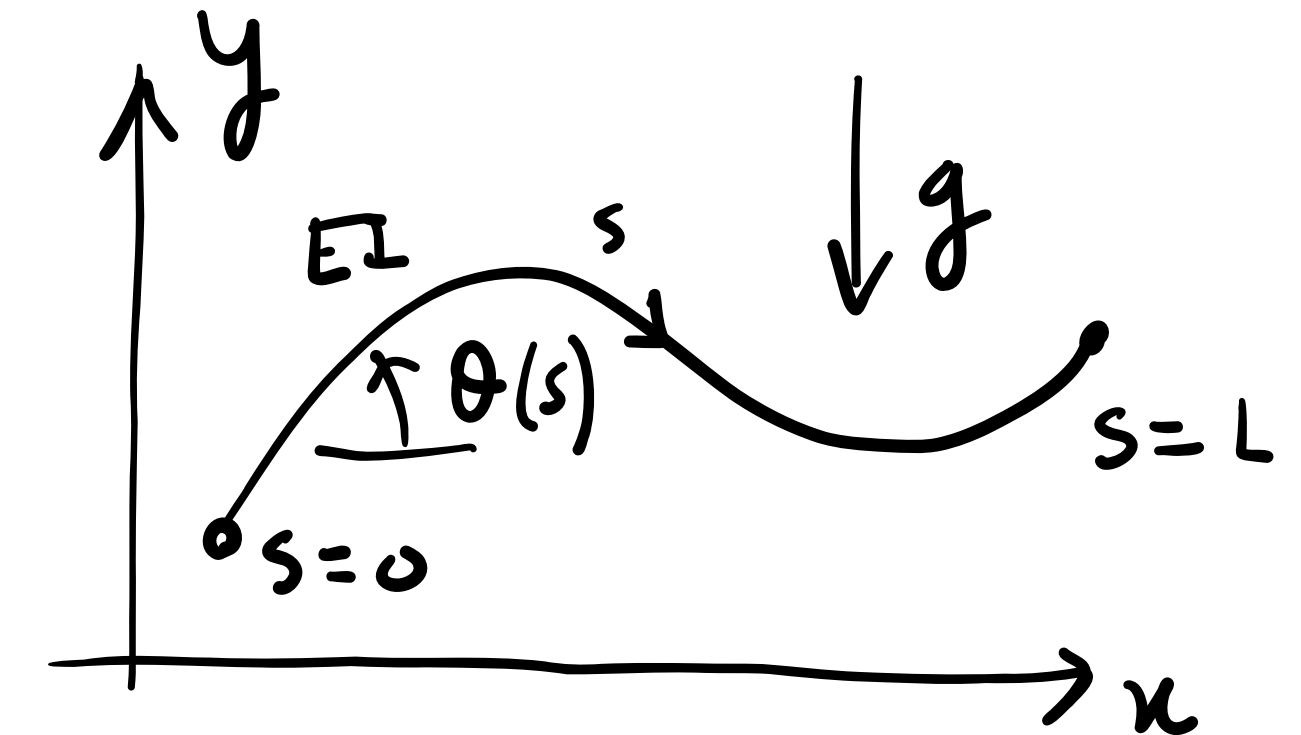
tension (axial force)

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial x'} = 0 \Rightarrow \lambda_1'(s) &= -\frac{\partial V(x, y)}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial y'} = 0 \Rightarrow \lambda_2'(s) &= -\frac{\partial V(x, y)}{\partial y} \end{aligned} \right\} \Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ force vector}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \theta'} = 0 \Rightarrow \lambda_3'(s) = \lambda_1 \sin \theta - \lambda_2 \cos \theta$$

$EI \theta''(s) = \text{shear force}$

constitutive relations



bound. cond.

$$\begin{aligned} x(0) &= \dots & x(L) &= \dots \\ y(0) &= \dots & y(L) &= \dots \\ \theta(0) &= \dots & \theta(L) &= \dots \end{aligned}$$

force and moment equilibrium

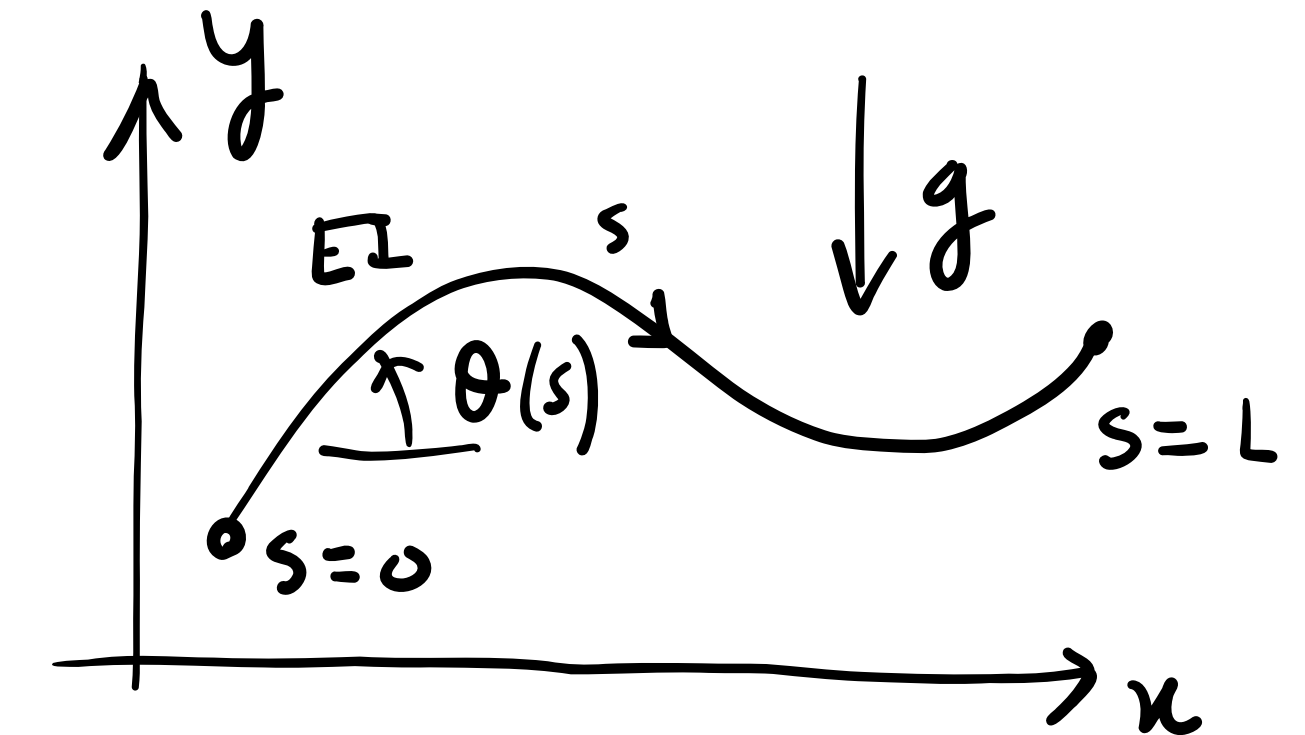
THE INVARIANT: WHERE DOES IT COME FROM?

force field: $-\nabla V$

$$\frac{\partial \mathcal{L}}{\partial \kappa} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \kappa'} = 0 \Rightarrow \overset{\substack{\text{bending moment} \\ \downarrow}}{\lambda_3} = EI (\kappa(s) - \hat{\kappa})$$

constitutive relations

$$\frac{\partial \mathcal{L}}{\partial e} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial e'} = 0 \Rightarrow \underbrace{\lambda_1 \cos \theta + \lambda_2 \sin \theta}_{\text{tension (axial force)}} = EA (e(s) - 1)$$



bound. cond.

$$\begin{aligned} x(0) &= \dots & x(L) &= \dots \\ y(0) &= \dots & y(L) &= \dots \\ \theta(0) &= \dots & \theta(L) &= \dots \end{aligned}$$

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial x'} = 0 \Rightarrow \lambda_1'(s) &= -\frac{\partial V(x, y)}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial y'} = 0 \Rightarrow \lambda_2'(s) &= -\frac{\partial V(x, y)}{\partial y} \end{aligned} \right\} \Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ force vector}$$

force and moment equilibrium

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \theta'} = 0 \Rightarrow \lambda_3'(s) = \underbrace{\lambda_1 \sin \theta - \lambda_2 \cos \theta}_{\text{shear force}}$$

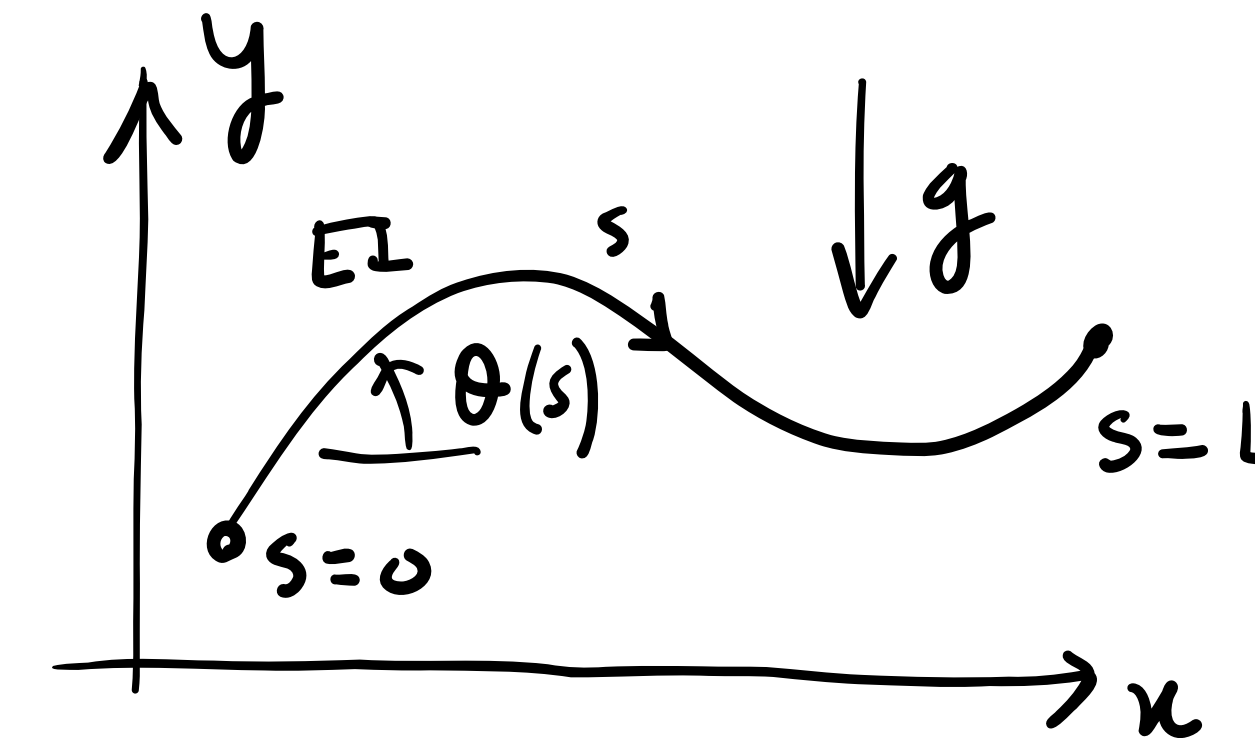
$$\downarrow$$

$$EI \theta''(s) = \text{shear force}$$

THE INVARIANT: WHERE DOES IT COME FROM?

force field: $-\nabla V$

in the case where $EI(s)$ and $EA(s)$ and $\hat{\kappa}(s)$ (no explicit s dependence)



Noether's theorem: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathcal{L}$ with $H'(s) \equiv 0$ at equilibrium $\mathbf{q} = \{x, y, e, \theta, \kappa\}$

Noether, 'Invariante Variationsprobleme'
Gottinger Nachrichten of 1918, pp. 235-257

$$H = \frac{\partial \mathcal{L}}{\partial x'} x' + \frac{\partial \mathcal{L}}{\partial y'} y' + \frac{\partial \mathcal{L}}{\partial e'} e' + \frac{\partial \mathcal{L}}{\partial \theta'} \theta' + \frac{\partial \mathcal{L}}{\partial \kappa'} \kappa' - \mathcal{L}$$

$$H = \frac{1}{2} EI [\kappa^2(s) - \hat{\kappa}^2] + \frac{1}{2} EA [e^2(s) - 1] - V(x, y)$$

not strain energies!

bound. cond.

$$x(0) = \dots \quad x(L) = \dots$$

$$y(0) = \dots \quad y(L) = \dots$$

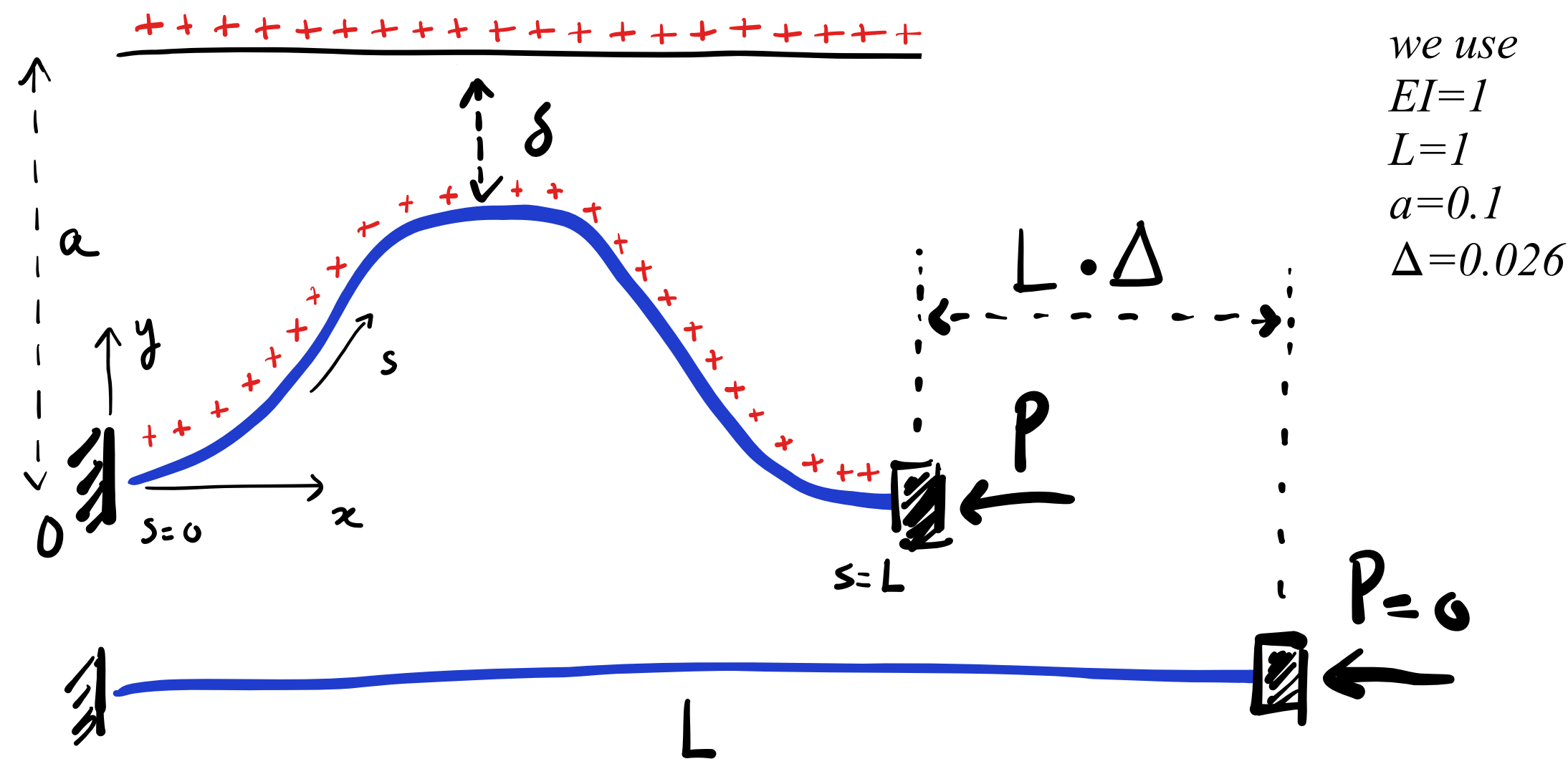
$$\theta(0) = \dots \quad \theta(L) = \dots$$

$$H = \underbrace{\frac{1}{2} EI \kappa^2(s) + tension}_{\text{inextensible naturally flat}} - V(x, y)$$

inextensible
naturally flat

force field: $-\nabla V$ $\left\{ \begin{array}{l} \text{Gravity} \\ \text{Electrostatics} \\ \text{Contact} \end{array} \right.$

CONSERVATIVE EXTERNAL POTENTIAL



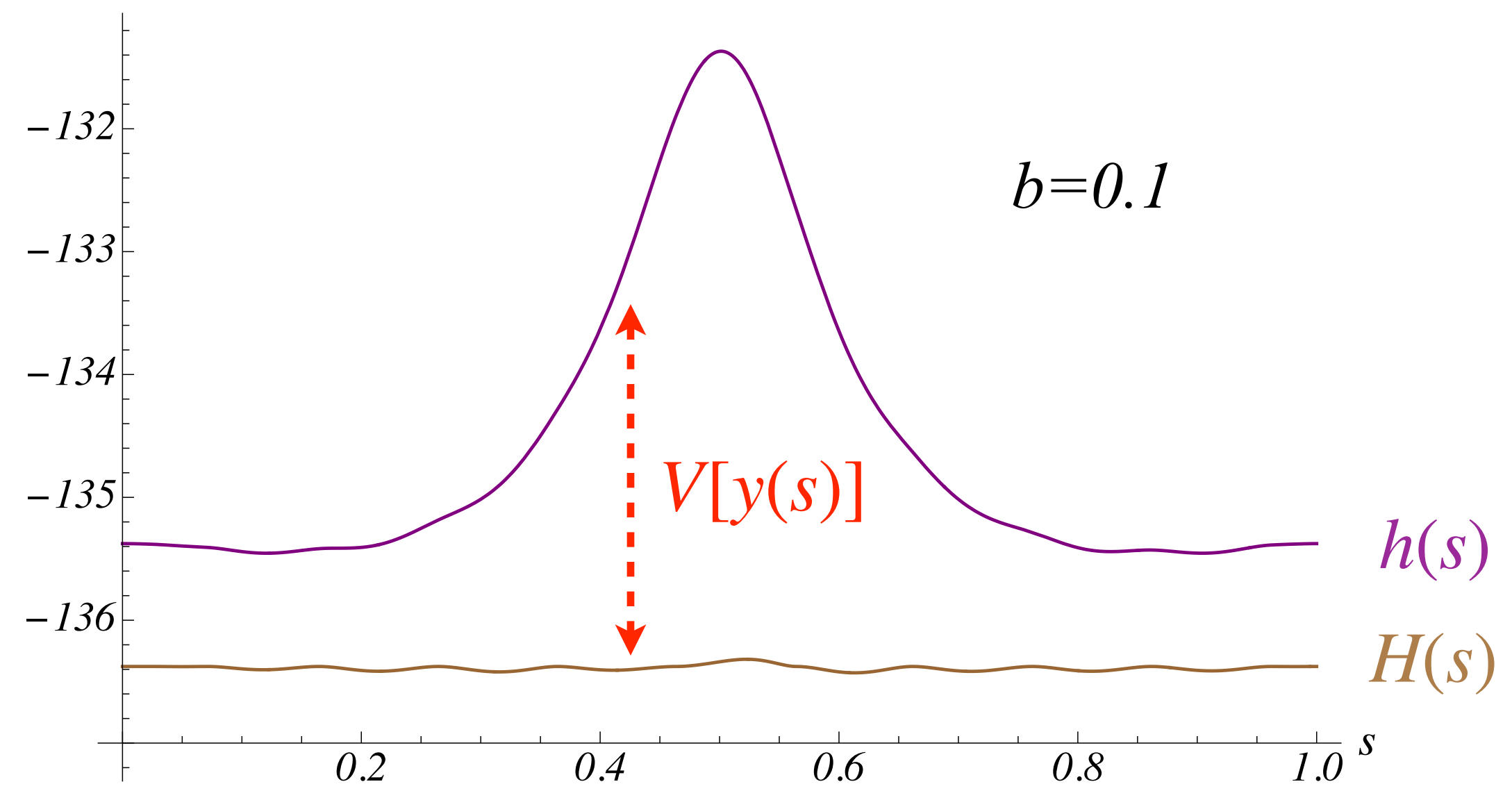
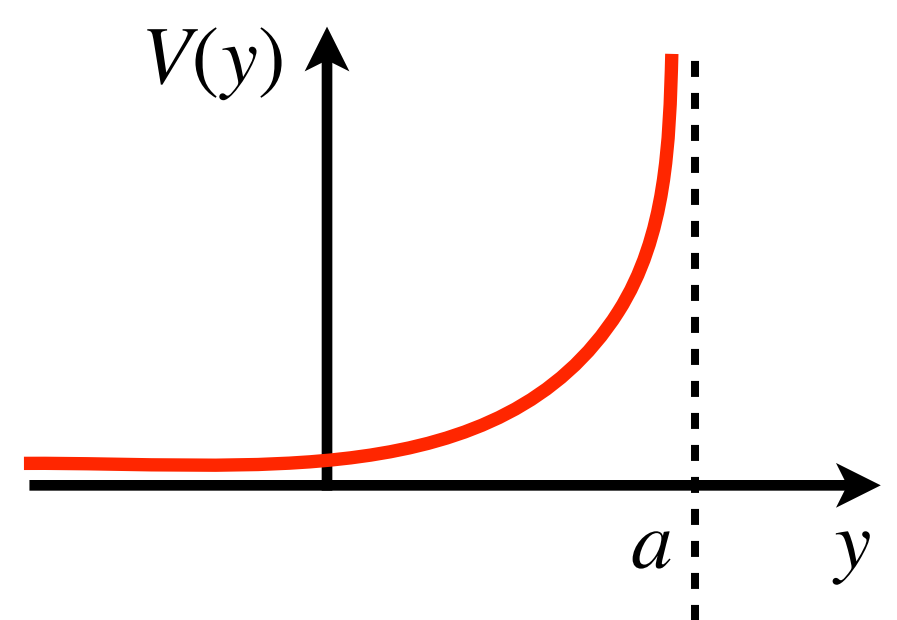
$$H = \underbrace{\frac{1}{2} EI \kappa^2(s) + \text{tension}}_{h(s)} - V[y(s)]$$

$$H'(s) = 0 \quad \forall s$$

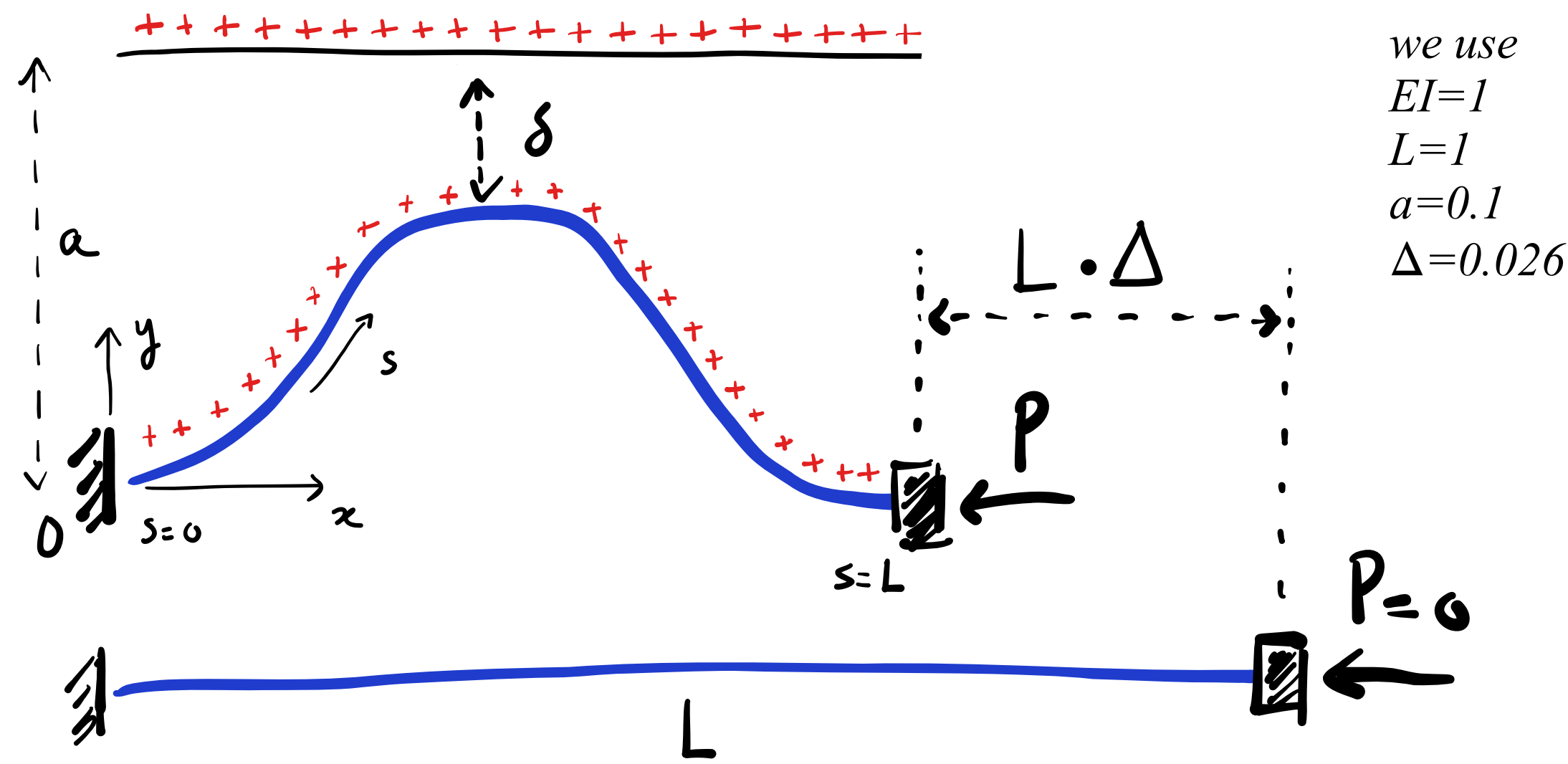
$$h'(s) \neq 0$$

soft-wall (or barrier) potential

$$V(y) = \frac{b}{a - y(s)}$$



CONSERVATIVE EXTERNAL POTENTIAL



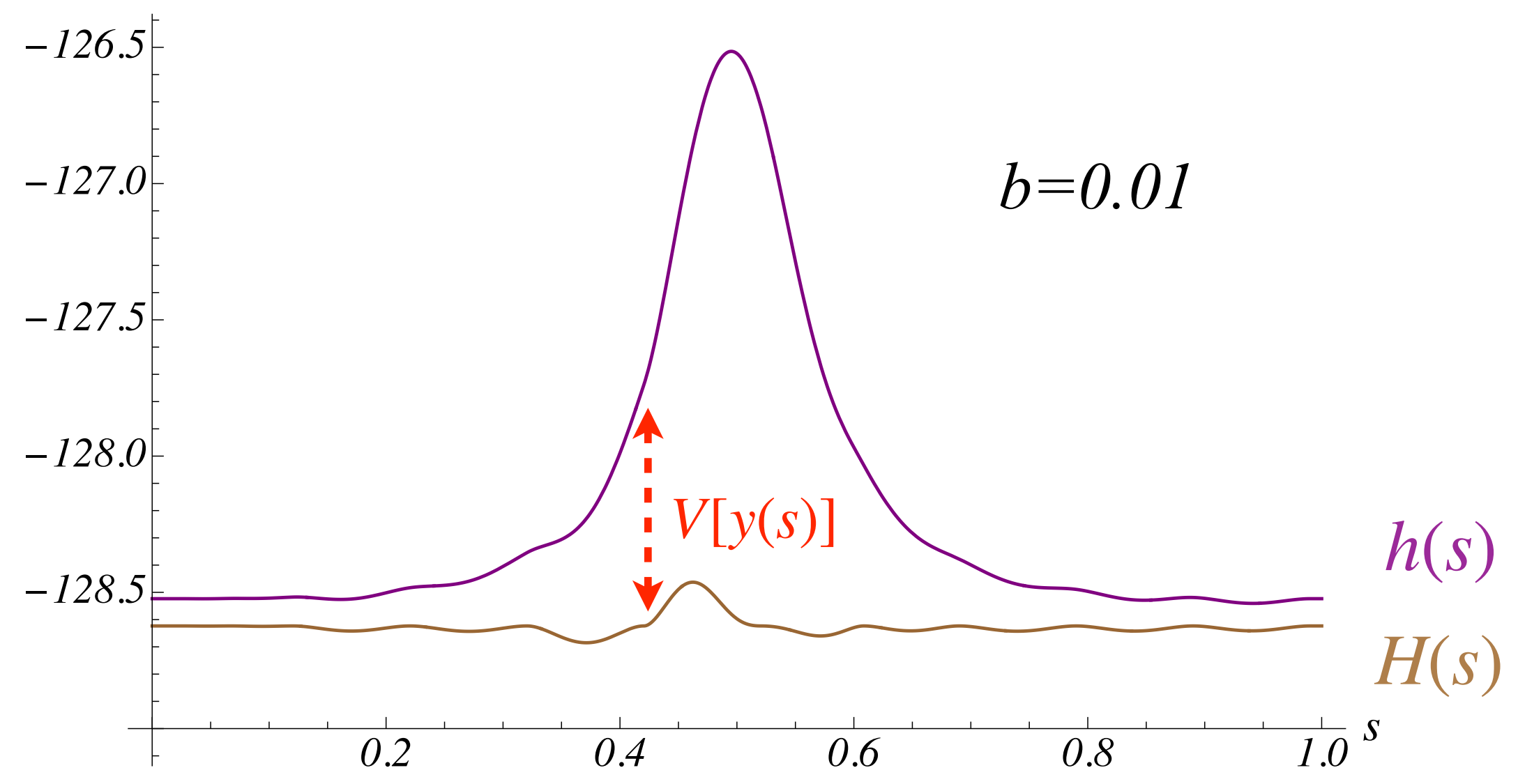
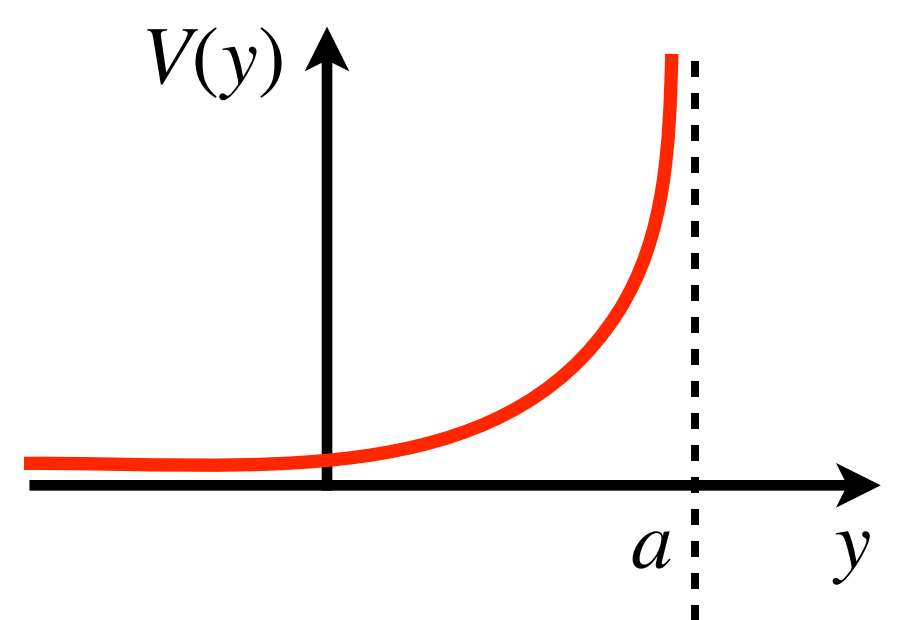
$$H = \underbrace{\frac{1}{2} EI \kappa^2(s) + \text{tension}}_{h(s)} - V[y(s)]$$

$$H'(s) = 0 \quad \forall s$$

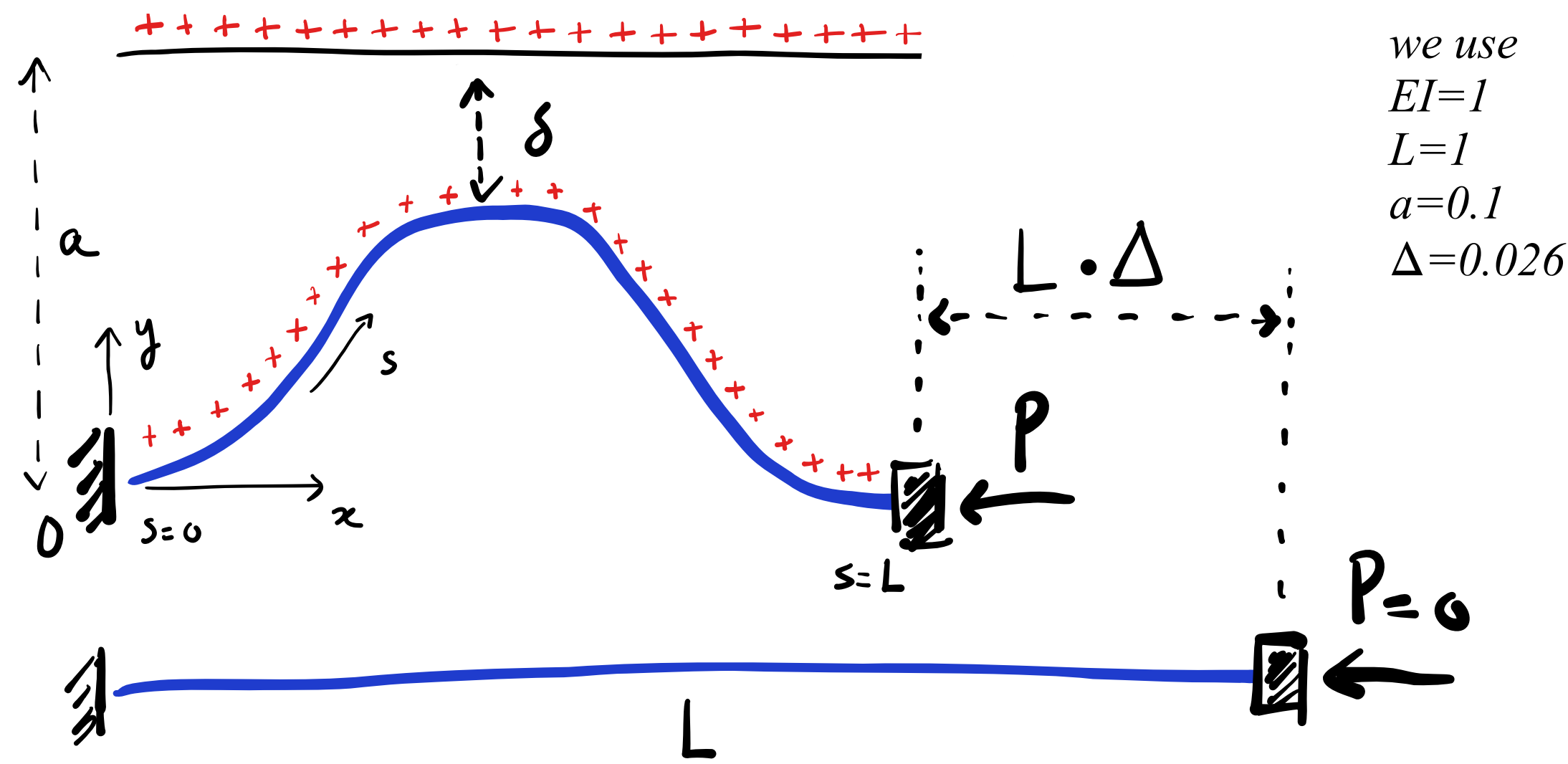
$$h'(s) \neq 0$$

soft-wall (or barrier) potential

$$V(y) = \frac{b}{a - y(s)}$$



CONSERVATIVE EXTERNAL POTENTIAL



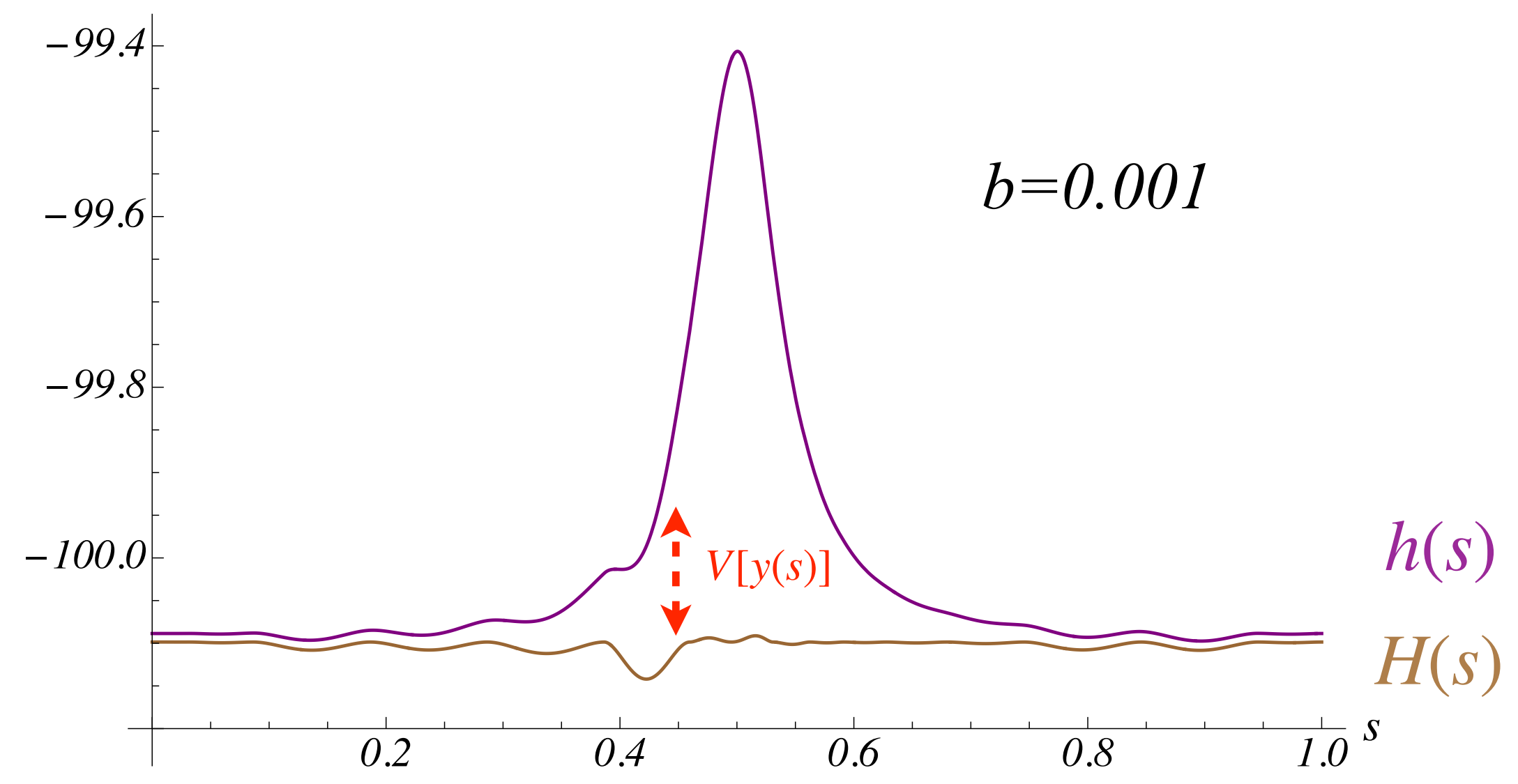
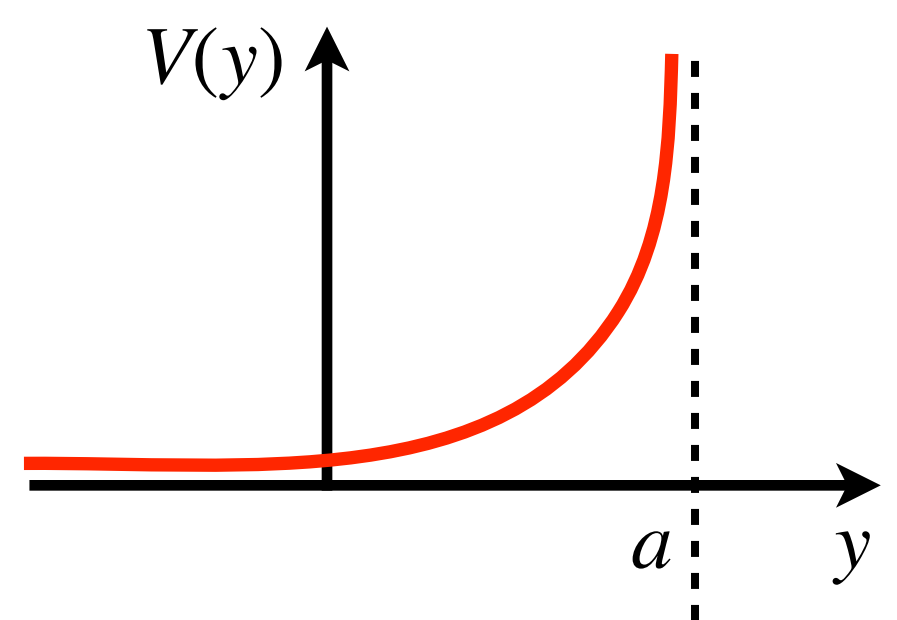
$$H = \underbrace{\frac{1}{2} EI \kappa^2(s) + \text{tension}}_{h(s)} - V[y(s)]$$

$$H'(s) = 0 \quad \forall s$$

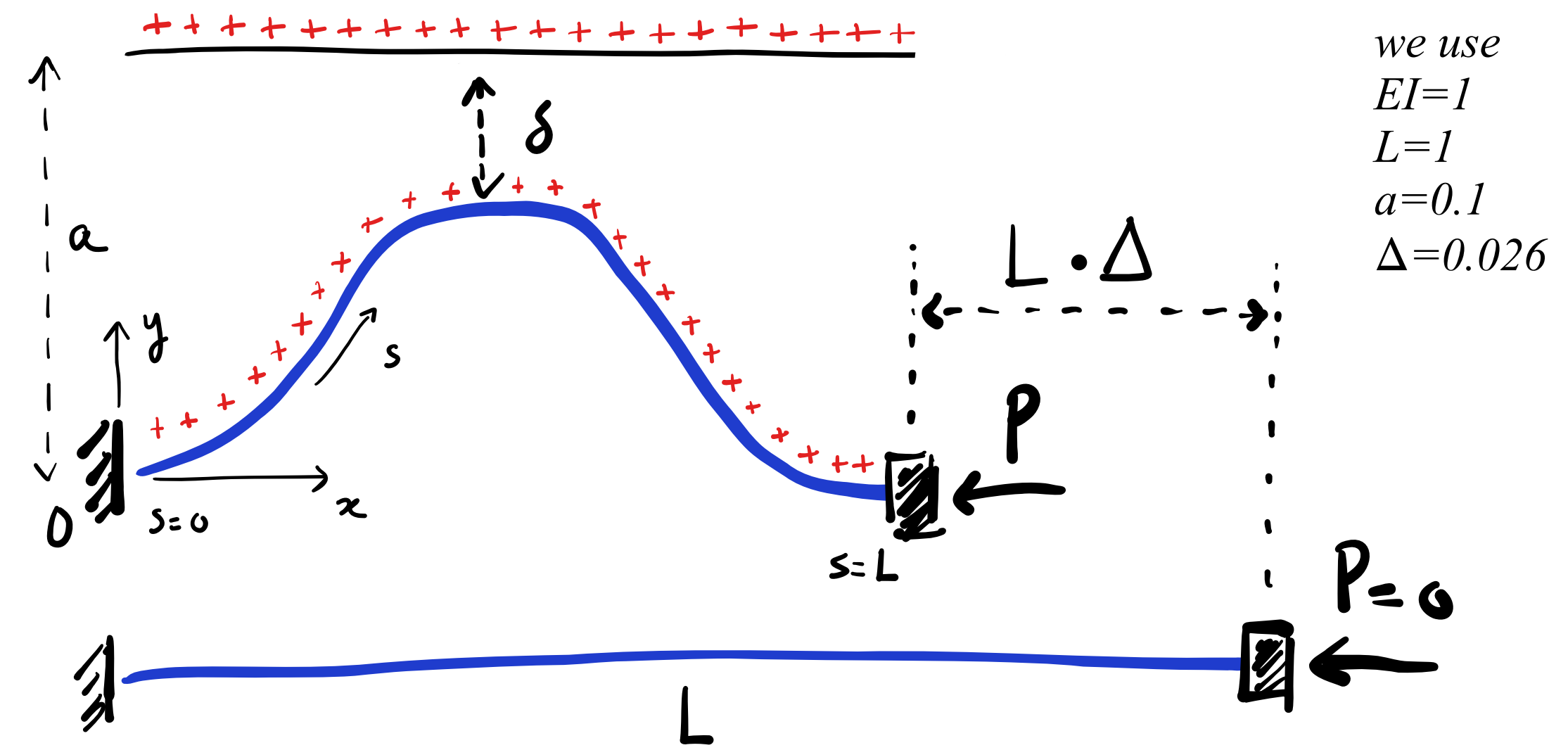
$$h'(s) \neq 0$$

soft-wall (or barrier) potential

$$V(y) = \frac{b}{a - y(s)}$$



CONSERVATIVE EXTERNAL POTENTIAL



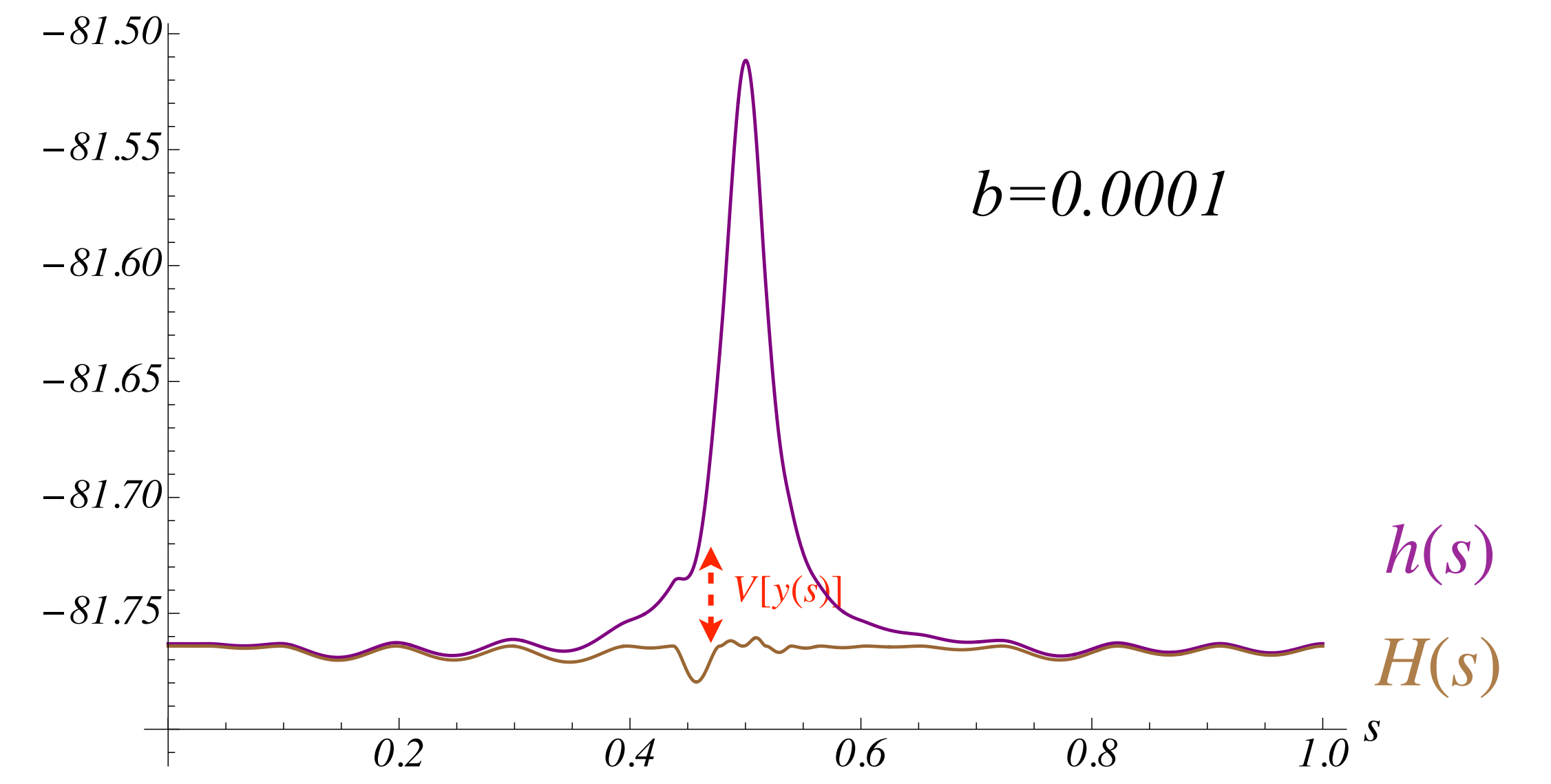
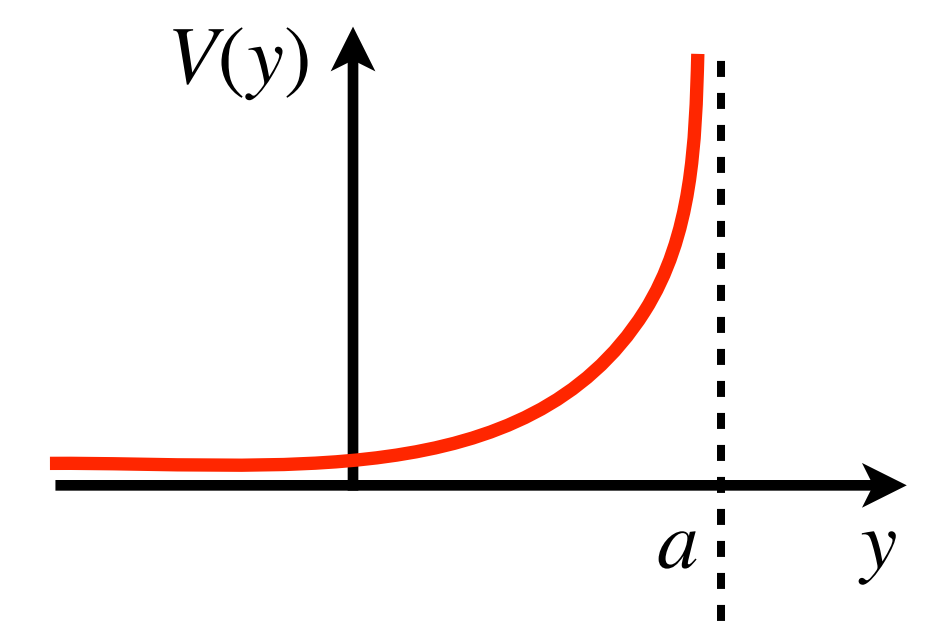
$$H = \underbrace{\frac{1}{2} EI \kappa^2(s) + \text{tension}}_{h(s)} - V[y(s)]$$

$$H'(s) = 0 \quad \forall s$$

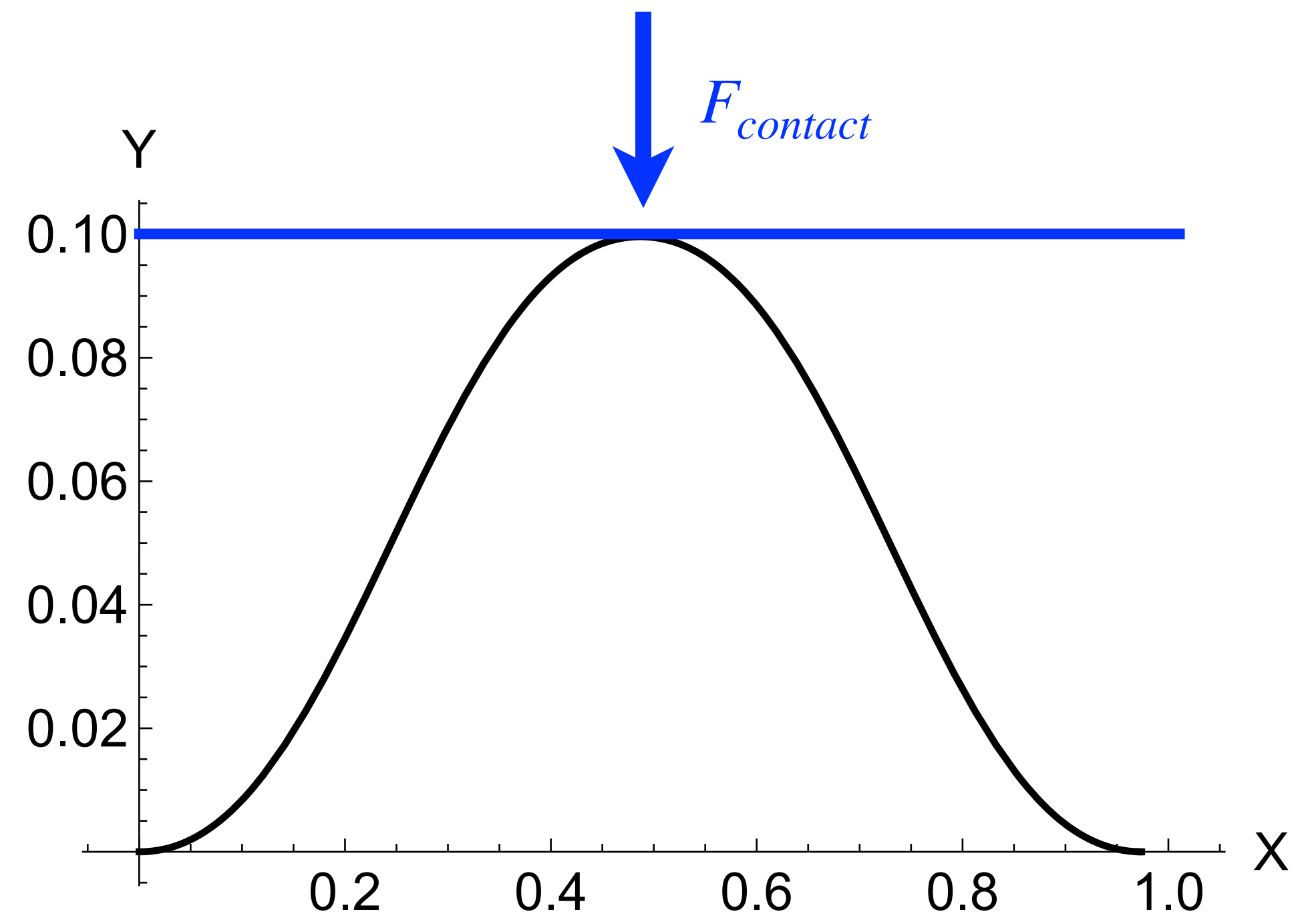
$$h'(s) \neq 0$$

soft-wall (or barrier) potential

$$V(y) = \frac{b}{a - y(s)}$$



CONSERVATIVE EXTERNAL POTENTIAL



we use
 $EI=1$
 $L=1$
 $a=0.1$
 $\Delta=0.026$

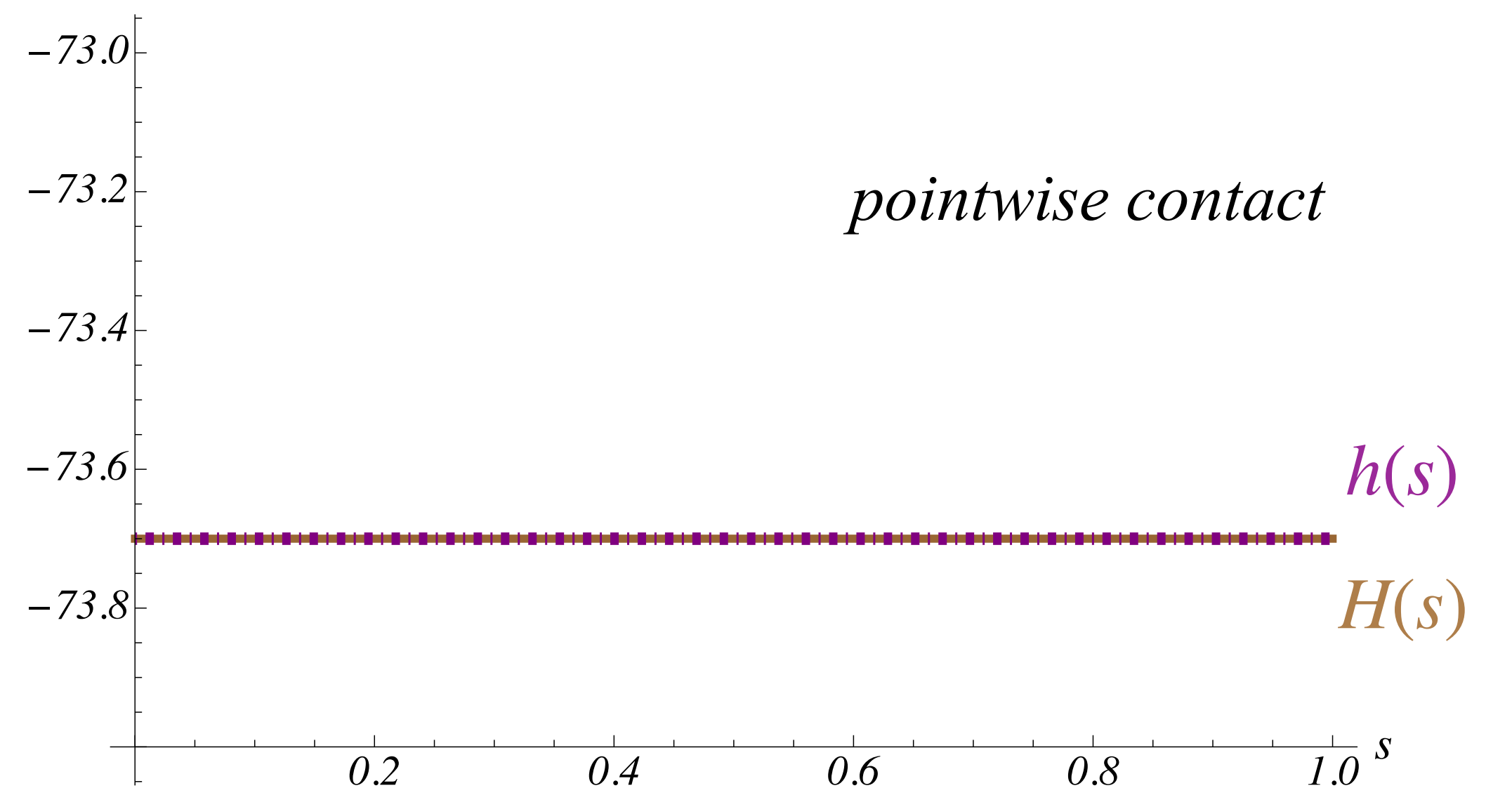
$$H = \underbrace{\frac{1}{2} EI \kappa^2(s) + tension}_{h(s)} - 0$$

$$H'(s) = 0 \forall s$$

$$h'(s) = 0 \forall s$$

potential $\int_0^L V(y) = \frac{b}{a - y(s)} ds \xrightarrow{b \rightarrow 0} 0$

force $\int_0^L \frac{\partial V}{\partial y} = \frac{b}{(a - y(s))^2} ds \xrightarrow{b \rightarrow 0} F_{contact}$

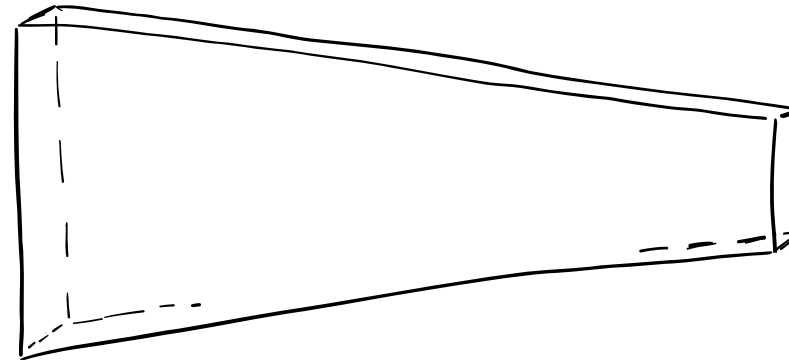


THE INVARIANT: WHEN DOES IT FAIL?

$$\mathcal{L} = \frac{1}{2} EI(s) (\kappa(s) - \hat{\kappa}(s))^2 + \frac{1}{2} EA(s) (e(s) - 1)^2 + V[x(s), y(s)] + \lambda_1(x' - e \cos \theta) + \lambda_2(y' - e \sin \theta) + \lambda_3(\theta' - \kappa)$$

explicit s dependence

Example: tapered beam



$$\mathcal{L} = \frac{1}{2} EI(s) \kappa^2(s) + \lambda_1(x' - \cos \theta) + \lambda_2(y' - \sin \theta) + \lambda_3(\theta' - \kappa)$$

Equilibrium: $[EI(s) \theta'(s)]' = -P \sin \theta(s)$

Invariant? $H(s) = \frac{1}{2} EI(s) \theta'(s)^2 - P \cos \theta(s)$

$$H'(s) = \dots = -\frac{1}{2} EI'(s) \theta'(s)^2 \neq 0$$

reminder

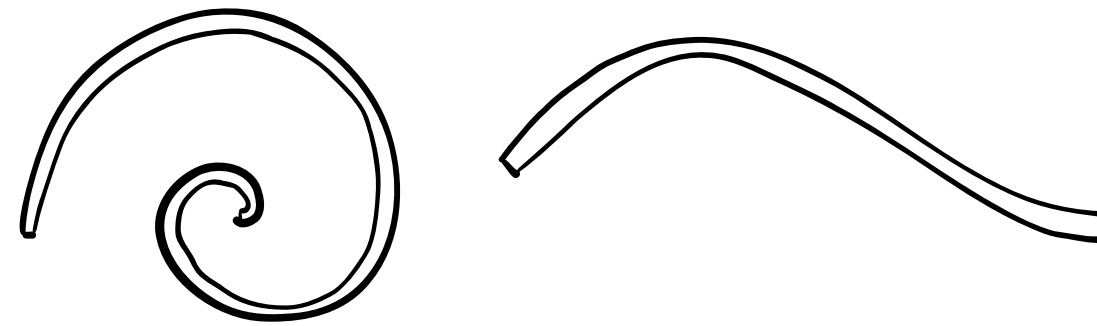
$$\kappa(s) \equiv \theta'(s)$$

THE INVARIANT: WHEN DOES IT FAIL?

$$\mathcal{L} = \frac{1}{2} EI(s) (\kappa(s) - \hat{\kappa}(s))^2 + \frac{1}{2} EA(s) (e(s) - 1)^2 + V[x(s), y(s)] + \lambda_1(x' - e \cos \theta) + \lambda_2(y' - e \sin \theta) + \lambda_3(\theta' - \kappa)$$

explicit s dependence

Example: curved beam



$$\mathcal{L} = \frac{1}{2} EI (\kappa(s) - \hat{\kappa}(s))^2 + \lambda_1(x' - \cos \theta) + \lambda_2(y' - \sin \theta) + \lambda_3(\theta' - \kappa)$$

Equilibrium: $EI [\theta'(s) - \hat{\kappa}(s)]' = -P \sin \theta(s)$

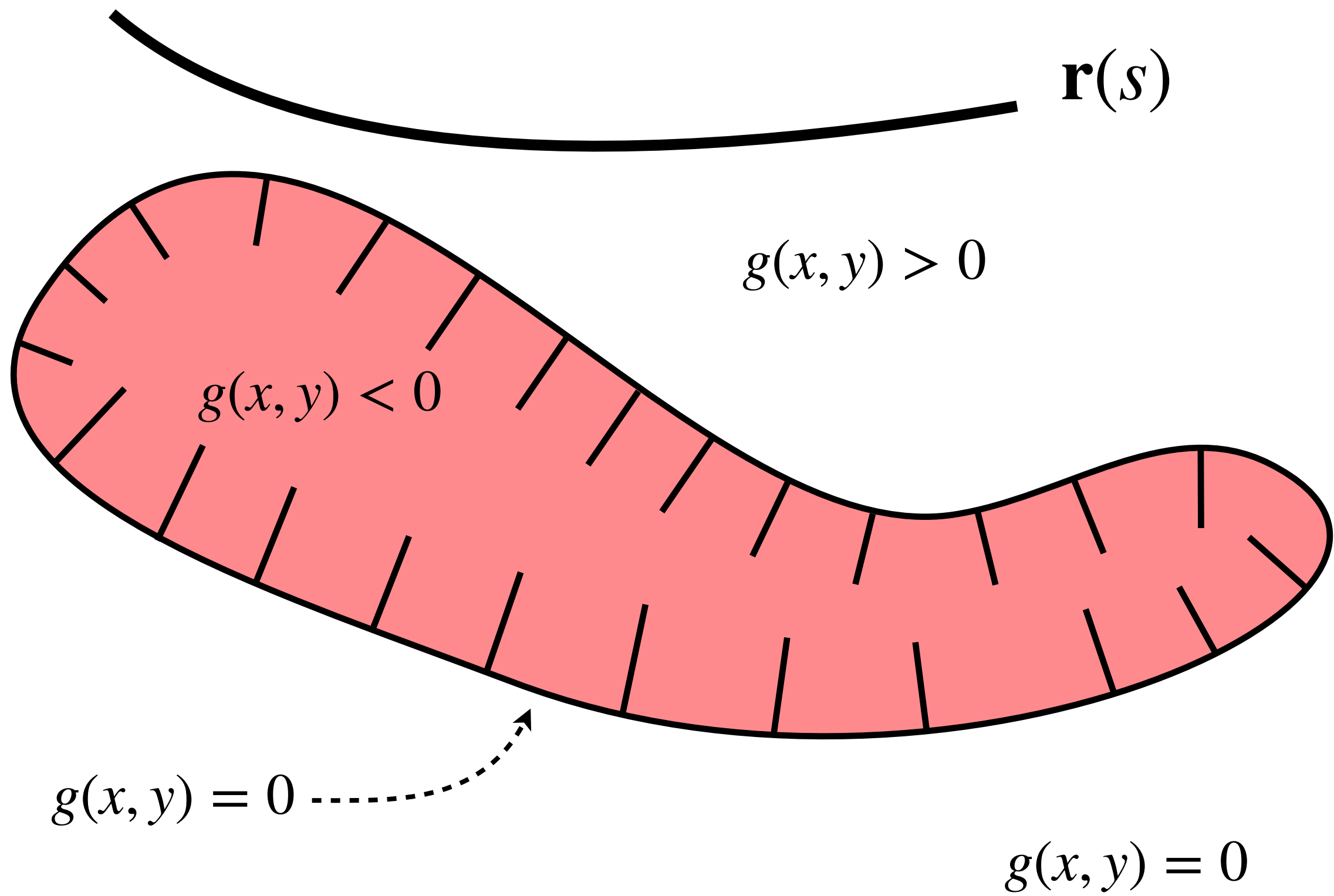
Invariant? $H(s) = \frac{1}{2} EI (\theta'(s)^2 - \hat{\kappa}^2(s)) - P \cos \theta(s)$

$$H'(s) = \dots = -\frac{1}{2} EI \hat{\kappa}'(s) (\theta' - \hat{\kappa}) \neq 0$$

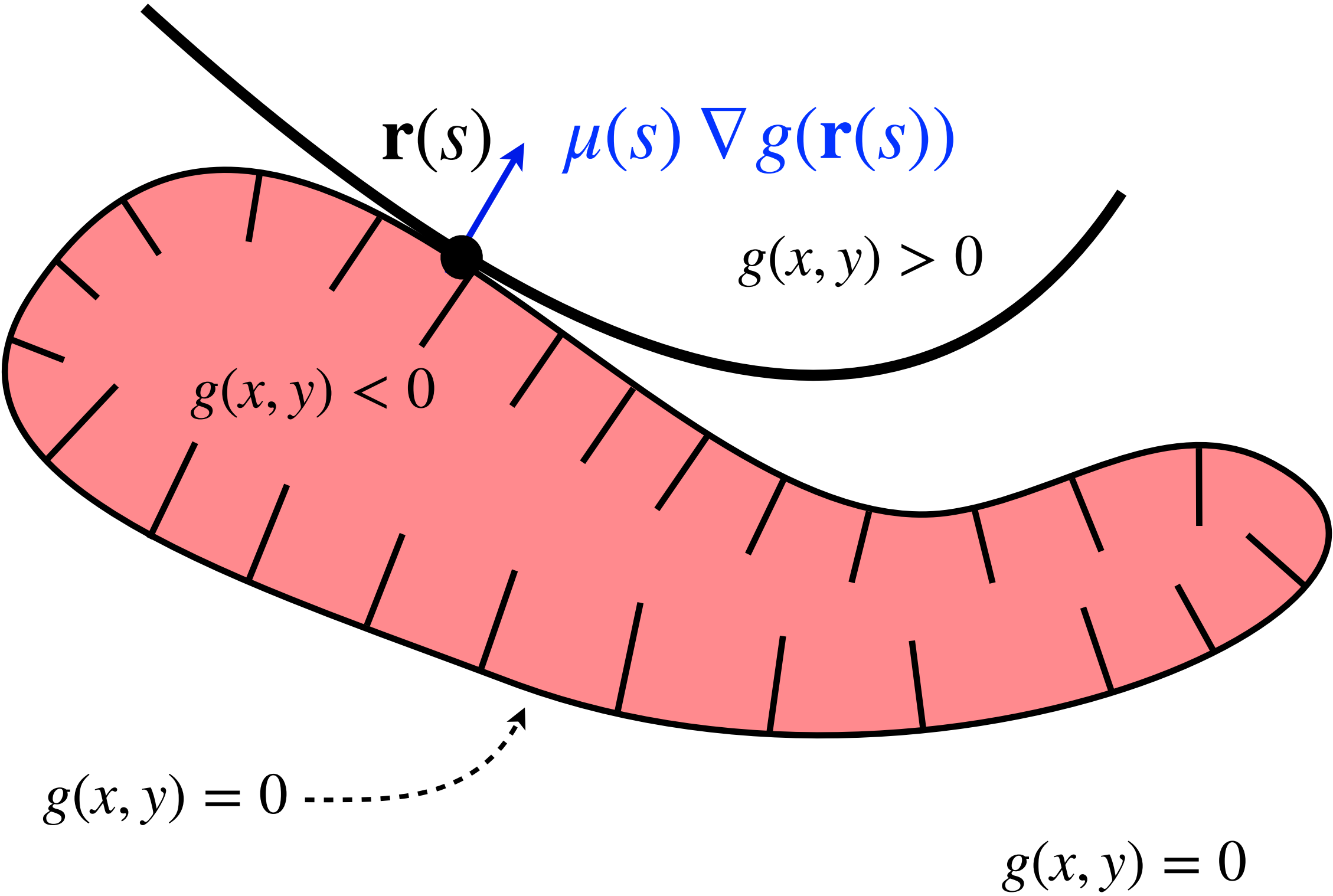
reminder

$$\kappa(s) \equiv \theta'(s)$$

BACK TO CONTACT

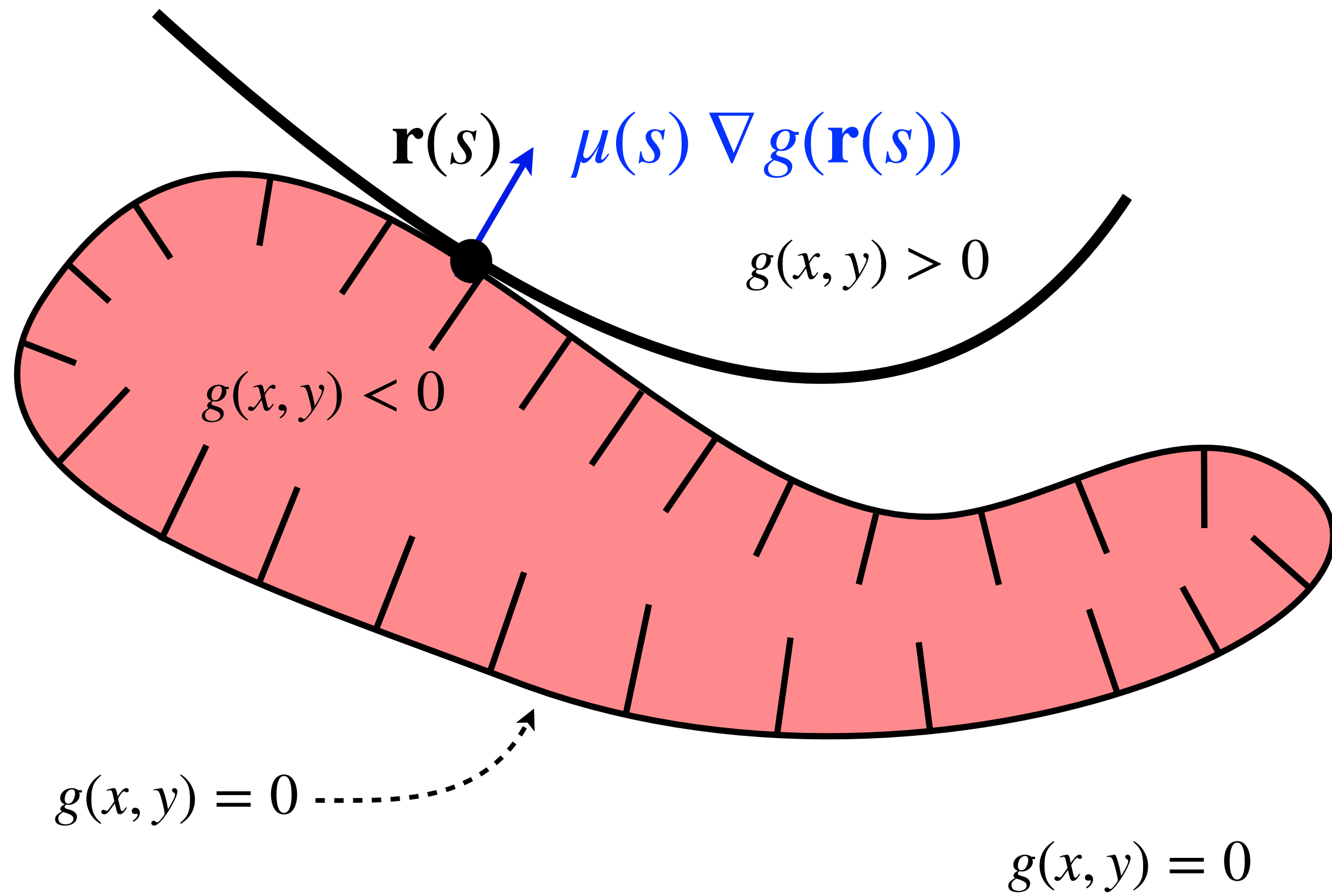


BACK TO CONTACT



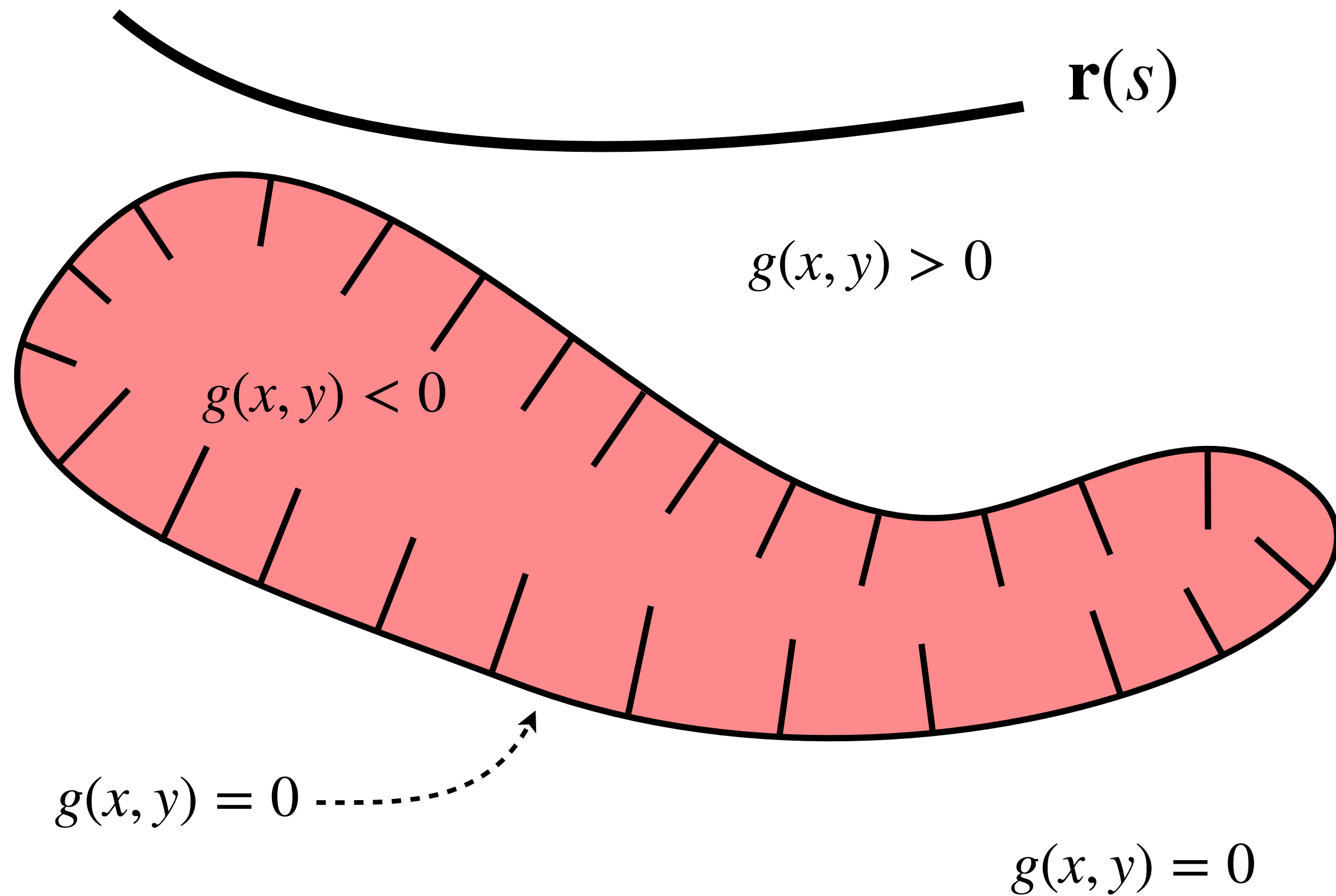
BACK TO CONTACT

New term in the Lagrangian: $\mathcal{L}_C = -\mu(s) g(\mathbf{r}(s))$



BACK TO CONTACT

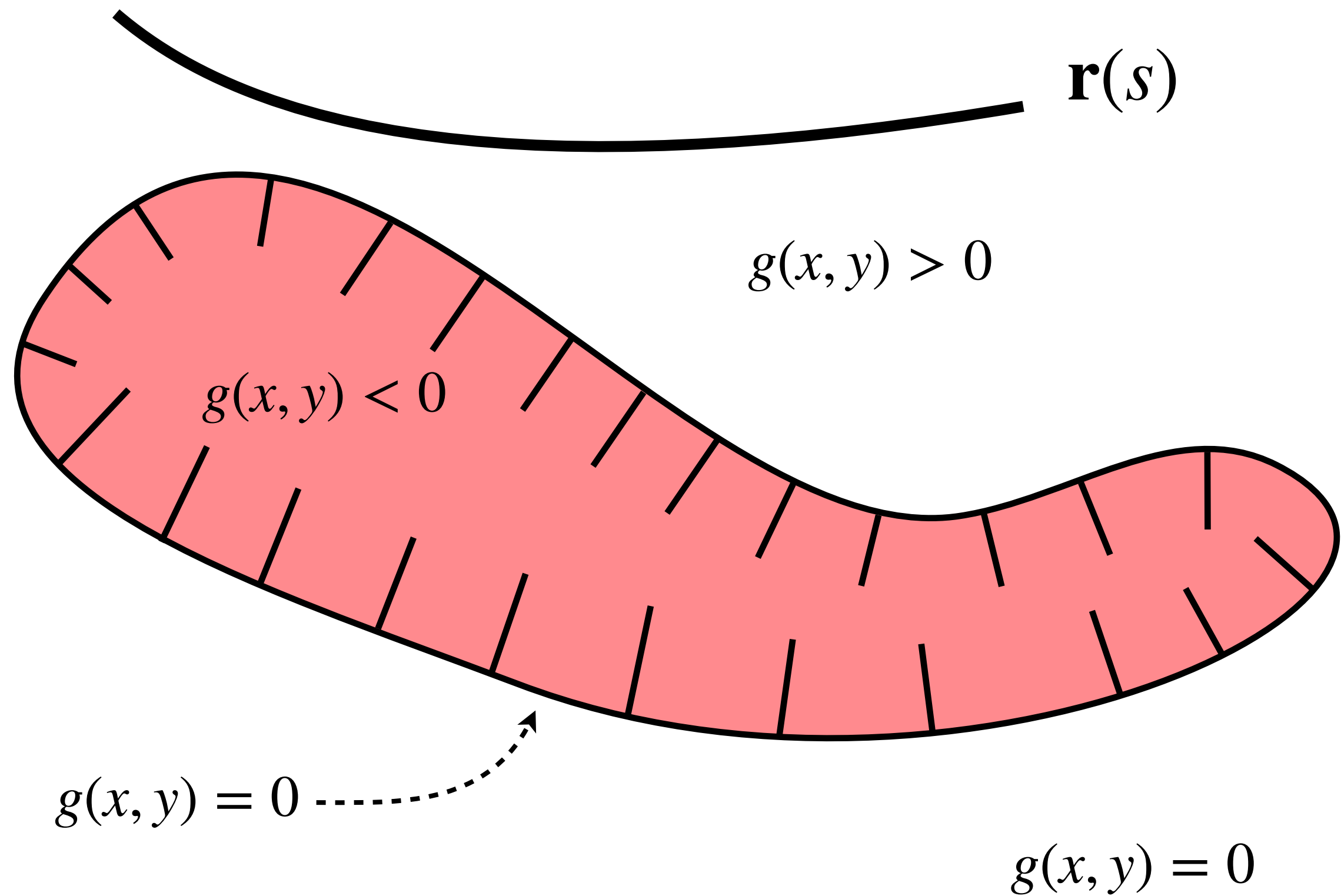
New term in the Lagrangian: $\mathcal{L}_c = -\mu(s) g(\mathbf{r}(s))$



Does it break Noether's invariant ?

BACK TO CONTACT

New term in the Lagrangian: $\mathcal{L}_C = -\mu(s) g(\mathbf{r}(s))$



Does it break Noether's invariant ?

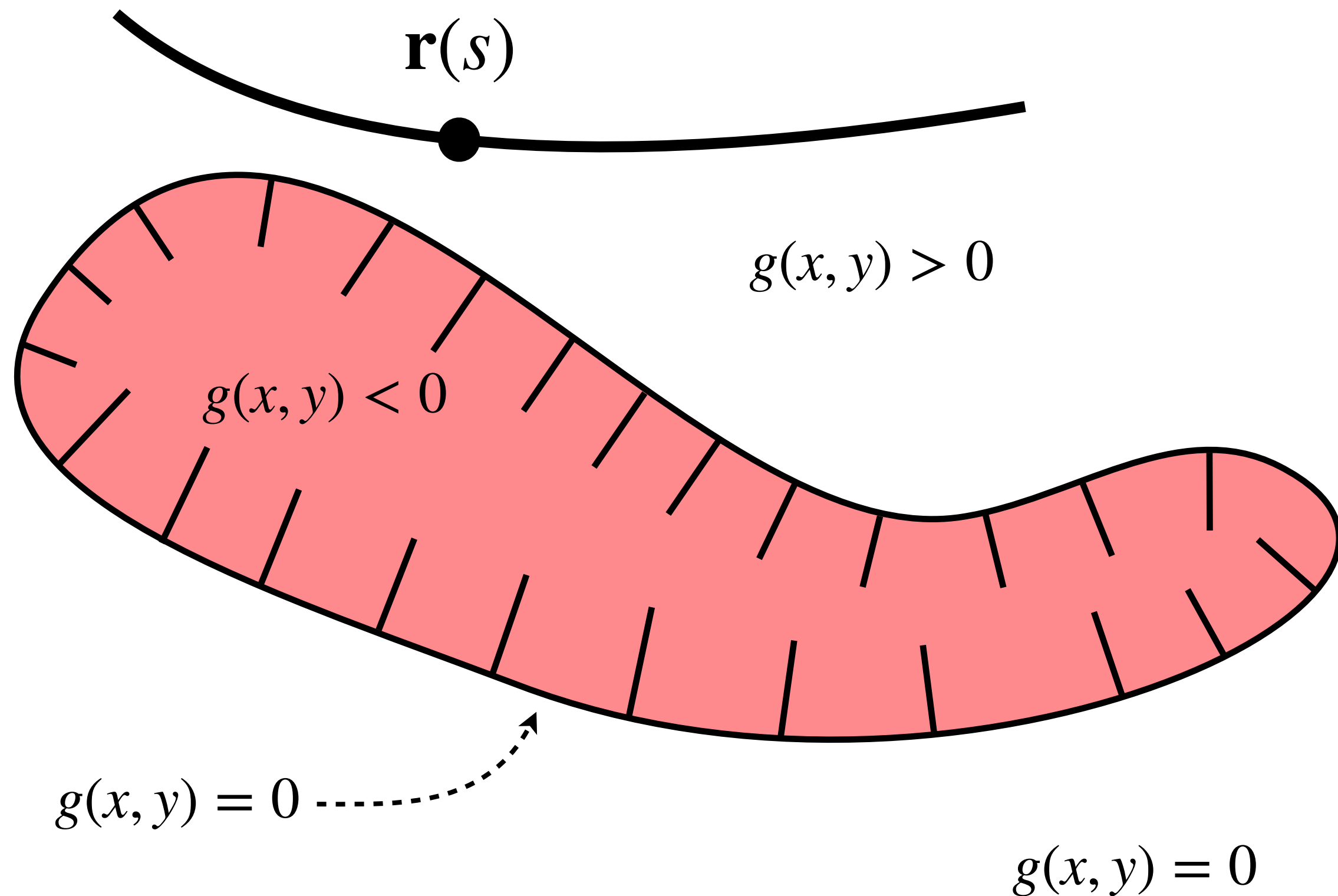
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

BACK TO CONTACT

New term in the Lagrangian: $\mathcal{L}_C = -\mu(s) g(\mathbf{r}(s))$



Does it break Noether's invariant ?

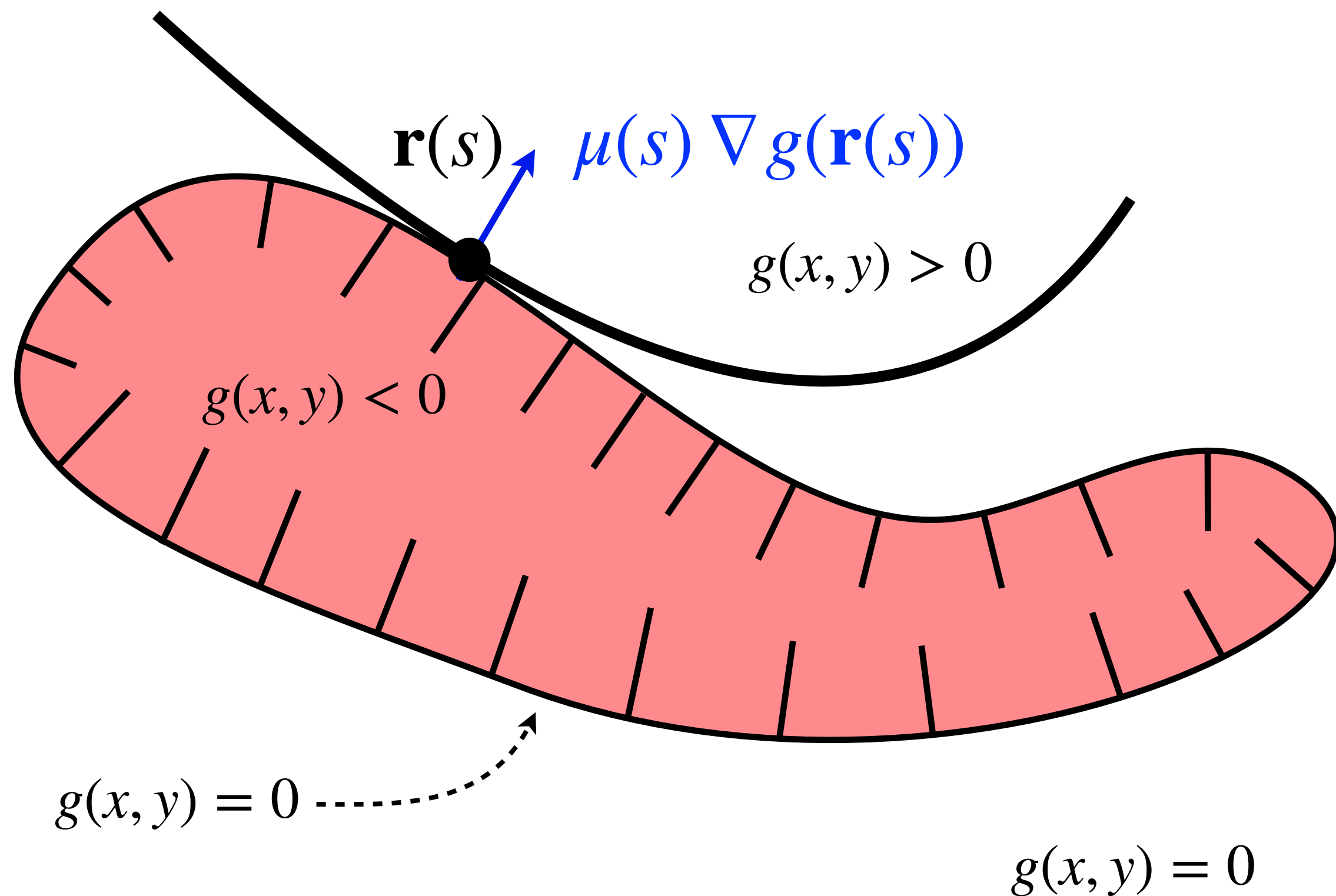
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

BACK TO CONTACT

New term in the Lagrangian: $\mathcal{L}_C = -\mu(s) g(\mathbf{r}(s))$



Does it break Noether's invariant ?

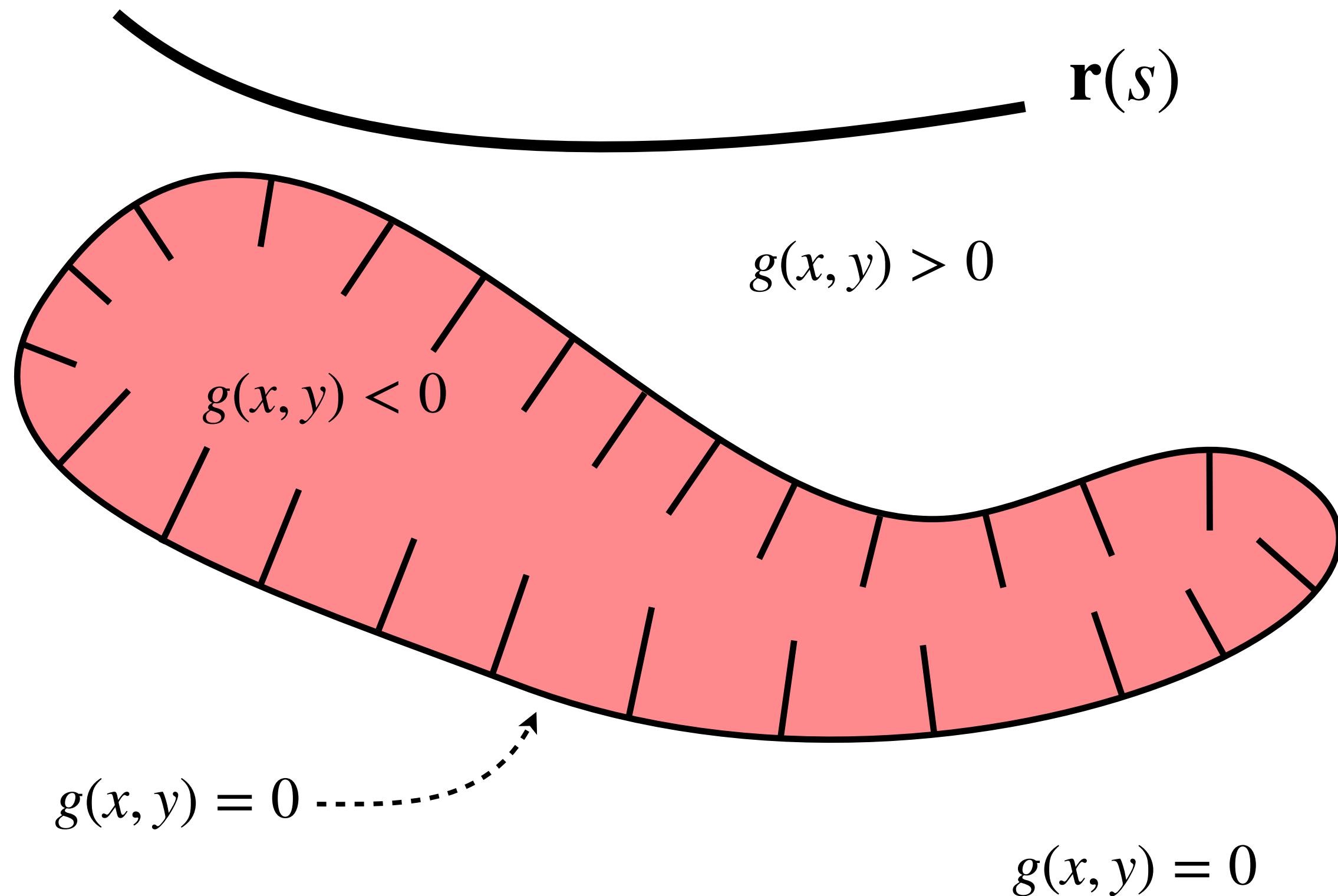
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

BACK TO CONTACT

New term in the Lagrangian: $\mathcal{L}_C = -\mu(s) g(\mathbf{r}(s))$



Does it break Noether's invariant ?

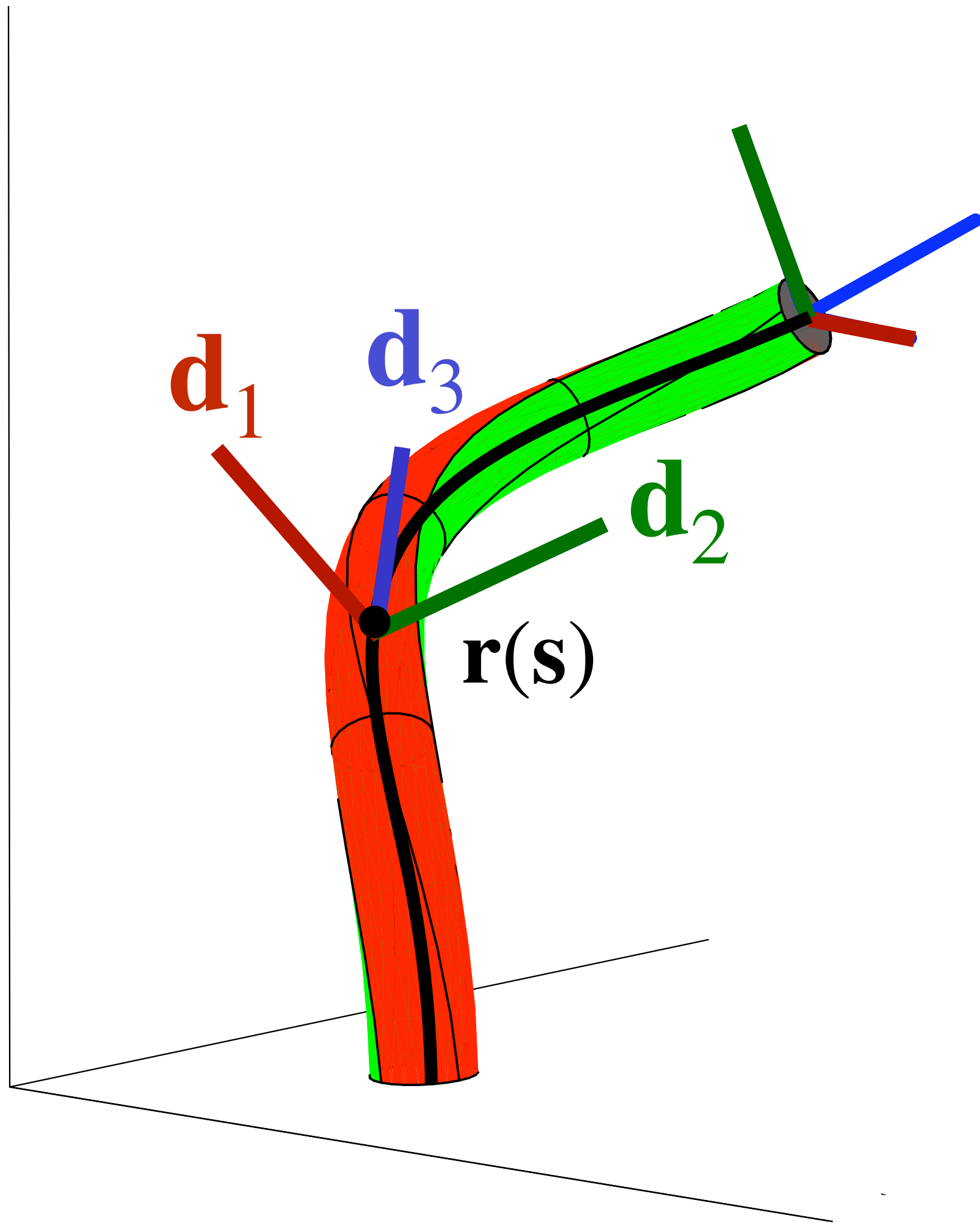
—> No!

Signorini law for frictionless contact

$$0 \leq g(\mathbf{r}(s)) \perp \mu(s) \geq 0$$

Moreover, H 's formula is unchanged !

3D RODS: KINEMATICS



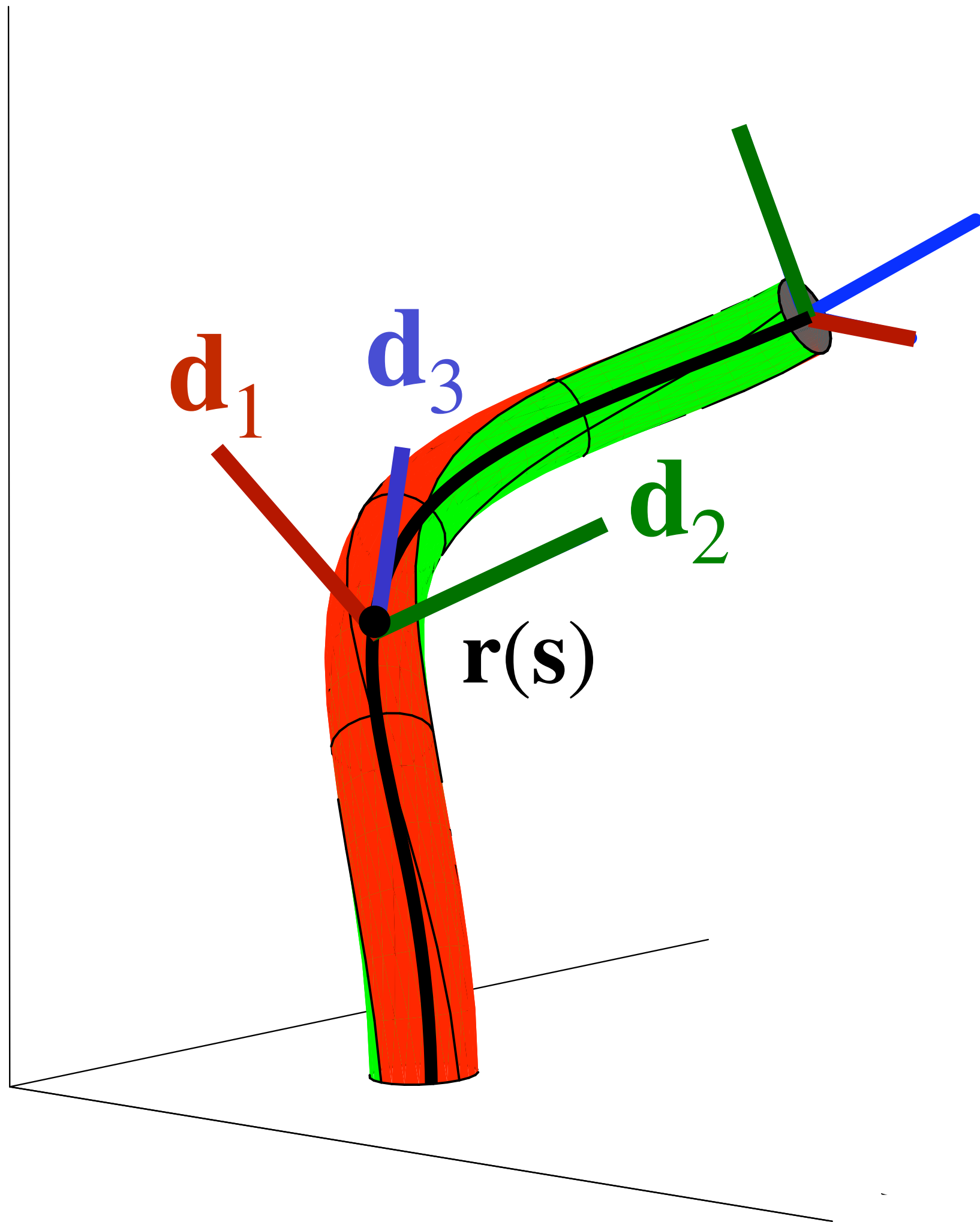
Centreline

$\mathbf{r}(s)$

Material frame

$\mathcal{R}(s) = \{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$

3D RODS: KINEMATICS



Centreline $\mathbf{r}(s)$

Material frame $\mathcal{R}(s) = \{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$

Two types of constraints:

1/ Coupling between \mathbf{r} and \mathcal{R}

e.g.,

$$\mathbf{r}' = \mathbf{d}_3 \quad (\text{Kirchhoff})$$

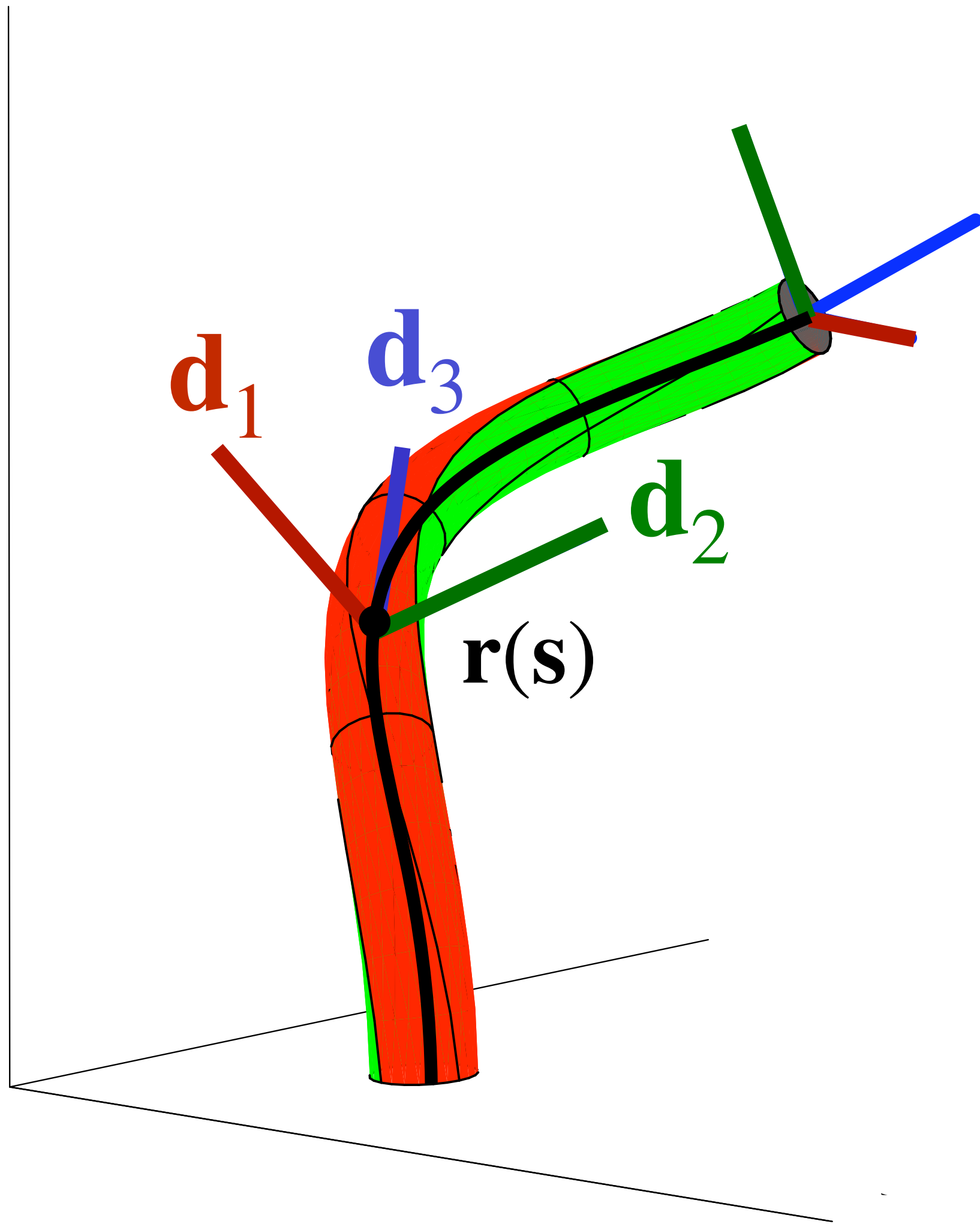
or

$$\mathbf{r}' = \nu_1 \mathbf{d}_1 + \nu_2 \mathbf{d}_2 + \nu_3 \mathbf{d}_3 \quad (\text{Reissner})$$

shear 1+extension
 ↙ ↘ ↙ ↘

→ General case handled here

3D RODS: KINEMATICS



Centreline $\mathbf{r}(s)$

Material frame $\mathcal{R}(s) = \{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$

Two types of constraints:

2/ $\mathcal{R} \in SO(3)$ Rotation group

→ Existence of a vector $\mathbf{u} = \{u_1, u_2, u_3\}_{d_j}$ such that

$$\begin{cases} \mathbf{d}'_1 = \mathbf{u} \times \mathbf{d}_1 \\ \mathbf{d}'_2 = \mathbf{u} \times \mathbf{d}_2 \\ \mathbf{d}'_3 = \mathbf{u} \times \mathbf{d}_3 \end{cases}$$

curvatures

twist

(Darboux equations)

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method \mathbb{R}^n ('flattened') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3, \dots)$

→ Add constraints 1 and 2 in \mathcal{L} with multipliers

Classical Noether invariant: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathcal{L}$

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method \mathbb{R}^n ('flattened') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3, \dots)$

→ Add constraints 1 and 2 in \mathcal{L} with multipliers

Classical Noether invariant: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathcal{L}$

Method $\mathbb{R}^n \times SO(3)$ ('compact') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \dots) \neq \mathbf{q}'$

→ Add only constraint 1 in \mathcal{L} with multipliers

New Noether invariant, valid on $\mathbb{R}^n \times SO(3)$: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \frac{\partial \mathcal{L}}{\partial \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} - \mathcal{L}$

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method \mathbb{R}^n ('flattened') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3, \dots)$

→ Add constraints 1 and 2 in \mathcal{L} with multipliers

Classical Noether invariant: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathcal{L}$

Method $\mathbb{R}^n \times SO(3)$ ('compact') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \dots) \neq \mathbf{q}'$

→ Add only constraint 1 in \mathcal{L} with multipliers

New Noether invariant, valid on $\mathbb{R}^n \times SO(3)$: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \frac{\partial \mathcal{L}}{\partial \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} - \mathcal{L}$ $\left\{ \begin{array}{l} \bar{\mathbf{u}} = \{u_1, u_2, u_3\} \\ \text{s.t. } \mathbf{u} = \mathcal{R} \bar{\mathbf{u}} \end{array} \right.$

THE INVARIANT IN THE 3D CASE: TWO METHODS

Method \mathbb{R}^n ('flattened') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3, \dots)$

→ Add constraints 1 and 2 in \mathcal{L} with multipliers

Classical Noether invariant: $H = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathcal{L}$

Method $\mathbb{R}^n \times SO(3)$ ('compact') :

$\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \dots) \neq \mathbf{q}'$

→ Add only constraint 1 in \mathcal{L} with multipliers

New Noether invariant, valid on $\mathbb{R}^n \times SO(3)$: $H = \underbrace{\frac{\partial \mathcal{L}}{\partial \mathbf{r}'}}_{\text{translation}} \cdot \mathbf{r}' + \underbrace{\frac{\partial \mathcal{L}}{\partial \tilde{\mathbf{u}}}}_{\text{rotation}} \cdot \tilde{\mathbf{u}} - \mathcal{L}$ $\left\{ \begin{array}{l} \tilde{\mathbf{u}} = \{u_1, u_2, u_3\} \\ \text{s.t. } \mathbf{u} = \mathcal{R} \tilde{\mathbf{u}} \end{array} \right.$

THE INVARIANT IN THE 3D CASE

$$\left. \begin{aligned} W_{bend} &= \frac{1}{2}B_1 (u_1(s) - \hat{u}_1)^2 + \frac{1}{2}B_2 (u_2(s) - \hat{u}_2)^2 + \frac{1}{2}B_3 (u_3(s) - \hat{u}_3)^2 \\ W_{shear} &= \frac{1}{2}A_1 (v_1(s) - \hat{v}_1)^2 + \frac{1}{2}A_2 (v_2(s) - \hat{v}_2)^2 + \frac{1}{2}A_3 (v_3(s) - \hat{v}_3)^2 \end{aligned} \right\} \text{Reissner model}$$

THE INVARIANT IN THE 3D CASE

$$\left. \begin{aligned} W_{bend} &= \frac{1}{2}B_1 (u_1(s) - \hat{u}_1)^2 + \frac{1}{2}B_2 (u_2(s) - \hat{u}_2)^2 + \frac{1}{2}B_3 (u_3(s) - \hat{u}_3)^2 \\ W_{shear} &= \frac{1}{2}A_1 (v_1(s) - \hat{v}_1)^2 + \frac{1}{2}A_2 (v_2(s) - \hat{v}_2)^2 + \frac{1}{2}A_3 (v_3(s) - \hat{v}_3)^2 \end{aligned} \right\} \text{Reissner model}$$

Gravity
Electrostatics
Soft contact

$$\mathcal{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) +$$

THE INVARIANT IN THE 3D CASE

$$\begin{aligned}
 W_{bend} &= \frac{1}{2}B_1 (u_1(s) - \hat{u}_1)^2 + \frac{1}{2}B_2 (u_2(s) - \hat{u}_2)^2 + \frac{1}{2}B_3 (u_3(s) - \hat{u}_3)^2 \\
 W_{shear} &= \frac{1}{2}A_1 (v_1(s) - \hat{v}_1)^2 + \frac{1}{2}A_2 (v_2(s) - \hat{v}_2)^2 + \frac{1}{2}A_3 (v_3(s) - \hat{v}_3)^2
 \end{aligned}
 \left. \vphantom{\begin{aligned} W_{bend} \\ W_{shear} \end{aligned}} \right\} \text{Reissner model}$$

Gravity
Electrostatics
Soft contact

$$\mathcal{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) +$$

constraints

$$\begin{aligned}
 &\lambda_{\mathbf{r}} \cdot (\mathbf{r}'(s) - v_1(s) \mathbf{d}_1 - v_2(s) \mathbf{d}_2 - v_3(s) \mathbf{d}_3) + \lambda_{u_1} (\mathbf{d}'_2 \cdot \mathbf{d}_3 - u_1) + \lambda_{u_2} (\mathbf{d}'_3 \cdot \mathbf{d}_1 - u_2) + \lambda_{u_3} (\mathbf{d}'_1 \cdot \mathbf{d}_2 - u_3) + \\
 &\lambda_{11} \frac{1}{2} (1 - \mathbf{d}_1 \cdot \mathbf{d}_1) + \lambda_{22} \frac{1}{2} (1 - \mathbf{d}_2 \cdot \mathbf{d}_2) + \lambda_{33} \frac{1}{2} (1 - \mathbf{d}_3 \cdot \mathbf{d}_3) + \lambda_{12} \mathbf{d}_1 \cdot \mathbf{d}_2 + \lambda_{23} \mathbf{d}_2 \cdot \mathbf{d}_3 + \lambda_{31} \mathbf{d}_3 \cdot \mathbf{d}_1
 \end{aligned}$$

where $\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \mathbf{q}'(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, u_1, u_2, u_3, v_1, v_2, v_3)$ and $\mathbf{q}' = (\mathbf{r}', \mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3, \dots)$

Method \mathbb{R}^n

$$H = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} \cdot \mathbf{q}' - \mathcal{L}$$

THE INVARIANT IN THE 3D CASE

$$\begin{aligned}
 W_{bend} &= \frac{1}{2}B_1 (u_1(s) - \hat{u}_1)^2 + \frac{1}{2}B_2 (u_2(s) - \hat{u}_2)^2 + \frac{1}{2}B_3 (u_3(s) - \hat{u}_3)^2 \\
 W_{shear} &= \frac{1}{2}A_1 (v_1(s) - \hat{v}_1)^2 + \frac{1}{2}A_2 (v_2(s) - \hat{v}_2)^2 + \frac{1}{2}A_3 (v_3(s) - \hat{v}_3)^2
 \end{aligned}
 \left. \vphantom{\begin{aligned} W_{bend} \\ W_{shear} \end{aligned}} \right\} \text{Reissner model}$$

Gravity
Electrostatics
Soft contact

$$\mathcal{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) + \lambda_{\mathbf{r}} \cdot (\mathbf{r}'(s) - v_1(s) \mathbf{d}_1 - v_2(s) \mathbf{d}_2 - v_3(s) \mathbf{d}_3)$$

constraints

where $\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \dots) \neq \mathbf{q}'$

Method $\mathbb{R}^n \times SO(3)$

$$H = \frac{\partial \mathcal{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \frac{\partial \mathcal{L}}{\partial \tilde{\mathbf{u}}} \cdot \tilde{\mathbf{u}} - \mathcal{L}$$

THE INVARIANT IN THE 3D CASE

$$\begin{aligned}
 W_{bend} &= \frac{1}{2}B_1 (u_1(s) - \hat{u}_1)^2 + \frac{1}{2}B_2 (u_2(s) - \hat{u}_2)^2 + \frac{1}{2}B_3 (u_3(s) - \hat{u}_3)^2 \\
 W_{shear} &= \frac{1}{2}A_1 (v_1(s) - \hat{v}_1)^2 + \frac{1}{2}A_2 (v_2(s) - \hat{v}_2)^2 + \frac{1}{2}A_3 (v_3(s) - \hat{v}_3)^2
 \end{aligned}
 \left. \vphantom{\begin{aligned} W_{bend} \\ W_{shear} \end{aligned}} \right\} \text{Reissner model}$$

Gravity
Electrostatics
Soft contact

$$\mathcal{L} = W_{bend} + W_{shear} + V(\mathbf{r}(s)) + \lambda_{\mathbf{r}} \cdot (\mathbf{r}'(s) - v_1(s) \mathbf{d}_1 - v_2(s) \mathbf{d}_2 - v_3(s) \mathbf{d}_3)$$

constraints

where $\mathcal{L} = \mathcal{L}[\mathbf{q}(s), \tilde{\mathbf{q}}(s)]$ with $\mathbf{q} = (\mathbf{r}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, v_1, v_2, v_3)$ and $\tilde{\mathbf{q}} = (\mathbf{r}', u_1, u_2, u_3, \dots) \neq \mathbf{q}'$

Method $\mathbb{R}^n \times SO(3)$

$$H = \frac{\partial \mathcal{L}}{\partial \mathbf{r}'} \cdot \mathbf{r}' + \frac{\partial \mathcal{L}}{\partial \tilde{\mathbf{u}}} \cdot \tilde{\mathbf{u}} - \mathcal{L}$$

Internal force
Internal moment

HOW DOES THE TENSION ENTER THE INVARIANT FORMULA (3D)

Noether invariant:

$$\begin{aligned} H &= \mathbf{n} \cdot \mathbf{v} + \mathbf{m} \cdot \mathbf{u} - W_{bend} - W_{shear} - V(\mathbf{r}) \\ &= \frac{1}{2} B_i [u_i^2(s) - \hat{u}_i^2] + \frac{1}{2} A_i [v_i^2(s) - \hat{v}_i^2] - V(\mathbf{r}) \\ &= \frac{1}{2} B_i [u_i(s) - \hat{u}_i]^2 + m_i \hat{u}_i + \frac{1}{2} A_i [v_i(s) - \hat{v}_i]^2 + n_i \hat{v}_i - V(\mathbf{r}) \end{aligned}$$

HOW DOES THE TENSION ENTER THE INVARIANT FORMULA (3D)

Noether invariant:

$$\begin{aligned} H &= \mathbf{n} \cdot \mathbf{v} + \mathbf{m} \cdot \mathbf{u} - W_{bend} - W_{shear} - V(\mathbf{r}) \\ &= \frac{1}{2} B_i [u_i^2(s) - \hat{u}_i^2] + \frac{1}{2} A_i [v_i^2(s) - \hat{v}_i^2] - V(\mathbf{r}) \\ &= \frac{1}{2} B_i [u_i(s) - \hat{u}_i]^2 + m_i \hat{u}_i + \frac{1}{2} A_i [v_i(s) - \hat{v}_i]^2 + n_i \hat{v}_i - V(\mathbf{r}) \end{aligned}$$

general case
not the energies !

HOW DOES THE TENSION ENTER THE INVARIANT FORMULA (3D)

Noether invariant:

$$H = \mathbf{n} \cdot \mathbf{v} + \mathbf{m} \cdot \mathbf{u} - W_{bend} - W_{shear} - V(\mathbf{r})$$

$$= \frac{1}{2} B_i [u_i^2(s) - \hat{u}_i^2] + \frac{1}{2} A_i [v_i^2(s) - \hat{v}_i^2] - V(\mathbf{r})$$

$$= \frac{1}{2} B_i [u_i(s) - \hat{u}_i]^2 + m_i \hat{u}_i + \frac{1}{2} A_i [v_i(s) - \hat{v}_i]^2 + n_i \hat{v}_i - V(\mathbf{r})$$

$$= \frac{1}{2} B_i u_i^2(s) + \frac{1}{2} A_i v_i^2(s) + n_3 - V(\mathbf{r})$$

$$= W_{bend} + W_{shear} + tension - V(\mathbf{r})$$

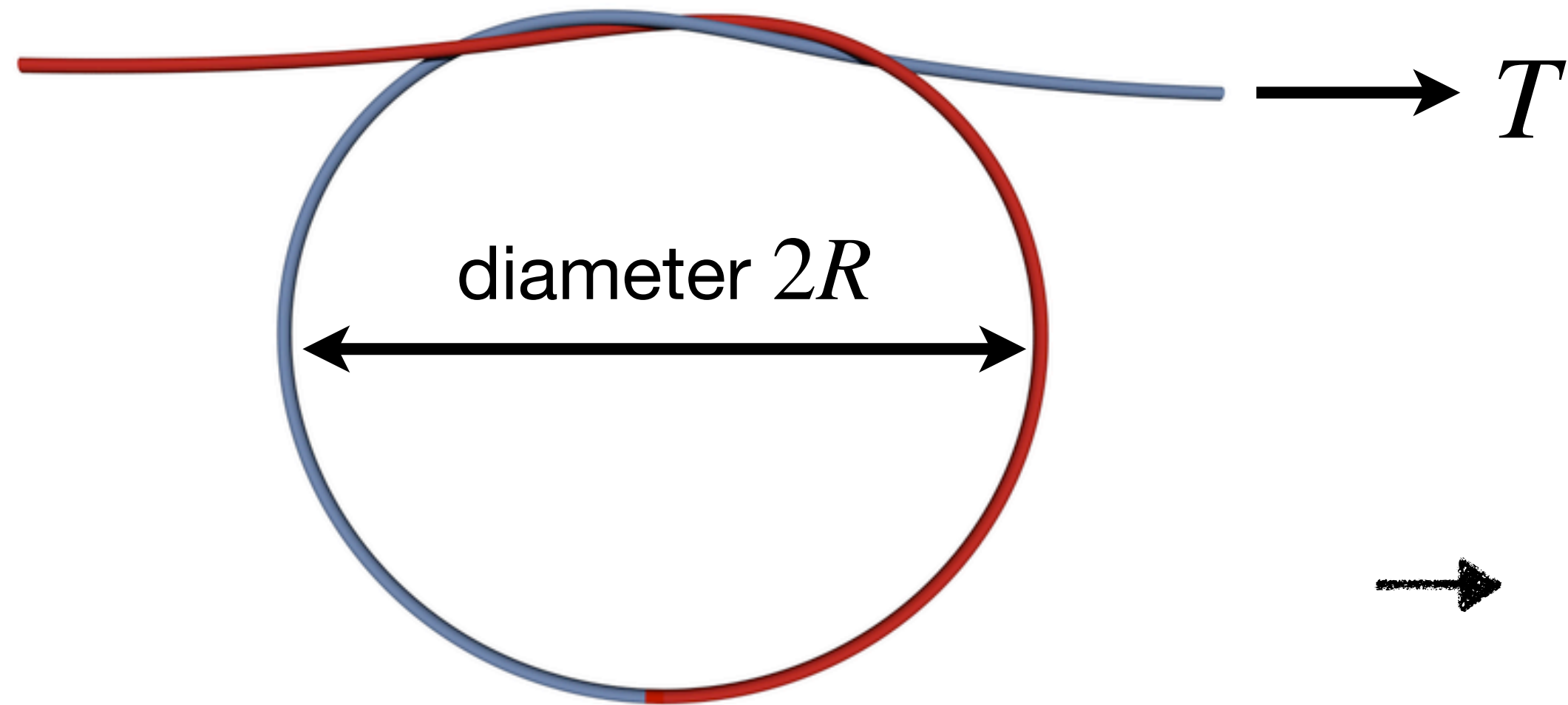
general case
not the energies !

if $\hat{u}_i = 0$ and $\hat{v}_{1,2} = 0$ and $\hat{v}_3 = 1$

(naturally flat rod, with naturally straight cross-section)

Maddocks 1994
(Intuited case $V = 0$)

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

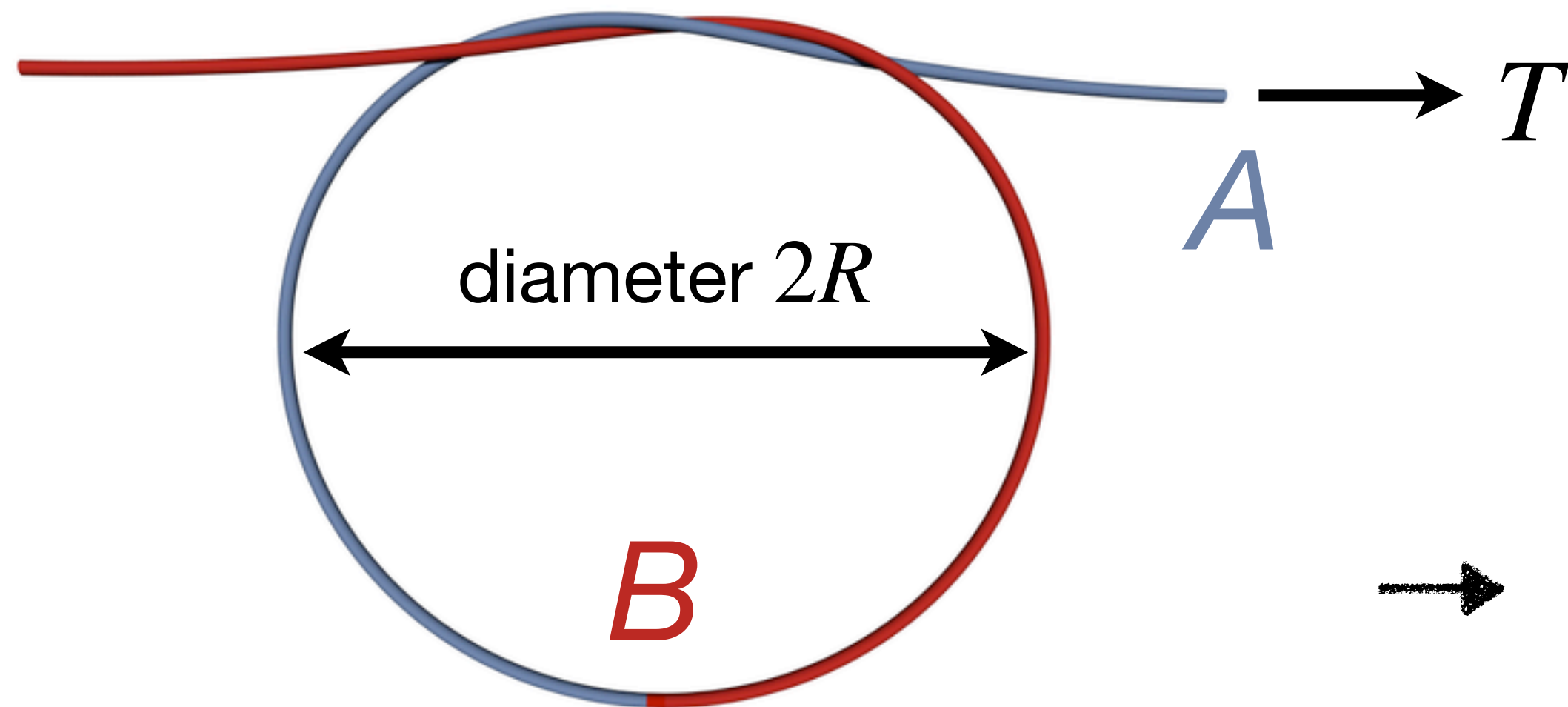


Limit case of vanishing thickness

$$H_{3D} = \textit{curvature} + \textit{twist} + \textit{tension}$$

→ A simple way to retrieve quickly a theory

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT



Limit case of vanishing thickness

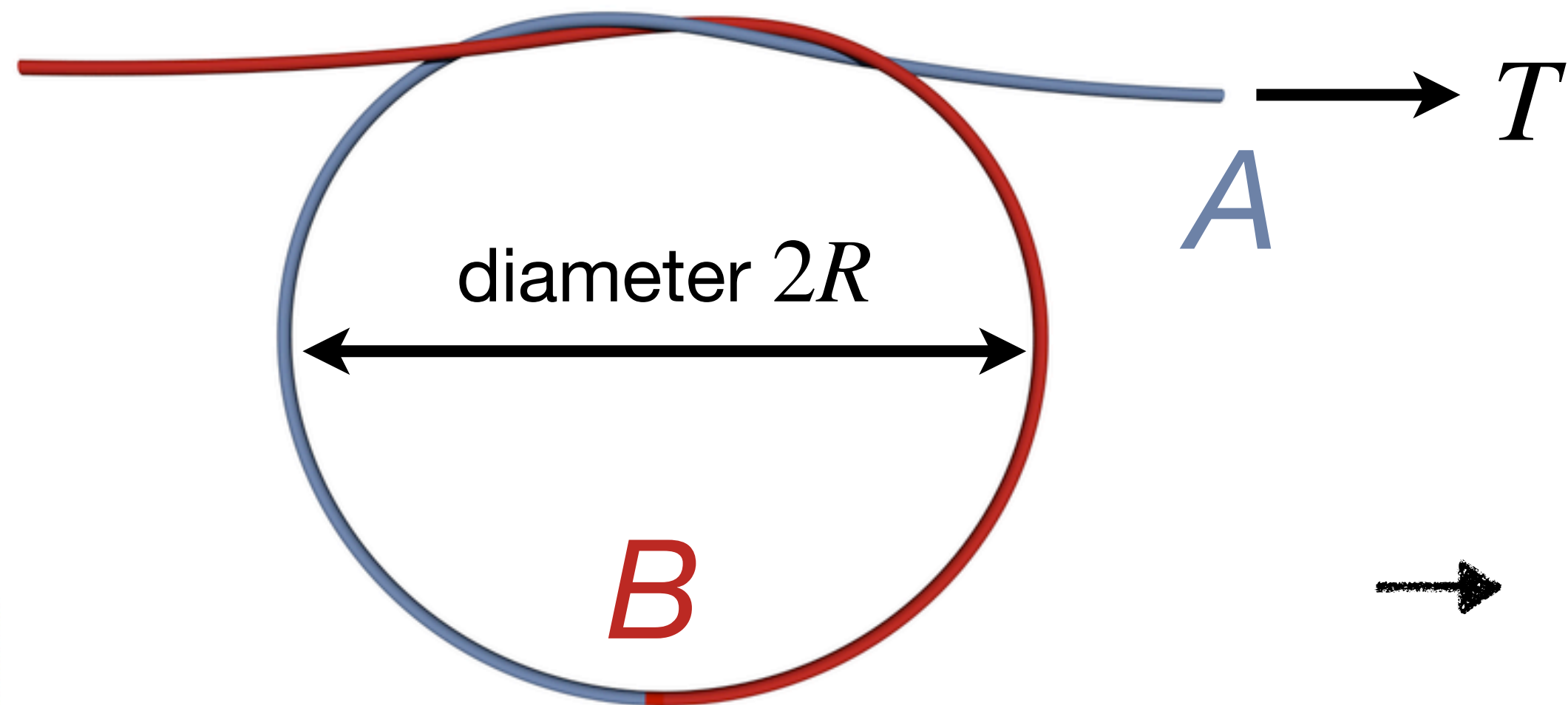
$$H_{3D} = \textit{curvature} + \textit{twist} + \textit{tension}$$

→ A simple way to retrieve quickly a theory

$$H_A = 0 + 0 + T$$

$$H_B = \frac{1}{2} \frac{EI}{R^2} + 0 + 0$$

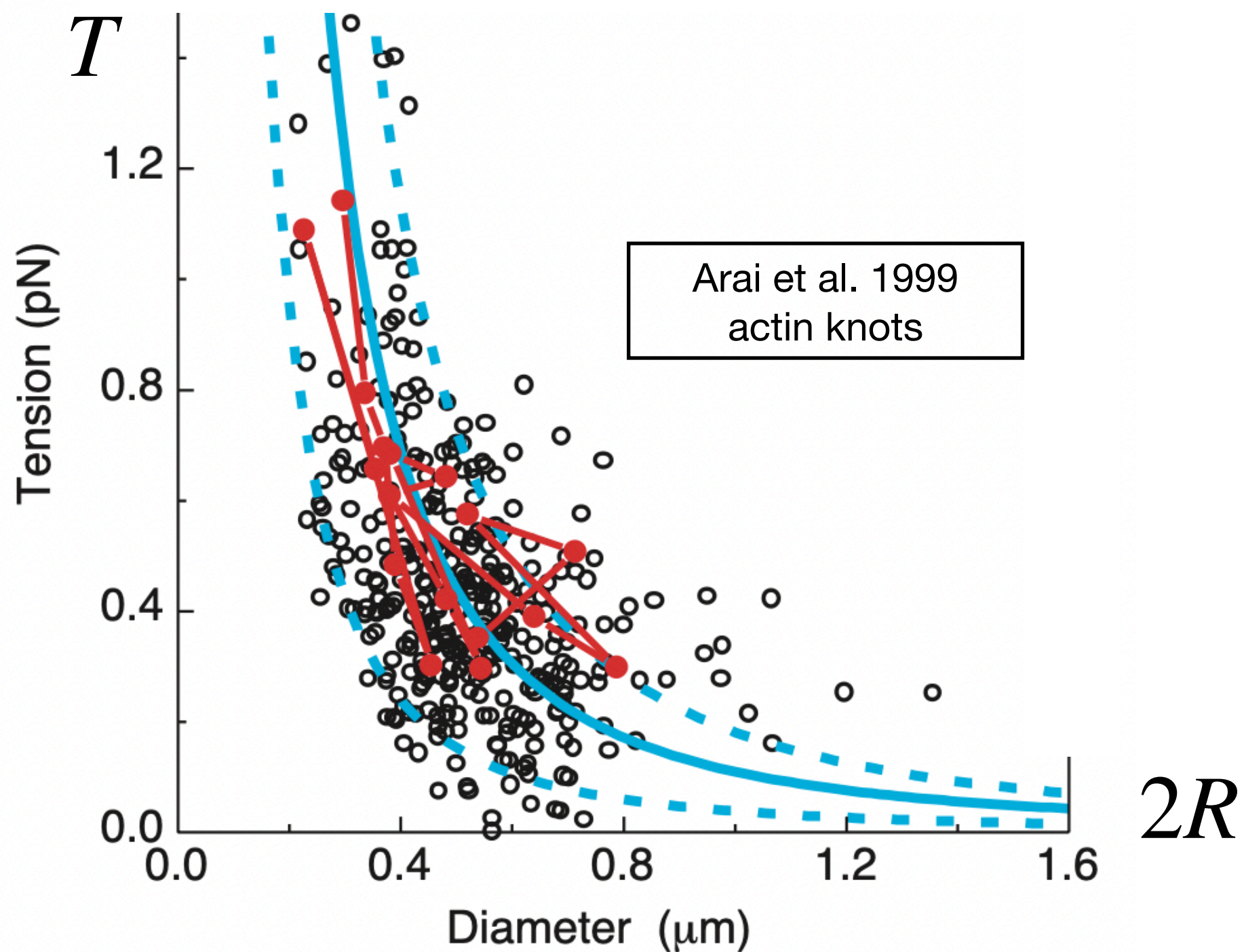
MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT



Limit case of vanishing thickness

$$H_{3D} = \textit{curvature} + \textit{twist} + \textit{tension}$$

→ A simple way to retrieve quickly a theory



$$H_A = 0 + 0 + T$$

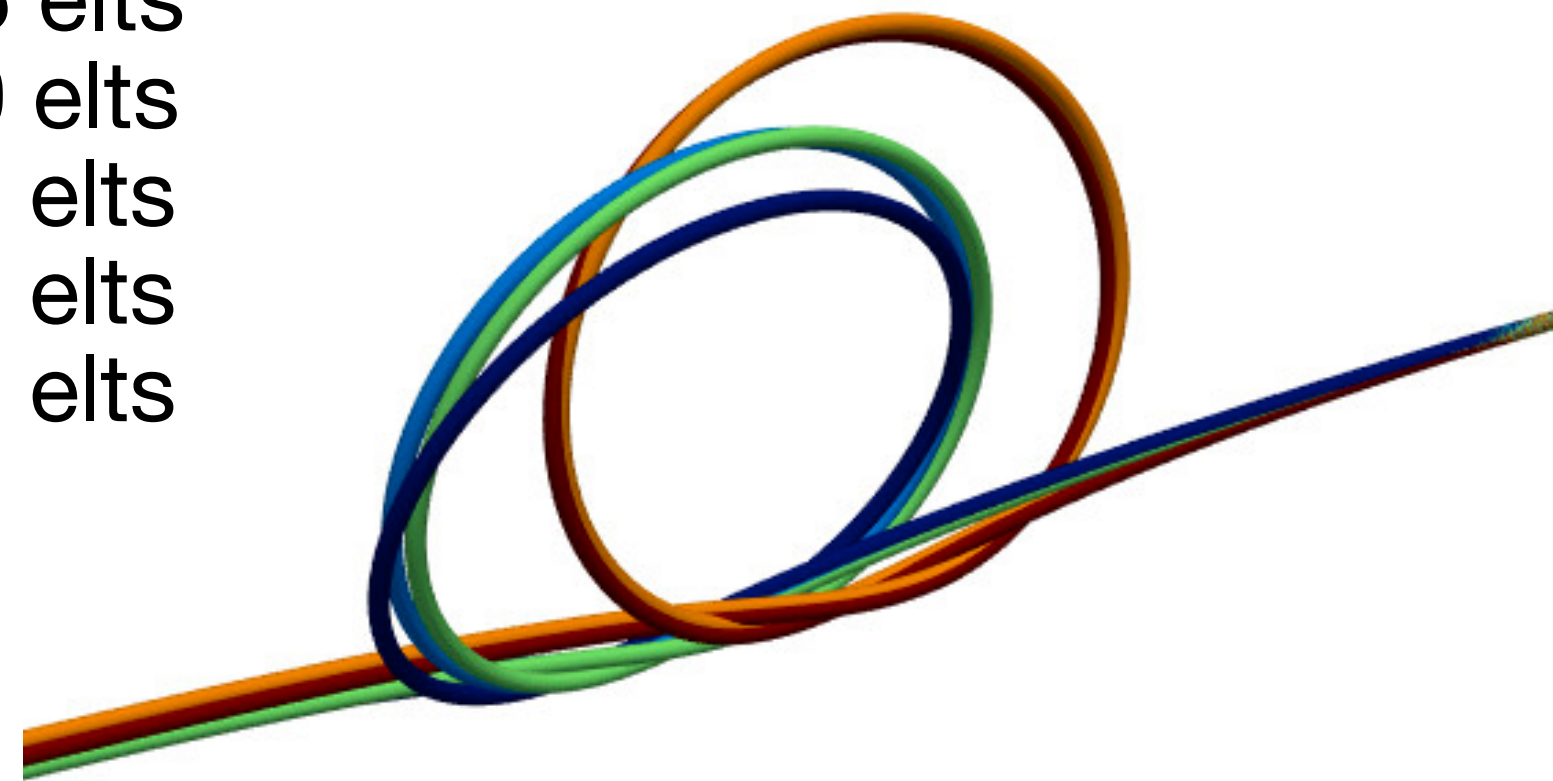
$$H_B = \frac{1}{2} \frac{EI}{R^2} + 0 + 0$$

$$H_A = H_B \Rightarrow T = \frac{1}{2} \frac{EI}{R^2}$$

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

Joint work with T. Metivet and V. Romero (Inria)

- 25 elts
- 50 elts
- 100 elts
- 200 elts
- 400 elts



Super-helix model
+ non-smooth contact solver

Case of finite thickness (simulation)

$$H_{3D} = \textit{curvature} + \textit{twist} + \textit{tension}$$

→ A simple way to **validate** a simulator

MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

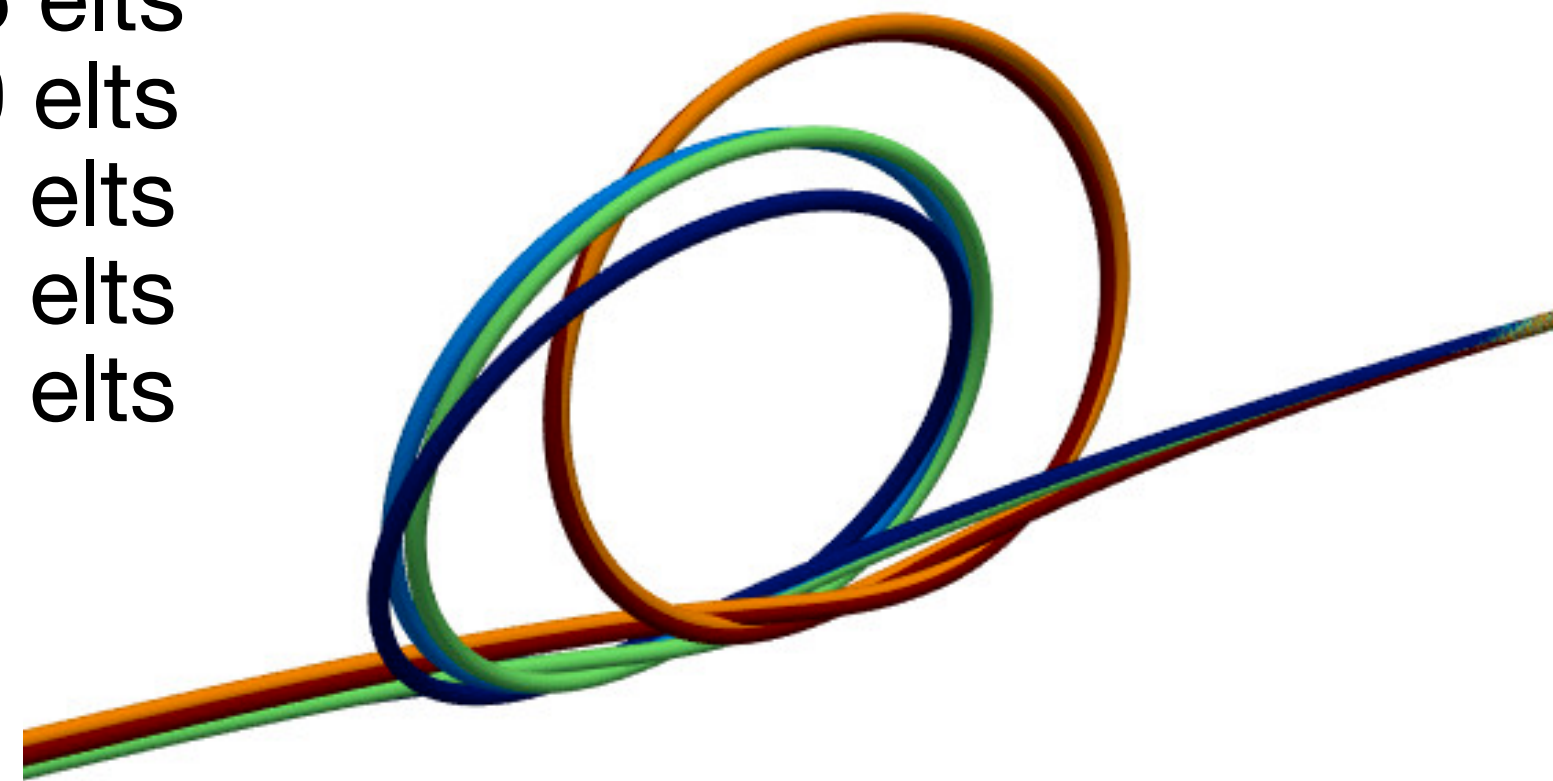
Joint work with T. Metivet and V. Romero (Inria)

Case of finite thickness (simulation)

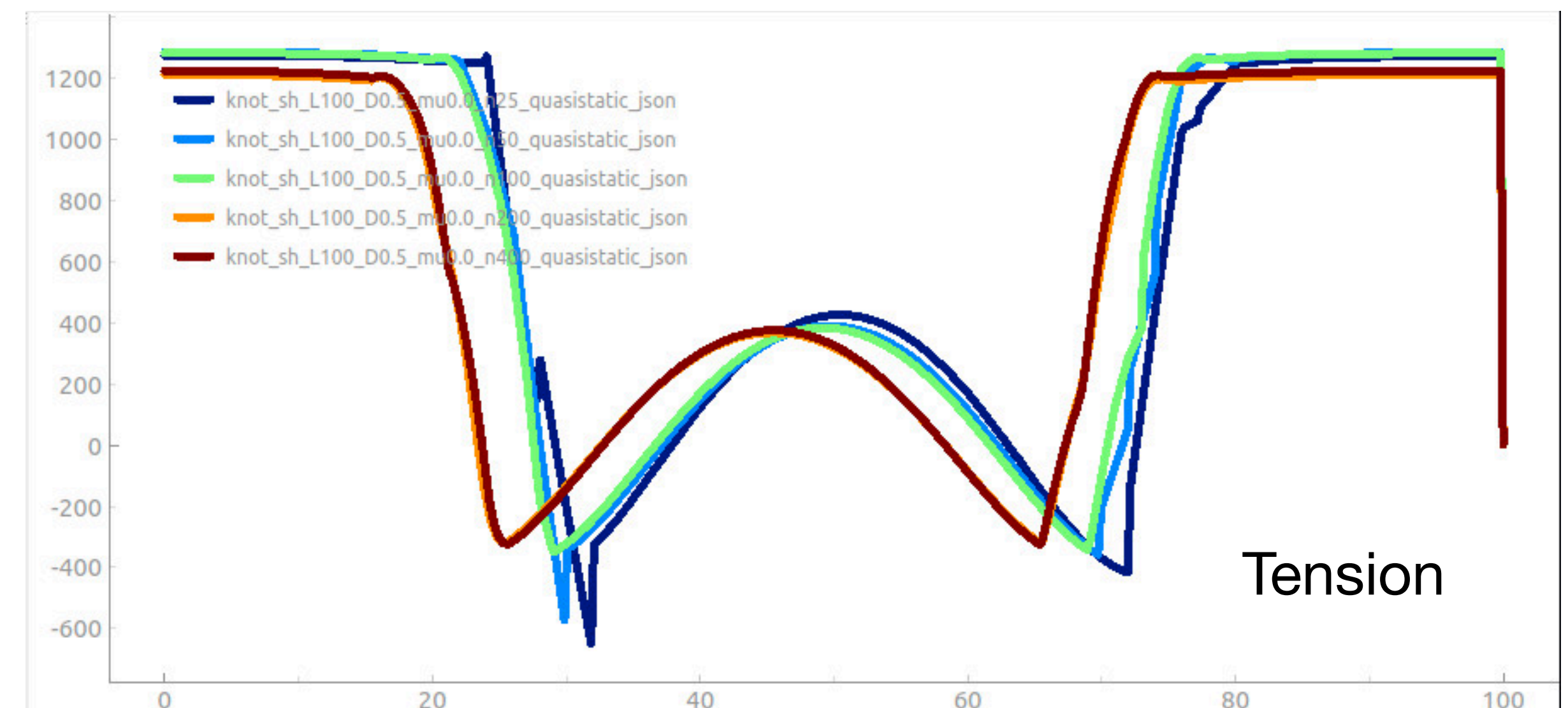
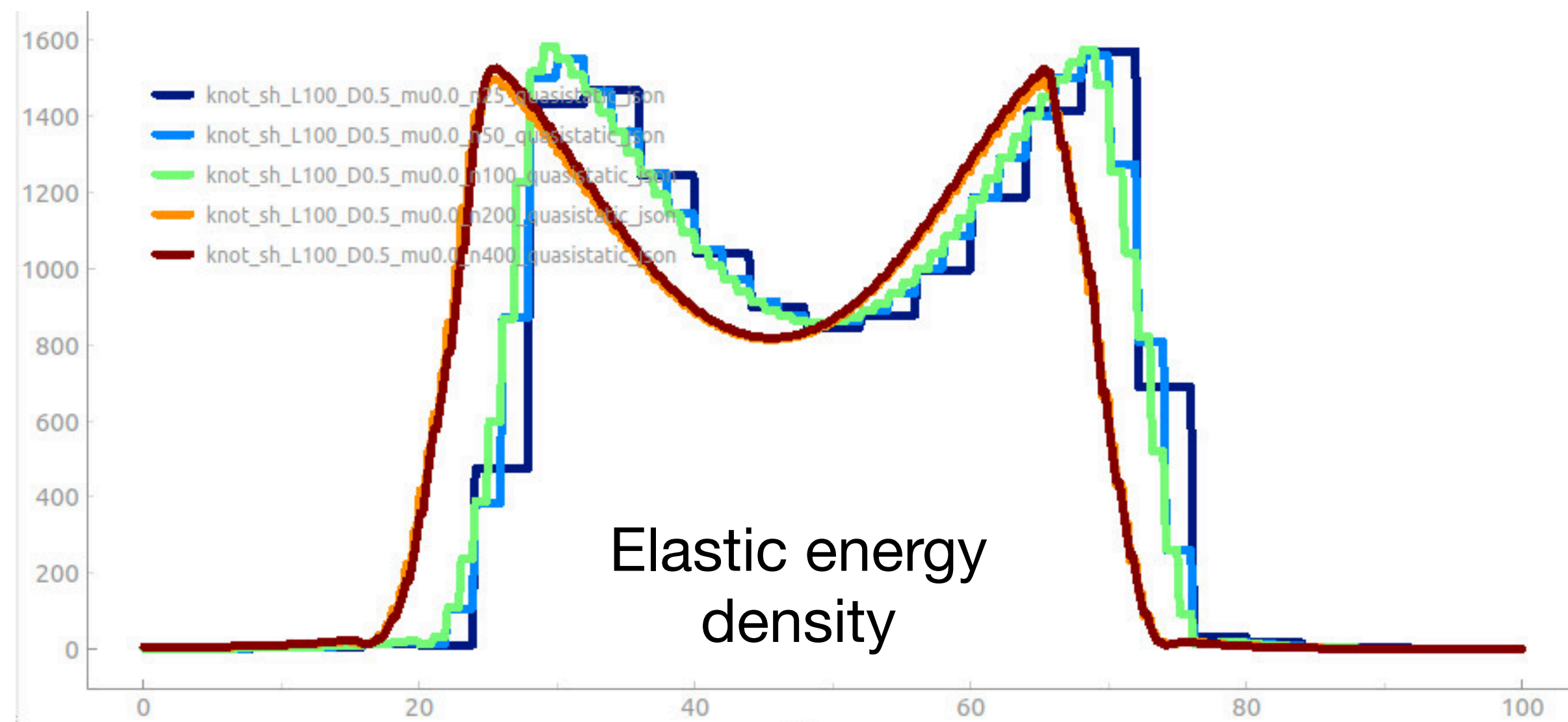
$$H_{3D} = \textit{curvature} + \textit{twist} + \textit{tension}$$

→ A simple way to **validate** a simulator

- 25 elts
- 50 elts
- 100 elts
- 200 elts
- 400 elts



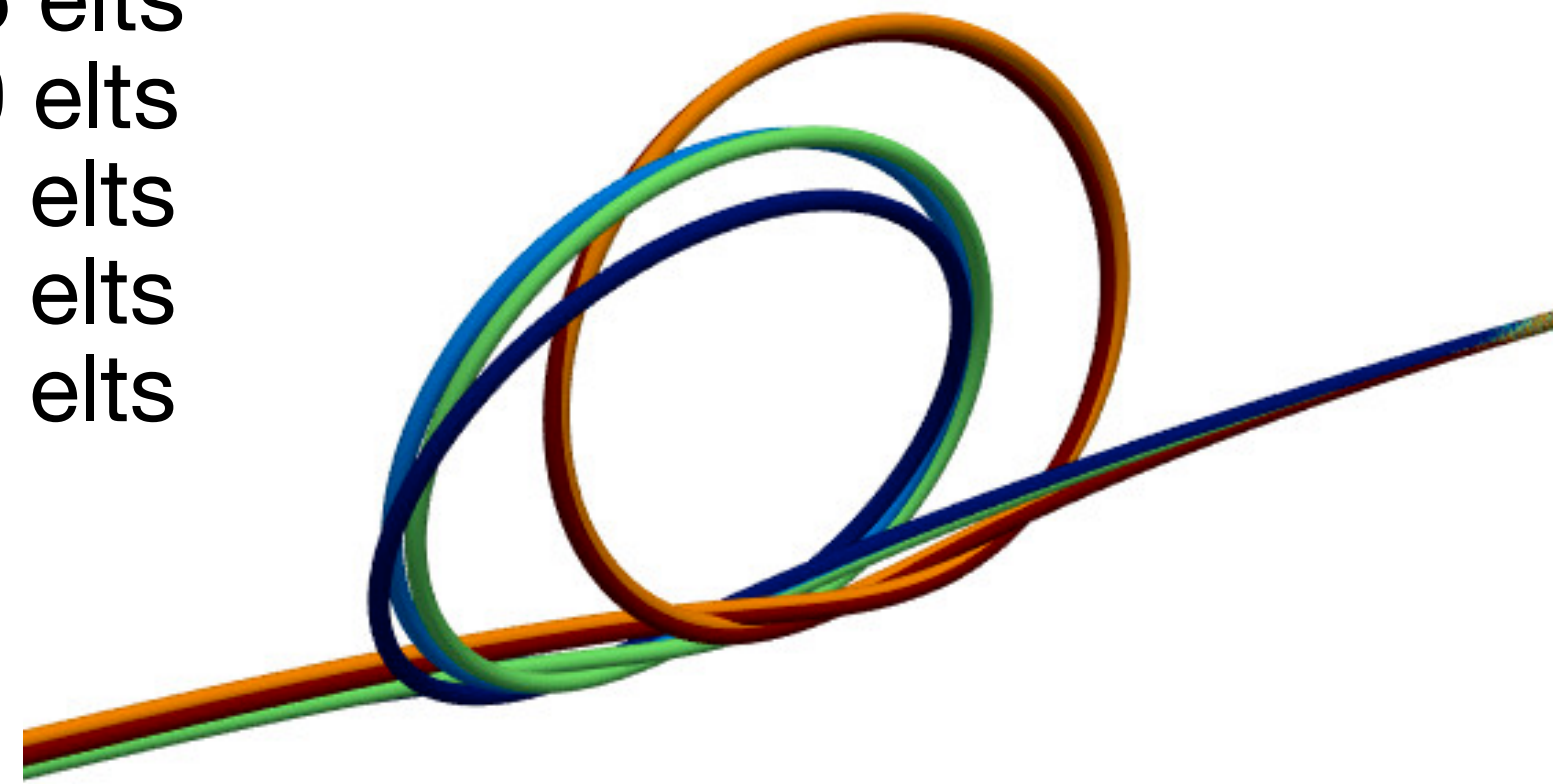
Super-helix model
+ non-smooth contact solver



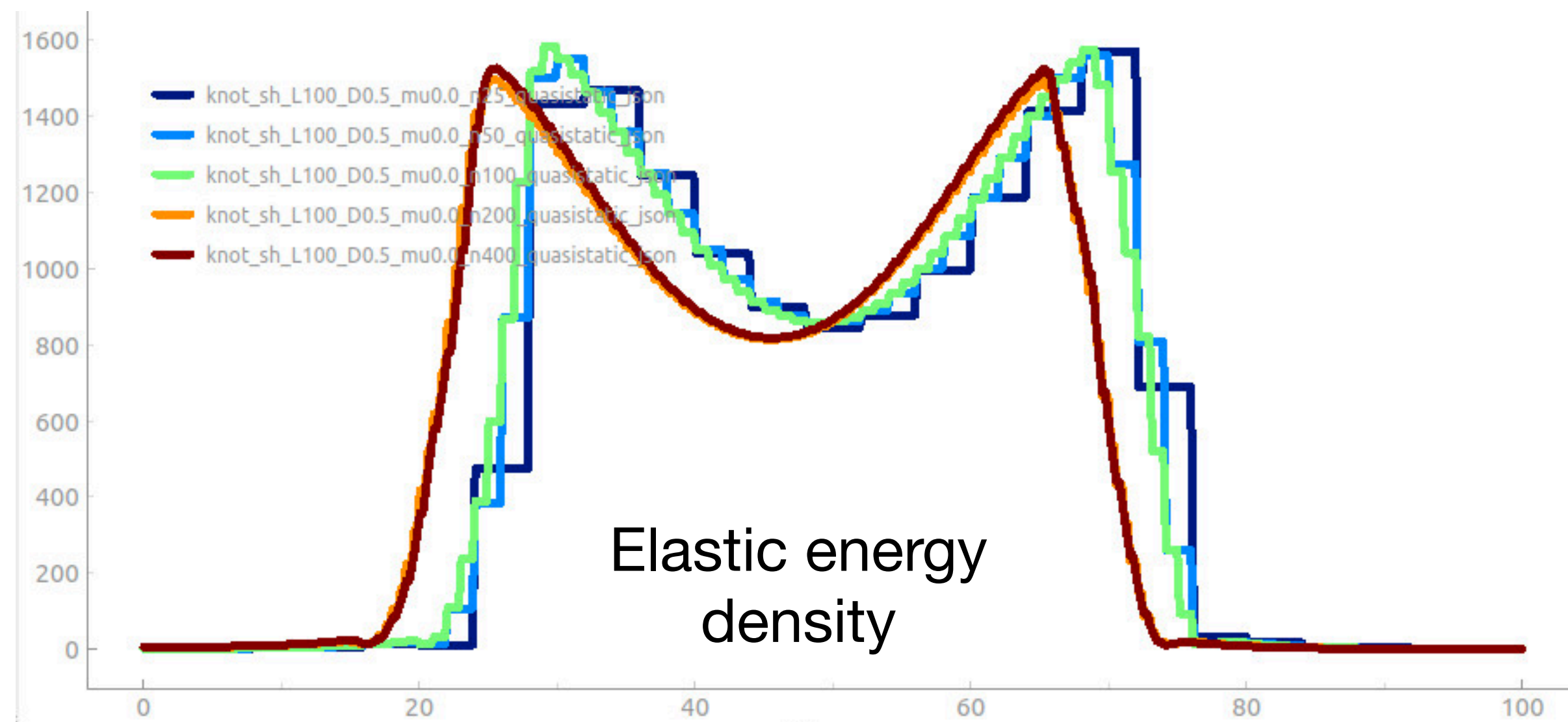
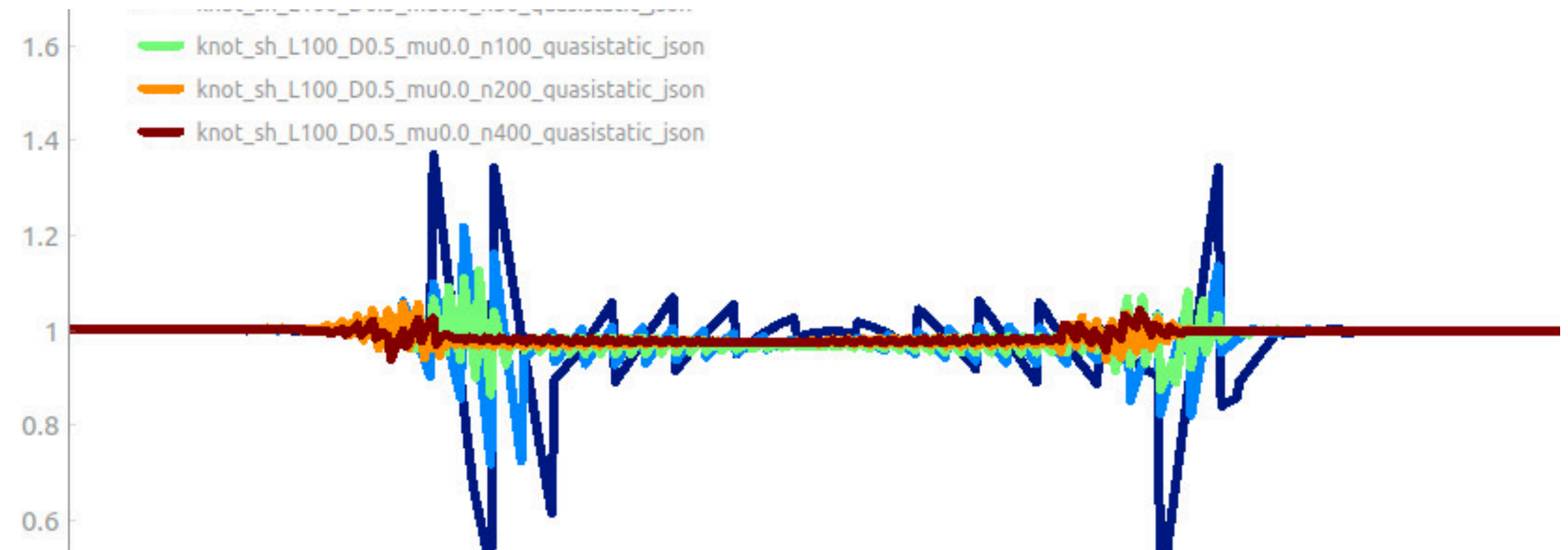
MAKING USE OF THE INVARIANT IN 3D: THE TREFOIL KNOT

Joint work with T. Metivet and V. Romero (Inria)

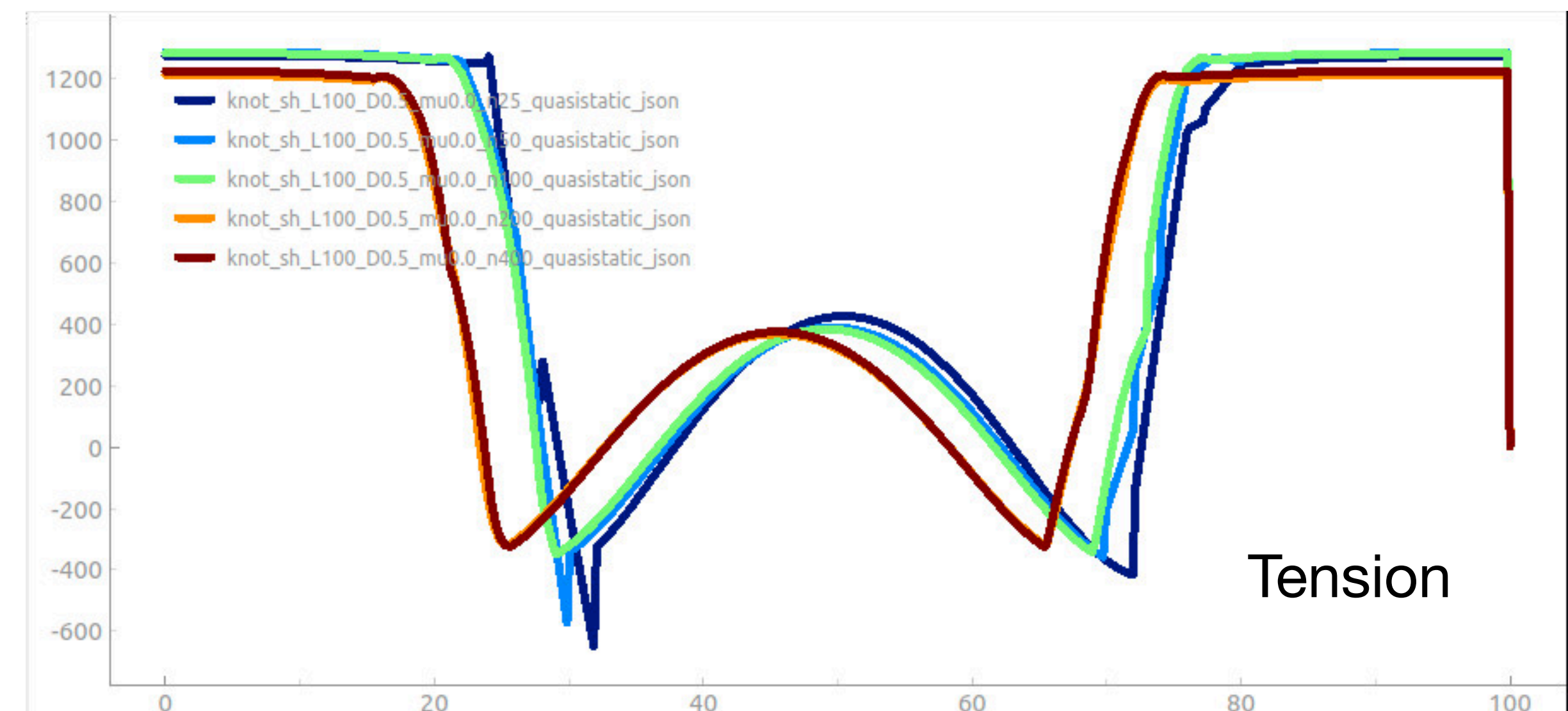
- 25 elts
- 50 elts
- 100 elts
- 200 elts
- 400 elts



Super-helix model
+ non-smooth contact solver

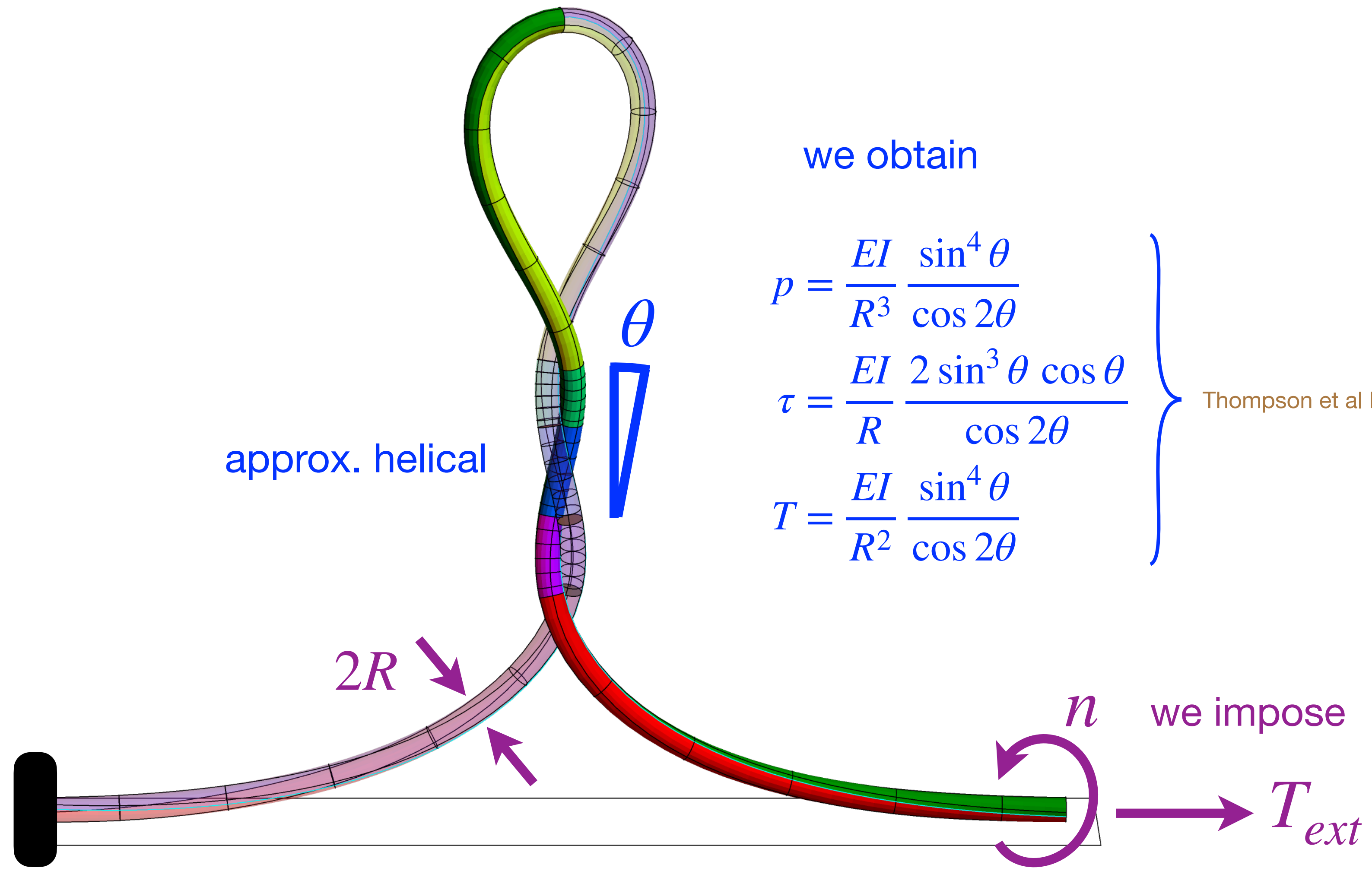


Elastic energy density



Tension

PLECTONEMES



we obtain

$$p = \frac{EI}{R^3} \frac{\sin^4 \theta}{\cos 2\theta}$$

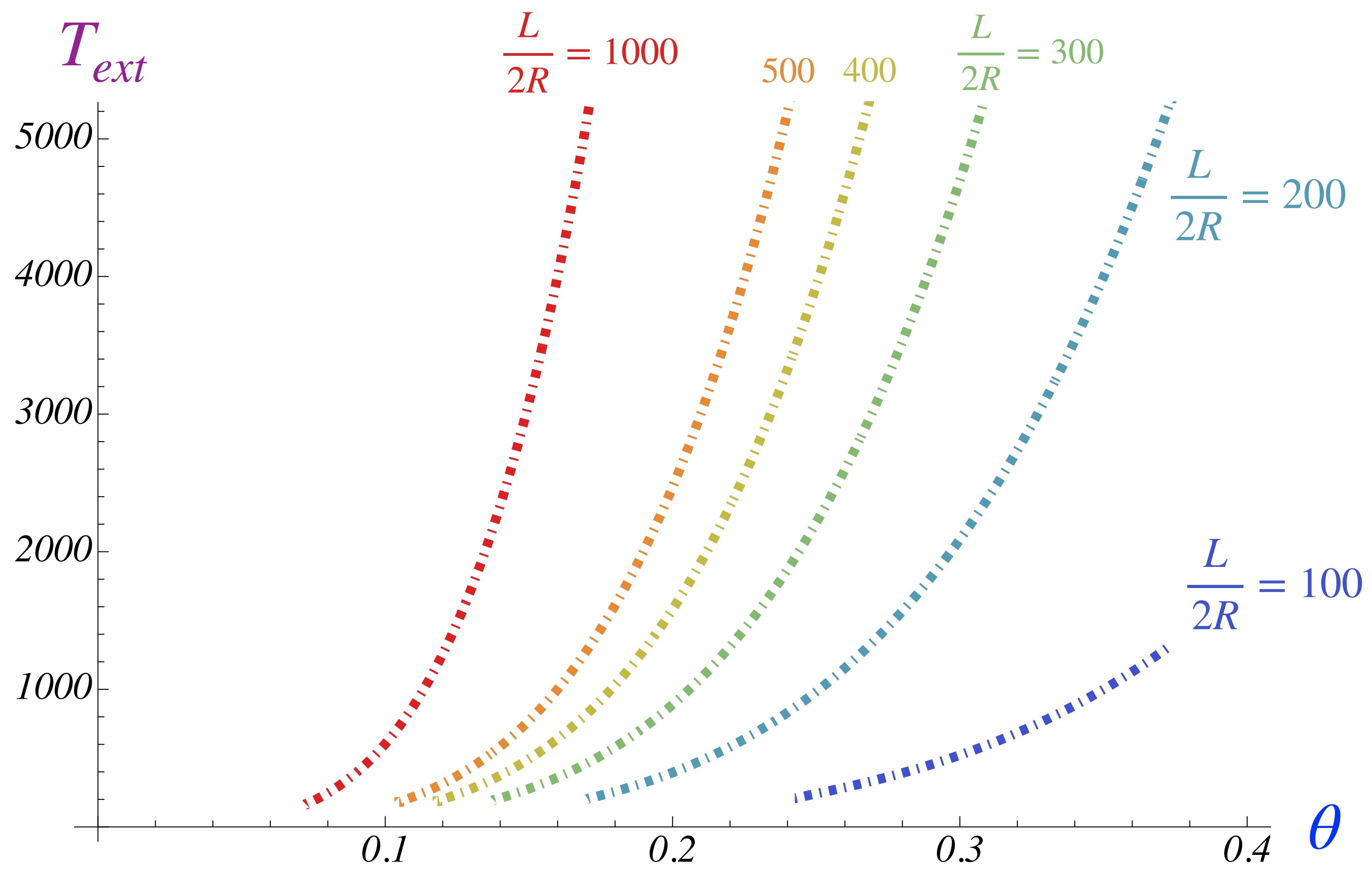
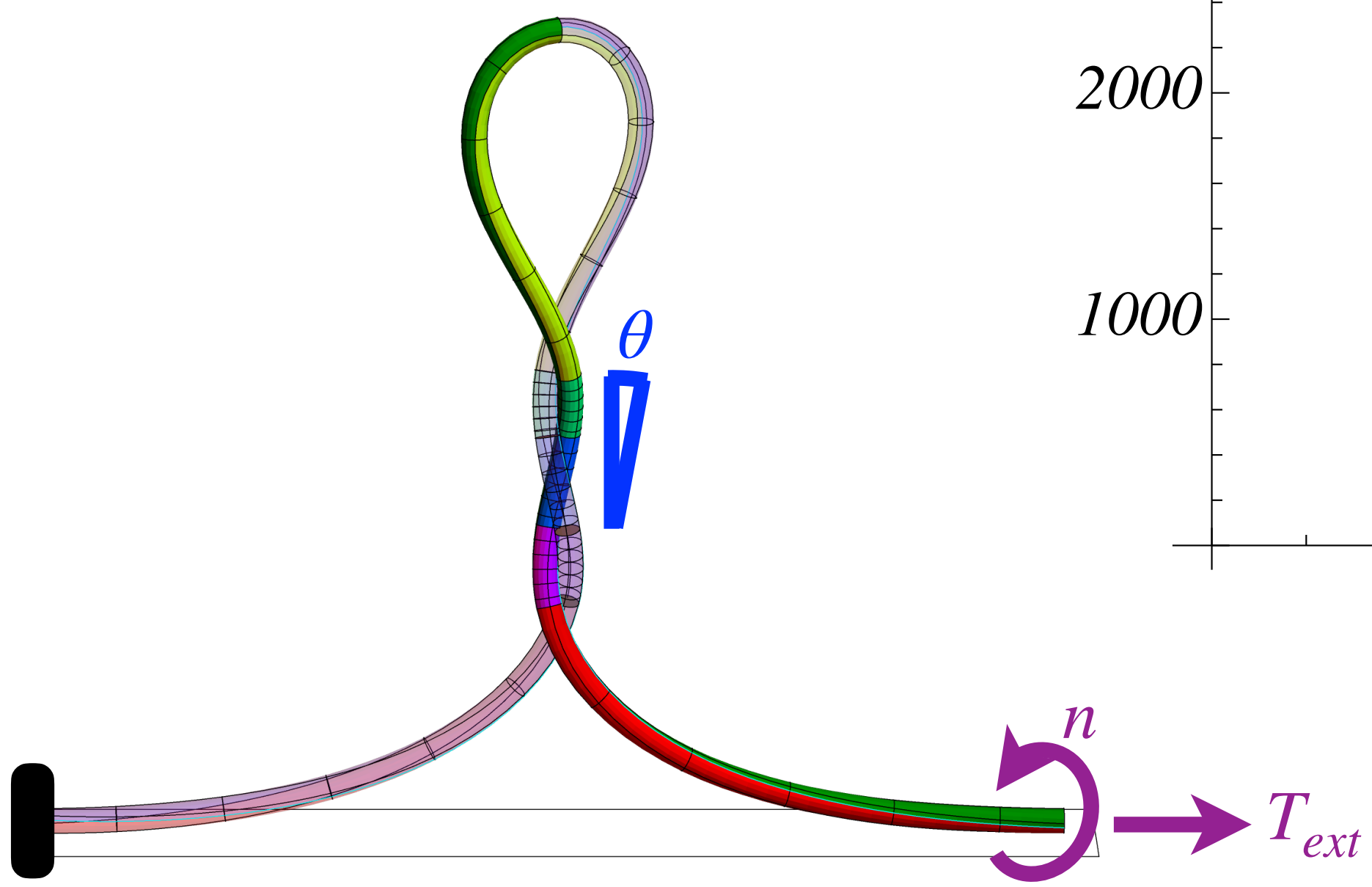
$$\tau = \frac{EI}{R} \frac{2 \sin^3 \theta \cos \theta}{\cos 2\theta}$$

$$T = \frac{EI}{R^2} \frac{\sin^4 \theta}{\cos 2\theta}$$

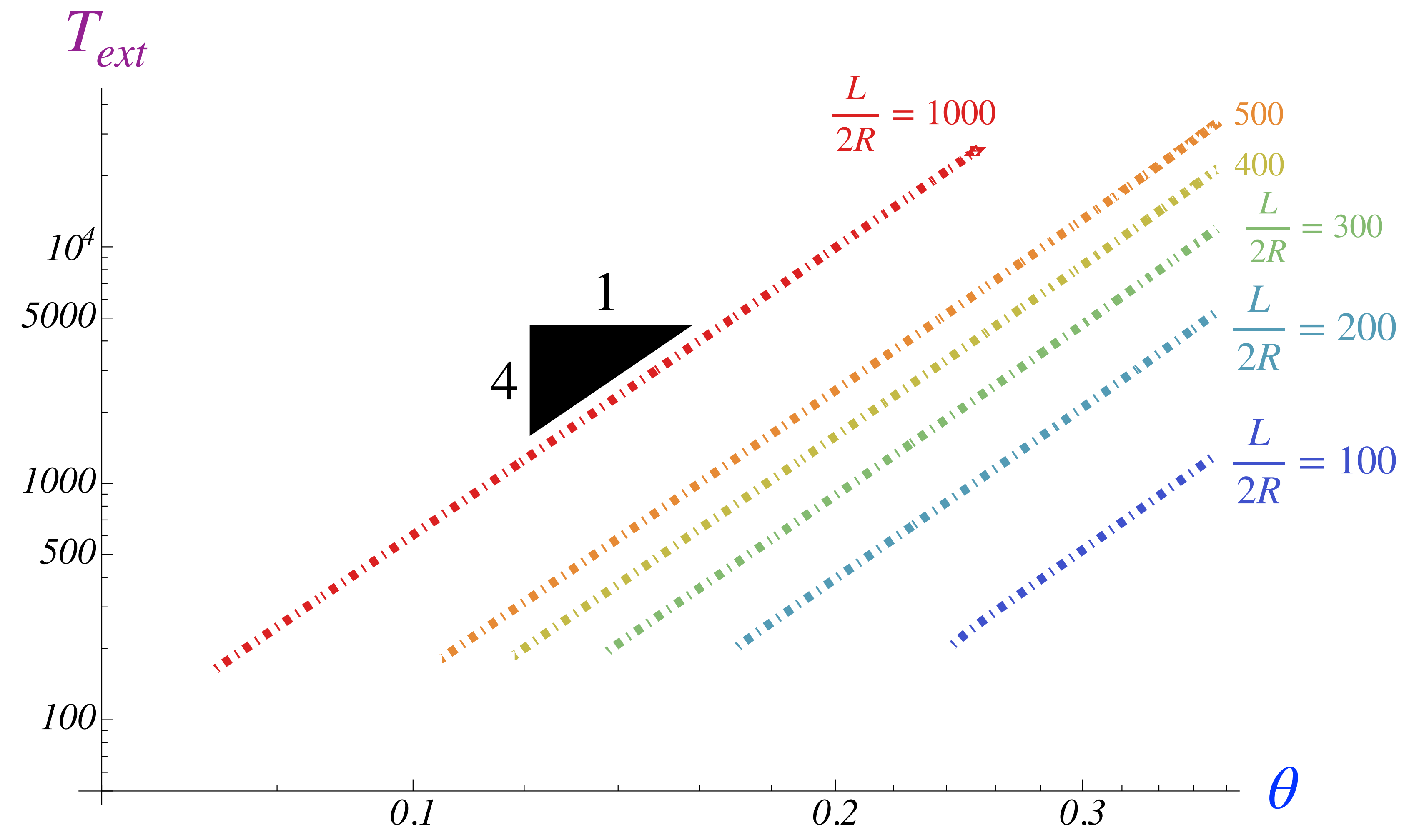
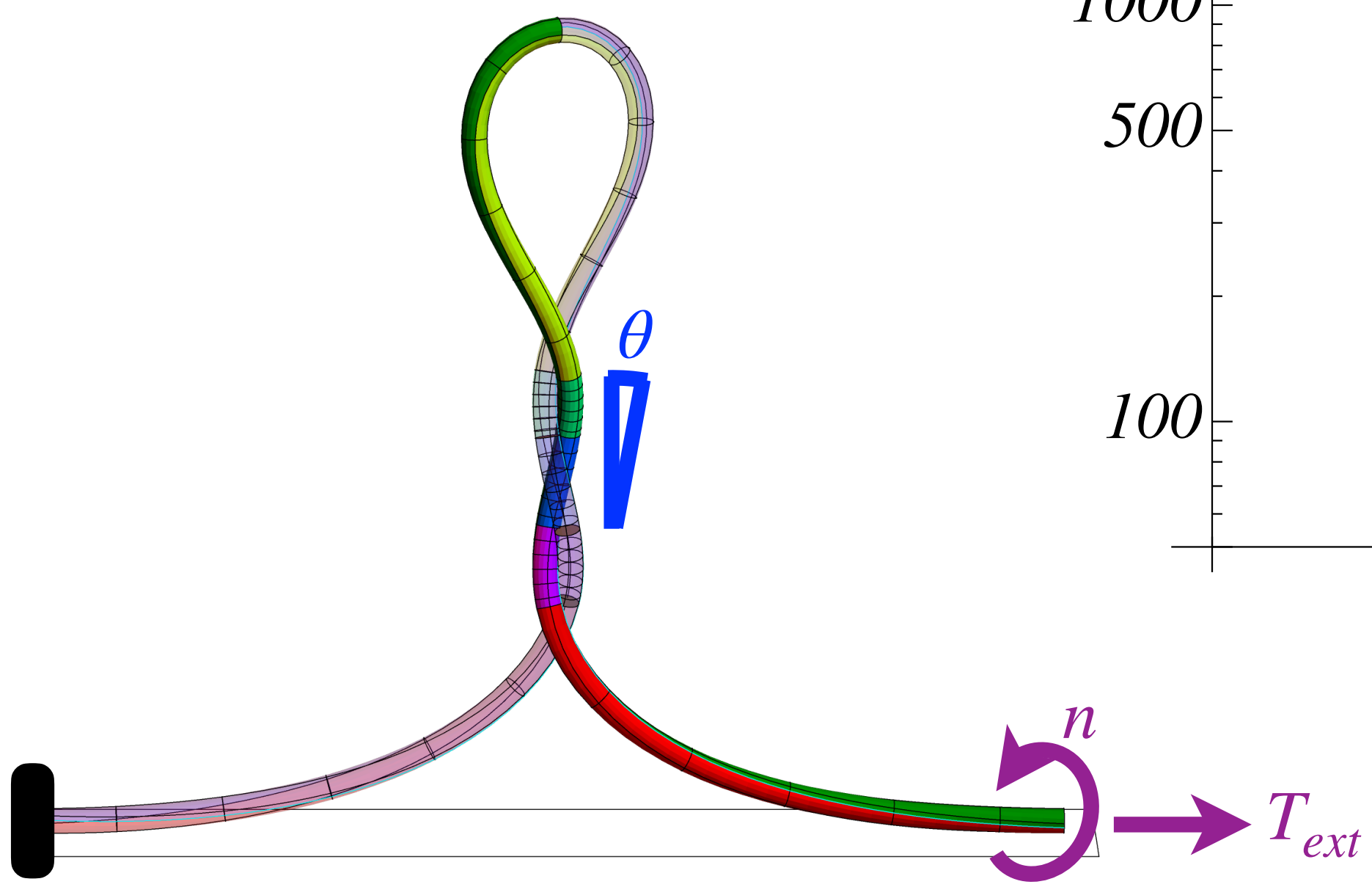
Thompson et al ProcSocA 2002

we impose

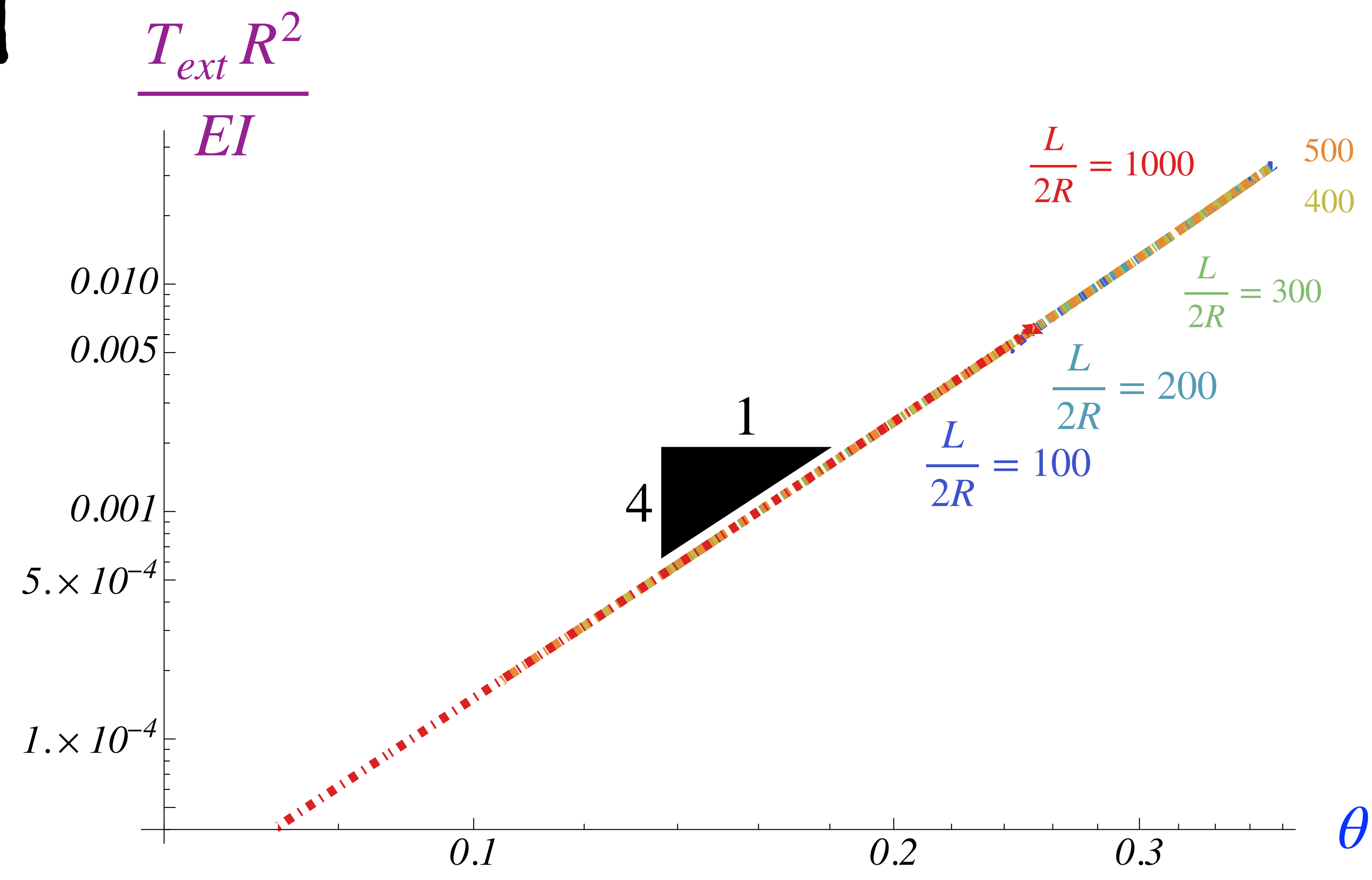
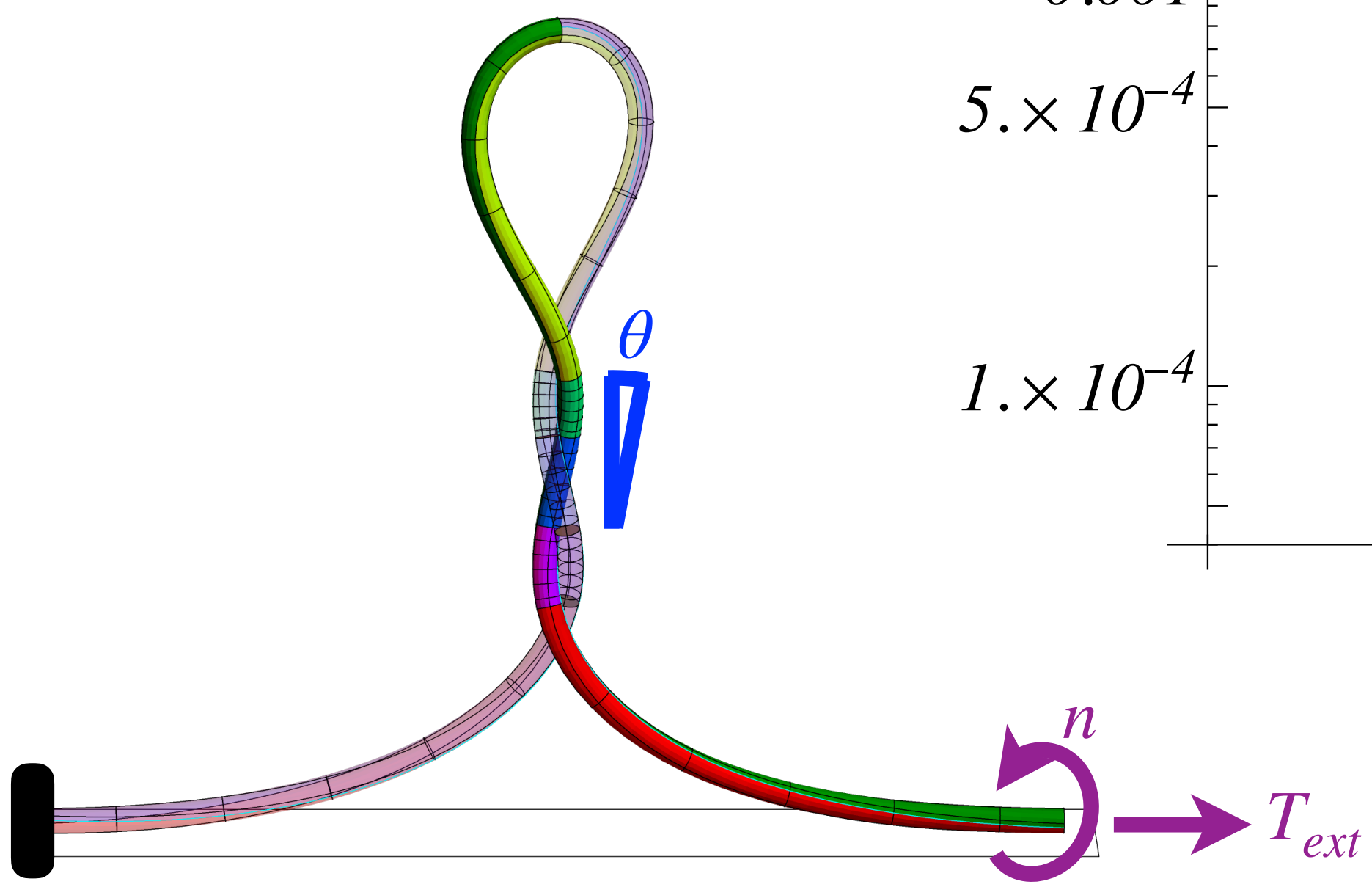
PLECTONEMES



PLECTONEMES

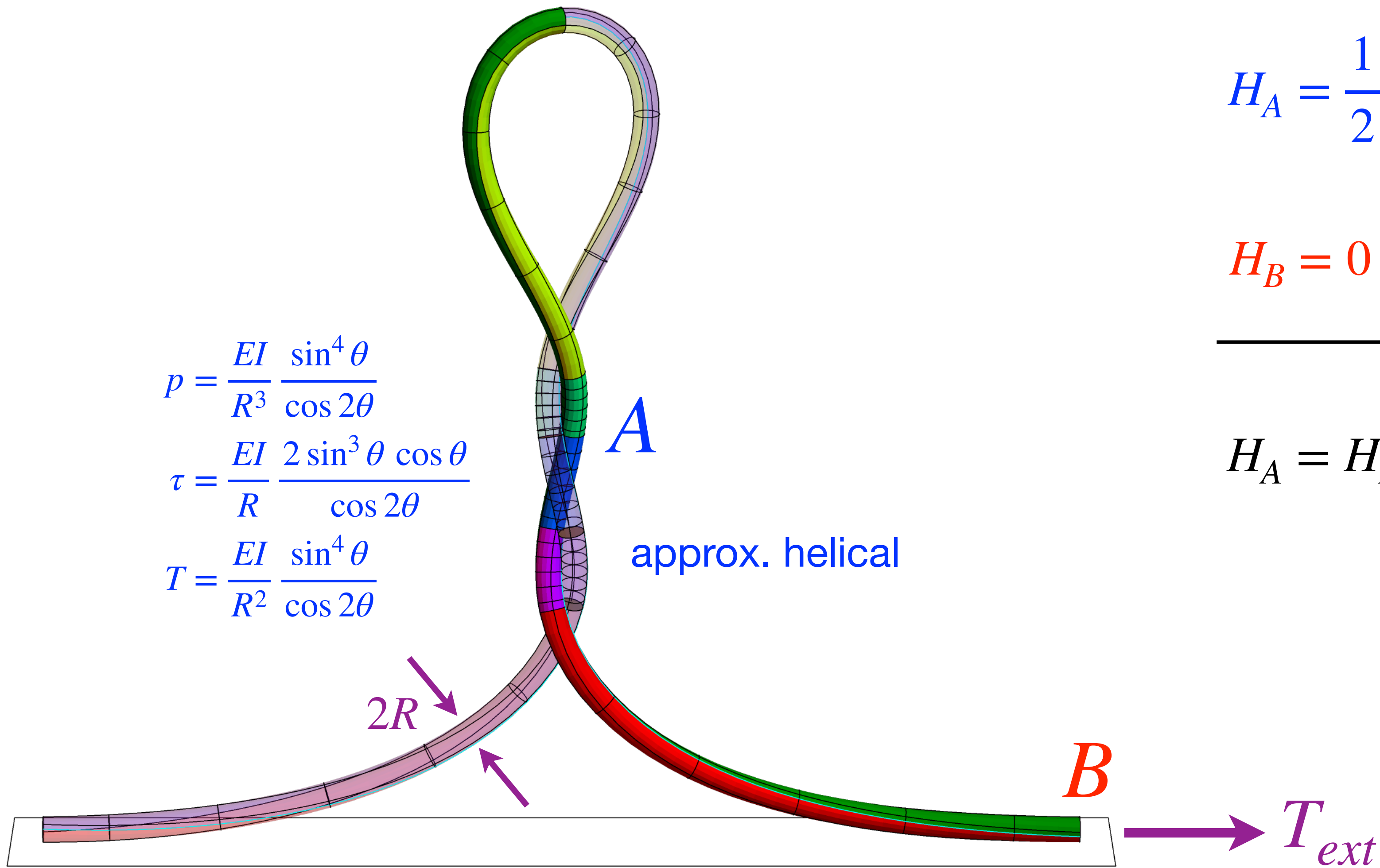


PLECTONEMES



PLECTONEMES

$H = \text{curvature} + \text{twist} + \text{tension}$

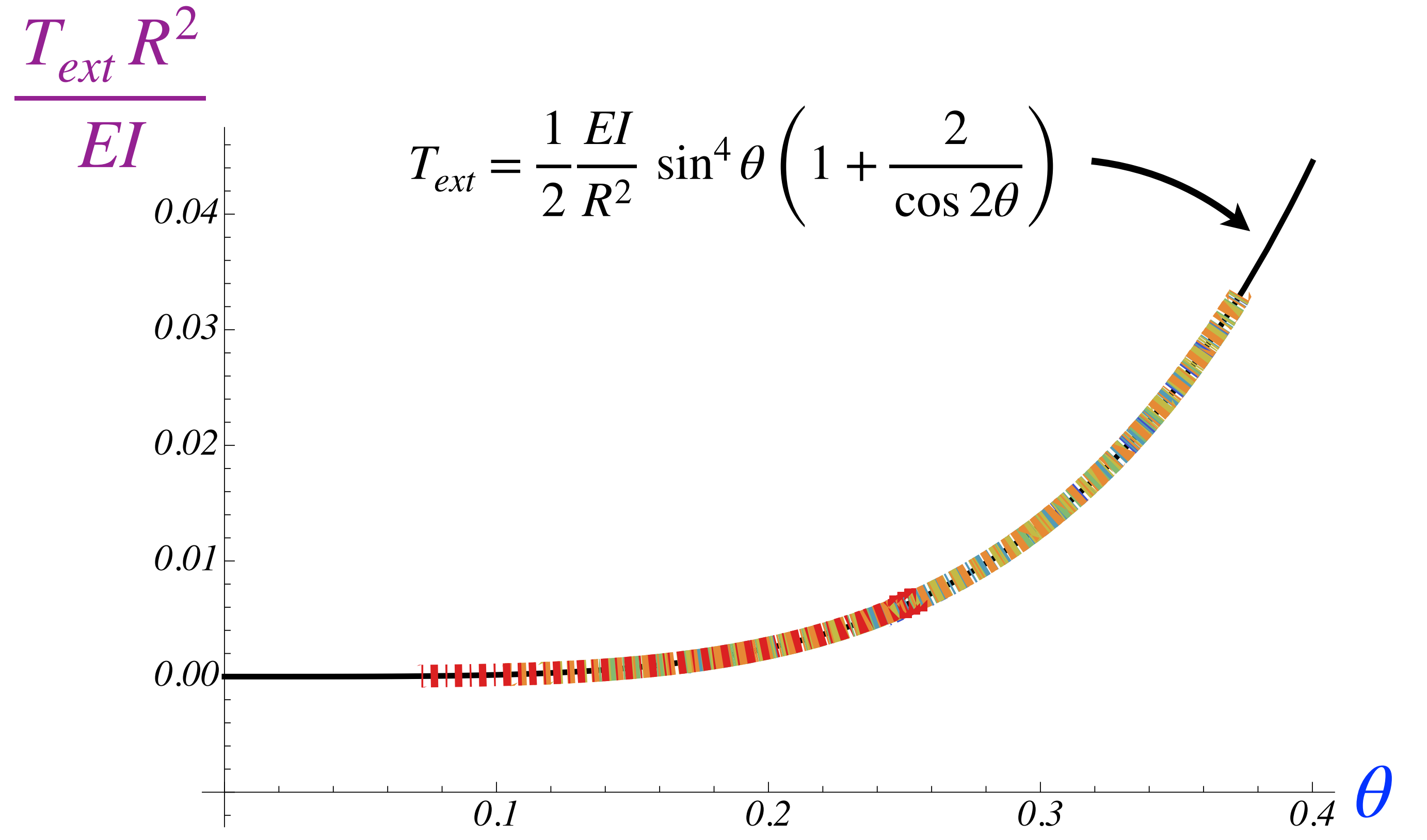
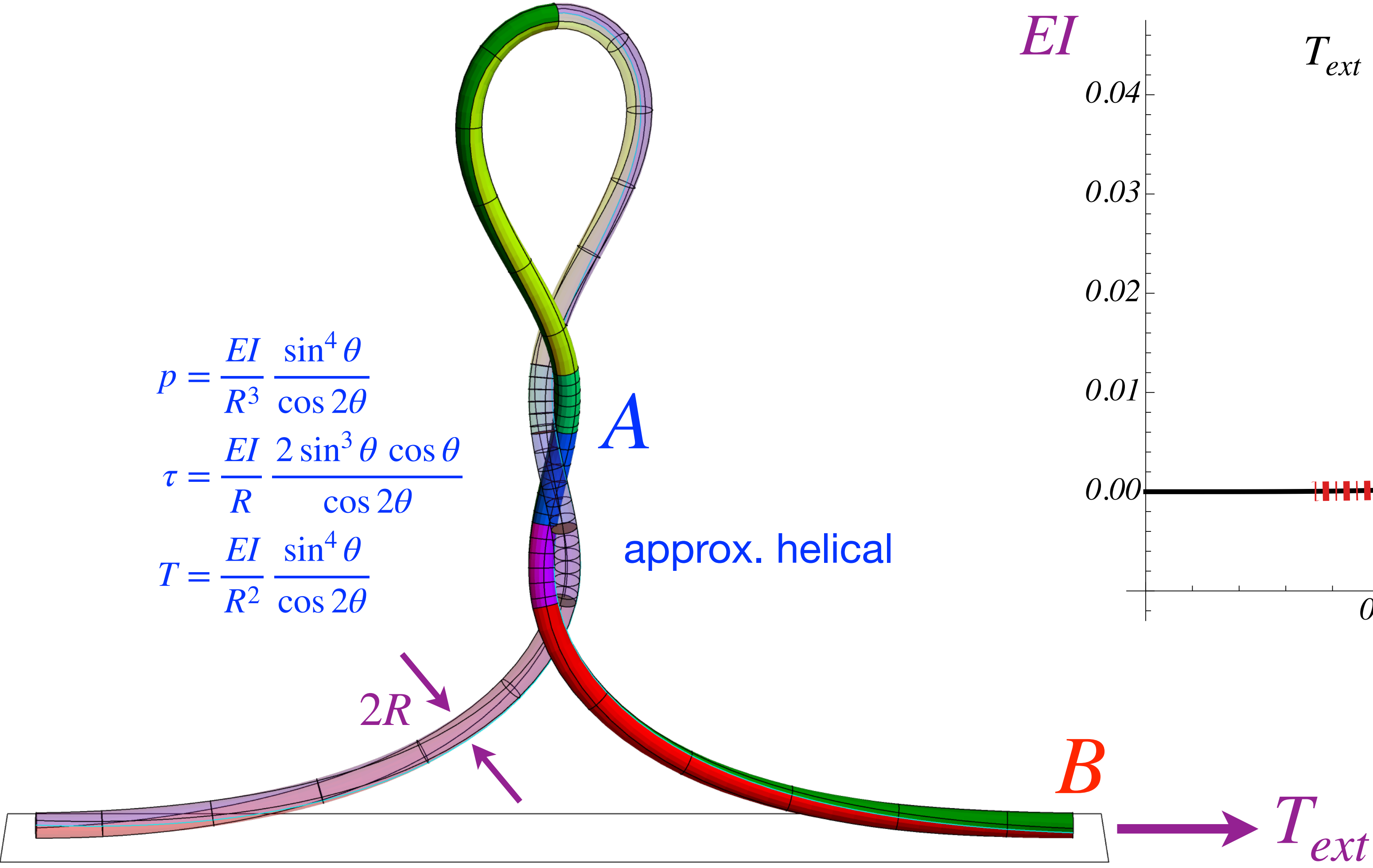


$$H_A = \frac{1}{2} \frac{EI}{R^2} \sin^4 \theta + \frac{1}{2} GJ\tau^2 + \frac{EI}{R^2} \frac{\sin^4 \theta}{\cos 2\theta}$$

$$H_B = 0 + \frac{1}{2} GJ\tau^2 + T_{ext}$$

$$H_A = H_B \Rightarrow T_{ext} = \frac{1}{2} \frac{EI}{R^2} \sin^4 \theta \left(1 + \frac{2}{\cos 2\theta} \right)$$

PLECTONEMES



CONCLUSION

- invariant for 2D and 3D Kirchhoff / Reissner elastic rods
- deduced from variational approach
- valid in case of (frictionless) contact / sleeves
- useful in numerical code validation
- dynamics: other invariants (statics/dynamics/vibrations)



preprint



CONCLUSION

- invariant for 2D and 3D Kirchhoff / Reissner elastic rods
- deduced from variational approach
- valid in case of (frictionless) contact / sleeves
- useful in numerical code validation
- dynamics: other invariants (statics/dynamics/vibrations)

preprint



EMMY NOETHER
(1882 - 1935)

- University of Erlangen, Göttingen then USA
- Exceptional mathematical contributions (algebra)
- Strongly discriminated against because was a woman
- Things have (fortunately) changed but... stay careful!