



**HAL**  
open science

## Evolution inclusion driven by fractional Brownian motion

Charles Castaing, Christiane Godet-Thobie, Le Xuan Truong

► **To cite this version:**

Charles Castaing, Christiane Godet-Thobie, Le Xuan Truong. Evolution inclusion driven by fractional Brownian motion. 2025. <hal-05157047>

**HAL Id: hal-05157047**

**<https://hal.science/hal-05157047v1>**

Preprint submitted on 10 Jul 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Evolution inclusion driven by fractional Brownian motion

Charles Castaing \* Christiane Godet-Thobie<sup>†</sup> Le Xuan Truong<sup>‡</sup>

May 20, 2025

## 1 Introduction

The paper is devoted to the research of  $\beta$ -Holder continuous and AC solutions for several types of fractional differential equation coupled by evolution inclusions with Sweeping Process or time dependent maximal monotone operator  $A_t$  driven by Volterra integral perturbation and fraction Brownian motion (fBM)  $B^H = \{B^H(t) : t \in [0, T]\}$  with Hurst parameter  $\frac{1}{2} < H < 1$ . In particular, if  $1-H < \beta < \frac{1}{2}$ , then there exist a pair  $(x, u)$  where  $x : [0, T] \rightarrow \mathbf{R}^e$  is  $\beta$ -Holder continuous and  $u : [0, T] \rightarrow \mathbf{R}^e$  is absolutely continuous, satisfying

$$\begin{cases} {}^c D^\beta x(t) = G(t, u(t)) \\ u(0) = u_0 \in C(0) \\ u(t) \in C(t), \forall t \in [0, T] \\ -\dot{u}(t) \in N_{C(t)} u(t) + \int_0^t f(t, s, u(s), \int_0^s b(\theta, x(\theta)) dB_\theta^H) ds, \quad \text{a.e.} \end{cases}$$

where  ${}^c D^\beta x$  denotes the Caputo fraction derivative of  $x$ ,  $G(t, \cdot)$  is linear continuous,  $f : [0, T] \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  and  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  are Caratheodory mapping,  $\int_0^s b(\theta, x(\theta)) dB_\theta^H$  is integral of  $b(\cdot, x(\cdot))$  with respect to  $dB^H$ . A quick glance shows that this problem is of great interest since that solutions  $(x, u)$  of the above dynamic are not classical. Furthermore, it is clear that the solution problem for such a system is quite general since it contains many results related to classical solutions of similar evolution equations. Several variants are available with application to Optimization problems. This research led to further ramifications by considering further types of fractional evolution inclusion and Operators, namely the Skorohod problem in **fractional** evolution inclusion driven by Sweeping process and Operators with perturbation  $g : [0, T] \times [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$ : there exist a pair  $(x, u)$  where  $x : [0, T] \rightarrow \mathbf{R}^e$  is  $\beta$ -Holder continuous and  $u : [0, T] \rightarrow \mathbf{R}^e$  is  $\beta$ -Holder continuous, satisfying to the dynamic

---

\*Departement de Mathematiques, Université de Montpellier, France. E-mail: charles.castaing@gmail.com

<sup>†</sup>Université de Bretagne Occidentale, Laboratoire de Mathématiques de Bretagne Atlantique, CNRS UMR 6205, 6, avenue Victor Le Gorgeu, CS 9387, F-29238 Brest Cedex 3, France. E-mail: christiane.godet-thobie@univ-brest.fr

<sup>‡</sup>Department of Mathematics and Statistics, University of Economics Ho Chi Minh City, Vietnam, Email:lxuantruong@gmail.com

$$\begin{cases} x(0) = u(0) = a \\ x(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H + \int_0^t g(t, s, x(s)) ds + u(t), \forall t \in I \\ u(t) \in C(t), \forall t \in I \\ -{}^c D^\beta u(t) \in N_{C(t)} u(t) + \int_0^t g(t, s, x(s)) ds, \text{ a.e. } t \in I, \end{cases}$$

$$\begin{cases} x(0) = u(0) = a \\ x(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H + \int_0^t g(t, s, x(s)) ds + u(t), \forall t \in I \\ u(t) \in D(A_t), \forall t \in I \\ -{}^c D^\beta u(t) \in A_t u(t) + \int_0^t g(t, s, x(s)) ds, \text{ a.e. } t \in I, \end{cases}$$

New result for the Skorohod problem driven by fractional inclusion by time and state dependent maximal monotone operator  $A_{(t,x)}$  is available.

$$\begin{cases} x(0) = u(0) = a \in D(A_{(0,0)}) \\ x(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H + \int_0^t g(t, s, x(s)) ds + u(t), \forall t \in I \\ u(t) \in D(A_{(t, \int_0^t b(\tau, x(\tau)) dB_\tau^H)}), \forall t \in I \\ -{}^c D^\alpha u(t) \in A_{(t, \int_0^t b(\tau, x(\tau)) dB_\tau^H)} u(t) + \int_0^t g(t, s, x(s)) ds, \text{ a.e. } t \in I, \end{cases}$$

where  $x$  and  $u$  are  $\alpha$ -Holder continuous with  $\alpha \in ]0, 1[$  and  ${}^c D^\alpha u$  is the Caputo fraction derivative of the  $\alpha$ -Holder  $u$ .

## 1.1 Integration with respect to fractional Brownian motion

A fractional Brownian motion (fBm) with Hurst parameter  $H \in ]0, 1[$  defined on a probability space  $\Omega, \mathcal{F}, P$ ) is a centered continuous Gaussian process  $B^H = \{B_t^H : t \in [0, T]\}$  with the covariance function

$$R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - (t-s)^{2H})$$

for all  $s, t \in [0, T]$ . It is well known that, by Kolmogorov's continuity criterion,  $B_t^H$  admits a version whose sample paths are almost surely Hölder continuous of order strictly less than  $H$ . To be specific, for every  $\varepsilon \in ]0, H[$ , there is a positive integrable random variable  $\eta_{\varepsilon, T}$  such that

$$|B_t^H - B_s^H| \leq \eta_{\varepsilon, T} |t - s|^{H-\varepsilon} \text{ a.s.}$$

See, for example, ([19] , Lemma 7.4).

For our purpose, we will work with the function spaces that are defined as follows: Let  $0 < \gamma < 1$ , we define

◇ The Hölder spaces:  $C^\gamma([a, b]; \mathbb{R}^e) = \{\phi \in C([a, b]; \mathbb{R}^d) \text{ such that } \|\phi\|_{\gamma, [a, b]} < \infty\}$  with  $\gamma \in (0, 1)$  and

$$\|\phi\|_{\gamma, [a, b]} := \sup_{a \leq t \leq b} \|\phi(t)\| + \sup_{a \leq s < t \leq b} \frac{\|\phi(t) - \phi(s)\|}{|t - s|^\gamma}.$$

◇ The spaces  $W_{\mathbf{R}^e}^{\gamma,1}[a,b] = \left\{ \text{measurable function } f : [a,b] \rightarrow \mathbb{R}^e \text{ such that } \|f\|_{W_{\mathbf{R}^e}^{\gamma,1}[a,b]} < \infty \right\}$ , where

$$\|f\|_{W_{\mathbf{R}^e}^{\gamma,1}[a,b]} := \int_a^b \frac{\|f(s)\|}{(s-a)^\gamma} ds + \int_a^b \int_a^s \frac{\|f(s) - f(\tau)\|}{(s-\tau)^{\gamma+1}} d\tau ds.$$

These function spaces are Banach spaces with the corresponding norms.

Next we discuss the integral of measurable function with respect to  $B^H$ . In this paper, we take  $H > \frac{1}{2}$  and  $\gamma$  such that  $1 - H < \gamma < \frac{1}{2}$ . For  $g \in W_{\mathbf{R}^e}^{\gamma,1}[0,T]$  and  $0 \leq a < b \leq T$ , we recall the definition and a property of its integral  $\int_a^b g(s)dB_s^H$  with respect to  $B^H$  by

$$\int_a^b g(s)dB_s^H = e^{-i\pi\gamma} \int_a^b (D_{a^+}^\gamma g)(D_{b^-}^{1-\gamma} B_{b^-}^H) dt$$

Herein for all  $t \in [a,b]$

$$B_{b^-}^H(t) = B^H(t) - \lim_{\varepsilon \rightarrow 0^+} B^H(b - \varepsilon)$$

wheras  $D_{a^+}^\gamma$  and  $D_{b^-}^{1-\gamma}$  stand for the Weyl derivative given by

$$D_{a^+}^\gamma g(t) := \frac{1}{\Gamma(1-\gamma)} \left( \frac{g(t)}{(t-a)^\gamma} + \gamma \int_a^t \frac{g(t) - g(s)}{(t-s)^{1+\gamma}} ds \right)$$

and

$$D_{b^-}^{1-\gamma} B^H(t) := \frac{(-1)^\gamma}{\Gamma(\gamma)} \left( \frac{B^H(t) - B^H(b)}{(b-t)^{1-\gamma}} + (1-\gamma) \int_t^b \frac{B^H(t) - B^H(s)}{(t-s)^\gamma} ds \right)$$

So, we have the inequality (\*):

$$\left\| \int_a^b g(s)dB_s^H \right\| \leq K(\gamma, [a,b]) \|g\|_{W_{\mathbf{R}^e}^{\gamma,1}[a,b]} \quad (*)$$

where

$$K(\gamma, [a,b]) := \frac{1}{\Gamma(1-\gamma)} \sup_{a < \tau < s < b} \|D_{s^-}^{1-\gamma} B_{s^-}^H(\tau)\| < \infty \text{ a.s}$$

For more details, we refer the reader to [19].

## 2 Fractional/evolution inclusion driven by fractional Brownian motion

We introduce in the following the definition and some properties of maximal monotone operators needed in the proofs of our results, and we refer the reader to [1] for their basic theory and more details.

Let  $A : D(A) \subset H \rightrightarrows H$  be a set-valued operator. We use classical definitions of the domain  $D(A)$ , the range  $R(A)$  and the graph  $gph(A)$  of  $A$ . We say that  $A : D(A) \subset H \rightrightarrows H$  is monotone, if  $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$  whenever  $(x_i, y_i) \in gph(A)$ ,  $i = 1, 2$ . It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all  $\lambda > 0$ ,  $R(I_H + \lambda A) = H$ , where  $I_H$  stands for the identity mapping of  $H$ .

If  $A$  is a maximal monotone operator then, for every  $x \in D(A)$ ,  $Ax$  is non-empty, closed and convex. So that, the projection of the origin into  $Ax$ ,  $A^0(x)$ , exists and is unique.

Let  $A : D(A) \subset H \rightrightarrows H$  and  $B : D(B) \subset H \rightrightarrows H$  be two maximal monotone operators, then we denote by  $dis(A, B)$  (see [20]) the pseudo-distance between  $A$  and  $B$  defined by

$$dis(A, B) = \sup \left\{ \frac{\langle y - y', x' - x \rangle}{1 + \|y\| + \|y'\|} : (x, y) \in gph(A), (x', y') \in gph(B) \right\}. \quad (2.1)$$

Clearly,  $dis(A, B) \in [0, +\infty]$ ,  $dis(A, B) = dis(B, A)$  and  $dis(A, B) = 0$  iff  $A = B$ .

If the maximal monotone operator is time dependent, it will be noted  $A_t$ . If it is time and space dependent, it will be noted  $A_{(t,x)}$ .

To prove our main results, we need the following lemmas (see [14]).

**Lemma 2.1** *Let  $A$  be a maximal monotone operator of  $H$ . If  $x \in \overline{D(A)}$  and  $y \in H$  are such that*

$$\langle A^0 z - y, z - x \rangle \geq 0 \quad \forall z \in D(A),$$

*then  $x \in D(A)$  and  $y \in Ax$ .*

**Lemma 2.2** *Let  $A_n$  ( $n \in \mathbb{N}$ ),  $A$  be maximal monotone operators of  $H$  such that  $dis(A_n, A) \rightarrow 0$ . Suppose also that  $x_n \in D(A_n)$  with  $x_n \rightarrow x$  and  $y_n \in A_n(x_n)$  with  $y_n \rightarrow y$  weakly for some  $x, y \in H$ . Then  $x \in D(A)$  and  $y \in Ax$ .*

**Lemma 2.3** *Let  $A_n$  ( $n \in \mathbb{N}$ ),  $A$  be maximal monotone operators of  $H$  such that  $dis(A_n, A) \rightarrow 0$  and  $\|A_n^0 x\| \leq c(1 + \|x\|)$  for some  $c > 0$ , all  $n \in \mathbb{N}$  and  $x \in D(A_n)$ . Then for every  $z \in D(A)$  there exists a sequence  $(z_n)$  such that*

$$z_n \in D(A_n), \quad z_n \rightarrow z \quad \text{and} \quad A_n^0 z_n \rightarrow A^0 z. \quad (2.2)$$

We recall the following result ([7], Corollary 1)

**Theorem 2.1** *Let  $\lambda = dt$  the Lebesgue measure on  $I = [0, 1]$ . Assume that for every  $t \in I$ ,  $A_t : D(A_t) \rightarrow ck(\mathbf{R}^e)$  is a maximal monotone operator satisfying  $(\mathcal{H}_1^A)$ ,  $(\mathcal{H}_2^A)$ ,  $(\mathcal{H}_3^A)$ ,  $(\mathcal{H}_4^A)$ .  $(\mathcal{H}_1^A)$  there exists a nonnegative real number  $c$  such that*

$$\|A_t^0 x\| = \|A^0(t, x)\| \leq c(1 + \|x\|) \quad \text{for } t \in I, x \in D(A_t)$$

$(\mathcal{H}_2^A)$   $\Gamma : t \mapsto D(A_t)$  has right closed graph,  $gph(\Gamma)$ ,

$(\mathcal{H}_3^A)$   $(t, x) \mapsto A_t(x) : gph(\Gamma) \rightarrow ck(\mathbf{R}^e)$  is scalar upper semicontinuous: for  $t_n \downarrow t$ , for  $x_n \rightarrow x$  with  $x_n \in D(A_{t_n})$  and  $x \in D(A_t)$ ,

$$\forall e \in \mathbf{R}^e, \limsup_n \delta^*(e, A_{t_n} x_n) \leq \delta^*(e, A_t)$$

$(\mathcal{H}_4^A)$  There exists a non decreasing and right continuous function  $r : I \rightarrow [0, \infty[$  such that  $r(T) < \infty$  with the Stieltjes measure  $dr$  such that, for  $t < \tau \subset I$ , for  $\lambda > 0$  and  $x \in D(A_t)$

$$\|x - J_\lambda^{A(\tau)}(x)\| \leq r(\tau) - r(t) + \lambda \|A_t^0 x\|$$

Let  $f : [0, T] \times E \rightarrow E$  such that

- (i)  $f(\cdot, x)$  is  $\mathcal{L}([0, T])$ -measurable on  $[0, T]$  for all  $x \in E$ ,
- (ii)  $\|f(t, x) - f(t, y)\| \leq M\|x - y\|$  for all  $t, x, y \in [0, T] \times E \times E$ ,
- (iii)  $\|f(t, x)\| \leq M(1 + \|x\|)$  for all  $(t, x) \in [0, T] \times E$ ,

for some constant  $M > 0$ . Let  $\nu = dr + \lambda$  and let  $\frac{d\lambda}{d\nu}(\cdot)$  be the density of  $\lambda$  relatively to the measure  $\nu$ . Assume further that there is  $\beta \in ]0, 1[$  such that  $\forall t \in [0, T]$ ,  $0 \leq 2M \frac{d\lambda}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$ . Then for all  $u_0 \in D(A(0))$  the evolution problem

$$-Du(t) \in A(t, u(t)) + f(t, u(t))$$

admits a unique BVRC solution  $u$  with  $u(0) = u_0$ , that is, there exists a unique BVRC function  $u : [0, T] \rightarrow E$  such that

$$\begin{cases} u(0) = u_0 \in D(A(0)) \\ u(t) \in D(A(t)), \forall t \in [0, T] \\ \frac{du}{d\nu}(t) \in L_E^\infty([0, T], d\nu) \\ -\frac{du}{d\nu}(t) \in A(t, u(t)) + f(t, u(t)) \frac{d\lambda}{d\nu}(t), \nu \text{ a.e.} \end{cases}$$

In this paper, to study the systems presented in the introduction, we detail here the hypotheses that will be used for the operator  $A$ , for the integral with respect to  $B_H$  and for diverse perturbations:

1- on the operators  $A_t (t \in I)$ :

( $\mathcal{A}_0$ ) The multimapping  $A : (t, x) \in I \times E \mapsto A_t x \in ck(E)$  is scalarly  $\mathcal{L}([0, T]) \otimes \mathcal{B}(E)$ -measurable, where  $ck(E)$  denotes the set of non empty convex compact subsets of  $E = \mathbf{R}^e$ .

( $\mathcal{H}_0^A$ ) There exists a real constant  $L > 0$  such that

$$dis(A_t, A_s) \leq L|t - s|^\beta \quad \forall t, s \in I.$$

( $\mathcal{H}_1^A$ ) there exists a nonnegative real number  $c$  such that

$$\|A_t^0 x\| = \|A^0(t, x)\| \leq c(1 + \|x\|) \quad \text{for } t \in I, x \in D(A_t)$$

( $\mathcal{H}_2^A$ )  $\Gamma : t \mapsto D(A_t)$  has right closed graph,  $gph(\Gamma)$ ,

( $\mathcal{H}_3^A$ )  $(t, x) \mapsto A_t(x) : gph(\Gamma) \rightarrow ck(\mathbf{R}^e)$  is scalar upper semicontinuous: for  $t_n \downarrow t$ , for  $x_n \rightarrow x$  with  $x_n \in D(A_{t_n})$  and  $x \in D(A_t)$ ,

$$\forall e \in \mathbf{R}^e, \limsup_n \delta^*(e, A_{t_n} x_n) \leq \delta^*(e, A_t)$$

( $\mathcal{H}_4^A$ ) There exists a non decreasing and right continuous function  $r : I \rightarrow [0, \infty[$  such that  $r(T) < \infty$  with the Stieltjes measure  $dr$  such that, for  $t < \tau \subset I$ , for  $\lambda > 0$  and  $x \in D(A_t)$

$$\|x - J_\lambda^{A(\tau)}(x)\| \leq r(\tau) - r(t) + \lambda \|A_t^0 x\|$$

2- for the integral with respect to the fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . with  $1 - H < \beta < \frac{1}{2}$ . we consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions ( $\mathcal{B}_1$ ), ( $\mathcal{B}_2$ ) and ( $\mathcal{B}_3$ ) or the group of weakened

conditions  $(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ :

- $(\mathcal{B}_1)$   $\|b(t, x)\| \leq M, \quad \forall x \in \mathbf{R}^e,$
- $(\mathcal{B}_2)$   $\|b(s, x) - b(t, x)\| \leq M|t - s|^\beta, \quad 0 \leq s \leq t \leq T, \quad \forall x \in \mathbf{R}^e,$
- $(\mathcal{B}_3)$   $\|b(t, x) - b(t, y)\| \leq M\|x - y\|, \quad \forall t \in [0, T], \quad \forall x, y \in \mathbf{R}^e \times \mathbf{R}^e.$
- $(\mathcal{B}_1^b)$   $\|b(t, x)\| \leq M_b, \quad \forall (t, x) \in I \times \mathbf{R}^e,$
- $(\mathcal{B}_2^b)$   $\|b(t, x) - b(s, y)\| \leq M_b|t - s|^\beta, \quad 0 \leq s \leq t \leq T, \quad \forall x, y \in \mathbf{R}^e \times \mathbf{R}^e,$
- $(\mathcal{B}_3^b)$   $\|b(t, x) - b(t, y)\| \leq M_b\|x - y\|, \quad \forall t \in [0, T], \quad \forall x, y \in \mathbf{R}^e \times \mathbf{R}^e$

3- For the perturbation  $f : I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$ :

- $(\mathcal{F}_1)$   $f(\cdot, x, y)$  is  $\mathcal{L}(I)$  measurable for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ ,
- $(\mathcal{F}_2)$   $f(t, \cdot, \cdot)$  is continuous on  $\mathbf{R}^e \times \mathbf{R}^e$  for all  $t \in I$ ,
- $(\mathcal{F}_3)$   $\|f(t, x, z) - f(t, y, z)\| \leq M\|x - y\|$  for all  $(t, x, y, z) \in I \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e$ ,
- $(\mathcal{F}_4)$   $\|f(t, x, y)\| \leq M$  for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ .

For the application  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  :

- $(\mathcal{G}_1)$   $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  is continuous,
- $(\mathcal{G}_2)$   $G_t : \mathbf{R}^e \rightarrow \mathbf{R}^e$  is linear continuous operator for all  $t \in I$ ,
- $(\mathcal{G}_3)$   $|G_t| \leq m_G$  for all  $t \in I$  where  $|G_t|$  is the norm of the operator  $G_t$  and  $m_G$  is a positive constant.

**Theorem 2.2** *Let  $I := [0, T]$ . Assume that for every  $t \in I$ ,  $A_t : D(A_t) \rightarrow \mathbf{R}^e$  is a time dependent maximal monotone operator satisfying  $(\mathcal{H}_1^A)$ ,  $(\mathcal{H}_2^A)$ ,  $(\mathcal{H}_3^A)$  and  $(\mathcal{H}_4^A)$ .*

*Let  $f : I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying  $(\mathcal{F}_1)$ ,  $(\mathcal{F}_2)$ ,  $(\mathcal{F}_3)$  and  $(\mathcal{F}_4)$ .*

*Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .*

*Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1)$ ,  $(\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .*

*Consider the measure  $\nu := dr + \lambda$  and assume that there is  $\delta > 0$  such that*

$$0 \leq 2M \frac{d\lambda}{d\nu}(t) \nu(t) \leq \delta < 1, \quad \forall t \in I.$$

*Then there exist a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $\zeta : [0, T] \rightarrow \mathbf{R}^e$  and a BVRC mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying the dynamic system with rough signal*

$$\begin{cases} \zeta(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, \quad t \in I \\ u(0) = u_0 \in D(A_0), u(t) \in D(A_t), \quad \forall t \in I \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, \zeta(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

**Proof. Step 1** Notice that for any continuous mapping  $g : I \rightarrow \mathbf{R}^e$ , by Theorem 2.1 there is a unique BVRC solution  $u_g$  to the differential inclusion

$$\begin{cases} u_g(0) = u_0 \in D(A_0) \\ u_g(t) \in D(A_t), \quad \forall t \in I, \\ du_g \nu \text{ -absolutely continuous} \\ -\frac{du_g}{d\nu}(t) \in A_t u(t) + f(t, u_g(t), g(t)) \frac{d\lambda}{d\nu} \nu \text{ a.e. } t \in I \end{cases}$$

and notice also that by the same theorem there is a real constant  $\eta > 0$  independent on

$g \in C_{\mathbf{R}^e}(I)$  such that  $u_g$  is uniformly bounded and equi-BVRC with

$$\left\| \frac{du_g}{d\nu}(t) \right\| \leq \eta, \nu \text{ a.e. } t \in I$$

So, for some constant  $K > 0$  independent on  $g \in C_{\mathbf{R}^e}(I)$  one has  $\|u_g(t)\| \leq K$  for all  $t \in I$  and for all continuous mapping  $g \in C_{\mathbf{R}^e}(I)$ . Now let us consider the set  $\mathcal{X}_G$  defined by

$$\mathcal{X}_G := \left\{ \zeta_q : I \rightarrow \mathbf{R}^e : \zeta_q(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} q(s) ds, t \in I, q \in S_G^1 \right\}$$

where  $\mathcal{G}$  is the convex compact valued multifunction  $t \rightarrow \mathcal{G}(t) := G(t, K\bar{B}_{\mathbf{R}^e}), t \in I$ . We note that  $\mathcal{X}_G$  is convex compact relative to the topology of uniform convergence and equi- $\beta$ -Holder continuous, i.e. for some constant  $N > 0$

$$\|h(t) - h(s)\| \leq N|t - s|^\beta$$

for all  $h \in \mathcal{X}_G$  and  $s, t \in I$ . Then, for any  $h \in \mathcal{X}_G$ , using  $(\mathcal{B}_2)$   $(\mathcal{B}_3)$

$$\begin{aligned} \|b(t, h(t)) - b(s, h(s))\| &\leq \|b(t, h(t)) - b(s, h(t))\| + \|b(s, h(t)) - b(s, h(s))\| \\ &\leq M|t - s|^\beta + MN|t - s|^\beta = (M + MN)|t - s|^\beta \end{aligned}$$

so  $b(\cdot, h(\cdot)) \in C^\beta([0, T]; \mathbf{R}^e)$  with  $\|b(\cdot, h(\cdot))\| \leq M$  by  $(\mathcal{B}_1)$ . Then the stochastic process

$$\mathcal{H} := \{b(\cdot, h(\cdot)) : h \in \mathcal{X}_G\}$$

belongs to the space  $\mathcal{W}_{\mathbf{R}^e}^{\gamma, 1}([0, T])$ , for any  $1 - H < \gamma < \beta_0 := \min\{\beta, 1 - \beta\}$ . Indeed

$$\int_0^T \frac{\|b(s, h(s))\|}{s^\gamma} ds \leq \int_0^T \frac{M}{s^\gamma} ds \leq \frac{T^{1-\gamma}}{1-\gamma} M$$

and

$$\int_0^T \int_0^t \frac{\|b(t, h(t)) - b(s, h(s))\|}{(t-s)^{\gamma+1}} ds dt \leq (M + MN) \int_0^T \int_0^t (t-s)^{\beta-\gamma-1} ds dt \leq \frac{(M + MN)T^{\beta-\gamma+1}}{(\beta-\gamma)(\beta-\gamma+1)}$$

From these estimates and the definition of  $\mathcal{W}_{\mathbf{R}^e}^{\gamma, 1}([0, T])$

$$\|b(\cdot, h(\cdot))\|_{\mathcal{W}_{\mathbf{R}^e}^{\gamma, 1}([0, T])} \leq \left( \frac{M}{1-\gamma} + \frac{(M + MN)}{(\beta-\gamma)(\beta-\gamma+1)} T^\beta \right) T^{1-\gamma}.$$

As consequence the set

$$\mathcal{Y}_G := \left\{ t \mapsto \int_0^t b(\theta, h(\theta)) dB_\theta^H : h \in \mathcal{X}_G \right\}$$

is compact for the topology of uniform convergence on  $I$ . Namely the set  $\mathcal{Y}_G$  is equi- $(1 - \gamma)$ -Holder continuous. Indeed, by the inequality (\*),

$$\left\| \int_s^t b(\theta, h(\theta)) dB_\theta^H \right\| \leq K(\gamma, [s, t]) \|b(\cdot, h(\cdot))\|_{\mathcal{W}_{\mathbf{R}^e}^{\gamma, 1}[s, t]}$$

$$\| \int_s^t b(\theta, h(\theta)) dB_\theta^H \| \leq K(\gamma, [s, t]) \left[ \int_s^t \frac{\|b(\theta, h(\theta))\|}{(\theta - s)^\gamma} d\theta + \int_s^t \left[ \int_s^\theta \frac{\|b(\theta, h(\theta)) - b(v, h(v))\|}{(\theta - v)^{\gamma+1}} dv \right] d\theta \right]$$

The first integral is estimated by

$$\int_s^t \frac{\|b(\theta, h(\theta))\|}{(\theta - s)^\gamma} d\tau \leq M \int_s^t (\theta - s)^{-\gamma} d\theta = \frac{M}{1 - \gamma} (t - s)^{1-\gamma}$$

The second integral is estimated by

$$\begin{aligned} & \int_s^t \left[ \int_s^\theta \frac{\|b(\theta, h(\theta)) - b(v, h(v))\|}{(\theta - v)^{\gamma+1}} dv \right] d\theta \\ & \leq (M + MN) \int_s^t \left[ \int_s^\theta \frac{(\theta - v)^\beta}{(\theta - v)^{\gamma+1}} dv \right] d\theta \leq \frac{(M + MN)|t - s|^{\beta-\gamma+1}}{(\beta - \gamma)(\beta - \gamma + 1)} \end{aligned}$$

So

$$\begin{aligned} \| \int_s^t b(\theta, h(\theta)) dB_\theta^H \| & \leq K(\gamma, [s, t]) \left[ \frac{M}{1 - \gamma} (t - s)^{1-\gamma} + \frac{(M + MN)|t - s|^{\beta-\gamma+1}}{(\beta - \gamma)(\beta - \gamma + 1)} \right] \\ & \leq K(\gamma, [0, T]) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^{1-\gamma} \end{aligned}$$

Then

$$\begin{aligned} \frac{\| \int_0^t b(\theta, h(\theta)) dB_\theta^H - \int_0^s b(\theta, h(\theta)) dB_\theta^H \|}{(t - s)^{1-\gamma}} & \leq (t - s)^{\gamma-1} \| \int_s^t b(\theta, h(\theta)) dB_\theta^H \| \\ & \leq K(\gamma, [0, T]) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \alpha)(\beta - \gamma + 1)} \right] \end{aligned}$$

In particular, if  $(h_n)_n$  in  $\mathcal{X}_G$  converge uniformly on  $I$  to a mapping  $h \in \mathcal{X}_G$ , the sequence  $(t \mapsto \int_0^t b(\theta, h_n(\theta)) dB_\theta^H)_n$  converges uniformly on  $I$  to the mapping  $t \mapsto \int_0^t b(\theta, h(\theta)) dB_\theta^H$ . This needs a concise examination. Set

$$g_n(\theta) = b(\theta, h_n(\theta)) - b(\theta, h(\theta)), \forall \theta \in I$$

$$\varepsilon_n = \sup_{s \in I} \|h_n(s) - h(s)\|$$

so that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\|g_n(\theta)\| \leq M \|h_n(\theta) - h(\theta)\| \leq M\varepsilon_n$  for all  $\theta \in I$ . Furthermore

$$\begin{aligned} \|g_n(\theta) - g_n(\tau)\| & \leq \|b(\theta, h_n(\theta)) - b(\theta, h(\theta)) - (b(\tau, h_n(\tau)) - b(\tau, h(\tau)))\| \\ & \leq M(\|h_n(\theta) - h(\theta)\| + \|h_n(\tau) - h(\tau)\|) \leq 2M\varepsilon_n \end{aligned}$$

for all  $\theta, \tau \in I$ . Then we have the estimation

$$\| \int_0^t g_n(\theta) dB_\theta^H \| \leq K(\gamma, [0, T]) \left[ \int_0^T \frac{\|g_n(\theta)\|}{\theta^\gamma} d\theta + \int_0^T \left[ \int_0^\theta \frac{\|g_n(\theta) - g_n(v)\|}{(\theta - v)^{\gamma+1}} dv \right] d\theta \right]$$

As

$$\int_0^T \frac{\|g_n(\theta)\|}{\theta^\gamma} d\theta \leq \int_0^T \frac{M\varepsilon_n}{\theta^\gamma} d\theta$$

and

$$\int_0^T \left[ \int_0^\theta \frac{\|g_n(\theta) - g_n(v)\|}{(\theta - v)^{\gamma+1}} dv \right] d\theta \leq \int_0^T \left[ \int_0^\theta \frac{2M\varepsilon_n}{(\theta - v)^{\gamma+1}} dv \right] d\theta$$

we deduce

$$\left( \left\| \int_0^t g_n(\theta) dB_\theta^H \right\| \right)_n$$

converges to 0 uniformly on  $I$ . Consequently  $(t \mapsto \int_0^t b(\theta, h_n(\theta)) dB_\theta^H)_n$  converges uniformly on  $I$  to the mapping  $t \mapsto \int_0^t b(\theta, h(\theta)) dB_\theta^H$ .

**Step 2** For each  $h \in \mathcal{X}_G$ , let us set

$$\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds, t \in I$$

where  $u_h$  is the unique BVRC solution to the differential inclusion

$$\begin{cases} u_h(0) = u_0 \in D(A_0) \\ u_h(t) \in D(A_t), \forall t \in I, \\ du_h \nu \text{-absolutely continuous} \\ -\frac{du_h}{d\nu}(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

Clearly we have  $\Phi(h) \in \mathcal{X}_G$ .

Now we check that  $\Phi$  is continuous relative to  $\mathcal{X}_G$ . It is enough to show that if  $(h_n)_n$  converges uniformly on  $I$  to  $h$  in  $\mathcal{X}_G$ , then the sequence  $(u_{h_n})_n$  when each  $u_{h_n}$  is the unique BVRC solution to

$$\begin{cases} u_{h_n}(0) = u_0 \in D(A_0) \\ u_{h_n}(t) \in D(A_t), \forall t \in I, \\ du_{h_n} \nu \text{-absolutely continuous} \\ -\frac{du_{h_n}}{d\nu}(t) \in A_t u_{h_n}(t) + f(t, u_{h_n}(t), \int_0^t b(s, h_n(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

converges uniformly on  $I$  to the unique BVRC solution  $u_h$  of the differential inclusion

$$\begin{cases} u_h(0) = u_0 \in D(A_0) \\ u_h(t) \in D(A_t), \forall t \in I, \\ du_h \nu \text{-absolutely continuous} \\ -\frac{du_h}{d\nu}(t) \in A_t u(t) + f(t, u_g(t), \int_0^t b(s, h(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

This need a careful look. Since  $(u_{h_n})_n$  is equi-BVRC with for each  $n \in \mathbf{N}$  the estimate  $\|\frac{du_{h_n}}{d\nu}(t)\| \leq \eta$ ,  $\nu$  a.e.  $t \in I$ , we may suppose that  $(u_{h_n})_n$  pointwise converge to a BVRC mapping  $w : I \rightarrow \mathbf{R}^e$  with  $w(t) = u_0 + \int_0^t \frac{dw}{d\nu}(s) d\nu(s)$  and the differential measure  $dw$  is  $\nu$ -absolutely continuous. We note by the closedness of  $D(A_t)$ ,  $w(t) \in D(A_t)$  for all  $t \in I$ . We also suppose that  $(\frac{du_{h_n}}{d\nu})_n$  weakly converge in  $L^1_{\mathbf{R}^e}(I)$  to  $\frac{dw}{d\nu}$ . Further for every  $t \in I$  we

have  $k_n(t) := f(t, u_{h_n}(t), \int_0^t b(s, h_n(s))dB_s^H) \rightarrow k(t) := f(t, w(t), \int_0^t b(s, h(s))dB_s^H)$  keeping in mind that

$$\|f(t, u_{h_n}(t), \int_0^t b(s, h_n(s))dz_s)\| \leq M, \|f(t, w(t), \int_0^t b(s, h(s))dB_s^H)\| \leq M$$

Now we show that  $w$  is solution of the differential inclusion

$$\begin{cases} w(0) = u_0 \in D(A_0) \\ w(t) \in D(A_t), \forall t \in I, \\ dw \text{ } \nu \text{-absolutely continuous} \\ -\frac{dw}{d\nu}(t) \in A_t w(t) + f(t, w(t), \int_0^t b(s, h(s))dB_s^H) \frac{d\lambda}{d\nu}(t) \text{ } \nu \text{ a.e. } t \in I \end{cases}$$

Indeed since  $(\frac{du_{h_n}}{d\nu} + k_n)_n$  weakly converge to  $\frac{dw}{d\nu} + k$  in  $L^1_{\mathbf{R}^e}(I)$ , we may suppose that  $(\frac{du_{h_n}}{d\nu} + k_n)_n$  Komlos converges to  $(\frac{dw}{d\nu} + k)$ . There is a  $\nu$ -negligible set  $Q$  of  $I$  such that

$$-\frac{du_{h_n}}{d\nu}(t) \in A_t u_{h_n}(t) + k_n(t)$$

for all  $n \in \mathbf{N}$  and  $t \in I \setminus Q$

$$\lim_n \sum_{j=1}^n \frac{du_{h_j}}{d\nu} + k_j(t) = \frac{dw}{d\nu}(t) + k(t)$$

for all  $t \in I \setminus Q$ . Let  $t \in I \setminus Q$  and  $\zeta \in D(A_t)$ . According to the inclusion

$$-\frac{du_{h_n}}{d\nu}(t) \in A_t u_{h_n}(t) + k_n(t)$$

and the monotonicity of  $A_t$  we have

$$\langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), u_{h_n}(t) - \zeta \rangle \leq \langle A_t^0 \zeta, \zeta - u_{h_n}(t) \rangle$$

Writting

$$\langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), w(t) - \zeta \rangle = \langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), u_{h_n}(t) - \zeta \rangle + \langle \frac{du_{h_n}}{d\nu}(t) + k_n(t), w(t) - u_{h_n}(t) \rangle$$

yields

$$\frac{1}{n} \sum_{j=1}^n \langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), w(t) - \zeta \rangle = \frac{1}{n} \sum_{j=1}^n \langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), u_{h_j}(t) - \zeta \rangle + \frac{1}{n} \sum_{j=1}^n \langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), w(t) - u_{h_j}(t) \rangle$$

so that

$$\frac{1}{n} \sum_{j=1}^n \langle \frac{du_{h_j}}{d\nu}(t) + k_j(t), w(t) - \zeta \rangle \leq \frac{1}{n} \sum_{j=1}^n \langle A_t^0 \zeta, \zeta - u_{h_j}(t) \rangle + (\eta + M) \frac{1}{n} \sum_{j=1}^n \|w(t) - u_{h_j}(t)\|$$

Passing to the limit as  $n \rightarrow \infty$  in the last inequalities give

$$\left\langle \frac{dw}{d\nu}(t) + k(t), w(t) - \zeta \right\rangle \leq \langle A_t^0 \zeta, \zeta - w(t) \rangle$$

Consequently  $w$  is a BVRC solution of the aforementioned differential inclusion, so by uniqueness  $w = u_h$ . Now let us write

$$\begin{aligned} \|\Phi(h_n)(t) - \Phi(h)(t)\| &= \left\| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_{h_n}(s)) ds - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds \right\| \\ &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m_G \|u_{h_n}(s) - u_h(s)\| ds \leq \int_0^t \frac{m_G}{\Gamma(\beta)} \|u_{h_n}(s) - u_h(s)\| ds \end{aligned}$$

Using for each  $s \in I$ , the convergence  $\|u_{h_n}(s) - u_h(s)\| \rightarrow 0$  as  $n \rightarrow \infty$  and using the inequality  $\|u_{h_n}(s) - u_h(s)\| \leq 2K$  for all  $s \in I$  we deduce from the Lebesgue convergence that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \frac{m_G}{\Gamma(\beta)} \|u_{h_n}(s) - u_h(s)\| ds \rightarrow 0$$

which entail that  $\Phi(h_n)$  converge uniformly on  $I$  to  $\Phi(h)$ . Then  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  is continuous. By Schauder theorem,  $\Phi$  has a fixed point  $h = \Phi(h) \in \mathcal{X}_G$ . This means

$$h(t) = \Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds, t \in I$$

with

$$\begin{cases} u_h(0) = u_0 \in D(A_0) \\ u_h(t) \in D(A_t), \forall t \in I, \\ du_h \nu \text{-absolutely continuous} \\ -\frac{du_h}{d\nu}(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

So, by putting  $x = h$ ,  $u = u_h$  we conclude that the pair  $(x, u)$  with  $x$   $\beta$ -Holder and  $u$  BVRC solves the fractional/evolution inclusion with Holder signal  $\int_0^t b(s, x(s)) dB_s^H$

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, t \in I \\ u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in I, \\ du \nu \text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, x(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

As consequence

$$\begin{cases} {}^c D^\beta x(t) = G(t, u(t)), t \in I \\ u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in I, \\ du \nu \text{-absolutely continuous} \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, x(s)) dB_s^H) \frac{d\lambda}{d\nu}(t) \nu \text{ a.e. } t \in I \end{cases}$$

The proof of the theorem is finished.

**Theorem 2.3** Let  $I := [0, T]$ . Let  $\varphi : I \times \mathbf{R}^e \rightarrow \mathbf{R} \cup \{+\infty\}$  be a normal convex integrand for which there are a mapping  $\zeta \in L^2_{\mathbf{R}^e}(I)$  such that  $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$  is in  $L^2_{\mathbf{R}_+}(I)$  and two non-negative functions  $\alpha_\varphi, \beta_\varphi : I \rightarrow \mathbf{R}$  with  $\alpha_\varphi, \sqrt{\beta_\varphi} \in L^2_{\mathbf{R}_+}(I)$  such that

$$-\alpha_\varphi(t)\|x\| - \beta_\varphi(t) \leq \varphi(t, x) \quad \text{for all } (t, x) \in I \times \mathbf{R}^e$$

(note that  $\sqrt{\beta_\varphi} \in L^2_{\mathbf{R}_+}(I)$  if  $\beta_\varphi \in L^2_{\mathbf{R}_+}(I)$ ).

Let  $f : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mappings such that  $f(\cdot, x)$  is Lebesgue measurable for all  $x \in \mathbf{R}^e$  and such that there are non-negative ,  $\alpha_f > 0$ , such that for all  $(t, x) \in I \times \mathbf{R}^e$

$$\|f(t, x)\| \leq \alpha_f$$

and for all  $t \in I$  and  $x, y \in \mathbf{R}^e$

$$\|f(t, x) - f(t, y)\| \leq \alpha_f \|x - y\|.$$

Let  $g : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mappings such that  $g(\cdot, x)$  is Lebesgue measurable for all  $x \in \mathbf{R}^e$  and such that there are non-negative functions ,  $\alpha_g, \beta_g$  in  $L^2_{\mathbf{R}_+}(I)$  along with  $\gamma_g$  in  $L^1_{\mathbf{R}_+}(I)$  such that for all  $(t, x) \in I \times \mathbf{R}^e$

$$\|g(t, x)\| \leq \alpha_g(t)\|x\| + \beta_g(t)$$

and for all  $t \in I$  and  $x, y \in \mathbf{R}^e$

$$\|g(t, x) - g(t, y)\| \leq \gamma_g(t)\|x - y\|.$$

Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1), (\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1), (\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .

Let  $\Lambda : \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a positive symmetric bijective linear continuous operator.

Then, for any  $u_0 \in \mathbf{R}^e$ , there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely continuous mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, & t \in I \\ u(0) = u_0 \in \mathbf{R}^e \\ f(t, \int_0^t b(s, x(s)) dB_s^H) - \Lambda \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t, u(t))), & \text{a.e.} \end{cases}$$

Proof. By Theorem 3.1 [8] there is a non-negative function  $\mu(\cdot) \in L^2_{\mathbf{R}_+}(I)$  such that for every mapping continuous mapping  $h : I \rightarrow \mathbf{R}^e$  the absolutely continuous solution  $\zeta(\cdot)$  of the differential inclusion

$$\begin{cases} f(t, h(t)) - \Lambda \frac{d\zeta}{dt}(t) \in \partial\varphi\left(t, \frac{d\zeta}{dt}(t) + g(t, \zeta(t))\right) & \text{a.e. } t \in I \\ \zeta(0) = u_0 \end{cases}$$

satisfies  $\|\dot{\zeta}(t)\| \leq \mu(t)$  a.e.  $t \in I$ . Let  $\Delta(t) = u_0 + (\int_0^t \mu(s)ds)\overline{B}_{\mathbf{R}^e}$  for all  $t \in I$ , then  $\zeta(t) \in \Delta(t)$  for all  $t \in I$  and the convex compact valued multimapping. Let us consider the set  $\mathcal{X}_G$  defined by

$$\mathcal{X}_G = \{\zeta_l : I \rightarrow \mathbf{R}^e : \zeta_l(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} l(s) ds, t \in I, l \in S_{\mathcal{G}}^1\}$$

where  $\mathcal{G}(t) = G(t, \Delta(t)), t \in I$ . Then  $\mathcal{X}_G$  is convex, compact in  $C^\beta([0, T]; \mathbf{R}^e)$  and equi- $\beta$ -Holder continuous with  $\|h(t) - h(s)\| \leq N|t - s|^\beta$  where  $N$  is a positive constant independant of  $h$ . Then, for any  $h \in \mathcal{X}_H$ , using  $(\mathcal{B}_2)(\mathcal{B}_3)$  we obtain

$$\|b(t, h(t)) - b(s, h(s))\| \leq \|b(t, h(t)) - b(s, h(t))\| + \|b(s, h(t)) - b(s, h(s))\| \leq (M + MN)|t - s|^\beta$$

so that for any  $h \in \mathcal{X}_G$ ,  $b(\cdot, h(\cdot)) \in C^\beta([0, T]; \mathbf{R}^e)$  and  $\|b(\cdot, h(\cdot))\| \leq M$  by  $(\mathcal{B}_1)$ . In particular, for each  $h \in \mathcal{X}_G$  and for each  $t \in I$  the integral  $\int_0^t b(s, h(s)) dB_s^H$  is well defined for all  $h \in \mathcal{X}_G$  and satisfies the properties given in the proof of Theorem 2.2. As consequence  $\mathcal{Y}_G = \{t \mapsto \int_0^t b(\theta, h(\theta)) dB_\theta^H, h \in \mathcal{X}_G\}$  is compact in  $C^{1-\gamma}([0, T]; \mathbf{R}^e)$  with  $1 - H < \gamma < \min\{\beta, 1 - \beta\}$ . In particular, if a sequence  $(h_n)_n$  in  $\mathcal{X}_G$  converges uniformly on  $I$  to a mapping  $h \in \mathcal{X}_G$ , then the sequence  $(t \mapsto \int_0^t b(\theta, h_n(\theta)) dB_\theta^H)_\theta$  converges uniformly on  $I$  to the mapping  $(t \mapsto \int_0^t b(\theta, h(\theta)) dB_\theta^H)$ . For each  $h \in \mathcal{X}_G$ , let us set  $\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, v_h(s)) ds$  for all  $t \in I$  where  $v_h$  is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ f(t, \int_0^t b(s, h(s)) dB_s^H) - \Lambda \frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t, v_h(t))\right), \text{ a.e. } t \in I \end{cases}$$

and this solution satisfies the inequality  $\|\frac{dv_h}{dt}(t)\| \leq \mu(t)$  a.e.  $t \in I$ . This ensures that the mapping  $t \mapsto \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, v_h(s)) ds$  belongs to  $\mathcal{X}_G$  by definition of  $\mathcal{X}_G$ . Then, we can consider the mapping  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  with  $\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, v_h(s)) ds$  for every  $h \in \mathcal{X}_G$  and for every  $t \in I$ . Let us show that  $\Phi$  is continuous with respect to the topology induced on  $\mathcal{X}_G$  by the topology of uniform convergence on  $I$ . For this purpose, consider any sequence  $(h_n)_n$  in  $\mathcal{X}_G$  converging uniformly on  $I$  to a mapping  $h \in \mathcal{X}_G$ . Denote by  $v_{h_n}$  the absolutely continuous solution of the differential inclusion

$$\begin{cases} v_{h_n}(0) = u_0 \\ f(t, \int_0^t b(s, h_n(s)) dB_s^H) - \Lambda \frac{dv_{h_n}}{dt}(t) \in \partial\varphi\left(t, \frac{dv_{h_n}}{dt}(t) + g(t, v_{h_n}(t))\right), \end{cases}$$

the inclusion being satisfied for a.e.  $t \in I$ . We are going to show that the sequence  $(v_{h_n})_n$  uniformly converges on  $I$  to the absolutely continuous solution  $v_h$  of the differential inclusion

$$\begin{cases} v_h(0) = u_0 \\ f(t, \int_0^t b(s, h(s)) dB_s^H) - \Lambda \frac{dv_h}{dt}(t) \in \partial\varphi\left(t, \frac{dv_h}{dt}(t) + g(t, v_h(t))\right), \end{cases}$$

The inequality  $\|\dot{v}_{h_n}(t)\| \leq \mu(t)$ , valid for a.e.  $t \in I$ , ensures that  $(v_{h_n})_n$  is equi-absolutely continuous, and these two properties allow us to suppose that  $(v_{h_n})_n$  uniformly converges on

$I$  to an absolutely continuous mapping  $v$  on  $I$ . Since  $v_{h_n}(t) = u_0 + \int_{]0,t]} \frac{dv_{h_n}}{ds}(s) ds$  for all  $t \in I$  and since  $\|\dot{v}_{h_n}(t)\| \leq \mu(t)$  a.e.  $t \in I$ , we may also suppose that  $(\frac{dv_{h_n}}{dt})_n$  weakly converges in  $L^2_{\mathbf{R}^e}(I)$  to some  $w \in L^2_{\mathbf{R}^e}(I)$  with  $\|w(t)\| \leq \mu(t)$  for all  $t \in I$ , so that

$$\lim_n v_{h_n}(t) = u_0 + \int_0^t w(s) ds \text{ for every } t \in I.$$

It follows that for every  $t \in I$

$$v(t) = u_0 + \int_0^t w(s) ds,$$

which gives in particular  $\dot{v} = w$ . Furthermore, it is readily seen that the sequence  $(t \mapsto \int_0^t b(s, h_n(s)) dB_s^H)_n$  uniformly converges on  $I$  to  $t \mapsto \int_0^t b(s, h(s)) dB_s^H$ . Therefore, by Lemma 4.7 [8] we obtain

$$f(t, \int_0^t b(s, h(s)) dB_s^H) - \Lambda \frac{dv}{dt}(t) \in \partial\varphi \left( t, \frac{dv}{dt}(t) + g(t, v(t)) \right) \text{ a.e. } t \in I$$

with  $v(0) = u_0$ , so that by uniqueness  $v = v_h$ . As desired, this confirms the uniform convergence on  $I$  of  $(v_{h_n}(\cdot))_n$  to  $v_h(\cdot)$ . Now let us write

$$\|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \|G(s, v_{h_n}(s)) - G(s, v_h(s))\| ds.$$

Since  $\|v_{h_n}(\cdot) - v_h(\cdot)\| \rightarrow 0$  uniformly on  $I$  as  $n \rightarrow \infty$ , we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m_G \|u_{h_n}(s) - u_h(s)\| ds \leq \int_0^1 \frac{m_G}{\Gamma(\beta)} \|v_{h_n}(s) - v_h(s)\| ds$$

which entails that  $\Phi(h_n) \rightarrow \Phi(h)$  uniformly on  $I$ . This translates the continuity  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  with respect to the topology induced by the topology of uniform convergence on  $I$ . This continuity of  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  on the compact convex set  $\mathcal{X}_G$  of  $C_{\mathbf{R}^e}(I)$  tells us that  $\Phi$  has a fixed point, say  $h = \Phi(h) \in \mathcal{X}_G$ . This means, according to the definition of  $\Phi$

$$h(t) = \Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, v_h(s)) ds \text{ for all } t \in I$$

and also

$$\begin{cases} v_h(0) = u_0 \\ f(t, \int_0^t b(s, h(s)) dB_s^H) - \Lambda \frac{dv_h}{dt}(t) \in \partial\varphi(t, \frac{dv_h}{dt}(t) + g(t, v_h(t))), \end{cases}$$

where the inclusion is satisfied for a.e.  $t \in I$ . So, by putting  $x = h$ ,  $u = v_h$  we conclude that the pair  $(x, u)$  solves the fractional/evolution inclusion with Holder signal

$$\begin{cases} {}^c D^\beta x(t) = G(t, u(t)), t \in I \\ u(0) = u_0 \\ f(t, \int_0^t b(s, x(s)) dB_s^H) - \Lambda \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + g(t, u(t))), \end{cases}$$

the inclusion being satisfied for a.e.  $t \in I$ . The proof of the theorem is finished.  $\blacksquare$

A direct application to **variational** sweeping process is available

**Corollary 2.1** Let  $I := [0, T]$ . Let  $C : [0, T] \rightarrow \mathbf{R}^e$  be a closed convex valued Lebesgue measurable multimapping that has a selection  $\zeta \in L^2_{\mathbf{R}^e}(I)$ , i.e.  $\zeta(t) \in C(t)$  for all  $t \in I$ .

Let  $f : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mappings such that  $f(\cdot, x)$  is Lebesgue measurable for all  $x \in \mathbf{R}^e$  and such that there are non-negative ,  $\alpha_f > 0$ , such that for all  $(t, x) \in I \times \mathbf{R}^e$

$$\|f(t, x)\| \leq \alpha_f$$

and for all  $t \in I$  and  $x, y \in \mathbf{R}^e$

$$\|f(t, x) - f(t, y)\| \leq \alpha_f \|x - y\|.$$

Let  $g : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mappings such that  $g(\cdot, x)$  is Lebesgue measurable for all  $x \in \mathbf{R}^e$  and such that there are non-negative functions ,  $\alpha_g$ , ,  $\beta_g$  in  $L^2_{\mathbf{R}^+}(I)$  along with  $\gamma_g$  in  $L^1_{\mathbf{R}^+}(I)$  such that for all  $(t, x) \in I \times \mathbf{R}^e$

$$\|g(t, x)\| \leq \alpha_g(t) \|x\| + \beta_g(t)$$

and for all  $t \in I$  and  $x, y \in \mathbf{R}^e$

$$\|g(t, x) - g(t, y)\| \leq \gamma_g(t) \|x - y\|.$$

Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1)$ ,  $(\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .

Let  $\Lambda : \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a positive symmetric bijective linear continuous operator.

Then, for any  $u_0 \in \mathbf{R}^e$ , there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely continuous mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, & t \in I \\ u(0) = u_0 \in \mathbf{R}^e \\ f(t, \int_0^t b(s, x(s)) dB_s^H) - \Lambda \frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t) + g(t, u(t))), & \text{a.e.} \end{cases}$$

**Theorem 2.4** Let  $I = [0, T]$ . Let  $r \in ]0, +\infty]$  and  $C : I \rightrightarrows \mathbf{R}^e$  be an absolutely continuous multimapping whose values are  $r$ -prox regular and closed.

Let  $f : I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying:  $(\mathcal{F}_1)$ ,  $(\mathcal{F}_2)$ ,  $(\mathcal{F}_3)$  and  $(\mathcal{F}_4)$ .

Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1)$ ,  $(\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .

Then, for any  $a \in C(0)$ , there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely continuous mapping  $u : [[0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, & t \in I \quad (\text{i.e. } {}^c D^\beta x(t) = G(t, u(t)), \quad t \in I) \\ u(0) = a \in C(0), u(t) \in C(t), & t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)} u(t) + f(t, u(t), \int_0^t b(s, x(s)) dB_s^H) & \text{a.e. } t \in I. \end{cases}$$

Proof. Given  $a \in C(0)$ , for any continuous mapping  $h : I \rightarrow \mathbf{R}^e$ , by Theorem 3.3 of [12] there is a **unique** AC solution  $u_h$  to the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + f(t, s, u_h(s), h(s)) \text{ a.e. } t \in I. \end{cases}$$

We claim that  $(u_h)_h$  is uniformly bounded and equi-absolutely continuous. Indeed,  $\frac{du_h}{dt}(t) \in K(t)\overline{B}_E$  for a.e.  $t \in I$ , where  $K$  is a non-negative integrable function on  $I$  (say  $K(t) = |\dot{v}(t)| + M$ , where  $v(\cdot)$  is a control function of absolute continuity of  $C(\cdot)$ ). Then  $\|u_h(t)\| \leq K(t)$  for a.e.  $t \in I$ , which ensures that  $(u_h)_h$  is equi-absolutely continuous. Putting  $L := \|a\| + \int_0^1 (|\dot{v}(t)| + 2M) dt$ , we see that  $\|u_h(t)\| \leq L$  for all  $t \in I$ , which confirms the uniform boundedness of  $(u_h)_h$  and finishes the justification of the claim.

Now let us consider the set  $\mathcal{X}_G$  defined by

$$\mathcal{X}_G := \{\zeta_l : I \rightarrow \mathbf{R}^e : l \in S_G^1\},$$

where  $\mathcal{G}(t) = G(t, L\overline{B}_{\mathbf{R}^e})$ ,  $t \in I$  each mapping  $\xi_f$  being given for every  $t \in I$  by

$$\zeta_l(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} l(s) ds, t \in I, l \in S_G^1$$

Then  $\mathcal{X}_G$  is convex, compact in  $C^\beta([0, T]; \mathbf{R}^e)$  and equi- $\beta$ -Holder continuous with  $\|h(t) - h(s)\| \leq N|t - s|^\beta$  where  $N$  is a positive constant independant of  $h$ . Then, for any  $h \in \mathcal{X}_G$ , using  $(\mathcal{B}_2)(\mathcal{B}_3)$  we obtain

$$\|b(t, h(t)) - b(s, h(s))\| \leq \|b(t, h(t)) - b(s, h(t))\| + \|b(s, h(t)) - b(s, h(s))\| \leq (M + MN)|t - s|^\beta$$

so that for any  $h \in \mathcal{X}_G$ ,  $b(\cdot, h(\cdot)) \in C^\beta([0, T]; \mathbf{R}^e)$  and  $\|b(\cdot, h(\cdot))\| \leq M$  by  $(\mathcal{B}_1)$ . In particular, for each  $h \in \mathcal{X}_G$  and for each  $t \in I$  the integral  $\int_0^t b(s, h(s)) dB_s^H$  is well defined for all  $h \in \mathcal{X}_G$  and satisfies the properties given in the proof of Theorem 2.2. As consequence  $\mathcal{Y}_G = \{t \mapsto \int_0^t b(s, h(s)) dB_s^H, h \in \mathcal{X}_G\}$  is compact in  $C_{\mathbf{R}^e}^{1-\gamma}(I)$  with  $1-H < \gamma \leq \min\{\beta, 1-\beta\}$ . In particular, if a sequence  $(h_n)_n$  in  $\mathcal{X}_G$  converges uniformly on  $I$  to a mapping  $h \in \mathcal{X}_G$ , then the sequence  $(t \mapsto \int_0^t b(s, h_n(s)) dB_s^H)_n$  converges uniformly on  $I$  to the mapping  $(t \mapsto \int_0^t b(s, h(s)) dB_s^H)$ . For each  $h \in \mathcal{X}_G$ , let us set  $\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds$  for all  $t \in I$  where  $u_h$  is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dB_s^H) \end{cases}$$

For each  $h \in \mathcal{X}_G$  let us consider again

$$\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds, \text{ for all } t \in I.$$

Then it is clear that  $\Phi(h) \in \mathcal{X}_G$ . Hence  $\Phi(\mathcal{X})$  is equicontinuous and relatively compact in the Banach space  $\mathcal{C}_{\mathbf{R}^e}(I)$  because  $\Phi(\mathcal{X}_G) \subset \mathcal{X}_G$ . Now we check that  $\Phi$  is continuous relative to  $\mathcal{X}_G$ . It is enough to show that, if  $(h_n)_n$  converges uniformly to  $h$  in  $\mathcal{X}_G$ , then the sequence  $(u_{h_n})_n$ , where each  $u_{h_n}$  is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a \in C(0) \\ u_{h_n}(t) \in C(t), \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + f(t, u_{h_n}(t), \int_0^t b(s, h_n(s))dB_s^H), ds \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution  $u_h$  of the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s))dB_s^H), ds \quad \text{a.e. } t \in I. \end{cases}$$

This needs a careful look. We note that  $(u_{h_n})_n$  is equicontinuous by what precedes (since for every  $n \in \mathbf{N}$  one has  $\|\dot{u}_{h_n}(t)\| \leq K(t)$  for almost all  $t \in I$ ). Further,  $\{u_{h_n}(t) : n \in \mathbf{N}\}$  is included in the compact set  $\Gamma(t)$  for every  $t \in I$ . The Arzelà-Ascoli theorem tells us that  $\{u_{h_n} : n \in \mathbf{N}\}$  is relatively compact in  $\mathcal{C}_{\mathbf{R}^e}(I)$ . So by extracting a subsequence, we may suppose that  $(u_{h_n})_n$  converges uniformly on  $I$  to some mapping  $\zeta : I \rightarrow E$  with

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s) ds \quad \text{for all } t \in I,$$

along with  $(\dot{u}_{h_n})_n$  converging weakly in  $L_{\mathbf{R}^e}^1(I)$  to  $\dot{\zeta}$  with  $\|\dot{\zeta}(t)\| \leq K(t)$  for a.e.  $t \in I$ . Note that

$$f(t, u_{h_n}(t), \int_0^t b(s, h_n(s))dB_s^H) \rightarrow f(t, \zeta(t), \int_0^t b(s, h(s))dB_s^H) \quad \text{for all } (t, s) \in I \times I.$$

For simplicity, denote  $z_n(t) := f(t, u_{h_n}(t), \int_0^t b(s, h_n(s))dB_s^H)$ , for all  $t \in I$ , and  $z(t) = f(t, \zeta(t), \int_0^t b(s, h(s))dB_s^H)$ , for all  $t \in I$ . First, we notice that these mappings are Lebesgue measurable. Second, by condition  $(\mathcal{F}_4)$  and the uniform boundedness of  $(u_{h_n})_n$  and  $u_h$ , we also notice that  $(z_n)_n$  and  $z$  are uniformly bounded, since

$$\|z_n(t)\| \leq M \quad \text{and} \quad \|z(t)\| \leq M, \quad \text{for all } n \in \mathbf{N} \text{ and } t \in I.$$

Consequently,  $(z_n)_n$  is a sequence of measurable and uniformly bounded mappings which converges pointwise to the measurable mapping  $z$ . Therefore, the sequence  $(\dot{u}_{h_n} + z_n)_n$  converges weakly to  $\dot{\zeta} + z$  in  $L_{\mathbf{R}^e}^1(I)$ . This combined with Lemma 3.4 in [12] and the uniqueness of solution of the latter differential inclusion gives

$$\begin{cases} -\frac{d\zeta}{dt}(t) \in N_{C(t)}\zeta(t) + z(t) \quad \text{a.e. } t \in I \\ \zeta(0) = a, \zeta(t) \in C(t) \quad \forall t \in I, \end{cases}$$

Now let us write

$$\|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \|G(s, u_{h_n}(s)) - G(s, v_h(s))\| ds.$$

Since  $\|u_{h_n}(\cdot) - u_h(\cdot)\| \rightarrow 0$  uniformly on  $I$  as  $n \rightarrow \infty$ , we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m_G \|u_{h_n}(s) - u_h(s)\| ds \leq \int_0^1 \frac{m_G}{\Gamma(\beta)} \|u_{h_n}(s) - u_h(s)\| ds$$

which entails that  $\Phi(h_n) \rightarrow \Phi(h)$  uniformly on  $I$ . This translates the continuity  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  with respect to the topology induced by the topology of uniform convergence on  $I$ . This continuity of  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  on the compact convex set  $\mathcal{X}_G$  of  $C_{\mathbf{R}^e}(I)$  tells us that  $\Phi$  has a fixed point, say  $h = \Phi(h) \in \mathcal{X}_G$ . This means, according to the definition of  $\Phi$

$$h(t) = \Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds \text{ for all } t \in I$$

and also

$$\begin{cases} u_h(0) = u_0 \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dB_s^H), dt \quad \text{a.e. } t \in I. \end{cases}$$

Now let us write

$$\|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \|G(s, u_{h_n}(s)) - G(s, u_h(s))\| ds.$$

Since  $\|u_{h_n}(\cdot) - u_h(\cdot)\| \rightarrow 0$  uniformly on  $I$  as  $n \rightarrow \infty$ , we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m_G \|u_{h_n}(s) - u_h(s)\| ds \leq \int_0^1 \frac{m_G}{\Gamma(\beta)} \|u_{h_n}(s) - u_h(s)\| ds$$

which entails that  $\Phi(h_n) \rightarrow \Phi(h)$  uniformly on  $I$ . This translates the continuity  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  with respect to the topology induced by the topology of uniform convergence on  $I$ . This continuity of  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  on the compact convex set  $\mathcal{X}_G$  of  $C_{\mathbf{R}^e}(I)$  tells us that  $\Phi$  has a fixed point, say  $h = \Phi(h) \in \mathcal{X}_G$ . This means, according to the definition of  $\Phi$

$$h(t) = \Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds \text{ for all } t \in I$$

and also

$$\begin{cases} u_h(0) = u_0 \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dB_s^H), dt \quad \text{a.e. } t \in I. \end{cases}$$

As consequence

$$\begin{cases} {}^c D^\beta h(t) = G(t, u_h(t)) \\ u_h(0) = u_0 \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + f(t, u_h(t), \int_0^t b(s, h(s)) dB_s^H), dt \quad \text{a.e. } t \in I. \end{cases}$$

So, by putting  $x = h$ ,  $u = v_h$  we conclude that the pair  $(x, u)$  solves the fractional/ differential inclusion with sweeping process and Holder signal

$$\begin{cases} {}^c D^\beta x(t) = G(t, u(t)), & t \in I \\ u(0) = u_0 \\ -\dot{u}(t) \in N_{C(t)}u(t) + f(t, u(t), \int_0^t b(s, x(s))dB_s^H), & \text{a.e. } t \in I. \end{cases}$$

The proof is complete

Applications to Optimization is given below.

## 2.1 Optimization

**Theorem 2.5** *Let  $I = [0, T]$  and let  $E = \mathbf{R}^e$ . Let  $r \in ]0, +\infty]$  and  $C : I \rightrightarrows \mathbf{R}^e$  be an absolutely continuous multimapping whose values are  $r$ -prox regular and closed.*

*Let  $f : I \times I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mapping such that*

- (i)  $f(\cdot, \cdot, x, y)$  is  $\mathcal{L}(I) \otimes \mathcal{L}(I)$  measurable for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ ;*
- (ii)  $f(t, s, \cdot, \cdot)$  is continuous on  $\mathbf{R}^e \times \mathbf{R}^e$  for all  $(t, s) \in I \times I$ ;*
- (iii)  $\|f(t, s, x, y)\| \leq \mu$  for all  $(t, s, x, y) \in I \times I \times \mathbf{R}^e \times \mathbf{R}^e$  for some constant  $\mu > 0$ ;*
- (iv)  $\|f(t, s, x, y) - f(t, s, z, y)\| \leq M\|x - z\|$ , for all  $(t, s, x, y, z) \in I \times I \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e$  for some constant  $M > 0$ .*

*Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .*

*Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1)$ ,  $(\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .*

*Let  $L : [0, T] \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow [0, \infty[$  be a lower semicontinuous integrand such that  $L(t, x, y, \cdot)$  is convex on  $\mathbf{R}^e$  for every  $(t, x, y) \in [0, T] \times \mathbf{R}^e \times \mathbf{R}^e$ . Then the problem of minimizing the cost function*

$$\int_0^1 L(t, \int_0^t b(s, x(s))dB_s^H, u(t), \dot{u}(t))dt$$

*over the pair of mapping  $(x, u)$  from  $I$  to  $\mathbf{R}^e$  with  $x \in C^\beta([0, T]; \mathbf{R}^e)$  and  $u \in W_{\mathbf{R}^e}^{1,1}(I)$  subject to*

$$\begin{cases} {}^c D^\beta x(t) = G(t, u(t)), & t \in I \\ u(0) = u_0 \in C(0), \\ u(t) \in C(t), & t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + f(t, u(t), \int_0^t b(s, x(s))dB_s^H), & dt \text{ a.e. } t \in I \end{cases}$$

*admits an optimal solution.*

**Proof.** Without loss of generality we may suppose that  $M \geq \mu$ . Define  $\sigma = \|u_0\| + \int_0^T |\dot{v}(t)| + 2M)dt$  and set  $\mathcal{X}_G$  by

$$\mathcal{X}_G := \{\zeta_t : I \rightarrow E : l \in S_G^1\},$$

where  $\mathcal{G}(t) = G(t, L\bar{B}_{\mathbf{R}^e})$ ,  $t \in I$  each mapping  $\xi_f$  being given for every  $t \in I$  by

$$\zeta_t(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} l(s)ds, t \in I, l \in S_G^1$$

Then  $\mathcal{X}_G$  is convex, compact in  $C^\beta([0, T]; \mathbf{R}^e)$  and equi- $\beta$ -Holder continuous with  $\|h(t) - h(s)\| \leq N|t - s|^\beta$  where  $N$  is a positive constant independant of  $h$ . Let  $(h_n, u_n)_n$  be a minimizing sequence for the minimizing problem, namely

$$\lim_{n \rightarrow \infty} \int_0^1 L(t, \int_0^t b(s, x(s))dB_s^H, u_n(t), \dot{u}_n(t))dt = \inf_{(k, w)} [\int_0^1 L(t, \int_0^t b(s, k(s))dB_s^H, w(t), \dot{w}(t))dt]$$

the infimum being taken over the pair  $(k, w)$  with  $k \in C^\beta([0, T]; \mathbf{R}^e)$  and  $w \in W_{\mathbf{R}^e}^{1,1}(I)$  such that

$$\begin{cases} {}^c D^\beta k(t) = G(t, w(t)), t \in I \\ w(0) = u_0 \in C(0), w(t) \in C(t), t \in I \\ -\frac{dw}{dt}(t) \in N_{C(t)}w(t) + f(t, w(t), \int_0^t b(s, k(s))dB_s^H), dt \text{ a.e. } t \in I \end{cases}$$

Note that

$$\begin{cases} {}^c D^\beta h_n(t) = G(t, u_n(t)), t \in I \\ u_n(0) = u_0 \in C(0), u_n(t) \in C(t), t \in I \\ -\frac{du_n}{dt}(t) \in N_{C(t)}u_n(t) + f(t, u_n(t), \int_0^t b(s, h_n(s))dB_s^H), dt \text{ a.e. } t \in I \end{cases}$$

By compactness of the solution set of the differential inclusion (as it can be see through the proof of Theorem 2.4) there is a non relabeled subsequence  $(h_n, u_n)$  with  $(h_n)_n$  converging uniformly on  $I$  to  $h \in \mathcal{X}_G$  and with  $(u_n)_n$  converging uniformly to  $u : I \rightarrow \mathbf{R}^e$  on  $I$  with  $\dot{u}_n \rightarrow \dot{u}$  weakly in  $L^1_{\mathbf{R}^e}(I)$ . We can ensure that  $\int_0^t b(s, h_n(s))dB_s^H \rightarrow \int_0^t b(s, h(s))dB_s^H$  on  $I$ . From the set

$${}^c D^\beta h_n(t) = G(t, u_n(t)), t \in I$$

it is equivalent to

$$h_n(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(t, u_n(t)), t \in I$$

Therefore passing to the limit

$$h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(t, u(t)), t \in I$$

So by repeating the argument as in Theorem 2.4 we obtain that  $u$  satisfies

$$-\frac{du}{dt}(t) \in N_{C(t)}u(t) + f(t, u(t), \int_0^t b(s, h(s))dB_s^H), dt \text{ a.e. } t \in I$$

According to the lower semicontinuity of the integral functional (see [13]) we obtain

$$\liminf_n \int_0^1 L(t, \int_0^t b(s, h_n(s))dB_s^H, u_n(t), \dot{u}_n(t))dt \geq \int_0^T L(t, \int_0^t b(s, h(s))dB_s^H, u(t), \dot{u}(t))dt$$

where the pair  $(h, u)$  is solution of the dynamic

$$\begin{cases} {}^c D^\beta h(t) = G(t, u(t)), t \in I \\ u(0) = u_0 \in C(0), u(t) \in C(t), t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + f(t, u(t), \int_0^t b(s, h(s))dB_s^H), dt \text{ a.e. } t \in I \end{cases}$$

Altogether we see that the pair  $(h, u)$  is an optimal solution.

**Theorem 2.6** Let  $I := [0, T]$ . Let  $C : [0, T] \rightarrow \mathbf{R}^e$  be a closed convex valued Lebesgue measurable multimapping that has a selection  $\zeta \in L^2_{\mathbf{R}^e}(I)$ , i.e.  $\zeta(t) \in C(t)$  for all  $t \in I$ . Let  $f : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mappings such that  $f(\cdot, x)$  is Lebesgue measurable for all  $x \in \mathbf{R}^e$  and such that there are non-negative ,  $\alpha_f > 0$ , such that for all  $(t, x) \in I \times \mathbf{R}^e$

$$\|f(t, x)\| \leq \alpha_f(t)$$

and for all  $t \in I$  and  $x, y \in \mathbf{R}^e$

$$\|f(t, x) - f(t, y)\| \leq \alpha_f(t)\|x - y\|.$$

Let  $g : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mappings such that  $g(\cdot, x)$  is Lebesgue measurable for all  $x \in \mathbf{R}^e$  and such that there are non-negative functions ,  $\alpha_g$ , ,  $\beta_g$  in  $L^2_{\mathbf{R}^+}(I)$  along with  $\gamma_g$  in  $L^1_{\mathbf{R}^+}(I)$  such that for all  $(t, x) \in I \times \mathbf{R}^e$

$$\|g(t, x)\| \leq \alpha_g(t)\|x\| + \beta_g(t)$$

and for all  $t \in I$  and  $x, y \in \mathbf{R}^e$

$$\|g(t, x) - g(t, y)\| \leq \gamma_g(t)\|x - y\|.$$

Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1)$ ,  $(\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .

Let  $\Lambda : \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a positive symmetric bijective linear continuous operator.

Let  $L : [0, T] \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow [0, \infty[$  be a lower semicontinuous integrand such that  $L(t, x, y, \cdot)$  is convex on  $\mathbf{R}^e$  for every  $(t, x, y) \in [0, T] \times \mathbf{R}^e \times \mathbf{R}^e$ . Then the problem of minimizing the cost function

$$\int_0^T L(t, \int_0^t b(s, x(s))dB_s^H, u(t), \dot{u}(t))dt$$

over the pair of mapping  $(x, u)$  from  $I$  to  $\mathbf{R}^e$  with  $x \in C^\beta([0, T]; \mathbf{R}^e)$  and  $u \in W_{\mathbf{R}^e}^{1,1}(I)$  subject to

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s))ds, & t \in I \\ u(0) = u_0 \in \mathbf{R}^e \\ f(t, \int_0^t b(s, x(s))dB_s^H) - \Lambda \frac{du}{dt}(t) \in N_{C(t)}\left(\frac{du}{dt}(t) + g(t, u(t))\right), & \text{a.e.} \end{cases}$$

admit an optimal solution.

*Proof.* By Theorem 3.1 [8] there is a non-negative function  $\mu(\cdot) \in L^2_{\mathbf{R}^+}(I)$  such that for every mapping continuous mapping  $h : I \rightarrow \mathbf{R}^e$  the absolutely continuous solution  $\zeta(\cdot)$  of the differential inclusion

$$\begin{cases} f(t, h(t)) - \Lambda \frac{d\zeta}{dt}(t) \in N_{C(t)}\left(t, \frac{d\zeta}{dt}(t) + g(t, \zeta(t))\right) & \text{a.e. } t \in I \\ \zeta(0) = u_0 \end{cases}$$

satisfies  $\|\dot{\zeta}(t)\| \leq \mu(t)$  a.e.  $t \in I$ . Let  $\Delta(t) = u_0 + (\int_0^t \mu(s)ds)\overline{B}_{\mathbf{R}^e}$  for all  $t \in I$ , then  $\zeta(t) \in \Delta(t)$  for all  $t \in I$  and the convex compact valued multimapping. Let us consider the set  $\mathcal{X}_G$  defined by

$$\mathcal{X}_G = \{\zeta_l : I \rightarrow \mathbf{R}^e : \zeta_l(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} l(s) ds, t \in I, l \in S_G^1\}$$

where  $\mathcal{G}(t) = G(t, \Delta(t)), t \in I$ . Then  $\mathcal{X}_G$  is convex, compact in  $C^\beta([0, T]; \mathbf{R}^e)$  and equi- $\beta$ -Holder continuous with  $\|h(t) - h(s)\| \leq N|t - s|^\beta$  where  $N$  is a positive constant independent of  $h$ . Then, for any  $h \in \mathcal{X}_G$ , using  $(\mathcal{B}_2)(\mathcal{B}_3)$  we obtain

$$\|b(t, h(t)) - b(s, h(s))\| \leq \|b(t, h(t)) - b(s, h(t))\| + \|b(s, h(t)) - b(s, h(s))\| \leq (M + MN)|t - s|^\beta$$

so that for any  $h \in \mathcal{X}_G$ ,  $b(\cdot, h(\cdot)) \in C^\beta([0, T]; \mathbf{R}^e)$  and  $\|b(\cdot, h(\cdot))\| \leq M$  by  $(\mathcal{B}_1)$ . In particular, for each  $h \in \mathcal{X}_G$  and for each  $t \in I$  the integral  $\int_0^t b(s, h(s)) dB_s^H$  is well defined for all  $h \in \mathcal{X}_G$  and satisfies the properties given in the proof of Theorem 2.2. As consequence  $\mathcal{Y}_G = \{t \mapsto \int_0^t b(\theta, h(\theta)) dB_\theta^H, h \in \mathcal{X}_G\}$  is compact in  $C^{1-\gamma}([0, T]; \mathbf{R}^e)$  with  $1 - H < \gamma < \min \beta, 1 - \beta$ . Let  $(h_n, u_n)_n$  be a minimizing sequence for the minimizing problem, namely

$$\lim_{n \rightarrow \infty} \int_0^T L(t, \int_0^t b(s, x(s)) dB_s^H, u_n(t), \dot{u}_n(t)) dt = \inf_{(k, w)} \left[ \int_0^1 L(t, \int_0^t b(s, k(s)) dB_s^H, w(t), \dot{w}(t)) dt \right]$$

the infimum being taken over the pair  $(k, w)$  with  $k \in C^\beta([0, T]; \mathbf{R}^e)$  and  $w \in W_{\mathbf{R}^e}^{1,1}(I)$  such that

$$\begin{cases} k(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, w(s)) ds, t \in I \\ w(0) = u_0 \in \mathbf{R}^e \\ f(t, \int_0^t b(s, k(s)) dB_s^H) - \Lambda \frac{dw}{dt}(t) \in N_{C(t)}(\frac{dw}{dt}(t) + g(t, w(t))), \text{ a.e.} \end{cases}$$

Note that

$$\begin{cases} h_n(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_n(s)) ds, t \in I \\ u_n(0) = u_0 \in \mathbf{R}^e \\ f(t, \int_0^t b(s, h_n(s)) dB_s^H) - \Lambda \frac{du_n}{dt}(t) \in N_{C(t)}(\frac{du_n}{dt}(t) + g(t, u_n(t))), \text{ a.e.} \end{cases}$$

By compactness of the solution set of the differential inclusion ( as it can be see through the proof of Theorem 2.5) there is a non relabeled subsequence  $(h_n, u_n)$  with  $(h_n)_n$  converging uniformly on  $I$  to  $h \in \mathcal{X}_G$  and with  $(u_n)_n$  converging uniformly to  $u$  on  $I$  with  $\dot{u}_n \rightarrow \dot{u}$  weakly in  $L_{\mathbf{R}^e}^1(I)$ . We can ensure that  $\int_0^t b(s, h_n(s)) dB_s^H \rightarrow \int_0^t b(s, h(s)) dB_s^H$  on  $I$ . From the set

$$h_n(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(t, u_n(t)), t \in I$$

Therefore passing to the limit

$$h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(t, u(t)), t \in I$$

So by repeating the argument as in Theorem 2.5 we obtain that  $u$  satisfies

$$f(t, \int_0^t b(s, u(s))dB_s^H) - \Lambda \frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t) + g(t, u(t))), \text{ a.e.}$$

According to the lower semicontinuity of the integral functional (see [13]) we obtain

$$\liminf_n \int_0^T L(t, \int_0^t b(s, h_n(s))dB_s^H, u_n(t), \dot{u}_n(t))dt \geq \int_0^T L(t, \int_0^t b(s, h(s))dB_s^H, u(t), \dot{u}(t))dt$$

where the pair  $(h, u)$  is solution of the dynamic

$$\begin{cases} h(t) == \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(t, u(t), t) \in I \\ u(0) = u_0 \in \mathbf{R}^e \\ f(t, \int_0^t b(s, u(s))dB_s^H) - \Lambda \frac{du}{dt}(t) \in N_{C(t)}(\frac{du}{dt}(t) + g(t, u(t))) \text{ a.e. } t \in I \end{cases}$$

Altogether we see that the pair  $(h, u)$  is an optimal solution.

**Theorem 2.7** *Let  $I = [0, T]$ . Let  $r \in ]0, +\infty]$  and  $C : I \rightrightarrows \mathbf{R}^e$  be an absolutely continuous multimapping whose values are  $r$ -prox regular. Let  $f : I \times I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a mapping such that*

- (i)  $f(\cdot, \cdot, x, y)$  is  $\mathcal{L}(I) \otimes \mathcal{L}(I)$  measurable for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ ;
- (ii)  $f(t, s, \cdot, \cdot)$  is continuous on  $\mathbf{R}^e \times \mathbf{R}^e$  for all  $(t, s) \in I \times I$ ;
- (iii)  $\|f(t, s, x, y)\| \leq \mu$  for all  $(t, s, x, y) \in I \times I \times \mathbf{R}^e \times \mathbf{R}^e$  for some constant  $\mu > 0$ ;
- (iv)  $\|f(t, s, x, y) - f(t, s, z, y)\| \leq M\|x - z\|$ ,  $\|f(t, s, x, y) - f(t, s, x, z)\| \leq M\|y - z\|$ , for all  $(t, s, x, y, z) \in I \times I \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e$  for some constant  $M > 0$ .

Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{G}_1)$ ,  $(\mathcal{G}_2)$  and  $(\mathcal{G}_3)$ .

Then there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely continuous mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} {}^c D^\beta x(t) = G(t, u(t)), t \in I \\ u(t) \in C(t), t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \int_0^t f(t, s, u(s), \int_0^s b(\theta, x(\theta))dB_\theta^H) ds \text{ a.e. } t \in I \\ u(0) = a \in C(0). \end{cases}$$

**Proof.** Without loss of generality we may suppose that  $M \geq \mu$ . For any continuous mapping  $h : I \rightarrow \mathbf{R}^e$ , the mapping  $f_h : I \times I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  defined by

$$f_h(t, s, x) := f(t, s, x, h(t)) \text{ for all } (t, s, x) \in I \times I \times \mathbf{R}^e$$

is  $\mathcal{L}(I) \otimes \mathcal{L}(I)$  measurable for every  $x \in \mathbf{R}^e$ , continuous on  $\mathbf{R}^e$  for every  $(t, s) \in I \times I$ , and satisfies

$$\|f_h(t, s, x)\| \leq M \text{ for all } (t, s, x) \in I \times I \times \mathbf{R}^e$$

and

$$\|f_h(t, s, x) - f_h(t, s, y)\| \leq M\|x - y\| \quad \text{for all } (t, s, x, y) \in I \times I \times \mathbf{R}^e \times \mathbf{R}^e.$$

Then, given  $a \in C(0)$ , by Theorem 3.3 [12] there is a **unique** AC solution  $u_h$  to the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + \int_0^t f(t, s, u_h(s), h(s)) ds \text{ a.e. } t \in I. \end{cases}$$

We claim that  $(u_h)_h$  is uniformly bounded and equi-absolutely continuous. Indeed,  $\frac{du_h}{dt}(t) \in K(t)\overline{B}_{\mathbf{R}^e}$  for a.e.  $t \in I$ , where  $K$  is a non-negative integrable function on  $I$  (say  $K(t) = |\dot{v}(t)| + M$ , where  $v(\cdot)$  is a control function of absolute continuity of  $C(\cdot)$ ). Then  $\|u_h(t)\| \leq K(t)$  for a.e.  $t \in I$ , which ensures that  $(u_h)_h$  is equi-absolutely continuous. Putting  $L := \|a\| + \int_0^1 (|\dot{v}(t)| + 2M) dt$ , we see that  $\|u_h(t)\| \leq L$  for all  $t \in I$ , which confirms the uniform boundedness of  $(u_h)_h$  and finishes the justification of the claim.  $\mathcal{X}_G$  by

$$\mathcal{X}_G := \{\zeta_l : I \rightarrow \mathbf{R}^e : l \in S_G^1\},$$

where  $G(t) = G(t, L\overline{B}_{\mathbf{R}^e})$ ,  $t \in I$  each mapping  $\xi_f$  being given for every  $t \in I$  by

$$\zeta_l(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} l(s) ds, t \in I, l \in S_G^1$$

Then  $\mathcal{X}_G$  is convex, compact in  $C^\beta([0, T]; \mathbf{R}^e)$  and equi- $\beta$ -Holder continuous with  $\|h(t) - h(s)\| \leq N|t - s|^\beta$  where  $N$  is a positive constant independent of  $h$ . Then, for any  $h \in \mathcal{X}_G$ , using  $(\mathcal{B}_2)(\mathcal{B}_3)$  we obtain

$$\|b(t, h(t)) - b(s, h(s))\| \leq \|b(t, h(t)) - b(s, h(t))\| + \|b(s, h(t)) - b(s, h(s))\| \leq (M + MN)|t - s|^\beta$$

so that for any  $h \in \mathcal{X}_G$ ,  $b(\cdot, h(\cdot)) \in C^\beta([0, T]; \mathbf{R}^e)$  and equi- $\beta$ -Holder continuous with  $\|h(t) - h(s)\| \leq N|t - s|^\beta$  where  $N$  is a positive constant independent of  $h$ . In particular, for each  $h \in \mathcal{X}_G$  and for each  $t \in I$  the integral  $\int_0^t b(s, h(s)) dB_s^H$  is well defined for all  $h \in \mathcal{X}_G$  and satisfies the properties given in the proof of Theorem 2.2. As consequence  $\mathcal{Y}_G = \{t \mapsto \int_0^t b(s, h(s)) dB_s^H, h \in \mathcal{X}_G\}$  is compact in  $C^{1-\gamma}([0, T]; \mathbf{R}^e)$  with  $1 - H < \gamma < \min\{\beta, 1 - \beta\}$ . In particular, if a sequence  $(h_n)_n$  in  $\mathcal{X}_G$  converges uniformly on  $I$  to a mapping  $h \in \mathcal{X}_G$ , then the sequence  $(t \mapsto \int_0^t b(s, h_n(s)) dB_s^H)_n$  converges uniformly on  $I$  to the mapping  $(t \mapsto \int_0^t b(s, h(s)) dB_s^H)$ . For each  $h \in \mathcal{X}_G$ , let us set  $\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds$  for all  $t \in I$  where  $u_h$  is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + \int_0^t f(t, s, u_h(s), \int_0^s b(\theta, h(\theta)) dB_\theta^H) ds \end{cases}$$

For each  $h \in \mathcal{X}_G$  let us consider again

$$\Phi(h)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds, \quad \text{for all } t \in I.$$

Then it is clear that  $\Phi(h) \in \mathcal{X}_G$ . Hence  $\Phi(\mathcal{X})$  is equicontinuous and relatively compact in the Banach space  $\mathcal{C}_{\mathbf{R}^e}(I)$  because  $\Phi(\mathcal{X}_G) \subset \mathcal{X}_G$ . Now we check that  $\Phi$  is continuous relative to  $\mathcal{X}_G$ . It is enough to show that, if  $(h_n)_n$  converges uniformly to  $h$  in  $\mathcal{X}$ , then the sequence  $(u_{h_n})_n$ , where each  $u_{h_n}$  is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a \in C(0) \\ u_{h_n}(t) \in C(t), \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + \int_0^t f(t, s, u_{h_n}(s), \int_0^s b(\theta, h_n(\theta))dB_\theta^H) ds \quad \text{a.e. } t \in I. \end{cases}$$

uniformly converges to the unique absolutely continuous solution  $u_h$  of the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + \int_0^t f(t, s, u_h(s), \int_0^s b(\theta, h(\theta))dB_\theta^H) ds \quad \text{a.e. } t \in I. \end{cases}$$

This needs a careful look. We note that  $(u_{h_n})_n$  is equicontinuous by what precedes (since for every  $n \in \mathbf{N}$  one has  $\|\dot{u}_{h_n}(t)\| \leq K(t)$  for almost all  $t \in I$ ). Further,  $\{u_{h_n}(t) : n \in \mathbf{N}\}$  is included in the compact set  $L\bar{B}_E$  for every  $t \in I$ . The Arzelà-Ascoli theorem tells us that  $\{u_{h_n} : n \in \mathbf{N}\}$  is relatively compact in  $\mathcal{C}_E(I)$ . So by extracting a subsequence, we may suppose that  $(u_{h_n})_n$  converges uniformly on  $I$  to some mapping  $\zeta : I \rightarrow E$  with

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s) ds \quad \text{for all } t \in I,$$

along with  $(\dot{u}_{h_n})_n$  converging weakly in  $L_E^1(I)$  to  $\dot{\zeta}$  with  $\|\dot{\zeta}(t)\| \leq K(t)$  for a.e.  $t \in I$ . Note that

$$f(t, s, u_{h_n}(s), \int_0^s b(\theta, h_n(\theta))dB_\theta^H) \rightarrow f(t, s, \zeta(s), \int_0^s b(\theta, h(\theta))dB_\theta^H) \quad \text{for all } (t, s) \in I \times I.$$

For simplicity, denote

$$z_n(t) := \int_0^t f(t, s, u_{h_n}(s), \int_0^s b(\theta, h_n(\theta))dB_\theta^H) ds, \quad \text{for all } t \in I,$$

$$z(t) = \int_0^t f(t, s, \zeta(s), \int_0^s b(\theta, h(\theta))dB_\theta^H) ds, \quad \text{for all } t \in I.$$

First, we notice that these mappings are Lebesgue measurable by the Fubini-Bochner property and the separability of the space  $E$ . Second, by condition (iii) and the uniform boundedness of  $(u_{h_n})_n$  and  $u_h$ , we also notice that  $(z_n)_n$  and  $z$  are uniformly bounded, since

$$\|z_n(t)\| \leq M \quad \text{and} \quad \|z(t)\| \leq M, \quad \text{for all } n \in \mathbf{N} \text{ and } t \in I.$$

Consequently,  $(z_n)_n$  is a sequence of measurable and uniformly bounded mappings which converges pointwise to the measurable mapping  $z$ . Therefore, the sequence  $(\dot{u}_{h_n} + z_n)_n$

converges weakly to  $\dot{\zeta} + z$  in  $L^1_E(I)$ . This combined with Lemma 3.4 in [12] and the uniqueness of solution of the latter differential inclusion

$$\begin{cases} -\frac{d\zeta}{dt}(t) \in N_{C(t)}\zeta(t) + z(t) \text{ a.e. } t \in I \\ \zeta(0) = a, \zeta(t) \in C(t) \forall t \in I, \end{cases}$$

we obtain that  $\zeta = u_h$ . Now let us write

$$\|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \|G(s, u_{h_n}(s)) - G(s, u_h(s))\| ds.$$

Since  $\|u_{h_n}(\cdot) - u_h(\cdot)\| \rightarrow 0$  uniformly on  $I$  as  $n \rightarrow \infty$ , we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} m_G \|u_{h_n}(s) - u_h(s)\| ds \leq \int_0^1 \frac{m_G}{\Gamma(\beta)} \|u_{h_n}(s) - u_h(s)\| ds$$

which entails that  $\Phi(h_n) \rightarrow \Phi(h)$  uniformly on  $I$ . This translates the continuity  $\Phi : \mathcal{X}_G \rightarrow \mathcal{X}_G$  with respect to the topology induced by the topology of uniform convergence on  $I$ , hence by the Schauder theorem  $\Phi$  has a fixed point, say  $h = \Phi(h) \in \mathcal{X}_G$ . This means that for every  $t \in I$

$$h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u_h(s)) ds, \forall t \in I$$

with

$$\begin{cases} u_h(0) \in C(0) \\ u_h(t) \in C(t), \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + \int_0^t f(t, s, u_h(s), \int_0^s b(\theta, h(\theta)) dB_\theta^H) ds, \text{ a.e. } t \in I. \end{cases}$$

Therefore we have just shown that there exists a mapping  $h \in C^\beta([0, T]; \mathbf{R}^e)$  and a AC mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} {}^c D^\beta h(t) = G(t, u(t)), \forall t \in I, \\ u(0) = u_0 \in C(0) \\ u(t) \in C(t), \forall t \in I \\ -\dot{u}(t) \in N_{C(t)}u(t) + \int_0^t f(t, s, u(s), \int_0^s b(\theta, h(\theta)) dB_\theta^H) ds, \text{ a.e. } t \in I. \end{cases}$$

The proof of the theorem is then complete.

**Theorem 2.8** *Let  $I := [0, T]$ . Let  $\varphi : I \times \mathbf{R}^e \rightarrow \mathbf{R} \cup \{+\infty\}$  be a normal convex integrand for which there are a mapping  $\zeta \in L^2_{\mathbf{R}^e}(I)$  such that  $t \mapsto \sqrt{(\varphi_t(\zeta(t)))^+}$  is in  $L^2_{\mathbf{R}_+}(I)$  and two non-negative functions  $\alpha_\varphi, \beta_\varphi : I \rightarrow \mathbf{R}$  with  $\alpha_\varphi, \sqrt{\beta_\varphi} \in L^2_{\mathbf{R}_+}(I)$  such that*

$$-\alpha_\varphi(t)\|x\| - \beta_\varphi(t) \leq \varphi(t, x) \text{ for all } (t, x) \in I \times \mathbf{R}^e$$

*(note that  $\sqrt{\beta_\varphi} \in L^2_{\mathbf{R}_+}(I)$  if  $\beta_\varphi \in L^2_{\mathbf{R}_+}(I)$ ). Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .*

*Let  $f : I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$ :*

- ( $\mathcal{F}_1$ )  $f(\cdot, x, y)$  is  $\mathcal{L}(I)$  measurable for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ ,  
( $\mathcal{F}_2$ )  $f(t, \cdot, \cdot)$  is continuous on  $\mathbf{R}^e \times \mathbf{R}^e$  for all  $t \in I$ ,  
( $\mathcal{F}_3$ )  $\|f(t, x, z) - f(t, y, z)\| \leq M\|x - y\|$  for all  $(t, x, y, z) \in I \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e$ ,  
( $\mathcal{F}_4$ )  $\|f(t, x, y)\| \leq M$  for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  :

- ( $\mathcal{G}_1$ )  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  is continuous,  
( $\mathcal{G}_2$ )  $G_t : \mathbf{R}^e \rightarrow \mathbf{R}^e$  is linear continuous operator for all  $t \in I$ ,  
( $\mathcal{G}_3$ )  $|G_t| \leq m_G$  for all  $t \in I$  where  $|G_t|$  is the norm of the operator  $G_t$  and  $m_G$  is a positive constant.

Let  $\Lambda : \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a positive symmetric bijective linear continuous operator. Let  $\mathcal{R} : C_{\mathbf{R}^e}(I) \rightarrow C_{\mathbf{R}^e}(I)$  be an almost history-dependent operator with time-dependent rates. Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ .

Then, for any  $u_0 \in \mathbf{R}^e$ , there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely continuous mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, & t \in I \\ u(0) = u_0 \in \mathbf{R}^e \\ -\Lambda \frac{du}{dt}(t) \in \partial\varphi(t, \frac{du}{dt}(t) + f(t, u(t), \int_0^t b(s, x(s)) dB_s^H) + \mathcal{R}(u)(t)), & \text{a.e.} \end{cases}$$

Proof. By careful application of Theorem 3.6 in [8] and by repetition of the argument of the proof of Theorem 2.3.

**Corollary 2.2** Let  $I := [0, T]$ . Let  $C : [0, T] \rightarrow \mathbf{R}^e$  be a closed convex valued Lebesgue measurable multimapping that has a selection  $\zeta \in L^2_{\mathbf{R}^e}(I)$ , i.e.  $\zeta(t) \in C(t)$  for all  $t \in I$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions ( $\mathcal{B}_1$ ), ( $\mathcal{B}_2$ ) and ( $\mathcal{B}_3$ ).

Let  $f : I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  :

- ( $\mathcal{F}_1$ )  $f(\cdot, x, y)$  is  $\mathcal{L}(I)$  measurable for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ ,  
( $\mathcal{F}_2$ )  $f(t, \cdot, \cdot)$  is continuous on  $\mathbf{R}^e \times \mathbf{R}^e$  for all  $t \in I$ ,  
( $\mathcal{F}_3$ )  $\|f(t, x, z) - f(t, y, z)\| \leq M\|x - y\|$  for all  $(t, x, y, z) \in I \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e$ ,  
( $\mathcal{F}_4$ )  $\|f(t, x, y)\| \leq M$  for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  :

- ( $\mathcal{G}_1$ )  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  is continuous,  
( $\mathcal{G}_2$ )  $G_t : \mathbf{R}^e \rightarrow \mathbf{R}^e$  is linear continuous operator for all  $t \in I$ ,  
( $\mathcal{G}_3$ )  $|G_t| \leq m_G$  for all  $t \in I$  where  $|G_t|$  is the norm of the operator  $G_t$  and  $m_G$  is a positive constant.

Let  $\Lambda : \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a positive symmetric bijective linear continuous operator. Let  $\mathcal{R} : C_{\mathbf{R}^e}(I) \rightarrow C_{\mathbf{R}^e}(I)$  be an almost history-dependent operator with time-dependent rates. Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ .

Then, for any  $u_0 \in \mathbf{R}^e$ , there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely

continuous mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, & t \in I \\ u(0) = u_0 \in \mathbf{R}^e \\ -\Lambda \frac{du}{dt}(t) \in N_{C(t)} \left( \frac{du}{dt}(t) + f(t, u(t)), \int_0^t b(s, x(s)) dB_s^H \right) + \mathcal{R}(u)(t), & \text{a.e.} \end{cases}$$

**Theorem 2.9** Let  $I := [0, T]$ . For each  $t \in I$  let  $A_t : D(A_t) \subset \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a maximal monotone operator such that the multimapping  $t \rightarrow \text{gph} A_t$  from  $I$  to  $\mathbf{R}^e \times \mathbf{R}^e$  is Lebesgue measurable and there  $\psi(\cdot) \in L^1_{\mathbf{R}^e}$  and  $\zeta 1_{\mathbf{R}^e}$  with  $\zeta(t) \in A_t \psi(t)$  for all  $t \in I$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1)$ ,  $(\mathcal{B}_2)$  and  $(\mathcal{B}_3)$ .

Let  $f : I \times \mathbf{R}^e \times \mathbf{R}^e \rightarrow \mathbf{R}^e$ :

$(\mathcal{F}_1)$   $f(\cdot, x, y)$  is  $\mathcal{L}(I)$  measurable for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ ,

$(\mathcal{F}_2)$   $f(t, \cdot, \cdot)$  is continuous on  $\mathbf{R}^e \times \mathbf{R}^e$  for all  $t \in I$ ,

$(\mathcal{F}_3)$   $\|f(t, x, z) - f(t, y, z)\| \leq M\|x - y\|$  for all  $(t, x, y, z) \in I \times \mathbf{R}^e \times \mathbf{R}^e \times \mathbf{R}^e$ ,

$(\mathcal{F}_4)$   $\|f(t, x, y)\| \leq M$  for all  $(x, y) \in \mathbf{R}^e \times \mathbf{R}^e$ .

Let  $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  :

$(\mathcal{G}_1)$   $G : I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  is continuous,

$(\mathcal{G}_2)$   $G_t : \mathbf{R}^e \rightarrow \mathbf{R}^e$  is linear continuous operator for all  $t \in I$ ,

$(\mathcal{G}_3)$   $|G_t| \leq m_G$  for all  $t \in I$  where  $|G_t|$  is the norm of the operator  $G_t$  and  $m_G$  is a positive constant.

Let  $\Lambda : \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a positive symmetric bijective linear continuous operator. Let  $\mathcal{R} : C_{\mathbf{R}^e}(I) \rightarrow C_{\mathbf{R}^e}(I)$  be an almost history-dependent operator with time-dependent rates. Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ .

Then, for any  $u_0 \in D(A_0)$ , there is a  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $x : [0, T] \rightarrow \mathbf{R}^e$  and an absolutely continuous mapping  $u : [0, T] \rightarrow \mathbf{R}^e$  satisfying

$$\begin{cases} x(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s, u(s)) ds, & t \in I \\ u(0) = u_0 \in D(A_0) \\ -\Lambda \frac{du}{dt}(t) \in A_t \left( \frac{du}{dt}(t) + f(t, u(t)), \int_0^t b(s, x(s)) dB_s^H \right) + \mathcal{R}(u)(t), & \text{a.e.} \end{cases}$$

Proof. By careful application of Theorem 3.16-3.18 in [8] and by repetition of the argument of the proof of Theorem 2.3 and Lemma 4.10 in [8].

### 3 Further ramification with integral equations/ fractional evolution inclusion driven by a fractional Brownian motion

Let us recall a fixed point theorem

**Theorem 3.1** (Krasnoselski) Let  $B_X$  be a closed convex subset of a Banach space  $(X, \|\cdot\|)$ . Let  $\mathcal{S} : B_X \rightarrow X$  and  $\mathcal{T} : B_X \rightarrow X$  satisfying the conditions:

(i)  $\mathcal{S}x + \mathcal{T}y \in B_X, \forall x, y \in B_X$

- (ii)  $\mathcal{S}$  is continuous and  $\{\mathcal{S}x : x \in B_X\}$  is relatively compact.  
(ii)  $\mathcal{T}$  is  $\beta_{\mathcal{T}} \in ]0, 1[$  contractive,  $\|\mathcal{T}x - \mathcal{T}y\| \leq \beta_{\mathcal{T}}\|x - y\|$ ,  $\forall x, y \in B_X$ .  
Then there exist  $x \in B_X$  such that  $x \in \mathcal{S}x + \mathcal{T}x$

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space endowed with the filtration  $\{\mathbb{F}_t : t \geq 0\}$ , and

$$B^H = \{B^H(t) : t \geq 0\}$$

be a real-valued fractional Brownian motion with  $H > 1/2$ . We will study the following stochastic integral equation

$$x(t) = x(0) + \int_0^t f(s, x(s))ds + \int_0^t g(s, x(s))dB^H(s), \quad t \in [0, T],$$

where the functions  $f, g : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$  are Carathéodory functions satisfying the following conditions

(F1) there exists a positive function  $\ell_f \in L^{1/(1-\beta)}(0, T)$  such that

$$\|f(t, x) - f(t, y)\| \leq \ell_f(t)\|x - y\| \text{ for any } (t, x), (t, y) \in [0, T] \times \mathbb{R}^e.$$

(F2) the function  $t \mapsto f(t, 0)$  belongs to the space  $L^{1/(1-\beta)}(0, T)$ .

(G1)  $\|g(t, x)\| \leq M_g^1\|x\|$  for any  $(t, x) \in [0, T] \times \mathbb{R}^e$ .

(G2)  $\|g(t, x) - g(t, y)\| \leq M_g^2\|x - y\|$  for any  $(t, x), (t, y) \in [0, T] \times \mathbb{R}^e$ .

(G3)  $\|g(t, x) - g(s, x)\| \leq M_g^3(t - s)^\beta\|x\|$  for any  $0 \leq s < t \leq T$  and  $x \in \mathbb{R}^e$  with  $1 - H < \beta < 1/2$ .

**Lemma 3.1** *Under the assumptions (G1)-(G3), if the process  $x$  belongs to  $\mathcal{C}^\beta([0, T]; \mathbb{R}^e)$ , the stochastic process*

$$G = \{g(s, x(s)) : 0 \leq s \leq T\},$$

*belongs to the space  $\mathcal{W}_{\mathbb{R}^e}^{\gamma, 1}[0, T]$ , for any  $\gamma$  such that  $1 - H < \gamma < \gamma_0 := \min\{\beta, 1 - \beta\}$ . Moreover, we have*

$$\|G\|_{\mathcal{W}_{\mathbb{R}^e}^{\gamma, 1}[0, T]} \leq \left( \frac{M_g^1}{1 - \gamma} + \frac{(M_g^2 + M_g^3)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right) \|x\|_{\beta, [0, T]} T^{1-\gamma}. \quad (3.1)$$

**Proof.** From (G1) we obtain

$$\int_0^T \frac{\|g(s, x(s))\|}{s^\gamma} ds \leq M_g^1 \int_0^T \frac{\|x(s)\|}{s^\gamma} ds \leq \frac{T^{1-\gamma}}{1 - \gamma} M_g^1 \|x\|_{\beta, [0, T]},$$

and by (G2) and (G3)

$$\begin{aligned}
\int_0^T \int_0^t \frac{\|g(t, x(t)) - g(s, x(s))\|}{(t-s)^{\gamma+1}} ds dt &\leq \int_0^T \int_0^t \frac{M_g^2 \|x(t) - x(s)\| + M_g^3 \|x\|_{\beta, [0, T]} (t-s)^\beta}{(t-s)^{\gamma+1}} ds dt \\
&\leq (M_g^2 + M_g^3) \|x\|_{\beta, [0, T]} \int_0^T \int_0^t (t-s)^{\beta-\gamma-1} ds dt \\
&\leq \frac{(M_g^2 + M_g^3) T^{\beta-\gamma+1} \|x\|_{\beta, [0, T]}}{(\beta-\gamma)(\beta-\gamma+1)}.
\end{aligned}$$

From the definition of the norm of  $W_{\mathbf{R}^e}^{\gamma, 1}[0, T]$ , it is clear that  $G \in W_{\mathbf{R}^e}^{\gamma, 1}[0, T]$  and that

$$\|G\|_{W_{\mathbf{R}^e}^{\gamma, 1}[0, T]} \leq \left( \frac{M_g^1}{1-\gamma} + \frac{(M_g^2 + M_g^3) T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right) \|x\|_{\beta, [0, T]} T^{1-\gamma}. \quad (3.1)$$

Now we consider the operator  $\mathcal{S}$  defined by

$$\mathcal{S}x(t) = \int_0^t g(s, x(s)) dB^H(s), \quad 0 \leq t \leq T.$$

It follows from above that  $\mathcal{S}x$  is well defined.

**Lemma 3.2** *The following results hold true:*

(a)  $\mathcal{S}x \in C^{1-\gamma}([0, T]; \mathbf{R}^e)$  and

$$\|\mathcal{S}x\|_{1-\gamma, [0, T]} \leq M \|x\|_{\beta, [0, T]}, \quad \text{for any } x \in \mathcal{C}^\beta. \quad (3.2)$$

where  $M$  is a positive constant depending only on  $\gamma, \beta, T$  and  $M_g^i, i \leq 3$ .

(b) If  $U$  is a bounded subset in  $\mathcal{C}^\beta([0, T]; \mathbf{R}^e)$  then  $\mathcal{S}(U)$  is relatively compact in  $\mathcal{C}^\beta([0, T]; \mathbf{R}^e)$ .

Proof. Let  $x \in \mathcal{C}^\beta([0, T]; \mathbf{R}^e)$ . Since the stochastic process  $\{g(s, x(s)) : 0 \leq s \leq T\}$  belongs to the space  $W_{\mathbf{R}^e}^{\gamma, 1}[0, T]$ , it implies by using Proposition 3.2 of [18] that

$$\mathcal{S}x \in C^{1-\gamma}([0, T]; \mathbf{R}^e)$$

Moreover, we have, by the inequalities (\*) and (3.1),

$$\begin{aligned}
\|\mathcal{S}x(t)\| &\leq K(\gamma; [0, T]) \|g(\cdot, x(\cdot))\|_{W^{\gamma, 1}; [0, T]} \\
&\leq K(\gamma; [0, T]) \left( \frac{M_g^1}{1-\gamma} + \frac{(M_g^2 + M_g^3) T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right) \|x\|_{\beta, [0, T]} T^{1-\gamma}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{S}x(t) - \mathcal{S}x(s)\| &= \left\| \int_s^t g(\tau, x(\tau)) dB^H(\tau) \right\| \\
&\leq K(\gamma, [s, t]) \left\{ \int_s^t \frac{\|g(\tau, x(\tau))\|}{(t-s)^\gamma} d\tau + \int_s^t \int_s^\theta \frac{\|g(\theta, x(\theta)) - g(\zeta, x(\zeta))\|}{(\theta-\zeta)^{\gamma+1}} d\zeta d\theta \right\} \\
&\leq K(\gamma, [0, T]) \left( \frac{M_g^1}{1-\gamma} + \frac{(M_g^2 + M_g^3)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right) \|x\|_{\beta, [0, T]} (t-s)^{1-\gamma}.
\end{aligned}$$

From these estimates we derive (3.2) where  $M = M(\gamma, \beta, T, M_g^1, M_g^2, M_g^3)$  is

$$M = K(\gamma, [0, T])(1 + T^{1-\gamma}) \left( \frac{M_g^1}{1-\gamma} + \frac{(M_g^2 + M_g^3)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right). \quad (3.3)$$

As consequence,  $\mathcal{S}(U)$  is relatively compact in  $C^{1-\gamma}([0, T]; \mathbb{R}^e)$ . Since  $C^{1-\gamma}([0, T]; \mathbb{R}^e)$  is embedded into  $C^\beta([0, T]; \mathbb{R}^e)$ , then  $\mathcal{S}(U)$  is relatively compact in  $C^\beta([0, T]; \mathbb{R}^e)$ . Furthermore, if  $M < 1$  and if  $U \subset C^\beta([0, T]; \mathbb{R}^e)$  is the ball with radius  $R$  centered at zero function, then  $\mathcal{S}(U) \subset U$ . The proof of this lemma is completed.  $\blacksquare$

**Theorem 3.2** *Let (G1)-(G3) hold true. Moreover, assume that  $E_{\gamma, \beta}M < 1$ , where  $E_{\gamma, \beta}$  is the embedding constant from  $C^{1-\gamma}([0, T]; \mathbb{R}^e)$  into  $C^\beta([0, T]; \mathbb{R}^e)$  and  $M$  the constant of (3.3). Then, the stochastic integral equation*

$$x(t) = \int_0^t g(s, x(s)) dB^H(s), \quad t \in [0, T], \quad (3.4)$$

has a solution  $x \in C^\beta([0, T]; \mathbb{R}^e)$ .

*Proof.* It is clear that the stochastic process  $x(t, \cdot)$  is a solution of (3.4) if and only if it is a fixed point of the operator  $\mathcal{S} : C^\beta([0, T]; \mathbb{R}^e) \rightarrow C^\beta([0, T]; \mathbb{R}^e)$  defined by

$$\mathcal{S}x(t) = \int_0^t g(s, x(s)) dB^H(s), \quad 0 \leq t \leq T.$$

Let  $U \subset C^\beta([0, T]; \mathbb{R}^e)$  be the ball with radius  $R$  centered at zero function. It is not difficult to see from Lemma 3.2 that  $\mathcal{S}x \in U$ , for each  $x \in U$  and  $\mathcal{S}$  is a continuous and compact operator. So, the Schauder fixed point theorem shows the existence of a fixed point of  $\mathcal{S}$  which is a solution to (3.4).  $\blacksquare$

**Lemma 3.3** *Let  $f : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$  be a Caratheodory mapping satisfying:*

- (i)  $\|f(t, x)\| \leq M_f$  for all  $(t, x) \in [0, T] \times \mathbb{R}^e$  where  $M_f$  is positive constant,
- (ii)  $\|f(t, x) - f(t, y)\| \leq \alpha_f \|x - y\|$  for all  $(t, x, y) \in [0, T] \times \mathbb{R}^e \times \mathbb{R}^e$  where  $\alpha_f \in ]0, 1[$ . For  $x \in C^\beta([0, T]; \mathbb{R}^e)$ , we put

$$\mathcal{I}x(t) = \int_0^t f(s, x(s)) ds, \quad t \in [0, T].$$

Then we have the estimation

$$\|\mathcal{I}x(t) - \mathcal{I}y(t)\| \leq \alpha_f \|x - y\|_{C^\beta([0, T]; \mathbb{R}^e)}$$

for all  $(x, y) \in C^\beta([0, T]; \mathbb{R}^e) \times C^\beta([0, T]; \mathbb{R}^e)$  and for all  $t \in [0, T]$ .

$$\|\mathcal{I}x(t)\| \leq M_f T$$

for all  $x \in C^\beta([0, T]; \mathbb{R}^e)$  and for all  $t \in [0, T]$ . As consequence the mapping  $\mathcal{I} : C^\beta([0, T]; \mathbb{R}^e) \rightarrow C^\beta([0, T]; \mathbb{R}^e)$  is bounded and contractive

$$\|\mathcal{I}x(t) - \mathcal{I}y(t)\| \leq \alpha_f \|x - y\|_{C^\beta([0, T]; \mathbb{R}^e)}$$

for all  $(x, y) \in C^\beta([0, T]; \mathbb{R}^e) \times C^\beta([0, T]; \mathbb{R}^e)$  and for all  $t \in [0, T]$ .

**Theorem 3.3** Let (G1)-(G3) hold true. Let  $f : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$  be a Caratheodory mapping satisfying:

(i)  $\|f(t, x)\| \leq M_f$  for all  $(t, x) \in [0, T] \times \mathbb{R}^e$  where  $M_f$  is positive constant,

(ii)  $\|f(t, x) - f(t, y)\| \leq \alpha_f \|x - y\|$  for all  $(t, x, y) \in [0, T] \times \mathbb{R}^e \times \mathbb{R}^e$  where  $\alpha_f \in ]0, 1[$ .

Moreover, assume that

$$M_f T + E_{\gamma, \beta} M < 1,$$

where  $E_{\gamma, \beta}$  is the embedding constant from  $C^{1-\gamma}([0, T]; \mathbb{R}^e)$  into  $C^\beta([0, T]; \mathbb{R}^e)$  and  $M$  the constant of (3.3).

Then, the stochastic integral equation

$$x(t) = \int_0^t g(s, x(s)) dB^H(s) + \int_0^t f(s, x(s)) ds, \quad t \in [0, T],$$

has a solution  $x \in C^\beta([0, T]; \mathbb{R}^e)$ .

Let  $U \subset C^\beta([0, T]; \mathbb{R}^e)$  be the ball with radius  $R$  centered at zero function. It is not difficult to see that  $\mathcal{S}x + \mathcal{I}y \in U$  for all  $(x, y) \in U \times U$ . On the other hand, it follows from Lemma 3.1 and Lemma 3.2 that  $\mathcal{S}$  is a continuous and compact operator while  $\mathcal{I}$  is a contraction operator. So, the Krasnosel'skii fixed point theorem shows the existence of a fixed point of  $\mathcal{A}$  which is a solution to the above stochastic integral equation.

More generally, let  $f$  satisfying (F1)-(F2) Now let  $x \in C^\beta([0, T]; \mathbb{R}^e)$ . We put

$$\mathcal{I}x(t) = \int_0^t f(s, x(s)) ds, \quad t \in [0, T].$$

From the assumptions (F1)-(F2) we can prove the following result.

**Lemma 3.4** (a) If  $x \in C^\beta([0, T]; \mathbb{R}^e)$ , then  $\mathcal{I}x \in C^\beta([0, T]; \mathbb{R}^e)$  and satisfies the estimate

$$\|\mathcal{I}x\|_{\beta; [0, T]} \leq (T^\beta + 1) \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} \|x\|_{\beta; [0, T]} + (T^\beta + 1) \|f(\cdot, 0)\|_{L^{1/(1-\beta)}(0, T)}.$$

(b) If  $x, y \in C^\beta([0, T]; \mathbb{R}^e)$  then one has

$$\|\mathcal{I}x - \mathcal{I}y\|_{\beta; [0, T]} \leq (T^\beta + 1) \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} \|x - y\|_{\beta; [0, T]}.$$

*Proof.* Let  $x, y \in C^\beta([0, T]; \mathbb{R}^e)$  and put  $\phi := \mathcal{I}x - \mathcal{I}y$ . By using (F1) and the definition of  $\mathcal{I}$ , we have

$$\|\phi(t)\| = \left\| \int_0^t (f(s, x(s)) - f(s, y(s))) ds \right\| \leq \|x - y\|_{\beta; [0, T]} \int_0^t \ell_f(s) ds \leq T^\beta \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} \|x - y\|_{\beta; [0, T]}.$$

On the other hand, for  $0 \leq \lambda < t \leq T$ ,

$$\|\phi(t) - \phi(\lambda)\| = \left\| \int_\lambda^t (f(s, x(s)) - f(s, y(s))) ds \right\| \leq \|x - y\|_{\beta; [0, T]} \int_\lambda^t \ell_f(s) ds.$$

This implies

$$\max_{0 \leq \lambda < t \leq T} \frac{\|\phi(t) - \phi(\lambda)\|}{(t - \lambda)^\beta} \leq \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} \|x - y\|_{\beta; [0, T]}.$$

Hence we derive the second estimate

$$\|\phi\|_{\beta; [0, T]} \leq (T^\beta + 1) \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} \|x - y\|_{\beta; [0, T]}.$$

In order to prove the first estimate, we note that

$$\|\mathcal{I}x\|_{\beta; [0, T]} \leq \|\mathcal{I}x - \mathcal{I}0\|_{\beta; [0, T]} + \|\mathcal{I}x - \mathcal{I}0\|_{\beta; [0, T]}.$$

By the same arguments, it derives

$$\|\mathcal{I}x\|_{\beta; [0, T]} \leq (T^\beta + 1) \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} \|x\|_{\beta; [0, T]} + (T^\beta + 1) \|f(\cdot, 0)\|_{L^{1/(1-\beta)}(0, T)}.$$

The proof is completed. ■

**Theorem 3.4** Let (F1), (F2) and (G1)-(G3) hold true. Moreover, assume that

$$(T^\beta + 1) \|\ell_f\|_{L^{1/(1-\beta)}(0, T)} + E_{\gamma, \beta} M < 1,$$

where  $E_{\gamma, \beta}$  is the embedding constant from  $C^{1-\gamma}([0, T]; \mathbb{R}^e)$  into  $C^\beta([0, T]; \mathbb{R}^e)$  and  $M$  the constant of (3.3).

Then, the stochastic integral equation

$$x(t) = x(0) + \int_0^t f(s, x(s)) ds + \int_0^t g(s, x(s)) dB^H(s), \quad t \in [0, T], \quad (3.5)$$

has a solution  $x \in C^\beta([0, T]; \mathbb{R}^e)$ .

*Proof.* It is clear that the stochastic process  $x(t, \cdot)$  is a solution of (3.4) if and only if it is a fixed point of the operator

$$\mathcal{A} := \mathcal{I}_0 + \mathcal{S}, \quad \text{with} \quad \mathcal{I}_0 x(t) = x_0 + \mathcal{I}x(t), \quad 0 \leq t \leq T.$$

Let  $U \subset C^\beta([0, T]; \mathbb{R}^e)$  be the ball with radius  $R$  centered at zero function. It is not difficult to see that

$$\mathcal{A}x = \mathcal{I}_0 x + \mathcal{S}x \in U, \quad \text{for each } x, y \in U.$$

On the other hand, it follows from Lemma 3.2 and Lemma 3.3 that  $\mathcal{S}$  is a continuous and compact operator while  $\mathcal{I}_0$  is a contraction operator. So, the Krasnosel'skii fixed point theorem shows the existence of a fixed point of  $\mathcal{A}$  which is a solution to (3.4).  $\blacksquare$

The next theorem presents a new result for the Skorohod problem driven by maximal monotone operator and the sweeping process in the line of Castaing, Monteiro Marques and Raynaud de Fitte [10], Castaing, Marie and Raynaud de Fitte [11], Maticiuc, Rascanu, Slominski and Topolewski [17], Castaing and Thibault [12].

**Theorem 3.5** *Let  $I = [0, T]$  and let  $E = \mathbb{R}^e$ . Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbb{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let  $A_t : D(A_t) \rightrightarrows ck(E)$  be a time dependent maximal monotone operator satisfying  $(\mathcal{H}_0^A)$  and  $(\mathcal{H}_1^A)$ .*

*Let us consider a class of continuous mapping  $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$  satisfying the conditions  $(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ .*

*Let also  $g : I \times I \times \mathbb{R}^e \rightarrow \mathbb{R}^e$  be a continuous mapping for which there is a real constant  $M_g > 0$  such that*

(c)  $\|g(t, s, x) - g(\tau, s, x)\|_{\mathbb{R}^e} \leq M_g |t - \tau|$  for all  $(t, \tau, s, x) \in I \times I \times I \times \mathbb{R}^e$

(d)  $\|g(t, s, x)\|_{\mathbb{R}^e} \leq M_g$  for all  $(t, s, x) \in I \times I \times \mathbb{R}^e$ ;

(e)  $\|g(t, s, x) - g(t, s, y)\|_{\mathbb{R}^e} \leq M_g \|x - y\|_{\mathbb{R}^e}$  for all  $(t, s) \in I \times I$ ,  $x, y \in \mathbb{R}^e$ .

*Given  $a \in D(A_0)$  there exist  $C^\beta([0, T]; \mathbb{R}^e)$  continuous mappings  $x : I \rightarrow \mathbb{R}^e$ ,  $h : I \rightarrow \mathbb{R}^e$ ,  $k : I \rightarrow \mathbb{R}^e$ ,  $u : I \rightarrow \mathbb{R}^e$  satisfying*

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \quad \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H, \quad \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s)) ds, \quad \forall t \in I \\ u(t) \in D(A_t), \quad \forall t \in I \\ {}^c D^\beta u(t) \in A_t u(t) + k(t), \quad \text{a.e. } t \in I, \end{cases}$$

*where  $h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H$  is the stochastic integral and where the mapping  $k$  with  $k(t) = \int_0^t g(t, s, x(s)) ds$  is the Volterra type perturbation; further, for  $M := \max\{M_b, M_g\}$  the  $C^\beta([0, T]; \mathbb{R}^e)$  mapping  $u$  satisfies*

$$\begin{cases} \|{}^c D^\beta u(t)\| \leq M, \quad \text{a.e. } t \in I \\ u(t) \in a + I^\beta(L_{\mathbb{R}^e}^\infty(I)) \quad t \in I \end{cases}$$

*Proof.* Denote  $M := \max\{M_b, M_g\}$ . For each  $t \in I$  set

$$x^0(t) = a, \quad h^1(t) = \int_0^t b(\tau, a) dB_\tau^H$$

and we see e.g Proposition 3.2 [18] that

$$\|h^1(t) - h^1(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M+MN)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right] (t-s)^{1-\gamma} \quad (3.6)$$

for all  $0 \leq s \leq t \leq 1$  with  $1-H < \gamma < \min\{\beta, 1-\beta\}$ , see e.g the property of the stochastic integral  $h(t) = \int_0^t b(\tau, a) dB^H_\tau$  given in the proof of Theorem 2.2. See also Lemma 3.2. So  $h^1 \in C^\beta([0, T]; \mathbf{R}^e)$  since  $C^{1-\gamma}([0, T]; \mathbf{R}^e) \subset C^\beta([0, T]; \mathbf{R}^e)$ . For each  $t \in I$  set

$$k^1(t) = \int_0^t g(t, s, x^0(s)) ds,$$

so  $k^1$  is continuous on  $I$ . Using conditions (c) and (d) we also have the estimates

$$\|k^1(t)\| \leq M \quad \forall t \in I, \quad \text{and} \quad \|k^1(t) - k^1(\tau)\| \leq M|t - \tau| \quad \forall \tau, t \in I.$$

$$\frac{\|k^1(t) - k^1(\tau)\|}{(t-\tau)^\beta} \leq MT^{1-\beta} := L \quad \forall \tau, t \in I.$$

So  $k^1$  is  $C^\beta([0, T]; \mathbf{R}^e)$  continuous.

By Theorem 3.4 [9] there is a  $\beta$ -Holder continuous mapping  $u^1 : I \rightarrow \mathbf{R}^e$  solution of the differential inclusion

$$\begin{cases} u^1(0) = a, \quad u^1(t) \in D(A_t), \quad \forall t \in I; \\ -{}^c D^\beta u^1(t) \in A_t u^1(t) + k^1(t), \quad \text{a.e. } t \in I \end{cases}$$

such that  $\|{}^c D^\beta u^1(t)\| \leq \kappa$  and  $u^1(t) \in a + I^\beta(L^\infty_{\mathbf{R}^e}(I))$  for all  $t \in I$ . Put for each  $t \in I$

$$x^1(t) = h^1(t) + k^1(t) + u^1(t) = \int_0^t b(\tau, a) dB^H_\tau + \int_0^t g(t, s, x^0(s)) ds + u^1(t).$$

Then  $x^1(\cdot) \in C^\beta([0, T]; \mathbf{R}^e)$  with  $x^1(0) = a$ . Continuing in this way we are going to construct by induction sequences of mappings  $(h^n(\cdot)), (k^n(\cdot)), (u^n(\cdot))$  and  $(x^n(\cdot))$  such that

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB^H_\tau$$

$$\|h^n(t) - h^n(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M+MN)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right] (t-s)^{1-\gamma}$$

for all  $0 \leq s \leq t \leq T$ .

$$\|k^n(t)\| \leq M \quad \text{and} \quad \|k^n(t) - k^n(\tau)\| \leq M|t - \tau|,$$

for all  $0 \leq s \leq t \leq T$ .

$$\frac{\|k^n(t) - k^n(\tau)\|}{(t-\tau)^\beta} \leq M|t - \tau|^{1-\beta} \leq MT^{1-\beta} := L \quad \forall \tau, t \in I.$$

So  $k^n \in C^\beta([0, T]; \mathbf{R}^e)$ .

$u^n$  is a  $\beta$ -Holder continuous solution of the differential inclusion

$$\begin{cases} u^n(0) = a, u^n(t) \in D(A_t), \forall t \in I; \\ -{}^c D^\beta u^n(t) \in A_t u^n(t) + k^n(t), \text{ a.e. } t \in I \end{cases}$$

such that  $\|{}^c D^\beta u^n(t)\| \leq M$  and  $u^n(t) \in a + I^\beta(L_{\mathbf{R}^e}^\infty(I))$  for all  $t \in I$  and:

$$x^n(t) = h^n(t) + k^n(t) + u^n(t), t \in I$$

We already defined  $x^0(\cdot) = a$  and we already constructed  $h^1(\cdot)$ ,  $k^1(\cdot)$ ,  $u^1(\cdot)$  and  $x^1(\cdot)$  fulfilling the required properties. Suppose that the desired mappings are constructed for  $j = 2, \dots, n-1$  with the required properties. For each  $t \in I$

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB_\tau^H$$

$$k^n(t) = \int_0^t g(t, s, x^{n-1}(s)) ds.$$

Clearly, for all  $t, \tau \in I$

$$\|k^n(t)\| \leq M \quad \text{and} \quad \|k^n(t) - k^n(\tau)\| \leq M|t - \tau|.$$

$$\frac{\|k^n(t) - k^n(\tau)\|}{(t - \tau)^\beta} \leq M|t - \tau|^{1-\beta} \leq MT^{1-\beta} := L \quad \forall \tau, t \in I.$$

So  $k^n \in C^\beta([0, T]; \mathbf{R}^e)$ .

By Theorem 3.4 [9] again there is  $\beta$ -Holder continuous solution  $u^n : I \rightarrow \mathbf{R}^e$  of the differential inclusion

$$\begin{cases} u^n(0) = a, u^n(t) \in D(A_t), \forall t \in I; \\ -{}^c D^\beta u^n(t) \in A_t u^n(t) + k^n(t), \text{ a.e. } t \in I \end{cases}$$

such that  $\|{}^c D^\beta u^n(t)\| \leq M$  and  $u^n(t) \in a + I^\beta(L_{\mathbf{R}^e}^\infty(I))$  for all  $t \in I$ . Put for each  $t \in I$

$$x^n(t) = h^n(t) + k^n(t) + u^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB_\tau^H + k^n(t) + u^n(t)$$

Then  $x^n(\cdot) \in C^\beta([0, T]; \mathbf{R}^e)$ ,  $u^n(t) \in a + I^\beta(L_{\mathbf{R}^e}^\infty(I))$  for all  $t \in I$ . The induction is then justified.

Our next step is a limit process. Since  $\|{}^c D^\beta u^n(t)\| \leq M$  for  $t \in I$ , we may ensure  $D^\beta u^n$  weakly converge to  $v \in L_{\mathbf{R}^e}^1(I)$  with  $\|v(t)\| \leq \kappa$  for all  $t \in I$ . It ensures that

$$\lim_n u^n(t) = a + \lim_n \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} D^\beta u^n(s) ds = a + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} v(s) ds := u(t)$$

Hence  $u^n$  pointwise converge to  $u \in C^\beta([0, T]; \mathbf{R}^e)$  and  ${}^c D^\beta u^n$  weakly converge in  $L_{\mathbf{R}^e}^1(I)$  to  ${}^c D^\beta u$ . As  $u^n(t) \in D(A_t)$  we deduce that  $u(t) \in D(A_t)$ . By the inequalities  $\|k^n(t) - k^n(s)\| \leq$

$L|t - s|^\beta$  and  $\|k_n(t)\| \leq M$  the Arzelà-Ascoli theorem allows us to suppose also that the sequence  $(k^n(\cdot))_n$  converges uniformly to a  $C^\beta([0, T]; \mathbf{R}^e)$  continuous mapping  $k$  satisfying  $\|k(t) - k(s)\| \leq L|t - s|^\beta$ . Keeping in mind that  $u^n$  is solution of the differential inclusion

$$\begin{cases} u^n(0) = a, u^n(t) \in D(A_t), \forall t \in I; \\ -{}^c D^\beta u^n(t) \in A_t u^n(t) + k^n(t), \text{ a.e. } t \in I, \end{cases}$$

the convergence of  $(u^n)_n$  and  $(k^n)_n$  on  $I$  to  $u$  and  $k$  respectively entails that  $u$  is solution of the differential inclusion

$$\begin{cases} u(0) = a, u(t) \in D(A_t) \forall t \in I \\ -{}^c D^\beta u(t) \in A_t u(t) + k(t). \end{cases}$$

Since  $(v^n(\cdot)) = ({}^c D^\beta u^n(\cdot))$  weakly converges in  $L^1_{\mathbf{R}^e}([0, T])$  to  $v(\cdot) = {}^c D^\beta u(\cdot)$ , one ensures that  $(v^n(\cdot) + k^n(\cdot))$  Komlós-converges to  $v(\cdot) + k(\cdot)$ . Let  $S$  be a negligible set such that for  $t \in [0, T] \setminus S$ ,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k (v^j(t) + k^j(t)) = v(t) + k(t), \quad (3.7)$$

and

$$-v^n(t) \in A_t u^n(t) + k^n(t), \forall n \in \mathbf{N}.$$

Let  $z \in D(A_t)$ . By monotonicity of  $A_t$  one has

$$\langle v^n(t) + k^n(t), u^n(t) - z \rangle \leq \langle A_t^0 z, z - u^n(t) \rangle. \quad (3.8)$$

Writting

$$\langle v^n(t) + k^n(t), u(t) - z \rangle = \langle v^n(t) + k^n(t), u^n(t) - z \rangle + \langle v^n(t) + k^n(t), u(t) - u^n(t) \rangle$$

yields

$$\frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u(t) - z \rangle = \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u^j(t) - z \rangle + \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u(t) - u^j(t) \rangle$$

so that

$$\frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u(t) - z \rangle \leq \frac{1}{n} \sum_{j=1}^n \langle A_t^0 z, z - u^j(t) \rangle + (\kappa + M) \frac{1}{n} \sum_{j=1}^n \|u(t) - u^j(t)\|$$

Passing to the limit as  $n \rightarrow \infty$  in the last inequalities give

$$\langle v(t) + k(t), u(t) - \zeta \rangle \leq \langle A_t^0 z, z - u(t) \rangle$$

Consequently  $u$  is a solution of the aforementioned differential inclusion,

$$\begin{cases} -{}^c D^\beta u(t) \in A_t u(t) + k(t) \\ u(0) = a \in D(A_0) \end{cases}$$

On the other hand, recall that

$$\|h^n(t) - h^n(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M+MN)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right] (t-s)^{1-\gamma}$$

for all  $s \leq t \leq T$  with  $1-H < \gamma < \min\{\beta, 1-\beta\}$ , so that by Ascoli theorem  $h^n$  converges uniformly to  $h$  with

$$\|h(t) - h(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M+MN)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right] (t-s)^{1-\gamma}$$

for all  $s \leq t \leq T$  with  $1-H < \gamma < \min\{\beta, 1-\beta\}$ . So  $h \in C^\beta([0, T]; \mathbf{R}^e)$  since  $C^{1-\gamma}([0, T]; \mathbf{R}^e) \subset C^\beta([0, T]; \mathbf{R}^e)$ . As consequence  $x^n(\cdot) = h^n(\cdot) + k^n(\cdot) + u^n(\cdot)$  pointwise converges on  $I$  to  $x(\cdot) := h(\cdot) + k(\cdot) + u(\cdot)$ . It is already seen that  $h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB_\tau^H$  uniformly converges to  $\int_0^t b(\tau, x(\tau)) dB_\tau^H$  so that  $h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H$ . For each  $t \in I$  by condition (c) the sequence  $(g(\cdot, \cdot, x^{n-1}(\cdot)))$  converges pointwise to  $g(\cdot, \cdot, x(\cdot))$ , hence by the Lebesgue dominated convergence theorem

$$\int_0^t g(t, s, x^{n-1}(s)) ds \rightarrow \int_0^t g(t, s, x(s)) ds \quad \text{as } n \rightarrow \infty,$$

which means that  $k(t) = \int_0^t g(t, s, x(s)) ds$  for all  $t \in I$ . Further, by identifying the limits we can write for each  $t \in I$

$$\begin{aligned} x(t) &= \lim_{n \rightarrow \infty} x^n(t) = \lim_{n \rightarrow \infty} h^n(t) + \lim_{n \rightarrow \infty} \int_0^t g(t, s, x^{n-1}(s)) ds + \lim_{n \rightarrow \infty} u^n(t) \\ &= \int_0^t b(\tau, x(\tau)) dB_\tau^H + \int_0^t g(t, s, x(s)) ds + u(t) \end{aligned}$$

with

$$\begin{cases} -^c D^\beta u(t) \in A_t u(t) + k(t) \\ u(0) = a \in D(A_0) \end{cases}$$

All the desired properties of the theorem are then established, so the proof is finished.

We now give a variant of this theorem in which the properties of the operator  $A_t$  are modified. The Lipschitz condition  $(\mathcal{H}_0^A)$  relating to the domain of  $A_t$  of the following type is removed:

$$\text{dis}(A_t, A_s) \leq |\beta(t) - \beta(s)|, \forall (t, s) \in [0, T] \times [0, T]$$

where  $\beta \in W^{1,2}([0, T])$  is nonnegative, nondecreasing with  $\beta(T) < \infty$ ,  $\beta(0) = 0$  and the measurability condition  $(\mathcal{A}_0)$  on  $A_t$  is imposed.

**Theorem 3.6** *Let  $I = [0, T]$  and let  $E = \mathbf{R}^e$ . Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1-H < \beta < \frac{1}{2}$ . Let  $A_t : E \rightrightarrows ck(E)$  be a time dependent maximal monotone operator satisfying  $(\mathcal{A}_0)$  and  $(\mathcal{H}_1^A)$ .*

*Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the weakened conditions  $(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ .*

Let also  $g : I \times I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a continuous mapping for which there is a real constant  $M_g > 0$  such that

(c)  $\|g(t, s, x) - g(\tau, s, x)\|_{\mathbf{R}^e} \leq M_g |t - \tau|$  for all  $(t, \tau, s, x) \in I \times I \times I \times \mathbf{R}^e$

(d)  $\|g(t, s, x)\|_{\mathbf{R}^e} \leq M_g$  for all  $(t, s, x) \in I \times I \times \mathbf{R}^e$ ,

(e)  $\|g(t, s, x) - g(t, s, y)\|_{\mathbf{R}^e} \leq M_g \|x - y\|_{\mathbf{R}^e}$  for all  $(t, s) \in I \times I$ ,  $x, y \in \mathbf{R}^e$ .

Given  $a \in \mathbf{R}^e$  there exist  $C^\beta([0, T]; \mathbf{R}^e)$  continuous mappings  $x : I \rightarrow \mathbf{R}^e$ ,  $h : I \rightarrow \mathbf{R}^e$ ,  $k : I \rightarrow \mathbf{R}^e$ ,  $u : I \rightarrow \mathbf{R}^e$ , satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H, \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s)) ds, \forall t \in I \\ u(t) \in D(A_t), \forall t \in I \\ -{}^c D^\beta u(t) \in A_t u(t) + k(t), \text{ a.e. } t \in I, \end{cases}$$

where  $h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H$  is the stochastic integral and where the mapping  $k$  with  $k(t) = \int_0^t g(t, s, x(s)) ds$  is the Volterra type perturbation; further, the  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $u$  satisfies

$$\begin{cases} \|{}^c D^\beta u(t)\| \leq \kappa, \text{ a.e. } t \in I \\ u(t) \in a + I^\beta(L^\infty_{\mathbf{R}^e}(I)), t \in I \end{cases}$$

Proof. The proof is similar to that of Theorem 3.5 using Theorem 3.2 in [9] instead of Theorem 3.4 in [9].

**Corollary 3.1** Let  $I = [0, T]$  and let  $E = \mathbf{R}^e$ . Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let  $C : I \rightrightarrows ck(E)$  be a convex compact valued mapping satisfying  $\text{Haus}(C(t), C(s)) \leq L|t - s|^\beta \forall t, s \in [0, T]$

Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ .

Let also  $g : I \times I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a continuous mapping for which there is a real constant  $M_g > 0$  such that

(c)  $\|g(t, s, x) - g(\tau, s, x)\|_{\mathbf{R}^e} \leq M_g |t - \tau|$  for all  $(t, \tau, s, x) \in I \times I \times I \times \mathbf{R}^e$

(d)  $\|g(t, s, x)\|_{\mathbf{R}^e} \leq M_g$  for all  $(t, s, x) \in I \times I \times \mathbf{R}^e$ ,

(e)  $\|g(t, s, x) - g(t, s, y)\|_{\mathbf{R}^e} \leq M_g \|x - y\|_{\mathbf{R}^e}$  for all  $(t, s) \in I \times I$ ,  $x, y \in \mathbf{R}^e$ .

Given  $a \in C(0)$  there exist  $C^\beta([0, T]; \mathbf{R}^e)$  continuous mappings  $x : I \rightarrow \mathbf{R}^e$ ,  $h : I \rightarrow \mathbf{R}^e$ ,  $k : I \rightarrow \mathbf{R}^e$ ,  $u : I \rightarrow \mathbf{R}^e$ , satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H, \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s)) ds, \forall t \in I \\ u(t) \in C(t), \forall t \in I \\ -{}^c D^\beta u(t) \in N_{C(t)} u(t) + k(t), \text{ a.e. } t \in I, \end{cases}$$

where  $h(t) = \int_0^t b(\tau, x(\tau)) dB^H \tau$  is the stochastic integral and where the mapping  $k$  with  $k(t) = \int_0^t g(t, s, x(s)) ds$  is the Volterra type perturbation; further, for  $M := \max\{M_b, M_g\}$  the  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $u$  satisfies

$$\begin{cases} \|{}^c D^\beta u(t)\| \leq \kappa, \text{ a.e. } t \in I \\ u(t) \in a + I^\beta(L^\infty_{\mathbf{R}^e}(I)), t \in I \end{cases}$$

A remarkable by product of Theorem 3.5 is available

**Theorem 3.7** *Let  $I = [0, T]$  and let  $E = \mathbf{R}^e$ . Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ . Let also  $g : I \times I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a continuous mapping for which there is a real constant  $M_g > 0$  such that*

(c)  $\|g(t, s, x) - g(\tau, s, x)\|_{\mathbf{R}^e} \leq M_g |t - \tau|$  for all  $(t, \tau, s, x) \in I \times I \times I \times \mathbf{R}^e$

(d)  $\|g(t, s, x)\|_{\mathbf{R}^e} \leq M_g$  for all  $(t, s, x) \in I \times I \times \mathbf{R}^e$ ,

(e)  $\|g(t, s, x) - g(t, s, y)\|_{\mathbf{R}^e} \leq M_g \|x - y\|_{\mathbf{R}^e}$  for all  $(t, s) \in I \times I, x, y \in \mathbf{R}^e$ .

Given  $a \in \mathbf{R}^e$  there exist  $C^\beta([0, T]; \mathbf{R}^e)$  continuous mappings  $x : I \rightarrow \mathbf{R}^e, h : I \rightarrow \mathbf{R}^e, k : I \rightarrow \mathbf{R}^e, u : I \rightarrow \mathbf{R}^e$ , satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H, \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s)) ds, \forall t \in I \\ {}^c D^\beta u(t) = k(t), \text{ a.e. } t \in I, \end{cases}$$

where  $h(t) = \int_0^t b(\tau, x(\tau)) dB^H \tau$  is the stochastic integral and where the mapping  $k$  with  $k(t) = \int_0^t g(t, s, x(s)) ds$  is the Volterra type perturbation; further, the  $C^\beta([0, T]; \mathbf{R}^e)$  mapping  $u$  satisfies

$$\begin{cases} \|{}^c D^\beta u(t)\| \leq \kappa, \text{ a.e. } t \in I \\ u(t) = a + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} [\int_0^s g(s, \tau, x(\tau)) d\tau] ds, t \in I \end{cases}$$

and the  $C^\beta([0, T]; \mathbf{R}^e)$  continuous mapping  $x : I \rightarrow \mathbf{R}^e$  satisfies

$$x(t) = a + \int_0^t b(\tau, x(\tau)) dB_\tau^H + \int_0^t g(t, s, x(s)) ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} [\int_0^s g(s, \tau, x(\tau)) d\tau] ds, t \in [0, T].$$

The following deal a Skorohod problem in time and space dependent maximal monotone operator  $A_{(t,x)}$  with fractional Brownian motion.

**Theorem 3.8** *Let  $I = [0, T]$  and let  $E = \mathbf{R}^e$ . Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1 - H < \beta < \frac{1}{2}$ . Let  $A_{(t,x)} : D(A_{(t,x)}) \rightrightarrows ck(E)$  be a time and space dependent maximal monotone operator satisfying*

$(A_1)$ : there exists a real constant  $L > 0$  such that

$$dis(A_{(t,x)}, A_{(s,y)}) \leq L(|t - s|^\beta + \|x - y\|), \forall t, s \in [0, T], \forall x, y \in \mathbf{R}^e \times \mathbf{R}^e.$$

$(A_2)$ : there exists a real constant  $c > 0$  such that  $|A_{(t,x)}^0 y| \leq c(1 + \|x\| + \|y\|)$  for all

$(t, x, y) \in [0, T] \times \mathbf{R}^e \times D(A_{(t,x)})$ .

Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions  $(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ .

Let also  $g : I \times I \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  be a continuous mapping for which there is a real constant  $M_g > 0$  such that

(c)  $\|g(t, s, x) - g(\tau, s, x)\|_{\mathbf{R}^e} \leq M_g |t - \tau|$  for all  $(t, \tau, s, x) \in I \times I \times I \times \mathbf{R}^e$

(d)  $\|g(t, s, x)\|_{\mathbf{R}^e} \leq M_g$  for all  $(t, s, x) \in I \times I \times \mathbf{R}^e$ ;

(e)  $\|g(t, s, x) - g(t, s, y)\|_{\mathbf{R}^e} \leq M_g \|x - y\|_{\mathbf{R}^e}$  for all  $(t, s) \in I \times I$ ,  $x, y \in \mathbf{R}^e$ .

Given  $a \in D(A_{(0,0)})$  there exist  $C^\alpha([0, T]; \mathbf{R}^e)$  continuous mappings  $x : I \rightarrow \mathbf{R}^e$ ,  $h : I \rightarrow \mathbf{R}^e$ ,  $k : I \rightarrow \mathbf{R}^e$ ,  $u : I \rightarrow \mathbf{R}^e$  with  $1 - H < \beta < \alpha < H < 1$  satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H, \forall t \in I \\ k(t) = \int_0^t g(t, s, x(s)) ds, \forall t \in I \\ u(t) \in D(A_{(t, h(t))}), \forall t \in I \\ {}^c D^\alpha u(t) \in A_{(t, h(t))} u(t) + k(t), \text{ a.e. } t \in I, \end{cases}$$

where  $h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H$  is the stochastic integral and where the mapping  $k$  with  $k(t) = \int_0^t g(t, s, x(s)) ds$  is the Volterra type perturbation; further, the  $C^\alpha([0, T]; \mathbf{R}^e)$  mapping  $u$  satisfies

$$\begin{cases} \|{}^c D^\alpha u(t)\| \leq \kappa, \text{ a.e. } t \in I \\ u(t) \in a + I^\alpha(L_{\mathbf{R}^e}^\infty(I)) t \in I \end{cases}$$

Proof. Denote  $M := \max\{M_b, M_g\}$ . For each  $t \in I$  set

$$x^0(t) = a, \quad h^1(t) = \int_0^t b(\tau, a) dB_\tau^H$$

and, as already done, we have

$$\|h^1(t) - h^1(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^{1 - \gamma} \quad (3.9)$$

for all  $0 \leq s \leq t \leq 1$  with  $1 - H < \gamma < \min\{\beta, 1 - \beta\}$ . So  $h^1 \in C^{1 - \gamma}([0, T]; \mathbf{R}^e)$  keeping in mind that  $1 - H < \beta < 1 - \gamma < H < 1$ . Furthermore,

$$\begin{aligned} \text{dis}(A_{(t, h^1(t))}, A_{(s, h^1(s))}) &\leq L|t - s|^\beta + LK(\gamma, [0, T]) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^{1 - \gamma} \\ &\leq (L + LK(\gamma, [0, T]) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right]) (t - s)^{1 - \gamma} \end{aligned}$$

For each  $t \in I$  set

$$k^1(t) = \int_0^t g(t, s, x^0(s)) ds,$$

so  $k^1$  is continuous on  $I$ . Using conditions (c) and (d) we also have the estimates

$$\|k^1(t)\| \leq M \quad \forall t \in I, \quad \text{and} \quad \|k^1(t) - k^1(\tau)\| \leq M|t - \tau| \quad \forall \tau, t \in I.$$

$$\frac{\|k^1(t) - k^1(\tau)\|}{(t - \tau)^{1-\gamma}} \leq \frac{\|k^1(t) - k^1(\tau)\|}{(t - \tau)^\beta} \leq MT^{1-\beta} \quad \forall \tau, t \in I.$$

So  $k^1$  is  $C^{1-\gamma}([0, T]; \mathbf{R}^e)$  continuous, taking a account into  $1 - H < \beta < 1 - \gamma < H < 1$ , by Theorem 3.4 [9] there is a  $(1 - \gamma)$ -Holder continuous mapping  $u^1 : I \rightarrow \mathbf{R}^e$  solution of the differential inclusion

$$\begin{cases} u^1(0) = a \in D(A_{(0,0)}), \quad u^1(t) \in D(A_{(t,h^1(t))}), \quad \forall t \in I; \\ -{}^c D^{1-\gamma} u^1(t) \in A_{(t,h^1(t))} u^1(t) + k^1(t), \quad \text{a.e. } t \in I \end{cases}$$

such that  $\|{}^c D^{1-\gamma} u^1(t)\| \leq \kappa$  and  $u^1(t) \in a + I^{1-\gamma}(L_{\mathbf{R}^e}^\infty(I))$  for all  $t \in I$ . Put for each  $t \in I$

$$x^1(t) = h^1(t) + k^1(t) + u^1(t) = \int_0^t b(\tau, a) dB_\tau^H + \int_0^t g(t, s, x^0(s)) ds + u^1(t).$$

Then  $x^1(\cdot) \in C^{1-\gamma}([0, T]; \mathbf{R}^e)$  with  $x^1(0) = a$ . Continuing in this way we are going to construct by induction sequences of  $C^{1-\gamma}([0, T]; \mathbf{R}^e)$  mappings  $(h^n(\cdot)), (k^n(\cdot)), (u^n(\cdot))$  and  $(x^n(\cdot))$  such that

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB_\tau^H$$

$$\|h^n(t) - h^n(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^{1-\gamma}$$

for all  $0 \leq s \leq t \leq T$ . So  $h^n \in C^{1-\gamma}([0, T]; \mathbf{R}^e)$  keeping in mind that  $1 - H < \beta < 1 - \gamma < H < 1$ . Furthermore,

$$\begin{aligned} \text{dis}(A_{(t,h^n(t))}, A_{(s,h^n(s))}) &\leq L|t - s|^\beta + LK(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^{1-\gamma} \\ &\leq (L + K(\gamma, [0, T])) \left[ \frac{M}{1-\gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^{1-\gamma} \end{aligned}$$

$$\|k^n(t)\| \leq M \quad \text{and} \quad \|k^n(t) - k^n(\tau)\| \leq M|t - \tau|,$$

for all  $0 \leq s \leq t \leq T$ .

$$\frac{\|k^n(t) - k^n(\tau)\|}{(t - \tau)^{1-\gamma}} \leq \frac{\|k^n(t) - k^n(\tau)\|}{(t - \tau)^\beta} \leq MT^{1-\beta} \quad \forall \tau, t \in I.$$

So  $k^n$  is  $C^{1-\gamma}([0, T]; \mathbf{R}^e)$  continuous, taking a account into  $1 - H < \beta < 1 - \gamma < H < 1$ , by Theorem 3.4 [9] there is a  $(1 - \gamma)$ -Holder continuous mapping  $u^n : I \rightarrow \mathbf{R}^e$  solution of the differential inclusion

$$\begin{cases} u^n(0) = a \in D(A_{(0,0)}), \quad u^n(t) \in D(A_{(t,h^n(t))}), \quad \forall t \in I; \\ -{}^c D^{1-\gamma} u^n(t) \in A_{(t,h^n(t))} u^n(t) + k^n(t), \quad \text{a.e. } t \in I \end{cases}$$

such that  $\|{}^c D^{1-\gamma} u^n(t)\| \leq \kappa$  and  $u^n(t) \in a + I^{1-\gamma}(L_{\mathbf{R}^e}^\infty(I))$  for all  $t \in I$ . Put for each  $t \in I$

$$x^n(t) = h^n(t) + k^n(t) + u^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB_\tau^H + k^n(t) + u^n(t)$$

Then  $x^n(\cdot) \in C^{1-\gamma}([0, T]; \mathbf{R}^e)$ ,  $u^n(t) \in a + I^{1-\gamma}(L_{\mathbf{R}^e}^\infty(I))$  for all  $t \in I$ . The induction is then justified.

Our next step is a limit process. For simplicity set  $\alpha = 1 - \gamma$  and recall

$$1 - H < \beta < \alpha = 1 - \gamma < H < 1$$

$$\begin{aligned} \text{dis}(A_{(t, h^n(t))}, A_{(s, h^n(s))}) &\leq L|t - s|^\beta + LK(\gamma, [0, T]) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^\alpha \\ &\leq (L + LK(\gamma, [0, T])) \left[ \frac{M}{1 - \gamma} + \frac{(M + MN)T^\beta}{(\beta - \gamma)(\beta - \gamma + 1)} \right] (t - s)^\alpha \end{aligned}$$

Since  $\|{}^c D^\alpha u^n(t)\| \leq \kappa$  for  $t \in I$ , we may ensure  ${}^c D^\alpha u^n$  weakly converge to  $v \in L_{\mathbf{R}^e}^1(I)$  with  $\|v(t)\| \leq \kappa$  for all  $t \in I$ . It ensures that

$$\lim_n u^n(t) = a + \lim_n \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} D^\alpha u^n(s) ds = a + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds := u(t)$$

Hence  $u^n$  pointwise converge to  $u \in C^\alpha([0, T]; \mathbf{R}^e)$  and  ${}^c D^\alpha u^n$  weakly converge in  $L_{\mathbf{R}^e}^1(I)$  to  ${}^c D^\alpha u$ . Further, we note that  $u(t) \in D(A_{(t, h(t))})$  for all  $t \in I$ . Indeed we have

$$\text{dis}(A_{(t, h_n(t))}, A_{(t, h(t))}) \leq L|h_n(t) - h(t)| \rightarrow 0$$

By the inequalities  $\|k^n(t) - k^n(s)\| \leq MT^{1-\beta}|t - s|^\alpha$  and  $\|k_n(t)\| \leq M$  the Arzelà-Ascoli theorem allows us to suppose also that the sequence  $(k^n(\cdot))_n$  converges uniformly to a  $C^\alpha([0, T]; \mathbf{R}^e)$  continuous mapping  $k$  satisfying  $\|k(t) - k(s)\| \leq MT^{1-\beta}|t - s|^\alpha$ . Keeping in mind that  $u^n$  is solution of the differential inclusion

$$\begin{cases} u^n(0) = a \in D(A_{(0,0)}), u^n(t) \in D(A_{(t, h^n(t))}), \forall t \in I; \\ -{}^c D^\alpha u^n(t) \in A_{(t, h^n(t))} u^n(t) + k^n(t), \text{ a.e. } t \in I, \end{cases}$$

the convergence of  $(u^n)_n$  and  $(k^n)_n$  on  $I$  to  $u$  and  $k$  respectively entails that  $u$  is solution of the differential inclusion

$$\begin{cases} u(0) = a \in D(A_{(0,0)}), u(t) \in D(A_{(t, h(t))}) \forall t \in I \\ -{}^c D^\alpha u(t) \in A_{(t, h(t))} u(t) + k(t). \end{cases}$$

Since  $v^n(\cdot) = {}^c D^\alpha u^n(\cdot)$  weakly converges in  $L_{\mathbf{R}^e}^1([0, T])$  to  $v(\cdot) = {}^c D^\alpha u(\cdot)$ , one ensures that  $(v^n(\cdot) + k^n(\cdot))$  Komlós-converges to  $v(\cdot) + k(\cdot)$ . We will show that

$$-v(t) - k(t) \in A_{(t, h(t))} u(t), \text{ dt a.e}$$

Further, we note that  $u(t) \in D(A_{(t,h(t))})$  for all  $t \in I$ . Indeed we have

$$\text{dis}(A_{(t,h_n(t))}, A_{(t,h(t))}) \leq L \|h_n(t) - h(t)\| \rightarrow 0$$

It is clear that  $(y_n = A_{(t,h_n(t))}^0 u^n(t))$  is bounded, hence relatively compact. By applying Lemma 2.2 to  $u^n(t) \rightarrow u(t)$  and to a convergent subsequence of  $(y_n)$  to show that  $u(t) \in D(A_{(t,h(t))})$ . There is a  $dt$ -negligible set  $N$  such that

$$-v^n(t) - k^n(t) \in A_{(t,h_n(t))} u^n(t), \quad t \in I \setminus N,$$

$$\lim_n \frac{1}{n} \sum_{j=1}^n v^j(t) + k^j(t) = v(t) + k(t), \quad t \in I \setminus N.$$

Let  $t \in [0, 1] \setminus N$ . Let  $\eta \in D(A_{(t,h(t))})$ . Apply Lemma 2.3 to  $A_{(t,h_n(t))}$  and  $A_{(t,h(t))}$  to find a sequence  $(\eta_n)$  such that  $\eta_n \in D(A_{(t,h_n(t))})$ ,  $\eta_n \rightarrow \eta$ ,  $A_{(t,h_n(t))}^0 \eta_n \rightarrow A_{(t,h(t))}^0 \eta$ . From

$$-v^n(t) - k^n(t) \in A_{(t,h_n(t))} u_{h_n}(t)$$

by monotonicity

$$\langle v^n(t) + k^n(t), u^n(t) - \eta_n \rangle \leq \langle A_{(t,h_n(t))}^0 \eta_n, \eta_n - u(t) \rangle.$$

From

$$\langle v^n(t) + k^n(t), u(t) - \eta \rangle = \langle v^n(t) + k^n(t), u^n(t) - \eta_n \rangle + \langle v^n(t) + k^n(t), u(t) - u^n(t) - (\eta - \eta_n) \rangle,$$

let us write

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u(t) - \eta \rangle &= \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u^j(t) - \eta_j \rangle \\ &+ \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u(t) - u^j(t) \rangle + \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), \eta_j - \eta \rangle, \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \langle v^j(t) + k^j(t), u(t) - \eta \rangle &\leq \frac{1}{n} \sum_{j=1}^n \langle A_{(t,h_j(t))}^0 \eta_j, \eta_j - u^j(t) \rangle + (\kappa + M) \frac{1}{n} \sum_{j=1}^n \|u(t) - u^j(t)\| \\ &+ (\kappa + M) \frac{1}{n} \sum_{j=1}^n \|\eta_j - \eta\|. \end{aligned}$$

Passing to the limit using when  $n \rightarrow \infty$ , this last inequality gives immediately

$$\langle v(t) + k(t), u(t) - \eta \rangle \leq \langle A_{(t,h(t))}^0 \eta, \eta - u(t) \rangle$$

As a consequence, by Lemma 2.1 we get  $-v(t) \in A_{(t,h(t))}u(t) + k(t)$ ,  $dp$  a.e. with  $u(t) \in D(A_{(t,h(t))})$  for all  $t \in I$ . Consequently  $u$  is a solution of the aforementioned differential inclusion,

$$\begin{cases} -{}^c D^\alpha u(t) \in A_{(t,h(t))}u(t) + k(t) \\ u(0) = a \in D(A_{(0,0)}) \end{cases}$$

On the other hand, recall that

$$\|h^n(t) - h^n(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M+MN)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right] (t-s)^{1-\gamma}$$

for all  $s \leq t \leq T$  with  $1-H < \gamma < \min\{\beta, 1-\beta\}$ , so that by Ascoli theorem  $h^n$  converges uniformly to  $h$  with

$$\|h(t) - h(s)\| \leq K(\gamma, [0, T]) \left[ \frac{M}{1-\gamma} + \frac{(M+MN)T^\beta}{(\beta-\gamma)(\beta-\gamma+1)} \right] (t-s)^{1-\gamma}$$

for all  $s \leq t \leq T$  with  $1-H < \gamma < \min\{\beta, 1-\beta\}$ . So  $h \in C^\alpha([0, T]; \mathbf{R}^e)$ . As consequence  $x^n(\cdot) = h^n(\cdot) + k^n(\cdot) + u^n(\cdot)$  pointwise converges on  $I$  to  $x(\cdot) := h(\cdot) + k(\cdot) + u(\cdot)$ . It is already seen that  $h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau)) dB_\tau^H$  uniformly converges to  $\int_0^t b(\tau, x(\tau)) dB_\tau^H$  so that  $h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H$ . For each  $t \in I$  by condition (c) the sequence  $(g(\cdot, \cdot, x^{n-1}(\cdot)))$  converges pointwise to  $g(\cdot, \cdot, x(\cdot))$ , hence by the Lebesgue dominated convergence theorem

$$\int_0^t g(t, s, x^{n-1}(s)) ds \rightarrow \int_0^t g(t, s, x(s)) ds \quad \text{as } n \rightarrow \infty,$$

which means that  $k(t) = \int_0^t g(t, s, x(s)) ds$  for all  $t \in I$ . Further, by identifying the limits we can write for each  $t \in I$

$$\begin{aligned} x(t) &= \lim_{n \rightarrow \infty} x^n(t) = \lim_{n \rightarrow \infty} h^n(t) + \lim_{n \rightarrow \infty} \int_0^t g(t, s, x^{n-1}(s)) ds + \lim_{n \rightarrow \infty} u^n(t) \\ &= \int_0^t b(\tau, x(\tau)) dB_\tau^H + \int_0^t g(t, s, x(s)) ds + u(t) \end{aligned}$$

with

$$\begin{cases} -{}^c D^\alpha u(t) \in A_{(t,h(t))}u(t) + k(t) \\ u(0) = a \in D(A_{(0,0)}) \end{cases}$$

All the desired properties of the theorem are then established, so the proof is finished.

By repetition of the arguments of the proof of Theorem 3.8 we have a remarkable SKP problem with fractional sweeping process driven by fBM.

**Corollary 3.2** *Let  $I = [0, T]$  and let  $E = \mathbf{R}^e$ . Consider a fractional Brownian motion  $B^H = (B_t^H)_{t \in I}$  of Hurst index  $\frac{1}{2} < H < 1$  in  $\mathbf{R}^e$ . Let  $1-H < \beta < \frac{1}{2}$ . Let  $C : I \rightrightarrows ck(E)$  be a convex compact valued mapping satisfying*

$$\text{Haus}(C(t), C(s)) \leq L|t-s|^\beta \quad \forall t, s \in [0, T]$$

*Let us consider a class of continuous mapping  $b : [0, T] \times \mathbf{R}^e \rightarrow \mathbf{R}^e$  satisfying the conditions*

$(\mathcal{B}_1^b)$ ,  $(\mathcal{B}_2^b)$  and  $(\mathcal{B}_3^b)$ .

Given  $a \in C(0)$  there exist  $C^\alpha([0, T]; \mathbf{R}^e)$  with  $1 - H < \beta < \alpha < H < 1$  continuous mappings  $x : I \rightarrow \mathbf{R}^e$ ,  $h : I \rightarrow \mathbf{R}^e$ ,  $u : I \rightarrow \mathbf{R}^e$ , satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + u(t), \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H, \forall t \in I \\ u(t) \in C(t) - h(t), \forall t \in I \\ -{}^c D^\alpha u(t) \in N_{C(t)-h(t)} u(t) = N_{C(t)} x(t), \text{ a.e. } t \in I, \end{cases}$$

where  $h(t) = \int_0^t b(\tau, x(\tau)) dB_\tau^H$  is the stochastic integral ; further, the  $C^\alpha([0, T]; \mathbf{R}^e)$  mapping  $u$  satisfies

$$\begin{cases} \|{}^c D^\alpha u(t)\| \leq \kappa, \text{ a.e. } t \in I \\ u(t) \in a + I^\alpha(L_{\mathbf{R}^e}^\infty(I)), t \in I \end{cases}$$

The above results are new, in particular the last results dealing with Skorohod problem for a new class of **fractional** evolution inclusion driven by a  $fBm$  motion. Several variants are available by considering diverse class of fractional and operators. See [6, 8, 10, 11, 12, 15] and the references therein. We refer to [2, 3, 4, 5, 16, 18, 19, 21, 22] for related results concerning the differential inclusion driven by fractional Brownian motion.

## References

- [1] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff Int. Publ., Leyden, 1976.
- [2] A. Boudaoui, T. Caraballo and A. Ouahab, Impulsive stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay. Math. Methods Appl. Sci. 39(6) (2016), 1435–1451.
- [3] B. Boufoussi and S. Hajji, Functional differential equations driven by a fractional Brownian motion. Comput. Math. Appl. 62 (2011), 746–754.
- [4] B. Boufoussi, S. Hajji and E. H. Lakhel, Functional differential equations in Hilbert spaces driven by a fractional Brownian motion. Afr. Mat. 23 (2012), 173–194.
- [5] R. Buckdahn, M. Quincampoix, C. Rainer and A. Rascanu, Viability of moving sets for stochastic differential equation. Adv. Differ. Equ. 7(9) (2002), 1045–1072.
- [6] Castaing, C.; Godet-Thobie, C.; Monteiro Marques, M.D.P. BV Solutions to Evolution Inclusion with a Time- and Space-Dependent Maximal Monotone Operator. Mathematics 2024, 12, 896. <https://doi.org/10.3390/math12060896>.
- [7] Castaing, C., Godet-Thobie, C. Monteiro Marques, M.D.P. Salvadori, A. Evolution Problems with  $m$ -Accretive Operators and Perturbations. Mathematics 2022, 10, 317. <https://doi.org/10.3390/math10030317>.

- [8] C. Castaing, C. Godet-Thobie, M.D.P. Monteiro Marques, L. Thibault, Evolution inclusions with and without rough signals, *J. Convex. Anal.*, Vol 31(4) (2024), 1057-1138.
- [9] C. Castaing, C. Godet-Thobie, S. Saidi, The Fractional Evolution Inclusion Involving Time-Dependent Maximal Monotone Operators, *Siberian Mathematical Journal*, Vol 66(2) (2025), 423-485.
- [10] C. Castaing, M.D.P. Monteiro Marques, P. Raynaud de Fitte, A Skorohod problem governed by a closed convex moving set, *J. Convex Analysis*, Volume 23 (2016), No 2, 387-423.
- [11] C. Castaing, N. Marie, P. Raynaud de Fitte, Sweeping process perturbed by rough signal, in: C. Donati-Martin, A. Lejay, A. Rouault, *Séminaire de Probabilités LI. Lecture Notes in Mathematics*, Vol. 2301, Springer, Cham.
- [12] C. Castaing and L. Thibault, Various Perturbations and Relaxations of the Sweeping Process. *J. Convex Anal.* 30 (2) (2023), 659-742.
- [13] C. Castaing, P. Raynaud de Fitte, M. Valadier, *Young Measures on Topological Spaces with Applications in Control Theory and Probability Theory*. Kluwer Academic Publishers, Dordrecht (2004).
- [14] M. Kunze and M.D.P. Monteiro Marques, BV solutions to evolution problems with time-dependent domains, *Set-Valued Anal.*, 5 (1997), 57-72.
- [15] N. Marie, Multifunction estimation in a time-discretized Skorohod reflexion problem, arXiv: submit/ 6157203 [math.ST] 26 Jan 2025.
- [16] B. Maslowski and D. Nualart, Evolution equations driven by a fractional Brownian motion. *J. Funct. Anal.* 202(1) (2003), 277–305.
- [17] L.Maticiuc, A. Rascanu, L. Slominski and M.Topolewski, Cadlag Skorohod problem driven by Maximal monotone operator, *JMAA*, 429 (2015), 1305-1346.
- [18] NG Ngo Trong, LE Xuan Truong, TA Duc Do, Viability for impulsive stochastic differential inclusion driven by fractional Brownian motion, NG. Ngoc Trong, LX. Truong, TD. Co, *Stochastics and Dynamics*, World Scientific Publishing Company.
- [19] D. Nualart and A. Rascanu, Differential equations driven by fractional Brownian motion. *Collect. Math.* 53(1) (2002), 55–81.
- [20] A.A. Vladimirov, Nonstationary dissipative evolution equations in Hilbert space, *Nonlinear Anal.*, 17 (1991), 499-518.
- [21] L. Xu and J. Luo, Viability for stochastic functional differential equations in Hilbert spaces driven by fractional Brownian motion. *Appl. Math. Comput.* 341 (2019), 93– 110.
- [22] L. Xu, Viability for stochastic functional differential equations with infinite memory driven by a fractional Brownian motion. *Physica A* 527 (2019), 121076.