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# Sub-game order preservation and values for TU-games\*

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## Abstract

In cooperative games with transferable utility (TU-games), a player’s contribution may vary depending on the coalition they join, reflecting different levels of synergy with other members. In this paper, we introduce a family of axioms referred to as *sub-game order preservation axioms*, which formalize the intuition that a player’s payoff should increase with the degree of synergy they exhibit within a coalition. We propose four distinct axioms, each of which captures a different interpretation of what constitutes synergy in the context of transferable utility. We demonstrate that one of these variants is incompatible with the classical Efficiency axiom, thereby giving rise to an impossibility result. The remaining three axioms, when combined with Efficiency, lead to unique characterizations of three well-known solution concepts: the Shapley value, the Center of the Imputation Set (CIS), and the Equal Allocation of Non-Separable Contributions (ENSC) value, respectively.

**Keywords:** TU-games; Sub-game order preservation; Shapley value, CIS value, ENSC value  
**JEL classification:** C71

## 1 Introduction

Consider a population of players  $P$  faced with the choice of cooperating with one of two distinct groups. The outcome of their cooperation depends not only on which group they join but also on how well they synergize with them. In particular, weak collaboration, where players’ abilities do not complement each other, results in little to no benefit, while strong synergy, where players and group members work together effectively, can lead to higher returns for all parties involved. Importantly, players in  $P$  expect to receive some payoff in exchange for their participation. It is reasonable to assume that this payoff depends on the degree of synergy achieved with the group they choose, reflecting the value that well-integrated players can contribute to and derive from the group’s collective effort.

Such scenarios naturally arise in real-world cooperative settings. A particularly illustrative example can be found in the context of professional sports. Consider a sports team that must decide which competition to enter, anticipating a monetary payoff primarily derived from broadcasting revenues (e.g., [Bergantiños and Moreno-Tertero, 2020a,b, 2022a,b, 2023a,b, 2024a,b](#)). The expected

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payoff for the team depends on the level of audience engagement it can generate in each event. Since higher viewership typically translates into greater broadcasting revenue, the team’s decision involves evaluating how appealing its participation would be to spectators. For instance, if the team enters a competition that also features its long-standing rival teams, the potential for heightened audience interest—and consequently higher viewership—is significantly increased.

Motivated by such scenarios, this paper introduces new axiomatic characterizations of well-known values in cooperative games with transferable utilities (TU-games), grounded in the principle that payoffs should increase with synergy. A TU-game consists of a set of players and a characteristic function assigning a worth to each coalition. A value is a rule that assigns, to each game, a payoff vector specifying how the total worth is allocated among the players. We focus on four such values: the Shapley value, the Egalitarian value, the Center of the imputation set, and the Equal allocation of non-separable contributions.

A foundational contribution to this line of research was made by [Navarro \(2025\)](#), who characterized the Egalitarian value using two axioms: Efficiency and Sub-game order preservation. Efficiency (E) requires that the total worth of the grand coalition be fully distributed among its members. Sub-game Order Preservation (SOP) posits that if a population  $P$  must choose between cooperating with one of two distinct but equally sized groups, then the players in  $P$  should prefer the group whose inclusion leads to a higher total worth for the grand coalition. In this framework, synergy is captured by the additional worth created through cooperation between  $P$  and the group they join. In this paper, we propose alternative ways to capture synergy, leading to new axioms inspired by (SOP) but grounded in different payoff criteria, which are summarized as follows.

**Sub-game order preservation w.r.t. coalitional surplus (SOPCs):** This axiom asserts that if  $P$  must choose between two equally sized groups of players, they should prefer the one where the surplus is higher. The surplus is defined as the difference between the worth of the coalition formed by  $P$  and a given group, and the sum of the singleton worth of the group’s members. Combined with Efficiency, this axiom characterizes the Center of the imputation set.

**Sub-game order preservation w.r.t. coalitional dual surplus (SOPCds):** This dual version of (SOPCs) leads, when combined with Efficiency, to a characterization of the Equal allocation of non-separable contributions. Essentially, this axiom replaces the singleton worth of the group’s members with their contribution to the grand coalition.

**Sub-game order preservation w.r.t. coalitional contribution (SOPCc):** This axiom focuses on the marginal contribution created by  $P$  when they join some group. Such marginal contribution is defined as the difference between the worth of the group with  $P$  and the worth of the group alone. If  $P$  has to choose between two equally sized groups of players, then  $P$  should prefer the one in which their marginal contribution is greater. While intuitive and aligned with the spirit of the Shapley value, we show that no value can satisfy both (SOPCc) and Efficiency simultaneously.

**Sub-game order preservation w.r.t. individual contribution (SOPIC):** This axiom refines (SOPCc) by considering individual marginal contributions. When  $P$  chooses between two equally sized groups, each player in  $P$  prefers the one where their personal contribution is greater. The equal size of the groups enables a natural one-to-one correspondence between players, allowing for meaningful comparisons. When paired with Efficiency, (SOPIC) yields a new axiomatic characterization of the Shapley value.

The remainder of the paper is organized as follows. Section 2 introduces the model and supporting preliminaries. Section 3 formally defines the new axioms, grounded in the synergy principle. Section 4 presents our main results: comparable axiomatic characterizations of the Shapley value, the Center of the imputation set, and the Equal allocation of non-separable contributions.

## 2 Formal framework

Let  $U$  be a non-empty and infinite set of players. A coalition is a non-empty and finite subset of  $U$ . A coalitional game with transferable utility (also called a TU-game) is a pair  $(N, v)$  where  $N \subset U$  is a coalition and  $v$  is a function satisfying  $v : 2^N \rightarrow \mathbb{R}$  and  $v(\emptyset) = 0$ . We sometimes refer to  $N$  as the *grand coalition*. We denote by  $\Gamma$  the set of all games.

Given a coalition  $S \subseteq N$ , we write  $s = |S|$  its cardinality. For any  $S \subsetneq N$ , we will write  $(S, v|_S)$  the sub-game restricted to coalition  $S$  with  $v|_S(T) = v(T)$ ,  $\forall T \subseteq S$ . Given a player  $i \in N$  and a coalition  $S \subseteq N$ , we say that  $\Delta_i^S(v) = v(S \cup \{i\}) - v(S)$  is the marginal contribution of player  $i$  to coalition  $S$ . We denote by  $\Delta_i(v) = (\Delta_i^S(v))_{S \subseteq N \setminus \{i\}} \in \mathbb{R}^{2^{n-1}}$  the vector of marginal contributions of player  $i$ . Two players  $i, j \in N$  are equals in the game  $(N, v)$ , denoted by  $i \stackrel{v}{\sim} j$ , if  $\Delta_i^S(v) = \Delta_j^S(v)$  for any  $S \subseteq N \setminus \{i, j\}$ .

A value on  $\Gamma$  is a function  $\psi$  which associates, to each game  $(N, v) \in \Gamma$ , a payoff vector  $\psi(N, v) \in \mathbb{R}^N$ . The Shapley value, first presented by Shapley (1953), rewards players based on their average marginal contributions to coalitions. It is defined, for each  $(N, v) \in \Gamma$ , as

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \Delta_i^S(v), \quad \forall i \in N.$$

The Egalitarian value divides equally the worth of the coalition  $N$  between all players. It is defined, for each  $(N, v) \in \Gamma$ , as

$$EG_i(N, v) = \frac{v(N)}{n}, \quad \forall i \in N.$$

The center of imputation set (CIS), introduced by Driessen and Funaki (1991), gives every player its standalone worth and one  $n^{\text{th}}$  of the surplus, which is defined as the difference between  $N$ 's worth and the standalone worth of the players.<sup>1</sup> It is defined, for each  $(N, v) \in \Gamma$ , as

$$CIS_i(N, v) = v(\{i\}) + \frac{1}{n} \left( v(N) - \sum_{j \in N} v(\{j\}) \right), \quad \forall i \in N.$$

The dual of the CIS value is known as the Equal allocation of non-separable contributions (ENSC) value.<sup>2</sup> This value assigns to each player their separable contribution, defined as the value they individually add to every other player when forming  $N$ . Beyond this, the non-separable contribution—which is the residual value remaining after accounting for all separable contributions—is distributed equally among all players. The value is defined, for each  $(N, v) \in \Gamma$ , as

$$ENSC_i(N, v) = \left( v(N) - v(N \setminus \{i\}) \right) + \frac{1}{n} \left( v(N) - (v(N) - v(N \setminus \{i\})) \right), \quad \forall i \in N.$$

Each of the above-mentioned values satisfies the Efficiency and Equal treatment of equals axioms. The former indicates that the worth of the grand coalition should be entirely redistributed among the players, whereas the latter states that two equal players should receive the same payoff.

<sup>1</sup>The Center of the imputation set is also known as the Equal Surplus Division value.

<sup>2</sup>Formally,  $ENSC(N, v) = CIS(N, v^*)$ , where  $v^*$  is the dual of  $v$  defined as  $v^*(S) = v(N) - v(N \setminus S)$ , for each  $S \subseteq N$ .

**Efficiency (E).** For each  $(N, v) \in \Gamma$ , it holds that  $\sum_{i \in N} \psi_i(N, v) = v(N)$ .

**Equal treatment of equals (ETE).** For each  $(N, v) \in \Gamma$  and each  $i, j \in N$ , such that  $i$  and  $j$  are equals in  $(N, v)$ , it holds that  $\psi_i(N, v) = \psi_j(N, v)$ .

### 3 Sub-games order preservation

Navarro (2025) introduced an axiom of sub-games order preservation, which was used to characterize the Egalitarian value. Let us consider a given population  $P$  facing the choice of cooperating with two different groups of the same size. This axiom states that no member of the population  $P$  can be worse off if the population chooses the group with whom it has the best synergy. To translate this concept in the context of cooperative games, this idea of “best synergy” here is measured by the worth of the coalition. The axiom states, therefore, that given two coalitions of players whose intersection is non-empty, no member of the intersection has a lower payoff in the coalition with the better worth.

**Sub-game order preservation (SOP).** For each  $(N, v)$  with  $n \geq 3$  and each  $S, T \subseteq N$  such that  $s = t$ ,  $P = S \cap T \neq \emptyset$ , if

$$v(S) \geq v(T)$$

then  $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$ , for each  $i \in P = S \cap T$ .

Navarro (2025) demonstrates that (E) and (SOP) imply (ETE), which allows the author to characterize the Egalitarian value with (E) and (SOP) only.

**Lemma 1** (Navarro, 2025) *If a value  $\psi$  on  $\Gamma$  satisfies Efficiency (E) and Sub-game order preservation (SOP), then it satisfies Equal treatment of equals (ETE).*

**Theorem 1** (Navarro, 2025)

*A value  $\psi$  on  $\Gamma$  satisfies Efficiency (E) and Sub-game order preservation (SOP) if and only if  $\psi = EG$ .*

One could argue that there are alternative approaches to measuring player synergy in cooperative settings. The above approach, in some sense, made the implicit assumption that the results of the cooperation is only determined by the ability of players, and that there are no other factor that can influence the outcome of cooperation. Therefore it is sufficient to look at the output produced by  $S \cup P$  and  $T \cup P$  to see with whom the cooperation is more fruitful. If, however, we consider that despite a fruitful and competent cooperation, the worth of a coalition might be affected by other exogenous elements, we might want to measure synergy with finer approaches.

In this work, we introduce several novel variants of sub-game order preservation axioms, each grounded in a distinct interpretation of player synergy.<sup>3</sup>

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<sup>3</sup>The concept of synergy between players has also been explored in another strand of the TU-game literature, particularly in the context of mutually dependent players (see, for instance, Nowak and Radzik, 1995; Casajus, 2018; Besner, 2022). These are players who are only productive when acting together—meaning that the contribution of each player to any coalition that does not include the other is zero. In this sense, mutually dependent players exhibit a specific form of synergy. However, the notion of mutual dependence is technically distinct from our conceptualization of synergy, and as such, the two concepts are not directly comparable.

**Sub-game order preservation w.r.t. coalitional surplus (SOPCs).** For each  $(N, v) \in \Gamma$  with  $n \geq 3$  and each  $S, T \subseteq N$  such that  $s = t$ ,  $P = S \cap T \neq \emptyset$ , if

$$v(S) - \sum_{j \in S \setminus P} v(\{j\}) \geq v(T) - \sum_{j \in T \setminus P} v(\{j\})$$

then  $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$ , for each  $i \in P = S \cap T$ .

**Sub-game order preservation w.r.t. coalitional dual surplus (SOPCds).** For each  $(N, v) \in \Gamma$  with  $n \geq 3$  and each  $S, T \subseteq N$  such that  $s = t$ ,  $P = S \cap T \neq \emptyset$ , if

$$v(S) - \sum_{j \in S \setminus P} (v(S) - v(S \setminus \{j\})) \geq v(T) - \sum_{j \in T \setminus P} (v(T) - v(T \setminus \{j\}))$$

and

$$v(S) - v(S \setminus \{j\}) = v(T) - v(T \setminus \{j\}), \quad \forall j \in P,$$

then  $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$ , for each  $i \in P = S \cap T$ .

**Sub-game order preservation w.r.t. coalitional contribution (SOPCcs).** For each  $(N, v) \in \Gamma$  with  $n \geq 3$  and each  $S, T \subseteq N$  such that  $s = t$ ,  $P = S \cap T \neq \emptyset$ , if

$$v(S) - v(S \setminus P) \geq v(T) - v(T \setminus P)$$

then  $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$ , for each  $i \in P = S \cap T$ .

Each axiom—(SOP), (SOPCs), (SOPCds), and (SOPCcs)—offers a distinct lens for assessing the synergy between members of  $P = S \cap T$  and those in  $S \setminus P$  or  $T \setminus P$ . Yet, they can be interpreted within a unified framework. Imagine a group of players  $P$  seeking to join one of two existing groups:  $S \setminus P$  or  $T \setminus P$ . The total worth generated through cooperation between  $P$  and each group is represented by  $v(S)$  and  $v(T)$ , which we denote by (i). At the same time, we assume that the groups  $S \setminus P$  and  $T \setminus P$  can each generate some standalone worth on their own (ii). Each axiom measures the synergy of  $P$  by comparing (i) and (ii)—that is, the added value  $P$  brings when joining a group. The key difference among the axioms lies in how they interpret the standalone worth (ii).

(SOP): The standalone worth of  $S \setminus P$  and  $T \setminus P$  is zero, meaning that these groups do not generate any worth on their own. The addition of  $P$  is necessary to realize the full worth of  $v(S)$  or  $v(T)$ . Thus, cooperation with  $P$  is a necessary condition for worth generation.

(SOPCs): The standalone worth of  $S \setminus P$  and  $T \setminus P$  equals the sum of their members' standalone worth. This reflects a scenario where players in  $S \setminus P$  and  $T \setminus P$  refuse to cooperate unless  $P$  joins them. In the absence of  $P$ , they only generate their individual (singleton) worth.

(SOPCds): The interpretation is similar to that of (SOPCs), but here the standalone worth of  $S \setminus P$  and  $T \setminus P$  is defined as the sum of the members' marginal contributions to the grand coalition—i.e., the dual of their singleton worth. The axiom also requires that the marginal contributions of the members of  $P$  to the grand coalition are the same in both games.

(SOPCcs): The standalone worth of  $S \setminus P$  and  $T \setminus P$  corresponds to the worth of the coalitions themselves. This implies that the members cooperate freely and unconditionally, even without the involvement of  $P$ .

We also introduce a synergy measure that emphasizes individual players' marginal contributions. Let  $i \in P = S \cap T$ . By construction, for any pair of coalitions  $S, T \subseteq N$  with  $|S| = |T|$ , a non-empty intersection  $P = S \cap T$ , and symmetric difference sizes  $|S \setminus T| = |T \setminus S|$ , there exists a bijection  $\pi : S \setminus T \rightarrow T \setminus S$ . This allows us to define a one-to-one correspondence between each subset  $E \subseteq S$  and a corresponding subset  $E' \subseteq T$ , where

$$E' = \bigcup_{j \in E \setminus P} \{\pi(j)\} \cup \bigcup_{j \in E \cap P} \{j\}.$$

For convenience, we denote this mapped set as  $\pi(E)$ . Using this correspondence, we define the following axiom.

**Sub-games order preservation w.r.t. individual contribution (SOPIc).** *For each  $(N, v) \in \Gamma$  with  $n \geq 3$ , each  $S, T \subseteq N$  such that  $s = t$ ,  $P = S \cap T \neq \emptyset$ , and each  $i \in P$ , if there exists a bijection  $\pi : S \setminus T \rightarrow T \setminus S$  such that*

$$v(E \cup \{i\}) - v(E) \geq v(\pi(E) \cup \{i\}) - v(\pi(E)), \quad \forall E \subseteq S \setminus \{i\},$$

*then it holds that  $\psi_i(S, v|_S) \geq \psi_i(T, v|_T)$ .*

(SOPIc) differs from the previously discussed axioms. Rather than comparing the worth generated with and without  $P$ , it focuses on a single player  $i \in P$  and measures the marginal contribution of  $i$  to each such sub-coalition of  $S$  and  $T$ . Using a one-to-one correspondence between sub-coalitions of  $S$  and  $T$ , the axiom can compare  $i$ 's contributions across both groups. The group in which  $i$  contributes more is the one where they can secure a higher payoff.

Conceptually, (SOPIc) is related to the *Strong monotonicity* axiom introduced by [Young \(1985\)](#) to characterize the Shapley value. This axiom requires that for any two games  $(N, v), (N, w) \in \Gamma$ , if a player  $i \in N$  has higher marginal contributions in  $(N, v)$  than in  $(N, w)$ , then  $\psi_i(N, v) \geq \psi_i(N, w)$ . The key distinction is that Young compares two games with the same player set and different characteristic functions, whereas (SOPIc) compares games with the same characteristic function but different player sets.

Similarly, (SOP) parallels the *Coalitional monotonicity* axiom proposed by [van den Brink \(2007\)](#) to characterize the Egalitarian Shapley Value. This axiom requires that for any two games  $(N, v), (N, w) \in \Gamma$ , if  $v(E) \geq w(E)$  for each  $E \ni i$ , then  $\psi_i(N, v) \geq \psi_i(N, w)$ . Again, Coalitional monotonicity fixes the player set and makes the characteristic function vary, whereas (SOP) does the converse. In the same vein, (SOPCs) resembles the *Individual monotonicity for equal surplus* axiom introduced by [Funaki and Koriyama \(2025\)](#) to characterize the CIS value, while (SOPCd) aligns with the dual version of that axiom, used to characterize the ENSC value.

In these three studies, the *monotonicity axioms* are combined with Efficiency and Equal treatment of equals to characterize the respective values. The authors establish the logical independence of these axioms, showing that Equal treatment of equals is not implied by the other two. In contrast, we demonstrate that Equal treatment of equals is implied by Efficiency together with each of the new variants of the sub-game order preservation axioms. This result extends [Lemma 1](#) and is proved in a unified manner.

### Proposition 1

*Suppose that a value  $\psi$  on  $\Gamma$  satisfies Efficiency (E). If, in addition,  $\psi$  satisfies one of the following four axioms:*

- Sub-game order preservation w.r.t. coalitional surplus (SOPCs),
- Sub-game order preservation w.r.t. coalitional dual surplus (SOPCds),
- Sub-game order preservation w.r.t. coalitional contribution (SOPCc),
- Sub-game order preservation w.r.t. individual contribution (SOPIc),

then it also satisfies Equal treatment of equals (ETE).

**Proof.** Take any game  $(N, v) \in \Gamma$  and let  $P \subset U \setminus N$  with  $p = n - 1$ . Also, take any  $i, j \in N$  and let  $N_i = P \cup \{i\}$  and  $N_j = P \cup \{j\}$ . Then, by (SOPCs), (SOPCds), (SOPCc), or (SOPIc), we will show that there exists a larger game  $(N^+, w)$  with  $N^+ = N \cup P$  that satisfies the following:

- (1)  $\psi_p(N_i, w|_{N_i}) = \psi_p(N_j, w|_{N_j}), \forall p \in P = N_i \cap N_j,$
- (2)  $\psi_i(N_i, w|_{N_i}) = \psi_i(N, w|_N),$
- (3)  $\psi_j(N_j, w|_{N_j}) = \psi_j(N, w|_N),$
- (4)  $w|_N = v.$

Once we have established the above, by efficiency of  $\psi$ , we have

$$w(N_i) - w(N_j) = \psi_i(N_i, w|_{N_i}) - \psi_j(N_j, w|_{N_j}) = \psi_i(N, v) - \psi_j(N, v) \dots \quad (*)$$

In the following, we show the above property by each of the additional axioms. For each case, we assume that  $i \overset{v}{\sim} j$  for any  $i, j \in N$ . Note that since  $U$  is infinite, all the constructions below are always possible.

(SOPCs): We consider the following  $(N^+, w)$ :

1.  $w(S) = v(S), \forall S \subseteq N,$
2.  $w(N_i) - w(\{i\}) = w(N_j) - w(\{j\}),$
3.  $w(N_i) - \sum_{k \in P} w(\{k\}) = w(N) - \sum_{k \in N \setminus \{i\}} w(\{k\}),$
4.  $w(N_j) - \sum_{k \in P} w(\{k\}) = w(N) - \sum_{k \in N \setminus \{j\}} w(\{k\}).$

Then, by applying (SOPCs) in both directions, we can see that the properties (1)–(3) are satisfied. In addition, by condition 1, property (4) also holds. Since  $i \overset{v}{\sim} j$  for any  $i, j \in N$ , and hence  $v(\{i\}) = v(\{j\})$ , by the condition 4 and (\*), we get

$$\begin{aligned} 0 &= \left( w(N) - \sum_{k \in N} w(\{k\}) \right) - \left( w(N) - \sum_{k \in N} w(\{k\}) \right) \\ &= \left( w(N_i) - \sum_{k \in N_i} w(\{k\}) \right) - \left( w(N_j) - \sum_{k \in N_j} w(\{k\}) \right) \\ &= w(N_i) - w(N_j) - \left( \sum_{k \in P} w(\{k\}) - \sum_{k \in P} w(\{k\}) \right) - \left( v(\{i\}) - v(\{j\}) \right) \\ &= w(N_i) - w(N_j) \\ &= \psi_i(N, v) - \psi_j(N, v), \end{aligned}$$

so that we obtain the desired condition  $\psi_i(N, v) = \psi_j(N, v)$ .

(SOPCds): We consider the following game  $(N^+, w)$ :

1.  $w(S) = v(S), \forall S \subseteq N$ ,
2.  $w(N_i) - w(N \setminus \{k\}) = w(N_j) - w(N_j \setminus \{j\}), \forall k \in P$ ,
3.  $w(N_l) - \sum_{k \in P} (w(N_l) - w(N_l \setminus \{k\})) = w(N) - \sum_{k \in N \setminus \{l\}} (w(N) - w(N \setminus \{k\})), \forall l = i, j$ ,
4.  $w(N_l) - w(N) = w(N) - w(N \setminus \{l\}), \forall l = i, j$ .

(SOPCc): We consider the following  $(N^+, w)$ :

1.  $w(S) = v(S)$  for all  $S \subseteq N$ ,
2.  $w(N_i) - w(\{i\}) = w(N_j) - w(\{j\})$ ,
3.  $w(N_i) - w(P) = v(N) - v(N \setminus \{i\})$ ,
4.  $w(N_j) - w(P) = v(N) - v(N \setminus \{j\})$ .

Then, by applying (SOPCds) or (SOPCc) in both directions accordingly, we can see the properties (1)–(3). In addition, by condition 1, property (4) also holds. Since  $i \overset{v}{\sim} j$ ,  $v(N \setminus \{i\}) = v((N \setminus \{i, j\}) \cup \{j\}) = v((N \setminus \{i, j\}) \cup \{i\}) = v(N \setminus \{j\})$ , by the condition 4 and (\*), we get the desired condition  $\psi_i(N, v) = \psi_j(N, v)$  by the similar way of (SOPCs).

(SOPIc): The case for (SOPIc) is somewhat involved, but we prove it by a similar logic. Since  $p = n - 1$ , given a  $\hat{k} \in P$ , there exists a bijection  $\pi^{ij} : N \setminus \{ij\} \rightarrow P \setminus \{k\}$ . In addition, it is obvious that there exist three other bijections

$$\begin{aligned} \pi^i : N \setminus \{j\} &\rightarrow P && \text{with } \pi^i(i) = \hat{k} \text{ and } \pi^i(l) = \pi^{ij}(l), \forall l \in N \setminus \{i, j\}; \\ \pi^j : N \setminus \{i\} &\rightarrow P, && \text{with } \pi^j(j) = \hat{k} \text{ and } \pi^j(l) = \pi^{ij}(l), \forall l \in N \setminus \{i, j\}; \\ \pi : N_i &\rightarrow N_j && \text{with } \pi(i) = j \text{ and } \pi(l) = l, \forall l \in P. \end{aligned}$$

Observe that  $N \cap N_i = \{i\}$  and  $N \cap N_j = \{j\}$ . Then, we consider the following  $(N^+, w)$ :

$$w(S) = \begin{cases} v(S) & \text{if } S \subseteq N, \\ v(\{i\} \cup (\pi^{ij})^{-1}(T)) = v(\{j\} \cup (\pi^{ij})^{-1}(T)) & \text{if } S = T \cup \{i\} \text{ or } S = T \cup \{j\}, \\ & T \subseteq P \setminus \{\hat{k}\}, \\ v(\{i, j\} \cup (\pi^{ij})^{-1}(T)) & \text{if } S = T \cup F, T \subseteq P \setminus \{\hat{k}\} \\ & \text{and } |F \cap \{i, j, \hat{k}\}| = 2, \\ v((\pi^i)^{-1}(S)) = v((\pi^j)^{-1}(S)) & \text{if } S \subseteq P, \end{cases}$$

Note that for  $S = T \cup F$  with  $|F \cap \{i, j, \hat{k}\}| \geq 3$ ,  $w(S)$  is unspecified in the above definition and can be arbitrarily chosen. This construction is always possible as there are no overlapping requirements and  $i \overset{v}{\sim} j$ . We hence have that

1.  $w(S) = v(S)$  for all  $S \subseteq N$ ,

2.  $\forall F \subseteq P, \forall E \subseteq N_i \setminus F$ ,

$$w(E \cup F) - w(E) = w(\pi(E) \cup F) - w(\pi(E)).$$

3.  $\forall E \subseteq N \setminus \{i\}$ ,

$$w(E \cup \{i\}) - w(E) = v(E \cup \{i\}) - v(E) = w((\pi^j)(E) \cup \{i\}) - w((\pi^j)(E)).$$

4.  $\forall E \subseteq N \setminus \{j\}$ ,

$$w(E \cup \{j\}) - w(E) = v(E \cup \{j\}) - v(E) = w(\pi^i(E) \cup \{j\}) - w(\pi^i(E)).$$

In particular, condition 2 because

$$\begin{aligned} & w(E \cup F) - w(E) \\ &= \begin{cases} v(\{i, j\} \cup (\pi^{ij})^{-1}(E \setminus \{i\}) \cup (\pi^{ij})^{-1}(F \setminus \{\hat{k}\})) - v(\{i\} \cup (\pi^{ij})^{-1}(E \setminus \{i\})) & \text{if } i \in E, \hat{k} \in F \\ v(\{i\} \cup (\pi^{ij})^{-1}(E \setminus \{i\}) \cup (\pi^{ij})^{-1}(F)) - v(\{i\} \cup (\pi^{ij})^{-1}(E \setminus \{i\})) & \text{if } i \in E, \hat{k} \notin F \\ v((\pi^i)^{-1}(E \cup F)) - v((\pi^i)^{-1}(E)) & \text{if } i \notin E \end{cases} \\ &= w(\pi(E) \cup F) - \pi(E). \end{aligned}$$

We will prove that  $\psi_i(N, w|_N) = \psi_j(N, w|_N)$ . Thanks to condition 2 in cases where  $|F| = 1$ , and conditions 3 and 4, we use (SOPIc) in both directions and obtain the following conditions on  $\psi$ :

$$\begin{aligned} \psi_k(N_i, w|_{N_i}) &= \psi_k(N_j, w|_{N_j}), \quad \forall k \in P = N_i \cap N_j, \\ \psi_i(N_i, w|_{N_i}) &= \psi_i(N, w|_N), \\ \psi_j(N_j, w|_{N_j}) &= \psi_j(N, w|_N). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} & \sum_{k \in P} \psi_k(N_j, w|_{N_j}) = \sum_{k \in P} \psi_k(N_i, w|_{N_i}) \\ & \Leftrightarrow \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) = \sum_{k \in P} \psi_k(N_i, w|_{N_i}) + \psi_i(N_i, w|_{N_i}) \\ & \Leftrightarrow \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) = w(N_i) \\ & \Leftrightarrow \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) - w(\{i\}) = w(N_i) - w(\{i\}) \\ & \Leftrightarrow \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) - w(\{i\}) = w(N_j) - w(\{j\}). \end{aligned}$$

The last equivalence follows when we replace  $F$  by  $P$  in condition 2. Since  $i \stackrel{v}{\sim} j$  and  $w(S) = v(S)$  for all  $S \subseteq N$  then  $w(\{i\}) = w(\{j\})$ . Consequently

$$\begin{aligned} & \Leftrightarrow \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) = w(N_j) \\ & \Leftrightarrow \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_i(N_i, w|_{N_i}) = \sum_{k \in P} \psi_k(N_j, w|_{N_j}) + \psi_j(N_j, w|_{N_j}) \\ & \Leftrightarrow \psi_i(N_i, w|_{N_i}) = \psi_j(N_j, w|_{N_j}). \end{aligned}$$

By definition of  $w$ , we can conclude that  $\psi_i(N, v) = \psi_j(N, v)$ . □

## 4 Axiomatic results

We demonstrate that sub-game order preservation-type axioms, when combined with Efficiency, are sufficient to characterize classical values for TU-games. By extending Theorem 1, our main result provides new axiomatic characterizations of the Shapley value, the CIS value, and the ENSC value, each based on only two axioms.<sup>4</sup>

### Theorem 2

The following results hold.

- (1) A value  $\psi$  on  $\Gamma$  satisfies Efficiency (E) and Sub-games order preservation w.r.t. coalitional surplus (SOPCs) if and only if  $\psi = CIS$ .
- (2) A value  $\psi$  on  $\Gamma$  satisfies Efficiency (E) and Sub-games order preservation w.r.t. coalitional dual surplus (SOPCds) if and only if  $\psi = ENSC$ .
- (3) A value  $\psi$  on  $\Gamma$  satisfies Efficiency (E) and Sub-games order preservation w.r.t. individual contribution (SOPIc) if and only if  $\psi = Sh$ .

**Proof.** For each point, the existence proof is direct; therefore, we focus on proving uniqueness.

(1) Consider a value  $\psi$  on  $\Gamma$  satisfying (E) and (SOPCs). Let us show that  $\psi = CIS$ . Consider a game  $(N, v) \in \Gamma$  and a player  $i \in N$ . We can build a game  $(N_i, w_i)$  with  $N_i$  such that

$$\begin{aligned} N &\subsetneq N_i, \\ \exists T \subsetneq N_i \text{ with } N \cap T &= \{i\} \text{ and } n = t. \end{aligned}$$

We also impose the following conditions on  $w_i$ :

$$\begin{aligned} w_i(P) &= v(P), \quad \forall P \subseteq N \\ w_i(T) &= v(N) - \sum_{j \in N} v(\{j\}) + n v(\{i\}), \\ w_i(P) &= v(\{i\}), \quad \forall P \subsetneq T, \quad P \neq \{i\}. \end{aligned}$$

Such a construction is always possible since  $U$  is infinite. By definition we have  $(w_i)_{|N}(S) = v(S)$  for all  $S \subseteq N$  hence  $\psi_i(N, (w_i)_{|N}) = \psi_i(N, v)$ . Using (E) and Proposition 1, we have

$$\begin{aligned} \psi_i(T, (w_i)_{|T}) &= \frac{(w_i)_{|T}(T)}{t} = \frac{1}{t} \left[ v(N) - \sum_{j \in N} v(\{j\}) + n v(\{i\}) \right] \\ &= v(\{i\}) + \frac{1}{n} \left[ v(N) - \sum_{j \in N} v(\{j\}) \right], \end{aligned}$$

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<sup>4</sup>It is worth mentioning that the four versions of the sub-games order preservation axiom we are using can be weakened to only apply to cases where  $|P| = 1$ . However, proposition 1 wouldn't hold with this weaker version. Hence, the following theorem would require the addition of the Equal treatment of equals axiom.

since  $i \stackrel{(w_i)|_T}{\sim} j$  for any  $i, j \in T$ . Furthermore, we have that

$$\begin{aligned}
w_i(T) - \sum_{j \in T \setminus \{i\}} w_i(\{j\}) &= v(N) - \sum_{j \in N} v(\{j\}) + n v(\{i\}) - \sum_{j \in T \setminus \{i\}} w(\{j\}) \\
&= v(N) - \sum_{j \in N} v(\{j\}) + n v(\{i\}) - (t-1)v(\{i\}) \\
&= v(N) - \sum_{j \in N} v(\{j\}) + n v(\{i\}) - (n-1)v(\{i\}) \\
&= v(N) - \sum_{j \in N} v(\{j\}) + v(\{i\}) \\
&= v(N) - \sum_{j \in N \setminus \{i\}} v(\{j\}) \\
&= w_i(N) - \sum_{j \in N \setminus \{i\}} w_i(\{j\}).
\end{aligned}$$

We can therefore apply (SOPCs) and we obtain that

$$\psi_i(T, (w_i)|_T) \geq \psi_i(N, (w_i)|_N) \quad \text{and} \quad \psi_i(N, (w_i)|_N) \geq \psi_i(T, (w_i)|_T),$$

which means that

$$\psi_i(N, (w_i)|_N) = \psi_i(T, (w_i)|_T) = v(\{i\}) + \frac{1}{n} \left[ v(N) - \sum_{j \in N} v(\{j\}) \right] = \psi_i(N, v).$$

Since  $U$  is an infinite set, it is possible to build a game  $(N_i, w_i)$  that satisfies our above conditions for every  $i \in N$ . Hence we have that  $\psi_i(N, v) = CIS_i(N, v)$ ,  $\forall i \in N$ , which concludes this part the proof.

**(2)** Consider a value  $\psi$  on  $\Gamma$  satisfying (E) and (SOPCds). Let us show that  $\psi = ENSC$ . Consider a game  $(N, v) \in \Gamma$  and a player  $i \in N$ . The construction is similar to (1). We build the following game  $(N_i, w_i)$  with  $N_i$  such that

$$\begin{aligned}
N &\subsetneq N_i, \\
\exists T &\subsetneq N_i \text{ with } N \cap T = \{i\} \text{ and } n = t.
\end{aligned}$$

We also impose the following conditions on  $w_i$ :

$$\begin{aligned}
w_i(P) &= v(P), \quad \forall P \subseteq N \\
w_i(T \setminus \{k\}) &= v(N \setminus \{i\}) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) + n(v(N) - v(N \setminus \{i\})), \quad \forall k \in T, \\
w_i(T) &= v(N) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) + n(v(N) - v(N \setminus \{i\})), \\
w_i(P) &= v(\{i\}), \quad \forall P \subsetneq T, \quad P \neq \{i\}, \quad |P| \leq n-2.
\end{aligned}$$

Such a construction is always possible since  $U$  is infinite. By definition we have  $(w_i)_{|_N}(S) = v(S)$  for all  $S \subseteq N$ , and hence,  $\psi_i(N, (w_i)_{|_N}) = \psi_i(N, v)$ . Using (E) and proposition 1, we have

$$\begin{aligned}\psi_i(T, (w_i)_{|_T}) &= \frac{(w_i)_{|_T}(T)}{t} = \frac{1}{t} \left[ v(N) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) + n(v(N) - v(N \setminus \{i\})) \right] \\ &= v(N) - v(N \setminus \{i\}) + \frac{1}{n} \left[ v(N) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) \right],\end{aligned}$$

since  $i \stackrel{(w_i)_{|_T}}{\sim} j$  for any  $i, j \in T$ . Furthermore, we have that

$$\begin{aligned}w_i(T) - \sum_{j \in T \setminus \{i\}} (w_i(T) - w_i(T \setminus \{j\})) &= v(N) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) + n(v(N) - v(N \setminus \{i\})) \\ &\quad - \sum_{j \in T \setminus \{i\}} (w_i(T) - w(T \setminus \{j\})) \\ &= v(N) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) + n(v(N) - v(N \setminus \{i\})) \\ &\quad - (n-1)(v(N) - v(N \setminus \{i\})) \\ &= v(N) - \sum_{j \in N \setminus \{i\}} (v(N) - v(N \setminus \{j\})) \\ &= w_i(N) - \sum_{j \in N \setminus \{i\}} (w_i(N) - w_i(N \setminus \{j\})),\end{aligned}$$

and  $w_i(T) - w_i(T \setminus \{i\}) = v(N) - v(N \setminus \{i\}) = w_i(N) - w_i(N \setminus \{i\})$ . We can therefore apply (SOPCds) and we obtain that

$$\psi_i(T, (w_i)_{|_T}) \geq \psi_i(N, (w_i)_{|_N}) \text{ and } \psi_i(N, (w_i)_{|_N}) \geq \psi_i(T, (w_i)_{|_T}),$$

which means that

$$\begin{aligned}\psi_i(N, (w_i)_{|_N}) &= \psi_i(T, (w_i)_{|_T}) \\ &= v(N) - v(N \setminus \{i\}) + \frac{1}{n} \left[ v(N) - \sum_{j \in N} (v(N) - v(N \setminus \{j\})) \right] \\ &= \psi_i(N, v).\end{aligned}$$

Since  $U$  is an infinite set, it is possible to build a game  $(N_i, w_i)$  that satisfies our above conditions for every  $i \in N$ . Hence we have that  $\psi_i(N, v) = ENSC_i(N, v)$ ,  $\forall i \in N$ , which concludes this part of the proof.

**(3)** Consider a value  $\psi$  on  $\Gamma$  satisfying (E) and (SOPIc). Let us show that  $\psi = Sh$ . Consider a game  $(N, v) \in \Gamma$  and a player  $i \in N$ . Let  $N \setminus \{i\} = \{i_1, \dots, i_{n-1}\}$ . By Proposition 3.6 of [Pintér \(2015\)](#), there exists a sequence of games  $(N, v^k)_{k=1}^{n-1}$  such that

- $i \stackrel{v^1}{\sim} i_1$ , and  $\Delta_{i_1}(v^1) = \Delta_{i_1}(v)$ ,

- For any  $k \geq 2$ ,  $j \stackrel{v^k}{\sim} l$  for any pair of  $j, l \in \{i, i_1, \dots, i_{k-1}, i_k\}$ , and  $\Delta_{i_k}(v^k) = \Delta_{i_k}(v^{k-1})$ .

That is, starting from  $v^0 = v$ , while keeping player  $i_k$ 's marginal contribution from the previous game  $v^{k-1}$ , we make any player  $j, l \in \{i, i_1, \dots, i_{k-1}, i_k\}$  equal each other in  $v^k$  for any  $k = 1, \dots, n-1$ .

In particular, note that  $i \stackrel{v^{n-1}}{\sim} j$  for any pair of  $i, j \in N$ .

Then, we construct a sequence of games  $(N^k, w^k)_{k=1}^{n-1}$  such that, for any  $k = 1, \dots, n-1$ ,

- $N^k = T^{k-1} \cap T^k$ ,  $T^{k-1} \cap T^k = \{j_{k-1}\}$ ,  $|T^k| = |T^{k-1}|$ , where  $j_0 = i$  and  $T^0 = N$ .
- $\pi^k : T^{k-1} \rightarrow T^k$  is a bijection with  $\pi^k(j_{k-1}) = j_{k-1}$  such that

$$w^k(E) = \begin{cases} v^{k-1}((\hat{\pi}^{k-1})^{-1}(E)) & \text{if } E \subseteq T^{k-1}, \\ v^k((\hat{\pi}^k)^{-1}(E)) & \text{if } E \subseteq T^k, \end{cases}$$

where  $v^0 = v$  and  $\hat{\pi}^k : N \rightarrow T^k$  is defined as the  $\hat{\pi}^k = \pi^k \circ \dots \circ \pi^1$ .

- $j_k = \hat{\pi}^k(i_k)$ .

We show that  $\psi(T_k, (w^k)_{|T^k}) = Sh(T_k, (w^k)_{|T^k})$  for any  $k = 1, \dots, n-1$ . The following arguments are adopted from the proof for Theorem 3.4 from [Pintér \(2015\)](#).

$k = n-1$ : By construction,  $j \stackrel{(w^{n-1})_{|T^{n-1}}}{\sim} l$  for any pair of  $j, l \in T^{n-1}$ . Therefore, we can see that  $\psi_j(T^{n-1}, (w^{n-1})_{|T^{n-1}}) = Sh_j(T^{n-1}, (w^{n-1})_{|T^{n-1}})$  for any  $j \in T^{n-1}$ . Moreover, since

$$\begin{aligned} w^{n-1}(E \cup \{j_{n-1}\}) - w^{n-1}(E) &= v^{n-1}((\hat{\pi}^{n-1})^{-1}(E \cup \{j_{n-1}\})) - v^{n-1}((\hat{\pi}^{n-1})^{-1}(E)) \\ &= v^{n-2}((\hat{\pi}^{n-1})^{-1}(E \cup \{j_{n-1}\})) - v^{n-1}((\hat{\pi}^{n-2})^{-1}(E)) \\ &= v^{n-2}((\hat{\pi}^{n-2})^{-1} \circ (\pi^{n-1})^{-1}(E \cup \{j_{n-1}\})) \\ &\quad - v^{n-1}((\hat{\pi}^{n-2})^{-1} \circ (\pi^{n-1})^{-1}(E)) \\ &= w^{n-1}((\pi^{n-1})^{-1}(E \cup \{j_{n-1}\})) - w^{n-1}((\pi^{n-1})^{-1}(E)) \end{aligned}$$

for any  $E \subseteq T^{n-1} \setminus \{j_{n-1}\}$ . Therefore, by (SOPIc), we have

$$\begin{aligned} \psi_{j_{n-1}}(T^{n-2}, (w^{n-1})_{|T^{n-2}}) &= \psi_{j_{n-1}}(T^{n-1}, (w^{n-1})_{|T^{n-1}}) \\ &= Sh_{j_{n-1}}(T^{n-1}, (w^{n-1})_{|T^{n-1}}) \\ &= Sh_{j_{n-1}}(T^{n-2}, (w^{n-2})_{|T^{n-2}}). \end{aligned}$$

Since  $j \stackrel{(w^{n-2})_{|T^{n-2}}}{\sim} l$  for any pair of  $j, l \in T^{n-2} \setminus \{j_{n-2}\}$ , by (ETE) and (E), we have  $\psi_j(T^{n-2}, (w^{n-1})_{|T^{n-2}}) = Sh_j(T^{n-2}, (w^{n-2})_{|T^{n-2}})$  for all  $j \in T^{n-2}$ .

$k = n-2$ : Similarly, by construction,  $j \stackrel{(w^{n-2})_{|T^{n-3}}}{\sim} l$  for any pair of  $j, l \in T^{n-3} \setminus \{\hat{\pi}^{n-3}(i_{n-1}), j_{n-2}\}$ . Moreover, since

$$\begin{aligned} w^{n-2}(E \cup \{j_{n-2}\}) - w^{n-2}(E) &= v^{n-2}((\hat{\pi}^{n-2})^{-1}(E \cup \{j_{n-2}\})) - v^{n-2}((\hat{\pi}^{n-2})^{-1}(E)) \\ &= v^{n-3}((\hat{\pi}^{n-2})^{-1}(E \cup \{j_{n-1}\})) - v^{n-3}((\hat{\pi}^{n-2})^{-1}(E)) \\ &= v^{n-3}((\hat{\pi}^{n-3})^{-1} \circ (\pi^{n-2})^{-1}(E \cup \{j_{n-2}\})) \\ &\quad - v^{n-3}((\hat{\pi}^{n-3})^{-1} \circ (\pi^{n-2})^{-1}(E)) \\ &= w^{n-2}((\pi^{n-2})^{-1}(E \cup \{j_{n-2}\})) - w^{n-2}((\pi^{n-2})^{-1}(E)) \end{aligned}$$

for any  $E \subseteq T^{n-2} \setminus \{j_{n-2}\}$ . Therefore, by (SOPIc), we have

$$\begin{aligned}\psi_{j_{n-2}}(T^{n-3}, (w^{n-2})|_{T^{n-3}}) &= \psi_{j_{n-2}}(T^{n-2}, (w^{n-2})|_{T^{n-2}}) \\ &= Sh_{j_{n-2}}(T^{n-2}, (w^{n-2})|_{T^{n-2}}) \\ &= Sh_{j_{n-2}}(T^{n-3}, (w^{n-3})|_{T^{n-3}}).\end{aligned}$$

Since  $j_{n-2}$  and  $\hat{\pi}^{n-3}(i_{n-1})$  can be arbitrarily chosen corresponding to the choice of  $i_{n-2}$  and  $i_{n-1}$  and  $j \stackrel{(w^{n-3})|_{T^{n-3}}}{\sim} l$  for any pair of  $j, l \in T^{n-3} \setminus \{\hat{\pi}^{n-3}(i_{n-1}), j_{n-2}\}$ , by (ETE) and (E), we have  $\psi_j(T^{n-3}, (w^{n-2})|_{T^{n-3}}) = Sh_j(T^{n-3}, (w^{n-2})|_{T^{n-3}})$  for all  $j \in T^{n-3}$ .

$k \leq n-3$ : Inductively, we can see that  $\psi_i(T^1, w^1|_{T^1}) = Sh_i(T^1, w^1|_{T^1}) = Sh_i(T^1, v^1 \circ (\pi^1)^{-1})$ .

Finally, by construction, note that  $(N, (w^1)|_{T^0}) = (N, v)$ ,  $(T^1, (w^1)|_{T^1}) = (T^1, v^1 \circ (\pi^1)^{-1})$ , and

$$\begin{aligned}w^1(E \cup \{i\}) - w^1(E) &= v(E \cup \{i\}) - v(E) \\ &= v^1((\pi^1)^{-1}(E \cup \{i\})) - v^1((\pi^1)^{-1}(E)) \\ &= w^1((\pi^1)^{-1}(E \cup \{i\})) - w^1((\pi^1)^{-1}(E))\end{aligned}$$

for any  $E \subseteq T^1 \setminus \{i\}$ . Therefore, by (SOPIc), we obtain

$$\begin{aligned}\psi_i(N, v) &= \psi_i(T^1, v^1 \circ (\pi^1)^{-1}) \\ &= Sh_i(T^1, w^1|_{T^1}) \\ &= Sh_i(T^1, v^1 \circ (\pi^1)^{-1}) \\ &= Sh_i(N, v^1) \\ &= Sh_i(N, v),\end{aligned}$$

which completes the proof. □

In contrast to the above characterization result, we obtain the following impossibility result for (SOPCc).

### Theorem 3

*There is no value on  $\Gamma$  that satisfies both Efficiency (E) and Sub-games order preservation w.r.t. coalitional contribution (SOPCc).*

**Proof.** Let us consider a game  $(N, v)$  and a player  $i \in N$ . We can build a game  $(N_i, w_i)$  with  $N_i$  such that

$$\begin{aligned}N &\subsetneq N_i, \\ \exists T &\subsetneq N_i \text{ with } N \cap T = \{i\} \text{ and } n = t.\end{aligned}$$

We also put the following conditions on  $w_i$ :

$$\begin{aligned}w_i(P) &= v(P), \quad \forall P \subseteq N \\ w_i(T) &= v(N), \\ w_i(P) &= v(\{i\}), \quad \forall P \subsetneq T, \quad |P| < n-1, \\ w_i(P) &= v(N \setminus \{i\}), \quad \forall P \subsetneq T, \quad |P| = n-1.\end{aligned}$$

Such a construction is always possible since  $U$  is infinite. By definition, we have  $(w_i)_{|_N}(S) = v(S)$  for all  $S \subseteq N$ , and hence  $\psi_i(N, (w_i)_{|_N}) = \psi_i(N, v)$ . Using efficiency and proposition 1, we have

$$\psi_i(T, (w_i)_{|_T}) = \frac{(w_i)_{|_T}(T)}{t} = \frac{v(N)}{n}$$

since  $i \stackrel{(w_i)_{|_T}}{\sim} j$  for any  $i, j \in T$ . Note that we have

$$w_i(T) - w_i(T \setminus \{i\}) = v(N) - v(N \setminus \{i\}).$$

Since  $\{i\} = N \cap T$  and  $n = t$  we can use (SOPCc) to obtain that

$$\psi_i(T, (w_i)_{|_T}) \geq \psi_i(N, (w_i)_{|_N}) \text{ and } \psi_i(N, (w_i)_{|_N}) \geq \psi_i(T, (w_i)_{|_T}),$$

which means that

$$\psi_i(N, (w_i)_{|_N}) = \psi_i(T, (w_i)_{|_T}) = \frac{v(N)}{n} = \psi_i(N, v).$$

Since  $U$  is an infinite set, it is possible to build a game  $(N_i, w_i)$  that satisfies our above conditions for every  $i \in N$ . Hence we have that  $\psi_i(N, v) = \frac{v(N)}{n}$ ,  $\forall i \in N$ .

Now let us consider the game  $(N, v)$  with  $n = 3$  and  $v$  such that  $v(\{2\}) = 0$ ,  $v(\{3\}) = 1.2$ ,  $v(\{1, 2\}) = 1$  and  $v(\{1, 3\}) = 2$ . We have  $\{1, 2\} \cap \{1, 3\} = \{1\} \neq \emptyset$ . We also have  $v(\{1, 2\}) - v(\{2\}) = 1$  and  $v(\{1, 3\}) - v(\{3\}) = 0.8$ . However we have  $\frac{v(\{1, 2\})}{2} = 0.5$  and  $\frac{v(\{1, 3\})}{2} = 1$  which violates (SOPCc). □

## 5 Conclusion

In this paper, we propose new axiomatic characterizations of the Shapley value, the CIS value, and the ENSC value. To this end, we introduce a family of four sub-game order preservation axioms. One of them—sub-game order preservation w.r.t. coalitional contribution—is incompatible with the Efficiency axiom, leading to an impossibility result. The remaining three, when combined with Efficiency, uniquely characterize the Shapley value, the CIS value, and the ENSC value, respectively.

Our contribution parallels existing characterizations of these values that rely on Efficiency, Equal treatment of equals, and various forms of monotonicity axioms, as in the works of [Young \(1985\)](#), [van den Brink \(2007\)](#), and [Funaki and Koriyama \(2025\)](#). While conceptually related to monotonicity axioms, our axioms differ in that they allow the player set to vary while keeping the characteristic function fixed. Moreover, Equal treatment of equals becomes redundant in our framework, as it is implied by the sub-game order preservation axioms when combined with Efficiency.

For future research, an interesting direction would be to explore how convex combinations of the studied values can be characterized using sub-game order preservation axioms. In works such as [van den Brink et al. \(2013\)](#), [Casajus and Huettner \(2013, 2014\)](#), [Yokote and Funaki \(2017\)](#), convex combinations of values are characterized using Efficiency, Equal treatment of equals, and weakened forms of monotonicity.<sup>5</sup> Drawing inspiration from these weakened monotonicity axioms could help in formulating suitable sub-game order preservation axioms, potentially leading to new characterization results.

<sup>5</sup>See also [Nakada \(2024\)](#) for additional discussions for the relationship between monotonicity and linearity.

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