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# Mixed 3D-2D asymptotics for the weak and strong solutions of the rotating magnetohydrodynamic system.

Frédéric Charve\*, Van-Sang Ngo†

## Abstract

In this article, we consider the 3D-rotating magnetohydrodynamic (MHD) system when the initial velocity and magnetic field both feature some 2D-part (i.e. depending only on the horizontal space variables).

We prove for weak and strong, solutions, that the limit system, when the Rossby number goes to zero (i.e. for strong rotation), is a 2D-MHD system *with three components*. Moreover we are able to provide explicit global-in-time convergence rates thanks to adapted Strichartz estimates and with the help of an additional 3D magnetic field transported by the 2D limit velocity.

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**Keywords:** Magnetohydrodynamics, rotating fluids, Strichartz estimates, Besov and Sobolev spaces.

## 1 Introduction

### 1.1 The magnetohydrodynamic system

The magnetohydrodynamic (MHD) systems describe the evolution of the flow of charged particles under the influence of the magnetic field induced by its own motion. We can find examples of these flows in the form of plasmas in nuclear fusion reactors or in the form of charged liquid metals in the core of the Earth, which are responsible for the Earth's magnetic field. These systems show the different ways to combine the Navier-Stokes equations, which describes the behavior of a viscous incompressible fluid, and the Maxwell equations, which form the the foundation of classical electromagnetism together with the Lorentz force law. We refer the reader to the short survey [3] and the full monograph [4] for the diverse couplings of these two systems of partial differential equations and for the rigorous derivation of MHD systems from the Vlasov-Maxwell-Boltzmann system.

Let us briefly recall the derivation of the MHD system which is considered in this paper. The evolution of a viscous fluid is governed by the classical Navier-Stokes equations

$$\begin{cases} \partial_t U + U \cdot \nabla U - \nu \Delta U = -\nabla P + F, \\ \operatorname{div} U = 0, \end{cases} \quad (1.1)$$

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where  $\nu > 0$  is the viscosity of the fluid, the velocity  $U(t, x)$ , the pressure  $p(t, x)$  are unknown, the forcing term  $F(t, x)$  is given and where  $t \in \mathbb{R}_+$  and  $x \in \mathbb{R}^3$  stand for the time and the space variables. We also suppose that the fluid evolves in the whole space  $\mathbb{R}^3$ . For a conducting fluid,  $F$  represents the Lorentz force produced by the movements of its charged particles

$$F = nE + j \wedge B,$$

where  $n(t, x)$  is the electric charge density,  $j(t, x)$  the electric current and where  $E$  and  $B$  are respectively the electric and the magnetic fields, which are determined by the quasi-static Maxwell's equations

$$\begin{cases} \partial_t B + \nabla \wedge E = 0, & \operatorname{div} B = 0, \\ \nabla \wedge B = j, & \operatorname{div} E = 0. \end{cases}$$

Note that, in this quasi-static approximation of classical Maxwell's equations, the displacement current density  $\partial_t E$  is neglected, which is reasonable in physics (see [27]). Together with this consideration, Ampère's law  $j = \nabla \wedge B$  implies that  $\operatorname{div} j = 0$  and that the electric charge density  $n$  is constant for all time  $t$ . Therefore, for the sake of simplicity, we suppose that  $n = 0$ . Finally, we recall that the current density  $j$  is also related to the electric and the magnetic fields through Ohm's law

$$j = \sigma(E + U \wedge B),$$

where  $\sigma > 0$  denotes the electrical conductivity.

Putting all the these considerations together, we obtain the following MHD equations

$$\begin{cases} \partial_t U - \nu \Delta U + U \cdot \nabla U = -\nabla P + (\nabla \wedge B) \wedge B, \\ \partial_t B - \frac{1}{\sigma} \Delta B = \nabla \wedge (U \wedge B), \\ \operatorname{div} U = 0, \operatorname{div} B = 0. \end{cases}$$

Now, classical identities show that

$$(\nabla \wedge B) \wedge B = B \cdot \nabla B - \frac{1}{2} \nabla |B|^2,$$

and

$$\nabla \wedge (U \wedge B) = B \cdot \nabla U - U \cdot \nabla B + (\operatorname{div} B)U - (\operatorname{div} U)B.$$

Thus, taking into account the fact that  $\operatorname{div} U = 0$  and  $\operatorname{div} B = 0$  and setting  $\nu' = \frac{1}{\sigma}$ , we can rewrite the MHD system as

$$\begin{cases} \partial_t U - \nu \Delta U + U \cdot \nabla U - B \cdot \nabla B = -\nabla \tilde{P}, \\ \partial_t B - \nu' \Delta B + U \cdot \nabla B - B \cdot \nabla U = 0, \\ \operatorname{div} U = 0, \operatorname{div} B = 0. \end{cases} \quad (\text{MHD})$$

**Remark 1.1** *The condition  $\operatorname{div} B = 0$  seems to make System (MHD) over-determined. However, thanks to the incompressibility condition  $\operatorname{div} U = 0$ , we remark that  $\operatorname{div} B$  satisfies a linear equation of the type*

$$\partial_t f - \nu' \Delta f + U \cdot \nabla f = 0, \quad (1.2)$$

and so, if  $\operatorname{div} B(0) = 0$  then the condition  $\operatorname{div} B(t) = 0$  automatically holds true for any time  $t$ .

There is a vast literature devoted to the study of various models taking into account the MHD coupling, especially to the study of the classical MHD system (MHD). Remark that like the Navier-Stokes system (1.1), (MHD) is also parabolic and due to a good symmetry of the nonlinear terms, one can easily prove the same type of energy inequality as for (1.1), say

$$\frac{1}{2} \left( \|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \right) + \int_0^t \left( \nu \|\nabla U(s)\|_{L^2}^2 + \frac{1}{\sigma} \|\nabla B(s)\|_{L^2}^2 \right) ds \leq \frac{1}{2} \left( \|U(0)\|_{L^2}^2 + \|B(0)\|_{L^2}^2 \right),$$

which enables the classical construction of Leray weak solutions. In [30], Duvaut and Lions proved the global existence of weak solutions to (MHD) if the initial data  $U_0, B_0 \in L^2(\mathbb{R}^3)$  and the local existence of a unique strong solution to (MHD) if  $U_0, B_0 \in H^m(\mathbb{R}^3)$ , with  $m \geq 3$ . In [43], the regularity of weak and strong solutions of (MHD) were studied by Sermange and Temam in  $H^1(\mathbb{R}^3)$ . For more results concerning the regularity of solutions of (MHD), we refer to [1, 9, 25, 34, 35, 39].

In the inviscid case where there are no diffusion ( $\nu = \nu' = 0$ ), the theory for hyperbolic systems applies to (MHD) and we can obtain similar results as in the case of Euler equations (System (1.1) with  $\nu = 0$ ). We refer to [8, 49] for more details concerning the inviscid case and the inviscid limit of the MHD system. Contrary to the case where there are full diffusion in both equations of  $U$  and of  $B$  or the case where there are no diffusion at all, the case where there are only partial diffusion in one or both equations is a challenging problem in both physical and mathematical points of view (see for instance [10, 11, 31, 39]).

## 1.2 The rotating MHD system

In this paper, we consider the case of MHD flows in fast rotation. Such models are interesting to describe the evolution of the plasmas in the core of the Earth or in the core of stars (see for instance [28] and the references therein for physical motivations). Recall that the evolution of a vector field  $V$  in an absolute reference frame and a rotating frame attached to the Earth (of angular velocity  $\Omega$ ) is given by the formula

$$(\partial_t V)_A = (\partial_t V)_R + \Omega \wedge V.$$

Direct calculations show that

$$U_A = U_R + \Omega \wedge r,$$

where  $r$  is the position of a fluid particle and  $U_A, U_R$  its velocity written respectively in the two references. The acceleration is then given by

$$(\partial_t U_A)_A = (\partial_t U_R)_R + 2\Omega \wedge U_R + \Omega \wedge (\Omega \wedge r),$$

where  $2\Omega \wedge U_R$  describes the Coriolis acceleration and  $\Omega \wedge (\Omega \wedge r) = -\frac{1}{2}\nabla |\Omega \wedge r|^2$  describes the centrifugal acceleration, which can be combined with the pressure. Let  $\varepsilon = \frac{1}{2|\Omega|}$  the Rossby number and suppose that  $\Omega = |\Omega| e_3$ , where  $e_3 = (0, 0, 1)$ . By replacing  $U_R$  by  $u_\varepsilon$ ,  $B$  by  $b_\varepsilon$ , we come to the following rotating MHD system

$$\begin{cases} \partial_t u_\varepsilon - \nu \Delta u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon \wedge e_3 + u_\varepsilon \cdot \nabla u_\varepsilon - b_\varepsilon \cdot \nabla b_\varepsilon = -\nabla p_\varepsilon, \\ \partial_t b_\varepsilon - \nu' \Delta b_\varepsilon + u_\varepsilon \cdot \nabla b_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon = 0, \\ \operatorname{div} u_\varepsilon = 0, \\ (u_\varepsilon, b_\varepsilon)|_{t=0} = (u_{0,\varepsilon}, b_{0,\varepsilon}). \end{cases} \quad (MHD_\varepsilon)$$

The unknowns are  $U_\varepsilon = (u_\varepsilon, b_\varepsilon) = (u_\varepsilon^1, u_\varepsilon^2, u_\varepsilon^3, b_\varepsilon^1, b_\varepsilon^2, b_\varepsilon^3)$ , where we recall that  $u_\varepsilon$  denotes the velocity of the fluid,  $b_\varepsilon$  the magnetic field, and  $p_\varepsilon$  the pressure, which gathers the classical pressure

term and the centrifugal force. The diffusion coefficients  $\nu, \nu'$  are positive ( $\nu$  is the kinematic viscosity).

Let us recall previous results focusing on the asymptotics when the Rossby number goes to zero. In the case where there is no magnetic field ( $b_\varepsilon \equiv 0$ ),  $(MHD_\varepsilon)$  reduces to the classical rotating fluid system (also called Navier-Stokes-Coriolis system)

$$\begin{cases} \partial_t u_\varepsilon - \nu \Delta u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon \wedge e_3 + u_\varepsilon \cdot \nabla u_\varepsilon = -\nabla p_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0, \\ u_\varepsilon|_{t=0} = u_{0,\varepsilon}. \end{cases} \quad (NSC_\varepsilon)$$

This system has been intensively studied over the last thirty years so we will only briefly recall some of the most significant results in our opinion. For a complete survey on the Navier-Stokes and the Navier-Stokes-Coriolis systems, we point the reader to the book [38, 24]. We remark that the skew-symmetric operator  $\cdot \wedge e_3$  does not contribute to classical energy estimates and so one can prove the global existence of Leray weak solutions and the local existence (global for small data) and uniqueness of Fujita-Kato strong solutions in a similar way as for the 3D Navier-Stokes system.

Let us remark that mathematically, the only way to deal with the singular perturbation  $\frac{1}{\varepsilon} u_\varepsilon \wedge e_3$  is to balance it with the pressure. This formally implies that at the limit  $\varepsilon \rightarrow 0$ , the pressure is independent of the vertical variable  $x_3$ , which in turn implies that the fluid velocity is independent of  $x_3$ , taking into account that the fluid is incompressible. The fluid has tendency to move in ‘‘columns’’, as predicted by what geophysicists call the Taylor-Proudman theorem. In the case of periodic domains, this singular perturbation was first studied by Babin, Mahalov and Nicolaenko [5] and Grenier [33] and then also by Gallagher [32] using Schochet’s method. In these works, it is proved that the fast rotation stabilizes the system and when  $\varepsilon \rightarrow 0$ ,  $(NSC_\varepsilon)$  tends towards a 2D Navier-Stokes system with 3 components (sometimes also called 2D +  $\frac{1}{2}$ ). Moreover, when  $\varepsilon$  is small enough, the strong solution of  $(NSC_\varepsilon)$  globally exists in time, even for large data.

In the case of the whole space  $\mathbb{R}^3$ , things are quite different. If we consider the wave equations associated to the Coriolis term  $\frac{1}{\varepsilon} u_\varepsilon \wedge e_3$

$$\partial_t W + \frac{1}{\varepsilon} \mathbb{P}(W \wedge e_3) = 0,$$

where  $\mathbb{P}$  stands for the Leray projection from  $L^2$  onto the subspace of divergence-free vector fields then, physically, this perturbation creates waves of non-vanishing amplitude and high speed  $\mathcal{O}(\varepsilon^{-1})$  which propagate through the medium (the so-called Rossby waves in physics). These waves go to infinity and carry out the energy of the system. In [22, 23], Chemin *et al.* proved similar result as in the periodic case by using Strichartz estimates on the associated wave system, which indeed characterizes the energy dispersion previously described. Note that in accordance to the Taylor-Proudman theorem, the fluid has a 2D behavior at the limit  $\varepsilon \rightarrow 0$ , and the only vector field, which is of  $L^2$  finite energy in  $\mathbb{R}^3$  and independent of  $x_3$ , is zero.

Thus, in [22, 23], to put in evidence the 2D limit system with 3 components, the authors considered initial data of the form  $2D + 3D$ , which we call **non-conventional** data. Similar results of convergence are obtained through the use of Strichartz estimates for various systems: among them the primitive system and the strongly stratified Boussinesq system. We refer to the following subsection for details.

The rotating MHD system  $(MHD_\varepsilon)$  is more difficult to deal with because the magnetic field  $b_\varepsilon$  does not *a priori* possess any dispersion. Inspired by the work of Desjardins, Dormy and Grenier [28], by considering the case where  $b_\varepsilon$  is a small perturbation of  $e_3$ , in [7], Benameur

*et al.* studied the convergence of strong solutions of  $(MHD_\varepsilon)$  when  $\varepsilon \rightarrow 0$  in both cases where the domain is  $\mathbb{T}^3$  or  $\mathbb{R}^3$  and for evanescent viscosity ( $\nu = \varepsilon$ ). In [41], Ngo considered  $(MHD_\varepsilon)$  with no vertical diffusion and only a small horizontal diffusion, in the case where  $b_\varepsilon$  is a small perturbation of  $e_3$  and proved that for  $\varepsilon$  close to 0,  $(MHD_\varepsilon)$  is globally well-posed for large initial data, using meticulous Strichartz estimates on the associated wave system. Also using Strichartz estimates, in [2], Ahn, Kim and Lee proved similar results as in [41] for system  $(MHD_\varepsilon)$  without the hypothesis that  $b_\varepsilon$  is a small perturbation of  $e_3$  (see also [36, 47]). We also refer to the work of Cobb and Fanelli [26] concerning the asymptotics of the rotating MHD system in the density-dependent case. Finally, we want to mention the works of Desjardins, Dormy and Grenier [28] and of Rousset [42] on the stability of the Ekman-Hartmann boundary layers for  $(MHD_\varepsilon)$ .

### 1.3 Statement of the results

#### 1.3.1 Hidden asymptotics

As was first observed in [22], systems (in the whole space  $\mathbb{R}^3$ ) featuring penalized terms and subject to dispersion may converge to a incomplete limit model when appropriate initial data are not considered.

The first example is the Rotating fluids system  $(NSC_\varepsilon)$ : when the initial data is a **conventional** data for the 3D-Navier-Stokes model (that is  $v_{0,\varepsilon}(x) = v_{0,\varepsilon}(x_1, x_2, x_3)$ ), then the solutions converge to zero. As proved in [22], richer asymptotics are reached for **non-conventional** initial data: if the initial data is of the form  $v_{0,\varepsilon}(x_h, x_3) + \tilde{u}_0(x_h)$  (where  $x_h = (x_1, x_2)$  stands for the horizontal space variables) the solutions converge, as the Rossby number goes to zero, to the unique solution of the 2D-Navier-Stokes system with three components (and whose initial data is  $\tilde{u}_0$ ). This is the reason why we can speak of hidden asymptotics. We refer to [23, 24, 18, 40] for refined studies in this direction.

Another example is the strongly stratified Boussinesq system which only takes into account the influence of gravity (the small parameter is called the Froude number and is linked to the buoyancy effect). For a conventional initial data  $U_{0,\varepsilon}(x) = (v_{0,\varepsilon}(x), \theta_{0,\varepsilon}(x))$  the solutions converge to the unique solution of the 3D-Navier-Stokes system with two components (see [44]). Surprisingly, this limit model does not depend anymore on the thermal diffusivity  $\nu'$  involved in the equation on the potential temperature  $\theta_\varepsilon$ , which suggests that there are some hidden asymptotics. It is the object of [19, 20] to study this phenomenon: getting a complete limit system requires us to consider **non-conventional** initial data of the form  $U_{0,\varepsilon}(x) + (0, 0, 0, \theta_{0,\varepsilon}(x_3))$  and the previous limit system is then completed by a 1D-heat equation on  $\theta_\varepsilon$  whose diffusion term involves  $\nu'$ .

Concerning the primitive system, which takes into account both the rotation of the Earth and the influence of gravity, there does not seem to be some hidden asymptotics as the limit system, called the 3D-quasigeostrophic system, genuinely depends on each parameter. In this case, any result will strongly rely on the structure of the limit system. We refer to [21, 12, 13, 14, 18, 16, 17, 18] for detailed studies.

We will not give much details about the analogous works in a torus or between planes because they do not involve dispersion but different mechanisms (and precise study of resonances).

In the present work, we focus on System  $(MHD_\varepsilon)$  and we wish to provide a limit system as general as possible. This will be possible when considering the following initial data:

$$(u_\varepsilon, b_\varepsilon)|_{t=0} = (\tilde{u}_0(x_h) + v_{0,\varepsilon}(x), \tilde{b}_0(x_h) + c_0(x)). \quad (1.3)$$

Assuming that  $v_{0,\varepsilon}$  is bounded we obtain as a limit the following 2D MHD system (depending on

the horizontal variables) with three components:

$$\begin{cases} \partial_t \tilde{u} - \nu \Delta_h \tilde{u} + \tilde{u}^h \cdot \nabla_h \tilde{u} - \tilde{b}^h \cdot \nabla_h \tilde{b} = - \begin{pmatrix} \nabla_h \tilde{q}^0 \\ 0 \end{pmatrix}, \\ \partial_t \tilde{b} - \nu' \Delta_h \tilde{b} + \tilde{u}^h \cdot \nabla_h \tilde{b} - \tilde{b}^h \cdot \nabla_h \tilde{u} = 0, \\ \operatorname{div}_h \tilde{u} = 0, \\ (\tilde{u}, \tilde{b})|_{t=0} = (\tilde{u}_0, \tilde{b}_0), \end{cases} \quad (2D - MHD^3)$$

completed by a second magnetic field transported by the previous velocity  $\tilde{u}$ :

$$\begin{cases} \partial_t c - \nu' \Delta c + \tilde{u} \cdot \nabla c - c \cdot \nabla \tilde{u} = 0, \\ c|_{t=0} = c_0. \end{cases} \quad (3D - M)$$

**Remark 1.2** *As in Remark 1.1, as  $\operatorname{div} c_0 = 0$ , then  $\operatorname{div} c(t) = 0$  for all time.*

Let us first introduce the spaces we will use in this article: for  $s \in \mathbb{R}$ ,  $d \in \{2, 3\}$  and  $T > 0$  we define:

$$\dot{E}_T^s(\mathbb{R}^3) = \left[ \mathcal{C}_T(\dot{H}^s(\mathbb{R}^3)) \cap L_T^2(\dot{H}^{s+1}(\mathbb{R}^3)) \right]^3, \quad \dot{F}_T^s(\mathbb{R}^d) = \left[ \mathcal{C}_T(\dot{H}^s(\mathbb{R}^d)) \cap L_T^2(\dot{H}^{s+1}(\mathbb{R}^d)) \right]^6$$

whose norms are defined as follows:

$$\begin{cases} \|u\|_{\dot{E}_T^s}^2 \stackrel{def}{=} \|u\|_{L_T^\infty \dot{H}^s}^2 + \nu \int_0^T \|u(\tau)\|_{\dot{H}^{s+1}}^2 d\tau, \\ \|(u, b)\|_{\dot{E}_T^s}^2 \stackrel{def}{=} \|u\|_{L_T^\infty \dot{H}^s}^2 + \|b\|_{L_T^\infty \dot{H}^s}^2 + \nu \int_0^T \|u(\tau)\|_{\dot{H}^{s+1}}^2 d\tau + \nu' \int_0^T \|b(\tau)\|_{\dot{H}^{s+1}}^2 d\tau, \end{cases} \quad (1.4)$$

where  $H^s(\mathbb{R}^d)$  and  $\dot{H}^s(\mathbb{R}^d)$  respectively denote the inhomogeneous and homogeneous Sobolev spaces of index  $s \in \mathbb{R}$ . When  $T = \infty$  we simply write  $\dot{E}^s$  or  $\dot{F}^s$  and the corresponding norms are understood as taken over  $\mathbb{R}_+$  in time.

We can now state the main results of the present article in the following sections.

### 1.3.2 Existence and convergence for Weak solutions (in the sense of Leray)

**Theorem 1.1** *(Existence of Leray weak solutions) Let  $\varepsilon > 0$  be fixed. For any  $\tilde{u}_0, \tilde{b}_0 \in L^2(\mathbb{R}^2)$ , any  $v_{0,\varepsilon} \in L^2(\mathbb{R}^3)$  and any  $c_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$  (with  $\operatorname{div}_h \tilde{u}_0 = \operatorname{div}_h \tilde{b}_0 = \operatorname{div} v_{0,\varepsilon} = \operatorname{div} c_0 = 0$ ), the respective solutions  $(\tilde{u}, \tilde{b})$  and  $c$  of Systems (2D - MHD<sup>3</sup>) and (3D - M) globally exist in  $\dot{F}^0(\mathbb{R}^2)$  and  $\dot{E}^0 \cap \dot{E}^{\frac{1}{2}}(\mathbb{R}^3)$  and there exists a global weak solution  $U_\varepsilon = (u_\varepsilon, b_\varepsilon) \in \dot{F}^0(\mathbb{R}^3)$  of (MHD <sub>$\varepsilon$</sub> ) with initial data as in (1.3). The pressure satisfies  $p_\varepsilon \in L^{\frac{8}{3}}(\mathbb{R}_+, L^2(\mathbb{R}^3)) + L^2(\mathbb{R}_+, L^2(\mathbb{R}^3)) + L^2(\mathbb{R}_+, L^2(\mathbb{R}^2)) + \dot{E}^1(\mathbb{R}^3)$ .*

Moreover there exists a constant  $\mathbb{D}_0 > 0$  depending on  $\nu, \nu', \|\tilde{u}_0\|_{L^2(\mathbb{R}^2)}, \|\tilde{b}_0\|_{L^2(\mathbb{R}^2)}$  and  $\|c_0\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}$  such that if we define  $D_\varepsilon \stackrel{def}{=} U_\varepsilon - (\tilde{u}, \tilde{b} + c)$ , then for all  $t \geq 0$ ,

$$\|D_\varepsilon(t)\|_{\dot{E}^0} \leq \mathbb{D}_0 (\|v_{0,\varepsilon}\|_{L^2}^2 + 1). \quad (1.5)$$

The convergence is described as follows:

**Theorem 1.2** (Convergence) For any  $\mathbb{C}_0 \geq 1$ , any divergence-free vectorfields  $\tilde{u}_0, \tilde{b}_0 \in L^2(\mathbb{R}^2)$ ,  $c_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$  and (for all  $\varepsilon > 0$ )  $v_{0,\varepsilon} \in L^2(\mathbb{R}^3)$  such that:

$$\max\left(\|(\tilde{u}_0, \tilde{b}_0)\|_{L^2(\mathbb{R}^2)}, \|c_0\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}, \sup_{\varepsilon>0} \|v_{0,\varepsilon}\|_{L^2(\mathbb{R}^3)}\right) \leq \mathbb{C}_0, \quad (1.6)$$

then any weak solution  $U_\varepsilon$  of  $(MHD_\varepsilon)$  (with initial data as in (1.3)) such that  $D_\varepsilon = U_\varepsilon - (\tilde{u}, \tilde{b} + c)$  satisfies (1.5) converges (as  $\varepsilon$  goes to zero) to  $(\tilde{u}, \tilde{b} + c)$  in the following sense:

1. For any  $r \in ]2, 6[$  there exists a constant  $\mathbb{F}_0$  (depending on  $\nu, \nu', \mathbb{C}_0, r$ ) such that:

$$\|u_\varepsilon - \tilde{u}\|_{\tilde{L}^{\max(2, \frac{4r}{5(r-2)})} \dot{B}_{r,2}^0 + \tilde{L}^{\max(1, \frac{2r}{3r-5})} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 \varepsilon^{\frac{1}{2r} \min(r-2, 3-\frac{r}{2})}. \quad (1.7)$$

Moreover we have the simpler (but local-in-time) estimates: for any  $r \in ]2, 6[$  there exists a similar constant  $\mathbb{F}_0$  such that for any  $t \geq 0$ ,

$$\|u_\varepsilon - \tilde{u}\|_{L_t^2 L^r} \leq \mathbb{F}_0 \max(1, t)^{\delta_r} \varepsilon^{m_r}, \quad (1.8)$$

where  $\delta_r \stackrel{def}{=} \frac{1}{4r} \max(10 - 3r, \frac{5}{7}(6 - r))$  and  $m_r \stackrel{def}{=} \frac{1}{2r} \min(r - 2, \frac{1}{7}(6 - r))$ .

2. Seconds,  $b_\varepsilon$  converges towards  $\tilde{b} + c$  in the following sense: for any  $r \in ]2, 6[$ ,

$$\|b_\varepsilon - \tilde{b} - c\|_{L_{loc}^2(\mathbb{R}_+, L_{loc}^r(\mathbb{R}^3))} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

**Remark 1.3** 1. We can say the previous results reach **hidden asymptotics** in the sense that when there are no 2D-parts in the initial data ( $\tilde{u}_0 = \tilde{v}_0 = 0$ ), the solution of system  $(2D - MHD^3)$  reduces to zero, and System  $(3D - M)$  reduces to the heat equation (transport by the 2D-limit velocity  $\tilde{u}$  vanish). This is the context of [7, 2, 36, 47], see also [7].

2. The previous results can be extended to initial data as follows:

$$(u_\varepsilon, b_\varepsilon)|_{t=0} = (\tilde{u}_0(x_h) + v_{0,\varepsilon}(x), \tilde{b}_0(x_h) + c_{0,\varepsilon}(x)). \quad (1.9)$$

If  $(c_{0,\varepsilon})_{\varepsilon>0}$  is uniformly bounded, existence remains unchanged and convergence concerns  $U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon)$  where  $c_\varepsilon$  solves  $(3D - M)$  with initial data  $c_{0,\varepsilon}$ .

3. We can also consider an initial 2D-part  $(\tilde{u}_{0,\varepsilon}, \tilde{b}_{0,\varepsilon})$  depending on  $\varepsilon$  and converging to  $(\tilde{u}_0, \tilde{b}_0)$ .
4. With slight adaptation, we can add in the first three lines of System  $(MHD_\varepsilon)$  an external force term of the form  $\tilde{f}(x_h) + f_\varepsilon(x)$  where  $\tilde{f} \in L^1 \dot{H}^{-1}(\mathbb{R}^2)$  and  $(f_\varepsilon)_{\varepsilon>0}$  uniformly bounded in  $\varepsilon$  in  $L^1 \dot{H}^{-1}(\mathbb{R}^3)$ .
5. With another slight adaptation of our proof, we can allow  $\|v_{0,\varepsilon}\|_{L^2} \leq \mathbb{C}_0 \varepsilon^{-\gamma}$  (with  $\gamma$  bounded by a constant depending on the chosen  $r$ ) and still have as convergence rate a power of  $\varepsilon$ .

### 1.3.3 Existence and convergence for Strong solutions (in the sense of Fujita and Kato)

In this part, we directly consider a general initial data as in (1.9), and  $c$  is now replaced by  $c_\varepsilon$ , solving:

$$\begin{cases} \partial_t c_\varepsilon - \nu' \Delta c_\varepsilon + \tilde{u} \cdot \nabla c_\varepsilon - c_\varepsilon \cdot \nabla \tilde{u} = 0, \\ c_\varepsilon|_{t=0} = c_{0,\varepsilon}. \end{cases} \quad (3D - M_\varepsilon)$$

**Theorem 1.3** For any  $\mathbb{C}_0 \geq 1$  (size), any  $\delta \in ]0, \frac{1}{6}]$  (extra regularity), any  $\gamma \in [0, \frac{5}{12}\delta]$ , there exists  $\varepsilon_0 > 0$  and  $\mathbb{K}_0, \mathbb{F}_0 \geq 1$  (depending on  $\nu, \nu', \mathbb{C}_0, \delta, \gamma, s$ ) such that for any  $\varepsilon \in ]0, \varepsilon_0]$  and any initial data as in (1.9) such that  $\tilde{u}_0, \tilde{b}_0 \in H^\delta(\mathbb{R}^2)$ ,  $v_{0,\varepsilon}, c_{0,\varepsilon} \in (\dot{H}^{\frac{1}{2}-\delta}(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)) \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)$  with (for some  $\mu \in [0, 1[$  as close to 1 as we wish):

$$\|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\mu\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma}, \text{ and } \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}} \leq (\mathbb{K}_0 |\ln \varepsilon|)^{\frac{1}{4}},$$

then  $T_\varepsilon^* = +\infty$  and we have the following convergence rate:

$$\|U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon)\|_{L^2 L^\infty} \leq \mathbb{F}_0 \varepsilon^{\frac{1}{18}(\frac{\delta}{2}-\gamma)}.$$

**Remark 1.4** 1. As previously, with slight adaptations of the proof, our result can be extended when:

- the initial 2D-part depends on  $\varepsilon$  and  $(\tilde{u}_{0,\varepsilon}, \tilde{b}_{0,\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} (\tilde{u}_0, \tilde{b}_0)$  in  $H^\delta(\mathbb{R}^2)$ .
  - the velocity part of System (MHD $_\varepsilon$ ) features an external force term of the form  $\tilde{f}(x_h) + f_\varepsilon(x)$  where  $\tilde{f} \in L^1 \dot{H}^{-1} \cap \dot{H}^{-1+\delta}(\mathbb{R}^2)$  and  $(f_\varepsilon)_{\varepsilon>0}$  uniformly bounded in  $\varepsilon$  in  $L^1 \dot{H}^{-\frac{1}{2}} \cap \dot{H}^{-\frac{1}{2}+\delta}(\mathbb{R}^3)$ .
2. It is natural to compare our results with [41, 2, 36, 47] which focus on global existence when  $\varepsilon$  is small enough (but do not study limits) of strong solutions for small **conventional** initial data. The first article focusses on a slightly different MHD model, with anisotropic evanescent viscosities.

Our results consider both weak and strong solutions and generalize the previous cited works as we get global existence and reach hidden limits for **non-conventional** large initial data.

More precisely, if we reword the result from [2] with our notations, without bidimensional parts in the initial data ( $\tilde{u}_0 = \tilde{b}_0 = 0$ ), for any  $\varepsilon > 0$ ,  $\delta \in ]0, \frac{2}{17}[$  there exists  $C > 0$  small enough such that if  $u_{0,\varepsilon} = u_0$  and  $b_{0,\varepsilon} = b_0$  are in  $H^{\frac{1}{2}+\delta}(\mathbb{R}^3)$  (inhomogeneous space) and satisfy the following bounds (we choose  $q = \frac{3}{1-\frac{\delta}{2}}$  and the conditions on  $q, s$  in Theorem 1 from [2] give the bound  $\frac{2}{17}$ )

$$\begin{cases} \|u_0\|_{\dot{H}^{\frac{1}{2}+\delta}} + \|b_0\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq C\varepsilon^{-\frac{\delta}{2}}, \\ (\|u_0\|_{L^2}^2 + \|u_0\|_{L^2}^2)(\|u_0\|_{\dot{H}^{\frac{1}{2}+\delta}}^{1+\frac{2}{\delta}} + \|b_0\|_{\dot{H}^{\frac{1}{2}+\delta}}^{2(1+\frac{2}{\delta})}) \leq C\varepsilon^{-\frac{1+\delta}{2}}, \end{cases}$$

then there exists a global mild solution. In [36] the previous conditions are improved as follows: for any  $\delta \in ]0, \frac{1}{2}]$ , there exists some  $\alpha \in ]0, 2\delta[$  such that for any  $u_0, b_0 \in H^{\frac{1}{2}+\delta}$  with

$$\begin{cases} \|(u_0, b_0)\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq C\varepsilon^{-\frac{\delta}{2}}, \\ \|(u_0, b_0)\|_{L^2}^\alpha \|(u_0, b_0)\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq C\varepsilon^{\frac{\alpha}{4}-\frac{\delta}{2}}, \end{cases}$$

there exists a unique global mild solution. They also provide time-estimates.

Finally, in [47] the authors prove (also by fixed point arguments) that for any  $u_0, b_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$  there exists  $\varepsilon_0$  such that for any  $\varepsilon \in ]0, \varepsilon_0]$ , we have a unique global strong solution. Note that as in [41], in order to reach global existence, the authors introduce an auxiliary system (following the ideas from [13, 41, 16, 18, 20]) that also features the heat equation on the magnetic field component (see also [7]). We refer to Remark 1.3 where we explain that the hidden part of this heat equation is System (3D - M $_\varepsilon$ ).

## 1.4 Outline of the article

The present article unfolds as follows: in the next section, we explain through a formal argument how we may have an intuition about the limit system and the more general initial data (see (1.9)) that has to be considered to reach this limit. Then we perform a change of unknown functions, get the system on which we will work, and study the limit systems.

Sections 3 and 4 are devoted to prove existence and convergence results related to weak solutions (Theorems 1.1 and 1.2). In Section 5, we first state a more precise version for the existence and convergence results for strong solutions (Theorems 5.1 and 5.2) and give their proofs.

In the appendix, we state general notations and results, and give proofs of several results used in the previous sections: estimates for external force terms, isotropic and anisotropic Strichartz estimates.

## 2 The limit system

### 2.1 A candidate for the limit

In this section we explain (in the same spirit as in [19]) how we are formally lead to consider Systems  $(2D - MHD^3)$  and  $(3D - M)$  as a limit. Let us begin with the pressure term: considering the divergence of the first three lines in  $(MHD_\varepsilon)$ , as  $\operatorname{div} u_\varepsilon = 0$  we can decompose  $p_\varepsilon$  and identify its truly penalized part:

$$p_\varepsilon = p_\varepsilon^0 + \frac{1}{\varepsilon} p_\varepsilon^1 \quad \text{with} \quad \begin{cases} \Delta p_\varepsilon^0 = -\sum_{i,j=1}^3 \partial_i \partial_j (u_\varepsilon^i u_\varepsilon^j - b_\varepsilon^i b_\varepsilon^j), \\ \Delta p_\varepsilon^1 = \partial_2 u_\varepsilon^1 - \partial_1 u_\varepsilon^2. \end{cases} \quad (2.10)$$

We can now rewrite System  $(MHD_\varepsilon)$  with well-identified penalized terms:

$$\begin{cases} \partial_t u_\varepsilon^1 - \nu \Delta u_\varepsilon^1 + u_\varepsilon \cdot \nabla u_\varepsilon^1 - b_\varepsilon \cdot \nabla b_\varepsilon^1 = -\partial_1 p_\varepsilon^0 - \frac{1}{\varepsilon} (\partial_1 p_\varepsilon^1 + u_\varepsilon^2), \\ \partial_t u_\varepsilon^2 - \nu \Delta u_\varepsilon^2 + u_\varepsilon \cdot \nabla u_\varepsilon^2 - b_\varepsilon \cdot \nabla b_\varepsilon^2 = -\partial_2 p_\varepsilon^0 - \frac{1}{\varepsilon} (\partial_2 p_\varepsilon^1 - u_\varepsilon^1), \\ \partial_t u_\varepsilon^3 - \nu \Delta u_\varepsilon^3 + u_\varepsilon \cdot \nabla u_\varepsilon^3 - b_\varepsilon \cdot \nabla b_\varepsilon^3 = -\partial_3 p_\varepsilon^0 - \frac{1}{\varepsilon} \partial_3 p_\varepsilon^1, \\ \partial_t b_\varepsilon - \nu' \Delta b_\varepsilon + u_\varepsilon \cdot \nabla b_\varepsilon - b_\varepsilon \cdot \nabla u_\varepsilon = 0. \end{cases}$$

The announced formal argument is the following: let us assume that  $(u_\varepsilon, b_\varepsilon, p_\varepsilon^0, p_\varepsilon^1) \xrightarrow{\varepsilon \rightarrow 0} (\tilde{u}, \tilde{b}, \tilde{p}^0, \tilde{p}^1)$  strongly enough so that we have convergence of the derivatives and non-linear terms on one hand and convergence of the penalized terms on the other hand:

$$\begin{cases} -\frac{1}{\varepsilon} (\partial_1 p_\varepsilon^1 + u_\varepsilon^2) \xrightarrow{\varepsilon \rightarrow 0} \tilde{A}, \\ -\frac{1}{\varepsilon} (\partial_2 p_\varepsilon^1 - u_\varepsilon^1) \xrightarrow{\varepsilon \rightarrow 0} \tilde{B}, \\ -\frac{1}{\varepsilon} \partial_3 p_\varepsilon^1 \xrightarrow{\varepsilon \rightarrow 0} \tilde{C}. \end{cases} \quad (2.11)$$

This immediately implies that the formal limits not only are solutions of the following limit system:

$$\begin{cases} \partial_t \tilde{u} - \nu \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} - \tilde{b} \cdot \nabla \tilde{b} = -\nabla \tilde{p}^0 + (\tilde{A}, \tilde{B}, \tilde{C}), \\ \partial_t \tilde{b} - \nu' \Delta \tilde{b} + \tilde{u} \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla \tilde{u} = 0, \end{cases} \quad (2.12)$$

but also satisfy:

$$\begin{cases} \partial_1 \tilde{p}^1 + \tilde{u}^2 = 0, \\ \partial_2 \tilde{p}^1 - \tilde{u}^1 = 0, \\ \partial_3 \tilde{p}^1 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{div} \tilde{u} = 0 = \operatorname{div} \tilde{b}, \\ \Delta \tilde{p}^0 = -\sum_{i,j=1}^3 \partial_i \partial_j (\tilde{u}^i \tilde{u}^j - \tilde{b}^i \tilde{b}^j), \\ \Delta \tilde{p}^1 = \partial_2 \tilde{u}^1 - \partial_1 \tilde{u}^2. \end{cases}$$

It follows that  $\tilde{p}^1$  and  $(\tilde{u}^1, \tilde{u}^2) = (\partial_2 \tilde{p}^1, -\partial_1 \tilde{p}^1)$  are independent of  $x_3$ . As  $\operatorname{div} \tilde{u} = 0$  we get that  $\tilde{u}$  is also independent of  $x_3$ . Moreover introducing the vorticity  $\omega(\tilde{u}) = \partial_1 \tilde{u}^2 - \partial_2 \tilde{u}^1$  we get  $\omega(\tilde{u}) = -\Delta_h \tilde{p}^1 = -\Delta \tilde{p}^1$ , so that  $\tilde{p}^1 = -\Delta_h^{-1} \omega(\tilde{u})$  and:

$$\tilde{u}^h = \begin{pmatrix} \tilde{u}^1 \\ \tilde{u}^2 \end{pmatrix} = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \Delta_h^{-1} \omega(\tilde{u}) = \nabla_h^\perp \Delta_h^{-1} \omega(\tilde{u}). \quad (2.13)$$

Computing  $\partial_1(2.11)_2 - \partial_2(2.11)_1$  we obtain in addition that

$$-\frac{1}{\varepsilon} \partial_3 u_\varepsilon^3 \xrightarrow{\varepsilon \rightarrow 0} \partial_1 \tilde{B} - \partial_2 \tilde{A} \stackrel{def}{=} \tilde{G}. \quad (2.14)$$

Next, computing  $\partial_1(2.11)_1 + \partial_2(2.11)_2 + \partial_3(2.11)_3$ , we obtain that (when  $\varepsilon \rightarrow 0$ ):

$$\partial_1 \tilde{A} + \partial_2 \tilde{B} + \partial_3 \tilde{C} = 0.$$

Formally solving the resulting system

$$\begin{cases} \partial_1 \tilde{A} + \partial_2 \tilde{B} = -\partial_3 \tilde{C}, \\ -\partial_2 \tilde{A} + \partial_1 \tilde{B} = \tilde{G}, \end{cases}$$

we end-up with:

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = -\nabla_h \Delta_h^{-1} \partial_3 \tilde{C} + \nabla_h^\perp \Delta_h^{-1} \tilde{G},$$

which turns the velocity equation from (2.12) into:

$$\begin{cases} \partial_t \tilde{u}^1 - \nu \Delta_h \tilde{u}^1 + \tilde{u}^h \cdot \nabla_h \tilde{u}^1 = -\partial_1 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^1 - \partial_1 \Delta_h^{-1} \partial_3 \tilde{C} - \partial_2 \Delta_h^{-1} \tilde{G}, \\ \partial_t \tilde{u}^2 - \nu \Delta_h \tilde{u}^2 + \tilde{u}^h \cdot \nabla_h \tilde{u}^2 = -\partial_2 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^2 - \partial_2 \Delta_h^{-1} \partial_3 \tilde{C} + \partial_1 \Delta_h^{-1} \tilde{G} \\ \partial_t \tilde{u}^3 - \nu \Delta_h \tilde{u}^3 + \tilde{u}^h \cdot \nabla_h \tilde{u}^3 = -\partial_3 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^3 + \tilde{C}. \end{cases} \quad (2.15)$$

Using that  $\partial_3 \tilde{u} = 0$ , we can make two useful observations: first, in the third equation we can write

$$\partial_3 (-\partial_3 \tilde{p}^0 + \tilde{b} \cdot \nabla \tilde{b}^3 + \tilde{C}) = 0.$$

Seconds, considering  $\partial_3(2.12)_2$ ,  $\partial_3 \tilde{b}$  obviously satisfies the following linear system:

$$\begin{cases} \partial_t \partial_3 \tilde{b} - \nu' \Delta \partial_3 \tilde{b} + \tilde{u} \cdot \nabla \partial_3 \tilde{b} - \partial_3 \tilde{b} \cdot \nabla \tilde{u} = 0, \\ \partial_3 \tilde{b}|_{t=0} = \partial_3 \tilde{b}_0. \end{cases} \quad (2.16)$$

As a consequence, if  $\partial_3 \tilde{b}_0 = 0$ , then for all time  $\partial_3 \tilde{b}(t) = 0$  and the previous condition turns into

$$\partial_3 (-\partial_3 \tilde{p}^0 + \tilde{C}) = 0. \quad (2.17)$$

This suggests us to impose  $\partial_3 \tilde{b}_0 = 0$  and  $(\tilde{C}, \tilde{G}) = (\partial_3 \tilde{p}^0, 0)$ , which turns the previous velocity equation into:

$$\partial_t \tilde{u} - \nu \Delta_h \tilde{u} + \tilde{u}^h \cdot \nabla_h \tilde{u} - \tilde{b}^h \cdot \nabla \tilde{b} = - \begin{pmatrix} \nabla^h (p^0 + \Delta_h^{-1} \partial_3^2 \tilde{p}^0) \\ 0 \end{pmatrix} = - \begin{pmatrix} \nabla^h \Delta_h^{-1} \Delta \tilde{p}^0 \\ 0 \end{pmatrix}.$$



*Propagation of regularity:* moreover, if in addition  $\tilde{u}_0, \tilde{b}_0 \in \dot{H}^s$  for some  $s \in ]-1, 1[$ , then for any  $t \geq 0$  we have (for  $C_{\nu, \nu'} = C(\frac{1}{\nu} + \frac{1}{\nu'})^2$  where  $C$  is a universal constant),

$$\begin{aligned} \|\tilde{u}(t)\|_{\dot{H}^s(\mathbb{R}^2)}^2 + \|\tilde{b}(t)\|_{\dot{H}^s(\mathbb{R}^2)}^2 + \int_0^t \left( \nu \|\nabla_h \tilde{u}(\tau)\|_{\dot{H}^s(\mathbb{R}^2)}^2 + \nu' \|\nabla_h \tilde{b}(\tau)\|_{\dot{H}^s(\mathbb{R}^2)}^2 \right) d\tau \\ \leq \left( \|\tilde{u}_0\|_{\dot{H}^s(\mathbb{R}^2)}^2 + \|\tilde{b}_0\|_{\dot{H}^s(\mathbb{R}^2)}^2 \right) e^{C_{\nu, \nu'} (\|\tilde{u}_0\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{b}_0\|_{L^2(\mathbb{R}^2)}^2)}, \end{aligned} \quad (2.21)$$

**Proof:** first we recall that the classical 2D-MHD system (without rotation and with values in  $\mathbb{R}^2$ ) can be solved using the same ideas as in the case of the Navier-Stokes system: a priori estimates and Friedrichs-type scheme coupled with the fact that for any functions  $f, g, h$ , if  $\operatorname{div} f = 0$  then

$$(f \cdot \nabla g, h)_{L^2} + (f \cdot \nabla h, g)_{L^2} = 0. \quad (2.22)$$

This is also what we will do in details for System (2.19) below. Similarly as for the limit system of the rotating fluids system, the first two lines of the velocity and magnetic field equations in System  $(2D - MHD^3)$  (2D-MHD with values in  $\mathbb{R}^3$ ) form the usual 2D-MHD system with  $(\tilde{u}_0^h, \tilde{b}_0^h)$  as initial data. Classical results immediately give unique global existence of  $(\tilde{u}^h, \tilde{b}^h)$  (in 2D the Leray and Fujita-Kato solutions coincide). Then, the third lines of each equation are linked through the following linear system (transported by  $\tilde{u}^h, \tilde{b}^h$ ) that we easily globally solve:

$$\begin{cases} \partial_t \tilde{u}^3 - \nu \Delta_h \tilde{u}^3 + \tilde{u}^h \cdot \nabla_h \tilde{u}^3 - \tilde{b}^h \cdot \nabla_h \tilde{b}^3 = 0, \\ \partial_t \tilde{b}^3 - \nu' \Delta_h \tilde{b}^3 + \tilde{u}^h \cdot \nabla_h \tilde{b}^3 - \tilde{b}^h \cdot \nabla_h \tilde{u}^3 = 0, \\ (\tilde{u}^3, \tilde{b}^3)|_{t=0} = (\tilde{u}_0^3, \tilde{b}_0^3), \end{cases}$$

The estimates in  $\dot{H}^s$  are obtained very similarly as for Navier-Stokes. ■

Let us now turn to System  $(3D - M)$ :

**Theorem 2.2** *Let  $\nu' > 0$  and  $\tilde{u} \in \dot{E}^0(\mathbb{R}^2)$ . For any  $s \in ]-1, 1[$  and any  $c_0 \in \dot{H}^s(\mathbb{R}^3)$ , there exists a unique global solution  $c \in \dot{E}^s(\mathbb{R}^3)$  of  $(3D - M)$ . Moreover there exists a constant  $C' > 0$  (depending on  $s$ ) such that  $c$  satisfies for any  $t \geq 0$ :*

$$\begin{aligned} \|c(t)\|_{\dot{H}^s(\mathbb{R}^3)}^2 + \nu' \int_0^t \|\nabla c(\tau)\|_{\dot{H}^s(\mathbb{R}^3)}^2 d\tau \\ \leq \|c_0\|_{\dot{H}^s(\mathbb{R}^3)}^2 \exp \left\{ \frac{C'}{\nu'} \int_0^t \left( 1 + \frac{1}{(\nu')^2} \|\tilde{u}(\tau)\|_{L^2(\mathbb{R}^2)}^2 \right) \|\nabla_h \tilde{u}(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \right\}. \end{aligned} \quad (2.23)$$

**Proof:** this linear equation is easily solved (see also Remark 1.2). To obtain the estimate, we compute the innerproduct in  $\dot{H}^s$  with  $c$ :

$$\frac{1}{2} \frac{d}{dt} \|c\|_{\dot{H}^s}^2 + \nu' \|\nabla c\|_{\dot{H}^s}^2 \leq A_1 + A_2 \stackrel{def}{=} |(\tilde{u} \cdot \nabla c|c)_{\dot{H}^s}| + |(c \cdot \nabla \tilde{u}|c)_{\dot{H}^s}|. \quad (2.24)$$

To deal with products of 2D functions (i.e. depending on  $x_h$ ) with 3D functions, we use Proposition 6.4 (that will impose restrictions for  $s$ , see also Remark (6.2)). Similarly to what we did in [18], we obtain that when  $s \in ]-1, \frac{3}{2}[$ :

$$\begin{aligned} A_1 &\leq \|\tilde{u} \cdot \nabla c\|_{\dot{H}^{s-1}} \|c\|_{\dot{H}^{s+1}} \leq C \|\tilde{u}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \|\nabla c\|_{\dot{H}^{s-\frac{1}{2}}(\mathbb{R}^3)} \|c\|_{\dot{H}^{s+1}(\mathbb{R}^3)} \\ &\leq C \|\tilde{u}\|_{L^2}^{\frac{1}{2}} \|\nabla_h \tilde{u}\|_{L^2}^{\frac{1}{2}} \|c\|_{\dot{H}^s}^{\frac{1}{2}} \|\nabla c\|_{\dot{H}^s}^{\frac{3}{2}} \leq \frac{\nu'}{4} \|\nabla c\|_{\dot{H}^s}^2 + \frac{C'}{(\nu')^3} \|\tilde{u}\|_{L^2}^2 \|\nabla_h \tilde{u}\|_{L^2}^2 \|c\|_{\dot{H}^s}^2, \end{aligned} \quad (2.25)$$

and when  $s \in ]-1, 1[$ :

$$\begin{aligned} A_2 &\leq \|c \cdot \nabla \tilde{u}\|_{\dot{H}^{s-1}} \|c\|_{\dot{H}^{s+1}} \leq C \|c\|_{\dot{H}^s(\mathbb{R}^3)} \|\nabla_h \tilde{u}\|_{L^2(\mathbb{R}^2)} \|\nabla c\|_{\dot{H}^s(\mathbb{R}^3)} \\ &\leq \frac{\nu'}{4} \|\nabla c\|_{\dot{H}^s}^2 + \frac{C'}{\nu'} \|\nabla_h \tilde{u}\|_{L^2}^2 \|c\|_{\dot{H}^s}^2. \end{aligned} \quad (2.26)$$

Using the Gronwall lemma implies (2.23). ■

### 3 Existence of global weak solutions

#### 3.1 A priori estimates

In this section we prove:

**Proposition 3.1** *Let  $\varepsilon > 0$  fixed. There exists a universal constant  $C > 0$  and a constant  $\mathbb{E}_0 = \mathbb{E}_0(\nu, \nu', \|\tilde{u}_0\|_{L^2}, \|\tilde{b}_0\|_{L^2}) > 0$  such that for any solution  $D_\varepsilon = (v_\varepsilon, d_\varepsilon) \in \dot{F}^0(\mathbb{R}^3)$  of (2.19), we have for all  $t \geq 0$ ,*

$$\begin{aligned} \|D_\varepsilon\|_{\dot{F}^0}^2 &\stackrel{def}{=} \|v_\varepsilon\|_{L_t^\infty L^2}^2 + \|d_\varepsilon\|_{L_t^\infty L^2}^2 + \nu \|\nabla v_\varepsilon\|_{L_t^2 L^2}^2 + \nu' \|\nabla d_\varepsilon\|_{L_t^2 L^2}^2 \\ &\leq \left[ \|v_{0,\varepsilon}\|_{L^2}^2 + C \left(1 + \frac{1}{\nu}\right) \left( (1 + \|\tilde{b}\|_{L_t^\infty L^2}^2) \|\nabla c\|_{L_t^2 L^2}^2 + \|\nabla_h \tilde{b}\|_{L_t^2 L^2}^2 \|c\|_{L_t^\infty L^2}^2 \right) \right] \\ &\quad \times \exp \left\{ C \left(1 + \frac{1}{\nu} + \frac{1}{\nu'}\right) \left( \|\nabla_h \tilde{u}\|_{L_t^2 L^2}^2 + \|\nabla_h \tilde{b}\|_{L_t^2 L^2}^2 + \|\nabla c\|_{L_t^2 \dot{H}^{\frac{1}{2}}}^2 \right) \right\} \\ &\leq \mathbb{E}_0 \left[ \|v_{0,\varepsilon}\|_{L^2}^2 + \|c_0\|_{L^2}^2 \right] e^{\mathbb{E}_0 \|c_0\|_{\dot{H}^{\frac{1}{2}}}^2} \end{aligned} \quad (3.27)$$

**Proof:** computing the innerproduct in  $L^2$  of the velocity part of (2.19) with  $v_\varepsilon$ , and of the magnetic part with  $d_\varepsilon$ , thanks to (2.22), we obtain that for all  $t \geq 0$ ,

$$\frac{1}{2} \frac{d}{dt} (\|v_\varepsilon\|_{L^2}^2 + \|d_\varepsilon\|_{L^2}^2) + \nu \|\nabla v_\varepsilon\|_{L^2}^2 + \nu' \|\nabla d_\varepsilon\|_{L^2}^2 \leq \sum_{i=1}^9 |B_i|, \quad (3.28)$$

where we put

$$\begin{cases} B_1 \stackrel{def}{=} (v_\varepsilon \cdot \nabla \tilde{u}|v_\varepsilon)_{L^2}, & B_2 \stackrel{def}{=} (d_\varepsilon \cdot \nabla \tilde{b}|v_\varepsilon)_{L^2}, & B_3 \stackrel{def}{=} (d_\varepsilon \cdot \nabla c|v_\varepsilon)_{L^2}, \\ B_4 \stackrel{def}{=} (d_\varepsilon \cdot \nabla \tilde{u}|d_\varepsilon)_{L^2}, & B_5 \stackrel{def}{=} (v_\varepsilon \cdot \nabla \tilde{b}|d_\varepsilon)_{L^2}, & B_6 \stackrel{def}{=} (v_\varepsilon \cdot \nabla c|d_\varepsilon)_{L^2}, \\ B_7 \stackrel{def}{=} (\tilde{b} \cdot \nabla c|v_\varepsilon)_{L^2}, & B_8 \stackrel{def}{=} (c \cdot \nabla \tilde{b}|v_\varepsilon)_{L^2}, & B_9 \stackrel{def}{=} (c \cdot \nabla c|v_\varepsilon)_{L^2}. \end{cases}$$

With the same  $2D \times 3D$  product laws used to prove (2.23) but for  $s = 0$ , we obtain that

$$\begin{aligned} |B_1| &\leq \|v_\varepsilon \cdot \nabla \tilde{u}\|_{\dot{H}^{-1}} \|v_\varepsilon\|_{\dot{H}^1} \leq \|v_\varepsilon\|_{L^2(\mathbb{R}^3)} \|\nabla_h \tilde{u}\|_{L^2(\mathbb{R}^2)} \|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^3)} \\ &\leq \frac{\nu}{6} \|\nabla v_\varepsilon\|_{L^2}^2 + \frac{C}{\nu} \|\nabla_h \tilde{u}\|_{L^2}^2 \|v_\varepsilon\|_{L^2}^2. \end{aligned} \quad (3.29)$$

Similarly, we obtain:

$$\begin{cases} |B_2| \leq \frac{\nu}{6} \|\nabla v_\varepsilon\|_{L^2}^2 + \frac{C}{\nu} \|\nabla_h \tilde{b}\|_{L^2}^2 \|d_\varepsilon\|_{L^2}^2, \\ |B_4| + |B_5| \leq \frac{\nu'}{3} \|\nabla d_\varepsilon\|_{L^2}^2 + \frac{C}{\nu'} \|\nabla_h \tilde{u}\|_{L^2}^2 \|d_\varepsilon\|_{L^2}^2 + \frac{C}{\nu'} \|\nabla_h \tilde{b}\|_{L^2}^2 \|v_\varepsilon\|_{L^2}^2, \\ |B_7| + |B_8| \leq \frac{\nu}{6} \|\nabla v_\varepsilon\|_{L^2}^2 + \frac{C}{\nu} \|\tilde{b}\|_{L^2}^2 \|\nabla c\|_{L^2}^2 + \frac{C}{\nu} \|\nabla_h \tilde{b}\|_{L^2}^2 \|c\|_{L^2}^2. \end{cases}$$

The last three terms do not feature products with 2D functions and will require additional regularity from  $c$ , we obtain:

$$\begin{cases} |B_3| + |B_6| \leq \|d_\varepsilon\|_{L^6} \|\nabla c\|_{L^3} \|v_\varepsilon\|_{L^2} \leq \frac{\nu'}{6} \|\nabla d_\varepsilon\|_{L^2}^2 + \frac{C}{\nu'} \|\nabla c\|_{\dot{H}^{\frac{1}{2}}}^2 \|v_\varepsilon\|_{L^2}^2, \\ |B_9| \leq \|c\|_{L^6} \|\nabla c\|_{L^3} \|v_\varepsilon\|_{L^2} \leq \frac{1}{2} \|\nabla c\|_{L^2}^2 + \frac{1}{2} \|\nabla c\|_{\dot{H}^{\frac{1}{2}}}^2 \|v_\varepsilon\|_{L^2}^2. \end{cases}$$

Gathering the previous estimates into (3.28), we obtain that:

$$\begin{aligned} \frac{d}{dt} (\|v_\varepsilon\|_{L^2}^2 + \|d_\varepsilon\|_{L^2}^2) + \nu \|\nabla v_\varepsilon\|_{L^2}^2 + \nu' \|\nabla d_\varepsilon\|_{L^2}^2 \leq \\ C \left(1 + \frac{1}{\nu} + \frac{1}{\nu'}\right) (\|v_\varepsilon\|_{L^2}^2 + \|d_\varepsilon\|_{L^2}^2) \left( \|\nabla_h \tilde{u}\|_{L^2}^2 + \|\nabla_h \tilde{b}\|_{L^2}^2 + \|\nabla c\|_{\dot{H}^{\frac{1}{2}}}^2 \right) \\ + C \left(1 + \frac{1}{\nu}\right) \left( \|\tilde{b}\|_{L^2}^2 \|\nabla c\|_{L^2}^2 + \|\nabla_h \tilde{b}\|_{L^2}^2 \|c\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 \right), \end{aligned} \quad (3.30)$$

which gives (3.27) thanks to the Gronwall lemma, (2.20) and (2.23). ■

### 3.2 Proof of Theorem 1.1

We will not give details here and refer to [19] where we adapt classical arguments to prove the existence of Leray-type weak solutions obtained as limits of a Friedrichs scheme: taking advantage of the divergence-free condition, we express the pressure term which allows to define and solve a scheme involving frequency truncation operators  $P_n$ . Using the a priori estimates, we are able to prove the solutions are bounded and global. Then we prove that the solution of the scheme converge (as  $n$  goes to infinity) towards the desired solution.

Compared to [19] where we had to deal with products of the type  $f(x_1, x_2, x_3)g(x_3)$  (requiring regularity indices  $s < \frac{1}{2}$  in product laws), the arguments are simpler as we multiply by functions of the type  $g(x_1, x_2)$  (which only requires  $s < 1$ ). In particular we outline that in [20] the restrictions on coefficients forced us to ask for additional low frequency assumptions on the initial data to obtain existence of strong Fujita-Kato solutions.

Let us outline that the change in the pressure comes from the fact that its expression features additional terms compared to the case of Navier-Stokes: the 2D-pressure (in  $L^2L^2(\mathbb{R}^2)$ ), the rotational term (in  $\dot{E}^1(\mathbb{R}^3)$ ), and the bilinear terms involving products of 3D by 2D-functions (in  $L^2L^2(\mathbb{R}^3)$ ).

## 4 Convergence

This section is devoted to the proof of Theorem 1.2. We first prove that the velocity strongly converges to zero. Then we use it to prove convergence of the magnetic field part.

### 4.1 Velocity

The equation satisfied by the velocity is the following:

$$\begin{aligned} \partial_t v_\varepsilon - \nu \Delta v_\varepsilon + \frac{1}{\varepsilon} \mathbb{P}(v_\varepsilon \wedge e_3) = \mathbb{P} \left[ -v_\varepsilon \cdot \nabla v_\varepsilon - v_\varepsilon \cdot \nabla \tilde{u} - \tilde{u} \cdot \nabla v_\varepsilon \right. \\ \left. + d_\varepsilon \cdot \nabla d_\varepsilon + d_\varepsilon \cdot \nabla (\tilde{b} + c) + (\tilde{b} + c) \cdot \nabla d_\varepsilon + \tilde{b} \cdot \nabla c + c \cdot \nabla \tilde{b} + c \cdot \nabla c \right] \end{aligned} \quad (4.31)$$

We wish here to adapt the argument from [19] and obtain a global-in-time estimate for  $v_\varepsilon$ . As in [18, 19] as the initial data and various external force terms do not share the same regularity, we will split  $v_\varepsilon = v_\varepsilon^1 + v_\varepsilon^2 + v_\varepsilon^3$  where the new functions satisfy:

$$\begin{cases} \partial_t v_\varepsilon^1 - \nu \Delta v_\varepsilon^1 + \frac{1}{\varepsilon} \mathbb{P}(v_\varepsilon^1 \wedge e_3) = \mathbb{P} \left[ -v_\varepsilon \cdot \nabla v_\varepsilon + d_\varepsilon \cdot \nabla d_\varepsilon + d_\varepsilon \cdot \nabla c + c \cdot \nabla d_\varepsilon + c \cdot \nabla c \right] \stackrel{def}{=} G_{\varepsilon,1} \\ v_\varepsilon^1|_{t=0} = 0, \end{cases} \quad (4.32)$$

$$\begin{cases} \partial_t v_\varepsilon^2 - \nu \Delta v_\varepsilon^2 + \frac{1}{\varepsilon} \mathbb{P}(v_\varepsilon^2 \wedge e_3) = \mathbb{P} \left[ -v_\varepsilon \cdot \nabla \tilde{u} - \tilde{u} \cdot \nabla v_\varepsilon + d_\varepsilon \cdot \nabla \tilde{b} + \tilde{b} \cdot \nabla d_\varepsilon + \tilde{b} \cdot \nabla c + c \cdot \nabla \tilde{b} \right] \stackrel{def}{=} G_{\varepsilon,2} \\ v_\varepsilon^2|_{t=0} = 0, \end{cases} \quad (4.33)$$

$$\begin{cases} \partial_t v_\varepsilon^3 - \nu \Delta v_\varepsilon^3 + \frac{1}{\varepsilon} \mathbb{P}(v_\varepsilon^3 \wedge e_3) = 0 \\ v_\varepsilon^3|_{t=0} = v_{0,\varepsilon}, \end{cases} \quad (4.34)$$

Thanks to the first point of Proposition 6.6 (with  $F_{ext} = G_{ext} = 0$ ,  $\theta \in [0, 1]$  and  $\sigma_1 = 0$ ), we obtain that for any  $r \geq 2$ , any  $p \in [1, \frac{2}{\theta(1-\frac{2}{r})}]$  such that

$$\frac{2}{p} = \frac{3}{2} - \frac{3}{r} + \theta \left(1 - \frac{2}{r}\right), \quad (4.35)$$

there exists a constant  $C_{\theta,r,p,\nu} > 0$  such that we have:

$$\|v_\varepsilon^3\|_{\tilde{L}^p \dot{B}_{r,2}^0} \leq C_{\theta,r,p,\nu} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \|v_{0,\varepsilon}\|_{L^2}.$$

Moreover for any  $r \geq 2$ ,  $\theta \in ]0, 1]$ , and any  $p$  satisfying (4.35), we have:

$$p \in [1, \frac{2}{\theta(1-\frac{2}{r})}] \iff \frac{\theta}{2} \left(1 - \frac{2}{r}\right) \leq \frac{1}{p} \leq 1 \iff r \geq 2 \text{ and } \frac{3+2\theta}{r} \geq \theta - \frac{1}{2}.$$

And finally, for any  $\theta \in ]0, 1]$  we have:

$$\|v_\varepsilon^3\|_{\tilde{L}^{\frac{4}{(3+2\theta)(1-\frac{2}{r})}} \dot{B}_{r,2}^0} \leq C_{\theta,r,\nu} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \|v_{0,\varepsilon}\|_{L^2}, \text{ for any } \begin{cases} r \in [2, \infty] & \text{if } \theta \in [0, \frac{1}{2}], \\ r \in [2, 2 + \frac{4}{\theta-\frac{1}{2}}] & \text{if } \theta \in ]\frac{1}{2}, 1]. \end{cases} \quad (4.36)$$

We now turn to  $v_\varepsilon^2$ : using the divergence-free condition and the modified product laws from Proposition 6.4, we have for some small  $\eta > 0$ :

$$\begin{aligned} \|v_\varepsilon \cdot \nabla \tilde{u}\|_{\dot{H}^{-\frac{1}{2}}} &= \|\operatorname{div}(v_\varepsilon \otimes \tilde{u})\|_{\dot{H}^{-\frac{1}{2}}} \leq \|v_\varepsilon \otimes \tilde{u}\|_{\dot{H}^{-\frac{1}{2}}} \leq C \|v_\varepsilon\|_{\dot{H}^{1-\eta}} \|\tilde{u}\|_{\dot{H}^{\frac{1}{2}+\eta}} \\ &\leq C \|v_\varepsilon\|_{L^2(\mathbb{R}^3)}^\eta \|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^3)}^{1-\eta} \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}-\eta} \|\nabla_h \tilde{u}\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}+\eta}. \end{aligned} \quad (4.37)$$

The second term in the product is  $L^{\frac{2}{1-\eta}}$  in time, the last one is  $L^{\frac{2}{\frac{1}{2}+\eta}}$ , the rest being  $L^\infty$ , we obtain (thanks to (3.27)) that for some constant depending on  $\nu$ :

$$\|v_\varepsilon \cdot \nabla \tilde{u}\|_{L^{\frac{4}{3}} \dot{H}^{-\frac{1}{2}}} \leq C \|v_\varepsilon\|_{L^\infty L^2}^\eta \|\nabla v_\varepsilon\|_{L^2 L^2}^{1-\eta} \|\tilde{u}\|_{L^\infty L^2}^{\frac{1}{2}-\eta} \|\nabla_h \tilde{u}\|_{L^2 L^2}^{\frac{1}{2}+\eta} \leq C \|D_\varepsilon\|_{\dot{F}^0(\mathbb{R}^3)} \|\tilde{u}\|_{\dot{B}^0(\mathbb{R}^2)}. \quad (4.38)$$

The other terms in  $G_{\varepsilon,2}$  are estimated similarly, so that for some constant  $\mathbb{E}_0 > 0$  (depending on  $\nu, \nu', \|\tilde{u}_0\|_{L^2}, \|\tilde{b}_0\|_{L^2}$ ) we have:

$$\|G_{\varepsilon,2}\|_{L^{\frac{4}{3}} \dot{H}^{-\frac{1}{2}}} \leq \mathbb{E}_0 \left[ \|v_{0,\varepsilon}\|_{L^2}^2 + \|c_0\|_{L^2}^2 \right]^{\frac{1}{2}} e^{\mathbb{E}_0 \|c_0\|_{\dot{H}^{\frac{1}{2}}}^2} \quad (4.39)$$

Thanks to the second point of Proposition 6.6 (with  $(F_{ext}, G_{ext}) = (0, G_{\varepsilon,2})$ ,  $\theta \in ]0, 1]$ ,  $k = \frac{4}{3}$  and  $\sigma_2 = -\frac{1}{2}$ ), we obtain that for any  $r \geq 2$ , any  $p \in [2, \frac{2}{\theta(1-\frac{2}{r})}]$  [also satisfying (4.35) there exists a constant  $C_{\theta,r,p,\nu} > 0$  such that:

$$\|v_\varepsilon^2\|_{\tilde{L}^p \dot{B}_{r,2}^0} \leq C_{\theta,r,p,\nu} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \|G_{\varepsilon,2}\|_{L^{\frac{4}{3}} \dot{H}^{-\frac{1}{2}}}.$$

Similarly, for any  $r \geq 2$ ,  $\theta \in ]0, 1]$ , and any  $p$  defined as in (4.35), we have:

$$p \in [2, \frac{2}{\theta(1-\frac{2}{r})}] \iff r \in ]2, 2 + \frac{2}{\theta + \frac{1}{2}}],$$

therefore, for any  $\theta \in ]0, 1]$ , any  $r \in ]2, 2 + \frac{2}{\theta + \frac{1}{2}}]$  we have:

$$\|v_\varepsilon^2\|_{\tilde{L}^{\frac{4}{(3+2\theta)(1-\frac{2}{r})}} \dot{B}_{r,2}^0} \leq C_{\theta,r,\nu} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \|G_{\varepsilon,2}\|_{L^{\frac{4}{3}} \dot{H}^{-\frac{1}{2}}}. \quad (4.40)$$

Gathering (4.40), (4.36) and (4.39) (and using the assumption (1.6)) we obtain that for any  $\theta \in ]0, 1]$ , any  $r \in ]2, 2 + \frac{2}{\theta + \frac{1}{2}}]$  there exists a constant  $\mathbb{F}_0$  (depending on  $\nu, \nu', \mathbb{C}_0, \theta, r$ ) such that:

$$\|v_\varepsilon^2 + v_\varepsilon^3\|_{\tilde{L}^{\frac{4}{(3+2\theta)(1-\frac{2}{r})}} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})}.$$

As  $0 < \theta \leq 1$ , the upper bound  $2 + \frac{2}{\theta + \frac{1}{2}}$  lies in  $[\frac{10}{3}, 6]$ , so for any  $r \in ]2, 6]$ , there exists some  $\theta \in ]0, 1]$  such that  $r \in ]2, 2 + \frac{2}{\theta + \frac{1}{2}}]$  if, and only if,  $\theta \leq \frac{6-r}{2(r-2)}$ . Of course, this latter quantity goes to infinity when  $r > 2$  goes to 2, so introducing

$$\theta_r \stackrel{def}{=} \min(1, \frac{6-r}{2(r-2)})$$

and the following quantity (lying in  $[2, \infty[$ ):

$$\alpha_r \stackrel{def}{=} \frac{4}{(3+2\theta_r)(1-\frac{2}{r})} = \max(2, \frac{4}{5(1-\frac{2}{r})}), \quad (4.41)$$

then for any  $r \in ]2, 6]$  there exists a constant  $\mathbb{F}_0$  (now depending on  $r, \nu, \nu', \mathbb{C}_0$ ) such that:

$$\|v_\varepsilon^2 + v_\varepsilon^3\|_{\tilde{L}^{\alpha_r} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 \varepsilon^{\frac{\theta_r}{2}(1-\frac{2}{r})} = \mathbb{F}_0 \varepsilon^{\frac{1}{2r} \min(r-2, 3-\frac{r}{2})}. \quad (4.42)$$

Turning to  $v_\varepsilon^1$ , thanks to the classical Sobolev product laws, we have:

$$\|v_\varepsilon \cdot \nabla v_\varepsilon\|_{\dot{H}^{-\frac{1}{2}}} \leq C \|v_\varepsilon\|_{\dot{H}^1} \|\nabla v_\varepsilon\|_{L^2} \leq C \|\nabla v_\varepsilon\|_{L^2}^2.$$

The other terms in  $G_{\varepsilon,1}$  are estimated similarly, and we obtain that:

$$\|G_{\varepsilon,1}\|_{L^1 \dot{H}^{-\frac{1}{2}}} \leq C (\|\nabla v_\varepsilon\|_{L^2 L^2}^2 + \|\nabla d_\varepsilon\|_{L^2 L^2}^2 + \|\nabla c\|_{L^2 L^2}^2) \leq \mathbb{E}_0 [\|v_{0,\varepsilon}\|_{L^2}^2 + \|c_0\|_{L^2}^2] e^{\mathbb{E}_0 \|c_0\|_{\dot{H}^{\frac{1}{2}}}^2}.$$

Using once more the first point of Proposition 6.6 (with  $(F_{ext}, G_{ext}) = (G_{\varepsilon,1}, 0)$ ,  $\theta \in ]0, 1]$  and  $\sigma_1 = -\frac{1}{2}$ ), we obtain that for any  $r \geq 2$ , any  $p \in [1, \frac{2}{\theta(1-\frac{2}{r})}]$  such that

$$\frac{2}{p} = 2 - \frac{3}{r} + \theta(1 - \frac{2}{r}), \quad (4.43)$$

there exists a constant  $C_{\theta,r,p,\nu} > 0$  such that we have:

$$\|v_\varepsilon^1\|_{\tilde{L}^p \dot{B}_{r,2}^0} \leq C_{\theta,r,p,\nu} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \|G_{\varepsilon,1}\|_{L^1 \dot{H}^{-\frac{1}{2}}}.$$

Similarly, for any  $r \geq 2$ ,  $\theta \in [0, 1]$  and  $p$  satisfying Condition (4.43), then

$$p \in [1, \frac{2}{\theta(1-\frac{2}{r})}] \iff r \in [2, 2 + \frac{3}{\theta}].$$

We can conclude similarly as before that for any  $\theta \in [0, 1]$  and  $r \in [2, 2 + \frac{3}{\theta}]$ , we have :

$$\|v_\varepsilon^1\|_{\tilde{L}^{\frac{4}{1+(3+2\theta)(1-\frac{2}{r})}} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})}, \quad (4.44)$$

Thus for any  $r > 2$ , there exists some  $\theta \in [0, 1]$  such that  $r \leq 2 + \frac{3}{\theta}$  if, and only if  $\theta \leq \frac{3}{r-2}$  so that defining

$$\theta'_r \stackrel{def}{=} \min(1, \frac{3}{r-2})$$

and the following quantity (lying in  $[1, 4]$ ):

$$\beta_r \stackrel{def}{=} \frac{4}{1 + (3 + 2\theta'_r)(1 - \frac{2}{r})} = \max(1, \frac{2}{3 - \frac{5}{r}}), \quad (4.45)$$

then for any  $r > 2$ , there exists a constant  $\mathbb{F}_0$  (depending on  $r, \nu, \nu', \mathbb{C}_0$ ) such that:

$$\|v_\varepsilon^1\|_{\tilde{L}^{\beta_r} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 \varepsilon^{\frac{\theta'_r}{2}(1-\frac{2}{r})} = \mathbb{F}_0 \varepsilon^{\frac{1}{2r} \min(r-2, 3)}. \quad (4.46)$$

Gathering (4.42) and (4.46) for  $r \in ]2, 6[$  gives (1.7). Let us now turn to the second simpler (and local-in-time) estimates from Theorem 1.2.

Coming back to (4.41) and (4.42), for any  $r \in ]2, 6[$  we have  $\alpha_r \geq 2$  and thanks to Propositions 6.2 and 6.1 we obtain that for any  $r \in ]2, 6[$ :

$$\|v_\varepsilon^2 + v_\varepsilon^3\|_{L^{\alpha_r} L^r} \leq C \|v_\varepsilon^2 + v_\varepsilon^3\|_{L^{\alpha_r} \dot{B}_{r,2}^0} \leq C \|v_\varepsilon^2 + v_\varepsilon^3\|_{\tilde{L}^{\alpha_r} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 \varepsilon^{\frac{1}{2r} \min(r-2, 3-\frac{r}{2})}, \quad (4.47)$$

which leads, for any  $r \in ]2, 6[$  and  $t \geq 0$  to:

$$\begin{aligned} \|v_\varepsilon^2 + v_\varepsilon^3\|_{L_t^2 L^r} &\leq t^{\frac{1}{2} - \frac{1}{\alpha_r}} \|v_\varepsilon^2 + v_\varepsilon^3\|_{L^{\alpha_r} L^r} \\ &\leq \mathbb{F}_0 t^{\frac{1}{2} - \frac{1}{\alpha_r}} \varepsilon^{\frac{1}{2r} \min(r-2, 3-\frac{r}{2})} = \mathbb{F}_0 t^{\max(0, \frac{5}{2r} - \frac{3}{4})} \varepsilon^{\frac{1}{2r} \min(r-2, 3-\frac{r}{2})}. \end{aligned} \quad (4.48)$$

Let us recall that, thanks to Proposition 6.1, the injection  $\tilde{L}_t^p \dot{B}_{r,2}^0 \hookrightarrow L_t^p \dot{B}_{r,2}^0$  is true if, and only if  $p \geq 2$ . Considering (4.45), we have to restrict the domain in  $r$ :  $\beta_r \geq 2 \iff r \in ]2, \frac{5}{2}[$ , and in this case we have  $\beta_r = \frac{2}{3-\frac{5}{r}}$  and thanks to (4.46),

$$\|v_\varepsilon^1\|_{L_t^2 L^r} \leq t^{\frac{1}{2} - \frac{1}{\beta_r}} \|v_\varepsilon^1\|_{L_t^{\beta_r} L^r} \leq t^{\frac{1}{2} - \frac{1}{\beta_r}} \|v_\varepsilon^1\|_{\tilde{L}^{\beta_r} \dot{B}_{r,2}^0} \leq \mathbb{F}_0 t^{\frac{5}{2r} - 1} \varepsilon^{\frac{1}{2} - \frac{1}{r}}.$$

Gathering this with (4.48), we obtain that for any  $r \in ]2, \frac{5}{2}[$ , there exists a constant  $\mathbb{F}_0$  (depending on  $\nu, \nu', \mathbb{C}_0, r$ ) such that for any  $t \geq 0$ ,

$$\begin{aligned} \|v_\varepsilon\|_{L_t^2 L^r} &\leq \|v_\varepsilon^1\|_{L_t^2 L^r} + \|v_\varepsilon^2 + v_\varepsilon^3\|_{L_t^2 L^r} \leq \mathbb{F}_0 t^{\frac{5}{2r} - 1} \varepsilon^{\frac{1}{2} - \frac{1}{r}} + \mathbb{F}_0 t^{\frac{5}{2r} - \frac{3}{4}} \varepsilon^{\frac{1}{2r} \min(r-2, 3-\frac{r}{2})} \\ &\leq \mathbb{F}_0 (t^{\frac{5}{2r} - 1} + t^{\frac{5}{2r} - \frac{3}{4}}) \varepsilon^{\frac{1}{2} - \frac{1}{r}} \leq \mathbb{F}_0 \max(1, t) \varepsilon^{\frac{5}{2r} - \frac{3}{4}} \varepsilon^{\frac{1}{2} - \frac{1}{r}} \end{aligned} \quad (4.49)$$

For  $r \in [\frac{5}{2}, 6[$ , we interpolate the previous estimates for  $r = \frac{5}{2}$ ,

$$\|v_\varepsilon\|_{L_t^2 L^{\frac{5}{2}}} \leq \mathbb{F}_0 \max(1, t) \varepsilon^{\frac{1}{4}} \varepsilon^{\frac{1}{10}},$$

with the following estimates (given by (3.27)):

$$\|v_\varepsilon\|_{L_t^2 L^6} \leq C \|v_\varepsilon\|_{L^2 \dot{H}^1} \leq \mathbb{F}_0,$$

and we obtain that for any  $r \in [\frac{5}{2}, 6]$ ,

$$\|v_\varepsilon\|_{L_t^2 L^r} \leq \mathbb{F}_0 \max(1, t)^{\frac{5}{28}(\frac{6}{r}-1)} \varepsilon^{\frac{1}{14}(\frac{6}{r}-1)}. \quad (4.50)$$

Combining (4.49) with (4.50), we finally obtain that for any  $r \in ]2, 6[$ ,

$$\|v_\varepsilon\|_{L_t^2 L^r} \leq \mathbb{F}_0 \max(1, t)^{\max(\frac{5}{27} - \frac{3}{4}, \frac{5}{28}(\frac{6}{r}-1))} \varepsilon^{\min(\frac{1}{2} - \frac{1}{r}, \frac{1}{14}(\frac{6}{r}-1))},$$

which gives (1.8). ■

## 4.2 Magnetic field

We now turn to the magnetic-field part: thanks to (3.27),  $d_\varepsilon$  is uniformly bounded (with respect to  $\varepsilon$ ) and we can extract a subsequence  $d_{\varepsilon'}$  that weakly converges to some  $d \in L^\infty L^2 \cap L^2 \dot{H}^1$  and satisfies the following system:

$$\begin{cases} \partial_t d - \nu' \Delta d + \tilde{u} \cdot \nabla d - d \cdot \nabla \tilde{u} = 0, \\ d|_{t=0} = 0 \end{cases}$$

**Remark 4.1** *We do not give details for the weak convergence of the nonlinear terms, for which the arguments are the same as for the classical Navier-Stokes system.*

Thanks to Theorem 2.2, we immediately obtain that  $d \equiv 0$ , and more generally, we obtained that the only possible weak limit of a subsequence is zero which ensures that the whole sequence itself weakly converges to zero. With similar arguments as in the classical proof for the Navier-Stokes system (see also [12] and the end of Section 4.1 in [19]), using once more (3.27) we have that for any  $r \in ]2, 6[$ ,

$$d_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} 0 \text{ in } L_{loc}^2 L_{loc}^r,$$

which ends the proof of Theorem 1.2. ■

## 5 Strong solutions

### 5.1 Precise statement of the results

We first recall that for an initial data as in (1.9), every term  $c$  in System (2.19) is now replaced by  $c_\varepsilon$ .

Let us begin with stating more precisely our main results concerning strong solutions (see the simplified Theorem 1.3 from the introduction): first local existence theorem, then global existence and convergence theorem for very large ill-prepared initial data.

**Theorem 5.1** *(Existence of local Fujita-Kato strong solutions) Let  $\varepsilon > 0$  fixed,  $\tilde{u}, \tilde{b} \in L^2(\mathbb{R}^3)$ ,  $v_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}}$  and  $c_{0,\varepsilon} \in H^{\frac{1}{2}}$ . For any initial data as in (1.9), there exists a unique local solution  $D_\varepsilon$  of (2.19) with lifespan  $T_\varepsilon^* > 0$  such that for any  $T < T_\varepsilon^*$ ,  $D_\varepsilon \in \dot{F}_T^{\frac{1}{2}}$ . Moreover, the following properties are true:*

- *Regularity propagation:* if in addition  $U_{0,\varepsilon}, c_{0,\varepsilon} \in \dot{H}^s$  for some  $s \in ]-1, 1[$  then for any  $t < T_\varepsilon^*$ ,  $D_\varepsilon \in \dot{H}_t^{\frac{1}{2}} \cap \dot{E}_t^s$  and for some  $C = C_{\nu, \nu', \|\tilde{u}_0\|_{L^2}, \|\tilde{b}_0\|_{L^2}} > 0$ ,

$$\|D_\varepsilon\|_{\dot{H}_t^s} \leq C \left( \|v_{0,\varepsilon}\|_{\dot{H}^s}^2 + \|c_{0,\varepsilon}\|_{\dot{H}^s}^2 \right) \times e^{C \left( 1 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^4 \right)} \times \exp \left\{ C \int_0^t (\|\nabla v_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla d_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}}^2) dt' \right\} \quad (5.51)$$

- *Continuation criterion:*  $\int_0^{T_\varepsilon^*} \|\nabla D_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau < \infty \implies T_\varepsilon^* = \infty$ .
- *Global existence for small data:* there exist a constant  $c_0 > 0$  such that if  $\|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}} \leq c_0 \nu$  (we recall that  $d_{\varepsilon|t=0} = 0$ ) then  $T_\varepsilon^* = +\infty$  and for any  $t \geq 0$  (for the same  $C$ ):

$$\|D_\varepsilon\|_{\dot{H}_t^{\frac{1}{2}}} \leq C \left( \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^2 \right) \times e^{C \left( 1 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^2 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}}}^4 \right)}$$

**Proof:** the proof involves estimates similar to what we do below (but simpler) so we will only point out that to obtain the estimate for the regularity propagation, we have to bound a little differently the quadratic terms from system (2.19), for instance:

$$|(v_\varepsilon \cdot \nabla v_\varepsilon | v_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu}{10} \|\nabla v_\varepsilon\|_{\dot{H}^s} + \frac{C}{\nu} \|v_\varepsilon\|_{\dot{H}^s}^2 \|\nabla v_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2. \blacksquare$$

If we wish to obtain convergence for strong solutions, as in the cases of the rotating fluids, primitive or stratified Boussinesq systems, any direct approach involving System (2.19) is bound to fail due to the following term appearing in the velocity equation:  $\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b} + c_\varepsilon \cdot \nabla c_\varepsilon$ . Indeed, this term is bounded but does not go to zero as  $\varepsilon$  goes to zero.

The usual way to reach convergence, first consists in introducing the oscillations  $W_\varepsilon$  that can absorb this constant term, and go to zero thanks to dispersive properties as they solve the following linear system:

$$\begin{cases} \partial_t W_\varepsilon - \nu \Delta W_\varepsilon + \frac{1}{\varepsilon} \mathbb{P}(W_\varepsilon \wedge e_3) = \mathbb{P}(\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b} + c_\varepsilon \cdot \nabla c_\varepsilon), \\ W_\varepsilon|_{t=0} = v_{0,\varepsilon}. \end{cases} \quad (5.52)$$

If we set:

$$\tilde{D}_\varepsilon \stackrel{def}{=} U_\varepsilon - \begin{pmatrix} \tilde{u} + W_\varepsilon \\ \tilde{b} + c_\varepsilon \end{pmatrix} = D_\varepsilon - \begin{pmatrix} W_\varepsilon \\ 0 \end{pmatrix} = \begin{pmatrix} v_\varepsilon - W_\varepsilon \\ d_\varepsilon \end{pmatrix} = \begin{pmatrix} \delta_\varepsilon \\ d_\varepsilon \end{pmatrix},$$

then it satisfies the following system

$$\begin{cases} \partial_t \delta_\varepsilon - \nu \Delta \delta_\varepsilon + \frac{1}{\varepsilon} \mathbb{P}(W_\varepsilon \wedge e_3) \delta_\varepsilon = \sum_{i=1}^{13} F_i, \\ \partial_t d_\varepsilon - \nu' \Delta d_\varepsilon = \sum_{i=1}^{14} G_i, \\ (\delta_\varepsilon, d_\varepsilon)|_{t=0} = (0, 0), \end{cases} \quad (5.53)$$

where we define:

$$\begin{cases} F_1 \stackrel{def}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla \delta_\varepsilon), & F_2 \stackrel{def}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla W_\varepsilon), & F_3 \stackrel{def}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla \delta_\varepsilon), \\ F_4 \stackrel{def}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla W_\varepsilon), & F_5 \stackrel{def}{=} -\mathbb{P}(\delta_\varepsilon \cdot \nabla_h \tilde{u}), & F_6 \stackrel{def}{=} -\mathbb{P}(W_\varepsilon \cdot \nabla_h \tilde{u}), \\ F_7 \stackrel{def}{=} -\mathbb{P}(\tilde{u} \cdot \nabla \delta_\varepsilon), & F_8 \stackrel{def}{=} -\mathbb{P}(\tilde{u} \cdot \nabla W_\varepsilon), & F_9 \stackrel{def}{=} \mathbb{P}(d_\varepsilon \cdot \nabla d_\varepsilon), \\ F_{10} \stackrel{def}{=} \mathbb{P}(d_\varepsilon \cdot \nabla_h \tilde{b}), & F_{11} \stackrel{def}{=} \mathbb{P}(d_\varepsilon \cdot \nabla c_\varepsilon), & F_{12} \stackrel{def}{=} \mathbb{P}(\tilde{b} \cdot \nabla d_\varepsilon), \\ F_{13} \stackrel{def}{=} \mathbb{P}(c_\varepsilon \cdot \nabla d_\varepsilon), & G_1 \stackrel{def}{=} -\delta_\varepsilon \cdot \nabla d_\varepsilon, & G_2 \stackrel{def}{=} -W_\varepsilon \cdot \nabla d_\varepsilon, & G_3 \stackrel{def}{=} d_\varepsilon \cdot \nabla \delta_\varepsilon, \\ G_4 \stackrel{def}{=} d_\varepsilon \cdot \nabla W_\varepsilon, & G_5 \stackrel{def}{=} -\tilde{u} \cdot \nabla d_\varepsilon, & G_6 \stackrel{def}{=} d_\varepsilon \cdot \nabla_h \tilde{u}, \\ G_7 \stackrel{def}{=} -\delta_\varepsilon \cdot \nabla_h \tilde{b}, & G_8 \stackrel{def}{=} -\delta_\varepsilon \cdot \nabla c_\varepsilon, & G_9 \stackrel{def}{=} -W_\varepsilon \cdot \nabla_h \tilde{b}, & G_{10} \stackrel{def}{=} -W_\varepsilon \cdot \nabla c_\varepsilon, \\ G_{11} \stackrel{def}{=} \tilde{b} \cdot \nabla \delta_\varepsilon, & G_{12} \stackrel{def}{=} c_\varepsilon \cdot \nabla \delta_\varepsilon, & G_{13} \stackrel{def}{=} \tilde{b} \cdot \nabla W_\varepsilon, & G_{14} \stackrel{def}{=} c_\varepsilon \cdot \nabla W_\varepsilon. \end{cases} \quad (5.54)$$

**Theorem 5.2** For any  $\mathbb{C}_0 \geq 1$  (size), any  $\delta \in ]0, \frac{1}{6}]$  (extra regularity), any  $\mu \in [0, 1[$  (as close to 1 as we wish), any  $\gamma \in [0, \frac{5}{12}\delta]$ , introducing  $\eta_0$  such that  $\gamma = \frac{\delta}{2}(1 - 2\eta_0)$  (it implies  $\eta_0 \in [\frac{1}{12}, \frac{1}{2}]$ ) there exists  $\varepsilon_0 > 0$  and  $\mathbb{K}_0, \mathbb{F}_0 \geq 1$  (depending on  $\nu, \nu', \mathbb{C}_0, \delta, \gamma$ ) such that for any  $\varepsilon \in ]0, \varepsilon_0]$  and any initial data as in (1.9) with  $\tilde{u}_0, \tilde{b}_0 \in H^\delta(\mathbb{R}^2)$ ,  $(v_{0,\varepsilon}, c_{0,\varepsilon}) \in (\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)) \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)$  satisfying:

$$\|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\mu\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq \mathbb{C}_0 \varepsilon^{-\gamma}, \text{ and } \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}} \leq (\mathbb{K}_0 |\ln \varepsilon|)^{\frac{1}{4}}, \quad (5.55)$$

then  $T_\varepsilon^* = +\infty$  and for any  $s \in [\frac{1}{2}, \frac{1}{2} + \frac{\eta_0}{10}\delta] = [\frac{1}{2}, \frac{1}{2} + \frac{1}{10}(\frac{\delta}{2} - \gamma)]$  we have:

$$\|\tilde{D}_\varepsilon\|_{\dot{E}^s} = \|(\delta_\varepsilon, d_\varepsilon)\|_{\dot{E}^s} \leq \mathbb{F}_0 \varepsilon^{\frac{\eta_0}{18}\delta} = \mathbb{F}_0 \varepsilon^{\frac{1}{18}(\frac{\delta}{2} - \gamma)}. \quad (5.56)$$

Moreover, if  $v_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}-\delta} \cap \dot{H}^{\frac{1}{2}+\delta}$ , the previous estimates remain true for any  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ , we can get rid of the oscillations  $W_\varepsilon$  and get:

$$\|D_\varepsilon\|_{L^2 L^\infty} = \|U_\varepsilon - (\tilde{u}, \tilde{b} + c_\varepsilon)\|_{L^2 L^\infty} \leq \mathbb{F}_0 \varepsilon^{\frac{\eta_0}{18}\delta} = \mathbb{F}_0 \varepsilon^{\frac{1}{18}(\frac{\delta}{2} - \gamma)}. \quad (5.57)$$

**Remark 5.1** 1. Due to the new term  $G_{14}$ , we cannot choose  $\gamma$  as close to  $\frac{\delta}{2}$  as we did in [17, 18, 20]. At best we are able to reach  $\gamma < \frac{3}{7}\delta$  (that is  $\eta_0 \in [\frac{1}{14}, \frac{1}{2}]$ ) but the price to pay is a much smaller power of  $\varepsilon$ . On the opposite, if we want the final convergence rate to be  $\varepsilon^{\frac{\eta_0}{2}\delta}$  (as in [17, 18] for instance) we can consider the case  $\gamma \in [0, \frac{3}{10}\delta]$  ( $\eta_0 \in [\frac{1}{5}, \frac{1}{2}]$ ). For the sake of simplicity we restrict to the case  $\eta_0 \geq \frac{1}{12}$  ( $\gamma \in [0, \frac{5}{12}\delta]$ ).

2. The constant  $\mathbb{K}_0$  is taylored for (5.93) to hold: with  $\mathbb{D}_0$  introduced in (5.87), we have:

$$\mathbb{K}_0 = \frac{\eta_0 \delta}{144 \mathbb{D}_0}.$$

3. The bound  $\varepsilon_0$  is specified in particular in (5.89) and (6.124).

## 5.2 Proof of Theorem 5.2

### 5.2.1 Estimates in $\dot{H}^s$ of external force terms

We know from Theorem 5.1 that there exists a local solution defined on  $[0, T_\varepsilon^*]$ . Assume by contradiction that  $T_\varepsilon^* < +\infty$ , then by the continuation criterion, we have:

$$\int_0^{T_\varepsilon^*} \|\nabla D_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 dt = \int_0^{T_\varepsilon^*} \left( \|\nabla v_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla d_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \right) dt = +\infty. \quad (5.58)$$

We can now define the following time (where the universal constant  $C$  is given below in the bounds involving  $F_1, F_9$  and  $G_1, G_3$ ):

$$T_\varepsilon = \sup\{t \in [0, T_\varepsilon^*[, \forall t' \leq t, \|\delta_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}} + \|d_\varepsilon(t')\|_{\dot{H}^{\frac{1}{2}}} \leq \frac{1}{8C} \min(\nu, \nu')\}. \quad (5.59)$$

As the initial data from System (5.53) is zero, we know that  $T_\varepsilon > 0$ . Let us now assume by contradiction that

$$T_\varepsilon < T_\varepsilon^*. \quad (5.60)$$

Now, performing innerproducts in  $\dot{H}^s$  of System (5.53) with  $(\delta_\varepsilon, d_\varepsilon)$ , we obtain that for all  $t \leq T_\varepsilon$ :

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu \|\nabla \delta_\varepsilon(t)\|_{\dot{H}^s}^2 \leq \sum_{k=1}^{13} |(F_k(t)|\delta_\varepsilon(t))_{\dot{H}^s}|, \\ \frac{1}{2} \frac{d}{dt} \|d_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu' \|\nabla d_\varepsilon(t)\|_{\dot{H}^s}^2 \leq \sum_{k=1}^{14} |(G_k(t)|d_\varepsilon(t))_{\dot{H}^s}|. \end{cases} \quad (5.61)$$

Most of the right-hand-side terms are treated like in the case of the rotating fluids, or very similarly (see [18] for details). We will give minimal details for these terms and focus on the new terms, among them  $F_8, G_{13}$  and especially  $G_{14}$  which will require most of the work.

**Remark 5.2** *Another change is that contrary to the case of rotating fluids, the oscillations  $W_\varepsilon$  now satisfy System (5.52) featuring several external force terms of different regularities, which forces us to systematically use the variant of our Strichartz estimates for time integration index  $k \in ]1, 2]$  (see the second point of Propositions 6.6 and 6.7)*

The following estimates directly come from [18] (see Sections 2.2 and 3.2) using classical Sobolev product laws:

$$\begin{cases} |(F_1|\delta_\varepsilon)_{\dot{H}^s}| \leq C \|\delta_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2, \\ |(F_2|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu} \|\nabla W_\varepsilon\|_{L^3}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2, \\ |(F_3|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^3} \|W_\varepsilon\|_{L^6}^4 \|\delta_\varepsilon\|_{\dot{H}^s}^2, \\ |(F_4|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^{\frac{2}{1-s}}} \|W_\varepsilon\|_{L^6}^{\frac{2}{1-s}} \|\delta_\varepsilon\|_{\dot{H}^s}^2 + C \|\nabla W_\varepsilon\|_{L^3}^2. \end{cases} \quad (5.62)$$

Next  $G_4$  and  $G_2$  are respectively treated like  $F_2$  and  $F_3$ , and  $F_9, G_1$  and  $G_3$  are dealt with as  $F_1$ :

$$\begin{cases} |(G_4|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu'} \|\nabla W_\varepsilon\|_{L^3}^2 \|d_\varepsilon\|_{\dot{H}^s}^2, \\ |(G_2|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{(\nu')^3} \|W_\varepsilon\|_{L^6}^4 \|d_\varepsilon\|_{\dot{H}^s}^2, \\ |(F_9|\delta_\varepsilon)_{\dot{H}^s}| + |(G_3|d_\varepsilon)_{\dot{H}^s}| \leq C \|d_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \left( \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 \right), \\ |(G_1|d_\varepsilon)_{\dot{H}^s}| \leq C \|\delta_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2. \end{cases} \quad (5.63)$$

Now, let us turn to terms involving products of functions depending on the whole space variable  $x = (x_1, x_2, x_3)$  by functions only depending on the horizontal variables  $x_h = (x_1, x_2)$ :  $F_{10}, G_6, G_7$  on one hand,  $G_5$  on the other hand are respectively dealt with as  $F_5$  and  $F_7$  which are the same as in [18] and we get that (using the  $2D \times 3D$  product laws from Proposition 6.4, which requires  $s \in ]-1, 1[$ ):

$$\begin{cases} |(F_5|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu} \|\nabla_h \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2, \\ |(G_6|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu'} \|\nabla_h \tilde{u}\|_{L^2}^2 \|d_\varepsilon\|_{\dot{H}^s}^2, \\ |(F_{10}|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu} \|\nabla_h \tilde{b}\|_{L^2}^2 \|d_\varepsilon\|_{\dot{H}^s}^2, \\ |(G_7|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu'} \|\nabla_h \tilde{b}\|_{L^2}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2, \end{cases} \quad (5.64)$$

and

$$\begin{cases} |(F_7|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^3} \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla_h \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2, \\ |(G_5|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{(\nu')^3} \|\tilde{u}\|_{L^2}^2 \|\nabla_h \tilde{u}\|_{L^2}^2 \|d_\varepsilon\|_{\dot{H}^s}^2. \end{cases} \quad (5.65)$$

Next, we estimate  $F_{12}$  with a similar method as we used for  $F_7$  (the coefficients in the final Young estimates being  $(4, 4, 2)$ ):

$$\begin{aligned} |(F_{12}|\delta_\varepsilon)_{\dot{H}^s}| &\leq \|\tilde{b} \cdot \nabla d_\varepsilon\|_{\dot{H}^{s-1}} \|\delta_\varepsilon\|_{\dot{H}^{s+1}} \leq C \|\tilde{b}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \|\nabla d_\varepsilon\|_{\dot{H}^{s-\frac{1}{2}}(\mathbb{R}^3)} \|\nabla \delta_\varepsilon\|_{\dot{H}^s} \\ &\leq C \|\tilde{b}\|_{L^2}^{\frac{1}{2}} \|\nabla_h \tilde{b}\|_{L^2}^{\frac{1}{2}} \|d_\varepsilon\|_{\dot{H}^s}^{\frac{1}{2}} \|d_\varepsilon\|_{\dot{H}^{s+1}}^{\frac{1}{2}} \|\nabla \delta_\varepsilon\|_{\dot{H}^s} \\ &\leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^2 \nu'} \|\tilde{b}\|_{L^2}^2 \|\nabla_h \tilde{b}\|_{L^2}^2 \|d_\varepsilon\|_{\dot{H}^s}^2. \end{aligned} \quad (5.66)$$

Similarly, we get

$$|(G_{11}|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu(\nu')^2} \|\tilde{b}\|_{L^2}^2 \|\nabla_h \tilde{b}\|_{L^2}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2. \quad (5.67)$$

We skip details for  $F_6, G_9$  which are dealt with as in [18] (we refer to the estimates of the term  $G_7$  from Section 3.2 therein, and to (6.96) for definition of anisotropic Lebesgue spaces) and we get that:

$$\begin{cases} |(F_6|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^{\frac{s}{1-s}}} \|\nabla_h \tilde{u}\|_{L^2}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2 + C \|W_\varepsilon\|_{L^{\infty,2}_{h,v}}^2 \|\nabla_h \tilde{u}\|_{L^2}^{2s}, \\ |(G_9|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{(\nu')^{\frac{s}{1-s}}} \|\nabla_h \tilde{b}\|_{L^2}^2 \|d_\varepsilon\|_{\dot{H}^s}^2 + C \|W_\varepsilon\|_{L^{\infty,2}_{h,v}}^2 \|\nabla_h \tilde{b}\|_{L^2}^{2s}. \end{cases} \quad (5.68)$$

We are now left with the new terms. Some are easily estimated using the previous arguments because they feature  $c_\varepsilon$ , which is very regular:

$$\begin{aligned} |(F_{11}|\delta_\varepsilon)_{\dot{H}^s}| &\leq \|d_\varepsilon \cdot \nabla c_\varepsilon\|_{\dot{H}^{s-1}} \|\delta_\varepsilon\|_{\dot{H}^{s+1}} \leq C \|d_\varepsilon\|_{\dot{H}^s} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \|\nabla \delta_\varepsilon\|_{\dot{H}^s} \\ &\leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|d_\varepsilon\|_{\dot{H}^s}^2. \end{aligned} \quad (5.69)$$

Similarly, we obtain that:

$$|(G_8|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu'} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2. \quad (5.70)$$

Using the classical Sobolev product laws (with  $(s_1, s_2) = (1, s - \frac{1}{2})$ ) instead of  $2D \times 3D$  laws in the method used for  $F_{12}$ , we obtain that:

$$\begin{cases} |(F_{13}|\delta_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^2 \nu'} \|c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|d_\varepsilon\|_{\dot{H}^s}^2, \\ |(G_{12}|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu(\nu')^2} \|c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2. \end{cases} \quad (5.71)$$

Slightly adapting the arguments used for  $F_6, G_9$  (using Young estimates with (6, 3), followed by Sobolev and interpolation estimates, and finally using an algebraic Young estimates with coefficients  $(2, \frac{2}{1-s}, \frac{2}{s})$ ) we obtain:

$$\begin{aligned} |(G_{10}|d_\varepsilon)_{\dot{H}^s}| &\leq \|W_\varepsilon \cdot \nabla c_\varepsilon\|_{L^2} \|d_\varepsilon\|_{\dot{H}^{2s}} \leq C \|W_\varepsilon\|_{L^6} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \|d_\varepsilon\|_{\dot{H}^s}^{1-s} \|d_\varepsilon\|_{\dot{H}^{s+1}}^s \\ &\leq C \left( \|W_\varepsilon\|_{L^6} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^s \right) \left( \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \|d_\varepsilon\|_{\dot{H}^s} \right)^{1-s} \|d_\varepsilon\|_{\dot{H}^{s+1}}^s \\ &\leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{(\nu')^{\frac{s}{1-s}}} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 \|d_\varepsilon\|_{\dot{H}^s}^2 + C \|W_\varepsilon\|_{L^6}^2 \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^{2s}. \end{aligned} \quad (5.72)$$

An analogous of  $F_8$  was already present in [18] but the estimates that we obtained in this article featured  $2s - 1$  as an exponent, which required  $s \geq \frac{1}{2}$ . In order to obtain estimates in the case  $s < \frac{1}{2}$  we had to use another method, which produced an estimate of  $W_\varepsilon$  with time integration index  $\frac{2}{2-s}$ . As outlined in Remark 5.2, contrary to its counterpart ( $LRF_\varepsilon$ ) from [18], System (5.52) now features several external force terms with different regularities, and we will have to use the modified Strichartz estimates from Section 6.3.4, which require time integration index greater than 2. As  $\frac{2}{2-s} \geq 2$  is false when  $s$  is close to  $\frac{1}{2}$  we need to change our estimates. Adapting the arguments used for  $F_6, F_9$  and  $G_{10}$ , and aiming to use the following Sobolev injection:

$$\dot{H}^{1-s}(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{s}}(\mathbb{R}^2),$$

we have (also using interpolation and the Young estimates with coefficients  $(2, \frac{2}{1-s}, \frac{2}{s})$ ):

$$\begin{aligned} |(F_8|\delta_\varepsilon)_{\dot{H}^s}| &\leq \|\tilde{u} \cdot \nabla W_\varepsilon\|_{L^2} \|\delta_\varepsilon\|_{\dot{H}^{2s}} \leq C \|\tilde{u}\|_{L^{\frac{2}{s}}(\mathbb{R}^2)} \|\nabla W_\varepsilon\|_{L^{\frac{2}{1-s}, 2}(\mathbb{R}^3)} \|\delta_\varepsilon\|_{\dot{H}^s}^{1-s} \|\delta_\varepsilon\|_{\dot{H}^{s+1}}^s \\ &\leq C \left( \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^s \|\nabla W_\varepsilon\|_{L^{\frac{2}{1-s}, 2}(\mathbb{R}^3)} \right) \left( \|\tilde{u}\|_{\dot{H}^1(\mathbb{R}^2)} \|\delta_\varepsilon\|_{\dot{H}^s} \right)^{1-s} \|\delta_\varepsilon\|_{\dot{H}^{s+1}}^s \\ &\leq \frac{\nu}{52} \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^{1-s}} \|\nabla_h \tilde{u}\|_{L^2}^2 \|\delta_\varepsilon\|_{\dot{H}^s}^2 + C \|\tilde{u}\|_{L^2}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{2}{1-s}, 2}}^2. \end{aligned} \quad (5.73)$$

**Remark 5.3** *This new estimates is valid independantly of the positions of  $s$  and  $\frac{1}{2}$  and would simplify a little the proofs from [18] in the case of rotating fluids.*

Similarly, we have:

$$|(G_{13}|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{(\nu')^{1-s}} \|\nabla_h \tilde{b}\|_{L^2}^2 \|d_\varepsilon\|_{\dot{H}^s}^2 + C \|\tilde{b}\|_{L^2}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{2}{1-s}, 2}}^2. \quad (5.74)$$

### 5.2.2 Estimating $G_{14}$

We will find how to correctly estimate the last term  $G_{14}$  under the guidance of several constraints:

- As outlined in Remark 5.2, we need to produce Strichartz estimates featuring a time index of integration  $p \geq 2$ .
- We need enough integrability on  $\|c_\varepsilon\|_{\dot{H}^\sigma}$  to be able to evacuate  $\|d_\varepsilon\|_{\dot{H}^s}^2$  through the Gronwall lemma.

First let us state the following result which, in the spirit of Proposition 2 from [18], details how time-integrable is  $\|c_\varepsilon(t)\|_{\dot{H}^\sigma}$  for a given  $\sigma$ .

**Proposition 5.1** *Let  $\delta > 0$ ,  $\tilde{u}, \tilde{b}$  satisfying (2.20), and  $c_\varepsilon$  be the unique global solution of  $(3D - M_\varepsilon)$  corresponding to the regular initial  $c_{0,\varepsilon} \in H^{\frac{1}{2}+\delta}$ . Then for any  $\sigma \in [0, \frac{3}{2} + \delta]$ , we have that  $t \mapsto \|c_\varepsilon(t)\|_{\dot{H}^\sigma}$  is in  $L^p(\mathbb{R}_+)$  for any  $p$  in:*

$$\begin{cases} [\frac{2}{\sigma}, +\infty] & \text{if } \sigma \in [0, \frac{1}{2} + \delta], \\ [\frac{2}{\sigma}, \frac{2}{\sigma - (\frac{1}{2} + \delta)}] & \text{if } \sigma \in ]\frac{1}{2} + \delta, 1], \\ [2, \frac{2}{\sigma - (\frac{1}{2} + \delta)}] & \text{if } \sigma \in ]1, \frac{3}{2} + \delta]. \end{cases} \quad (5.75)$$

Moreover in any case, we have for some constant  $C$  (depending on  $\nu, \nu', \|\tilde{u}_0\|_{L^2}, \|\tilde{b}_0\|_{L^2}$ )

$$\|c_\varepsilon\|_{L^p \dot{H}^\sigma} \leq C \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}.$$

**Proof:** thanks to (2.23) and by interpolation, there exists a constant  $C = C(\nu, \nu', \|\tilde{u}_0\|_{L^2}, \|\tilde{b}_0\|_{L^2}) > 0$  such that for any  $p \in [2, \infty]$ , any  $s \in [0, \frac{1}{2} + \delta]$ ,

$$\|c_\varepsilon\|_{L^p \dot{H}^{s+\frac{2}{p}}} \leq C \|c_0\|_{\dot{H}^s} \leq C \|c_0\|_{H^{\frac{1}{2}+\delta}}.$$

We note that with these assumptions,  $\sigma \stackrel{def}{=} s + \frac{2}{p} \in [0, \frac{3}{2} + \delta]$ . For any  $\sigma \in [0, \frac{3}{2} + \delta]$  and any  $p \in [2, \infty]$  there exists some  $s \in [0, \frac{1}{2} + \delta]$  such that  $\sigma = s + \frac{2}{p}$  if and only if

$$\frac{2}{p} \in [\sigma - (\frac{1}{2} + \delta), \sigma].$$

From this we have several cases:

- If  $\sigma \in [0, \frac{1}{2} + \delta]$ , the previous condition is equivalent to  $\frac{2}{p} \in [0, \sigma]$  that is  $p \in [\frac{2}{\sigma}, \infty]$ .
- If  $\sigma \in ]\frac{1}{2} + \delta, \frac{3}{2} + \delta]$ , the previous condition is equivalent to  $p \in [\frac{2}{\sigma}, \frac{2}{\sigma - (\frac{1}{2} + \delta)}]$ . We have to consider the lower bound of this interval keeping in mind that  $p \in [2, \infty]$ :  $\frac{2}{\sigma} \leq 2 \Leftrightarrow \sigma \geq 1$ , which gives the last two cases from the proposition. ■

Now we can turn to the estimate for  $G_{14}$ : let us begin as follows: for any  $q_1, q_2 \in [2, \infty]$  such that  $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$ , we have

$$|(G_{14}|d_\varepsilon)_{\dot{H}^s}| \leq \|c_\varepsilon \cdot \nabla W_\varepsilon\|_{L^2} \|d_\varepsilon\|_{\dot{H}^{2s}} \leq C \|c_\varepsilon\|_{L^{q_1}} \|\nabla W_\varepsilon\|_{L^{q_2}} \|d_\varepsilon\|_{\dot{H}^s}^{1-s} \|d_\varepsilon\|_{\dot{H}^{s+1}}^s.$$

Thanks to the Sobolev injections in  $\mathbb{R}^3$ , if we ask  $q_1 < \infty$  and put  $\sigma' = 3(\frac{1}{2} - \frac{1}{q_1})$ , then  $\sigma' \in [0, \frac{3}{2}[$  and  $q_2 = \frac{3}{\sigma'}$  so that (using for the third line the Young estimates with coefficients  $(2, \frac{2}{1-s}, \frac{2}{s})$ ) for some constant  $C = C_{s,\sigma}$ ,

$$\begin{aligned} |(G_{14}|d_\varepsilon)_{\dot{H}^s}| &\leq C \|c_\varepsilon\|_{\dot{H}^{\sigma'}} \|\nabla W_\varepsilon\|_{L^{\frac{3}{\sigma'}}} \|d_\varepsilon\|_{\dot{H}^s}^{1-s} \|d_\varepsilon\|_{\dot{H}^{s+1}}^s \\ &\leq C \left( \|c_\varepsilon\|_{\dot{H}^{\sigma'}}^s \|\nabla W_\varepsilon\|_{L^{\frac{3}{\sigma'}}} \right) (\|c_\varepsilon\|_{\dot{H}^{\sigma'}} \|d_\varepsilon\|_{\dot{H}^s})^{1-s} \|d_\varepsilon\|_{\dot{H}^{s+1}}^s \\ &\leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{(\nu')^{1-s}} \|c_\varepsilon\|_{\dot{H}^{\sigma'}}^2 \|d_\varepsilon\|_{\dot{H}^s}^2 + C \|c_\varepsilon\|_{\dot{H}^{\sigma'}}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{3}{\sigma'}}}^2. \end{aligned} \quad (5.76)$$

Notice that this estimate is useful when  $\sigma'$  is such that  $\|c_\varepsilon\|_{\dot{H}^{\sigma'}} \in L^2$  that is (thanks to Proposition 5.1) when  $\sigma' \in [1, \frac{3}{2}[$ . In this case we will have to bound the following integral according to:

$$\int_0^t \|c_\varepsilon\|_{\dot{H}^{\sigma'}}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{3}{\sigma'}}}^2 d\tau \leq \left( \int_0^t \|c_\varepsilon\|_{\dot{H}^{\sigma'}}^{2p_1 s} d\tau \right)^{\frac{1}{p_1}} \left( \int_0^t \|\nabla W_\varepsilon\|_{L^{\frac{3}{\sigma'}}}^{2p_2} d\tau \right)^{\frac{1}{p_2}},$$

where  $\frac{1}{p_1} + \frac{1}{p_2} = 1$ . If we wish to minimize  $p_2$  we need to maximize  $p_1$  so that thanks to Proposition 5.1 we will choose  $2p_1 s = \frac{2}{\sigma' - (\frac{1}{2} + \delta)}$  and we end-up, for some  $\sigma' \in [1, \frac{3}{2}[$  to specify, with:

$$\int_0^t \|c_\varepsilon\|_{\dot{H}^{\sigma'}}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{3}{\sigma'}}}^2 d\tau \leq \|c_\varepsilon\|_{L^{\frac{2s}{\sigma' - (\frac{1}{2} + \delta)}}, \dot{H}^{\sigma'}}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{2}{1-s(\sigma' - (\frac{1}{2} + \delta))}}, L^{\frac{3}{\sigma'}}}^2. \quad (5.77)$$

Thanks to Proposition 6.6, the regularity index that we have to consider applying the first point for  $(d, p, r) = (1, \frac{2}{\sigma' - (\frac{1}{2} + \delta)}, \frac{3}{\sigma'})$  (and for some  $\theta \in ]0, 1[$ ) is

$$\sigma_1 = \frac{3}{2} - \sigma' + s(\sigma' - (\frac{1}{2} + \delta)) + \theta(1 - \frac{2}{3}\sigma'). \quad (5.78)$$

Due to the regularities of the external force terms in  $(3D - M_\varepsilon)$ , we expect to use the bound  $c_\varepsilon \cdot \nabla c_\varepsilon$  in  $L^1 \dot{H}^{\frac{1}{2}+\delta}$  (we refer to the Appendix for details). As  $s$  is close to  $\frac{1}{2}$  and  $\theta$  will be small, taking  $(s, \theta) = (\frac{1}{2}, 0)$  we get that  $\sigma_1 = \frac{5}{4} - \frac{\sigma'}{2} - \frac{\delta}{2}$  so that

$$\sigma_1 \sim \frac{1}{2} + \delta \iff \sigma' \sim \frac{3}{2} - 3\delta,$$

which suggests us to choose  $\sigma' = \frac{3}{2} - \alpha\delta$ . Injecting this in (5.78) we obtain:

$$\sigma_1 = \alpha\delta + s\left(1 - (\alpha + 1)\delta\right) + \frac{2\alpha\delta}{3}\theta,$$

and we can find some  $\theta \in ]0, 1]$  such that  $\sigma_1 = \frac{1}{2} + \delta$  if and only if:

$$\alpha\delta + s\left(1 - (\alpha + 1)\delta\right) < \frac{1}{2} + \delta \leq \frac{5\alpha\delta}{3} + s\left(1 - (\alpha + 1)\delta\right),$$

which is possible for any  $s \in [\frac{1}{2} - \eta\delta, \frac{1}{2} + \eta\delta]$  if, and only if

$$\alpha\delta + \left(\frac{1}{2} + \eta\delta\right)\left(1 - (\alpha + 1)\delta\right) < \frac{1}{2} + \delta \leq \frac{5\alpha\delta}{3} + \left(\frac{1}{2} - \eta\delta\right)\left(1 - (\alpha + 1)\delta\right),$$

that is if, and only if

$$\frac{9 + 6\eta(1 - \delta)}{7 + 6\eta\delta} \leq \alpha < \frac{3 - 2\eta(1 - \delta)}{1 - 2\eta\delta},$$

which in turn is possible when  $\eta(20 - 56\delta) < 12$ , and true as soon as  $\eta < \frac{3}{5}$ . We could optimize the choice of  $\eta$  but, to simplify the presentation, from now on we will fix  $\eta = \frac{70}{10}$ . Moreover, we need  $\alpha$  to be as small as possible (we refer to the appendix for an explanation), so we also fix

$$\alpha = \alpha_0 = \frac{9 + \frac{3}{5}\eta_0(1 - \delta)}{7 + \frac{3}{5}\eta_0\delta}, \quad (5.79)$$

finally leading to the last estimates

$$|(G_{14}|d_\varepsilon)_{\dot{H}^s}| \leq \frac{\nu'}{56} \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 + \frac{C}{\nu^{\frac{s}{1-s}}} \|c_\varepsilon\|_{\dot{H}^{\frac{3}{2}-\alpha_0\delta}}^2 \|d_\varepsilon\|_{\dot{H}^s}^2 + C \|c_\varepsilon\|_{\dot{H}^{\frac{3}{2}-\alpha_0\delta}}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{3}{\frac{3}{2}-\alpha_0\delta}}}^2. \quad (5.80)$$

### 5.2.3 End of the proof

Injecting (5.62) to (5.74) and (5.80) into (5.61), we obtain that for any  $t \leq T_\varepsilon < T_\varepsilon^*$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 \right) + \frac{3}{4} \nu \|\nabla \delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \frac{3}{4} \nu' \|\nabla d_\varepsilon(t)\|_{\dot{H}^s}^2 \\ & \leq C \left( \|\delta_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \|d_\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \right) \left( \|\nabla \delta_\varepsilon\|_{\dot{H}^s}^2 + \|\nabla d_\varepsilon\|_{\dot{H}^s}^2 \right) + J_1(t) \left( \|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 \right) + J_2(t), \end{aligned} \quad (5.81)$$

where

$$\begin{aligned} J_1(t) = & C_{\nu, \nu', s} \left\{ (1 + \|\tilde{u}\|_{L^2(\mathbb{R}^2)}^2) \|\nabla_h \tilde{u}\|_{L^2(\mathbb{R}^2)}^2 + (1 + \|\tilde{b}\|_{L^2(\mathbb{R}^2)}^2) \|\nabla_h \tilde{b}\|_{L^2(\mathbb{R}^2)}^2 \right. \\ & \left. + (1 + \|c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2) \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}-\alpha_0\delta}}^2 + \|\nabla W_\varepsilon\|_{L^3}^2 + \|W_\varepsilon\|_{L^6}^4 + \|W_\varepsilon\|_{L^6}^{\frac{2}{1-2s}} \right\}, \end{aligned} \quad (5.82)$$

and

$$J_2(t) = C_{\nu, \nu', s} \left[ \|\nabla W_\varepsilon\|_{L^3}^2 + \|W_\varepsilon\|_{L_t^\infty L_{h,v}^2}^2 (\|\nabla_h \tilde{u}\|_{L^2}^{2s} + \|\nabla_h \tilde{b}\|_{L^2}^{2s}) + \|W_\varepsilon\|_{L^6}^2 \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}}}^{2s} \right. \\ \left. + (\|\tilde{u}\|_{L^2}^{2s} + \|\tilde{b}\|_{L^2}^{2s}) \|\nabla W_\varepsilon\|_{L_{h,v}^{\frac{2}{1-s}, 2}}^2 + \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2} - \alpha_0 \delta}}^{2s} \|\nabla W_\varepsilon\|_{L^{\frac{3}{\frac{3}{2} - \alpha_0 \delta}}}^2 \right]. \quad (5.83)$$

Thanks to the fact that  $t \leq T_\varepsilon$  (we refer to (5.59)), we can absorb the first term from the right-hand-side of (5.81), so that integrating and using the Gronwall estimates, we obtain that (thanks to (5.77) and the choice in (5.79)) for any  $t \leq T_\varepsilon$  (we recall the initial data is zero),

$$\|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau \\ \leq \left( \int_0^t J_2(t') dt' \right) e^{\int_0^t J_1(t') dt'}, \quad (5.84)$$

where (thanks to (2.20) and (2.23)) we have

$$\int_0^t J_1(t') dt' \leq C_{\nu, \nu', s} \left\{ (1 + \|\tilde{u}\|_{L_t^\infty L^2(\mathbb{R}^2)}^2) \|\nabla_h \tilde{u}\|_{L_t^2 L^2(\mathbb{R}^2)}^2 + (1 + \|\tilde{b}\|_{L_t^\infty L^2(\mathbb{R}^2)}^2) \|\nabla_h \tilde{b}\|_{L_t^2 L^2(\mathbb{R}^2)}^2 \right. \\ \left. + (1 + \|c_\varepsilon\|_{L_t^\infty \dot{H}^{\frac{1}{2}}}^2) \|\nabla c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{1}{2}}}^2 + \|\nabla c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{1}{2} - \alpha_0 \delta}}^2 + \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^4 L^6}^4 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^{\frac{2}{1-s}} \right\} \\ \leq \mathbb{D}_0 \left\{ 1 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^4 + \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^4 L^6}^4 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^{\frac{2}{1-s}} \right\} \quad (5.85)$$

and (thanks to Proposition 5.1)

$$\int_0^t J_2(t') dt' \leq C_{\nu, \nu', s} \left[ \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L_{h,v}^\infty}^2 (\|\nabla_h \tilde{u}\|_{L_t^2 L^2}^{2s} + \|\nabla_h \tilde{b}\|_{L_t^2 L^2}^{2s}) \right. \\ \left. + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^2 \|\nabla c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{1}{2}}}^{2s} + (\|\tilde{u}\|_{L_t^\infty L^2}^{2s} + \|\tilde{b}\|_{L_t^\infty L^2}^{2s}) \|\nabla W_\varepsilon\|_{L_t^2 L_{h,v}^{\frac{2}{1-s}, 2}}^2 \right. \\ \left. + \|c_\varepsilon\|_{L_t^{\frac{2}{1-(\alpha_0+1)\delta}} \dot{H}^{\frac{3}{2} - \alpha_0 \delta}}^{2s} \|\nabla W_\varepsilon\|_{L_t^{\frac{2}{1-s(1-(\alpha_0+1)\delta)}} L^{\frac{3}{\frac{3}{2} - \alpha_0 \delta}}}^2 \right] \\ \leq \mathbb{D}_0 \left[ \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L_{h,v}^\infty}^2 + \|\nabla W_\varepsilon\|_{L_t^2 L_{h,v}^{\frac{2}{1-s}, 2}}^2 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^{2s} \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^2 \right. \\ \left. + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^{2s} \|\nabla W_\varepsilon\|_{L_t^{\frac{2}{1-s(1-(\alpha_0+1)\delta)}} L^{\frac{3}{\frac{3}{2} - \alpha_0 \delta}}}^2 \right], \quad (5.86)$$

for some constant  $\mathbb{D}_0 = \mathbb{D}_0(\nu, \nu', C_0, s) \geq 1$ . We finally obtain that for any  $t \leq T_\varepsilon$ ,

$$\|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau \\ \leq \mathbb{D}_0 \left[ \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L_{h,v}^\infty}^2 + \|\nabla W_\varepsilon\|_{L_t^2 L_{h,v}^{\frac{2}{1-s}, 2}}^2 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^{2s} \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^2 \right. \\ \left. + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^{2s} \|\nabla W_\varepsilon\|_{L_t^{\frac{2}{1-s(1-(\alpha_0+1)\delta)}} L^{\frac{3}{\frac{3}{2} - \alpha_0 \delta}}}^2 \right] \\ \times \exp \mathbb{D}_0 \left\{ 1 + \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^4 + \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^4 L^6}^4 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^{\frac{2}{1-s}} \right\}, \quad (5.87)$$

### 5.3 Use of Strichartz estimates

The isotropic and anisotropic Strichartz estimates allow us to state the following result (whose proof is postponed to Sections 6.3.3 and 6.3.4 in the appendix):

**Proposition 5.2** *Under the notations from Theorem 5.2, there exist positive constants  $\varepsilon_0, \mathbb{E}_0$  (both of them depending on  $\nu, \nu', \mathbb{C}_0, \delta, \eta_0, \mu$ ) such that if  $\delta \leq \frac{1}{6}$ , for any  $\varepsilon \in ]0, \varepsilon_0]$  and any  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ , we have:*

$$\begin{cases} \|\nabla W_\varepsilon\|_{L^2 L^3} + \|W_\varepsilon\|_{L^4 L^6} \leq \mathbb{E}_0 \varepsilon^{\eta_0 \delta}, \\ \|W_\varepsilon\|_{L^2 L^\infty} \leq \mathbb{E}_0 \varepsilon^{\frac{\eta_0}{2} \delta}, \\ \|\nabla W_\varepsilon\|_{L_t^{\frac{2}{1-s(1-(\alpha_0+1)\delta)}} L^{\frac{3}{\frac{3}{2}-\alpha_0\delta}}} \leq \mathbb{E}_0 \varepsilon^{\frac{\eta_0}{16} \delta}, \\ \|W_\varepsilon\|_{L^{\frac{2}{1-s}} L^6} + \|W_\varepsilon\|_{L^{\frac{2}{1-s}} L^{\infty,2}_{h,v}} + \|\nabla W_\varepsilon\|_{L^2 L^{\frac{2}{1-s},2}_{h,v}} \leq \mathbb{E}_0 \varepsilon^{\frac{9}{10} \eta_0 \delta}. \end{cases} \quad (5.88)$$

Plugging this into (5.87), if in addition  $\varepsilon_0$  is so small that for any  $\varepsilon \in ]0, \varepsilon_0]$ :

$$\begin{aligned} \|\nabla W_\varepsilon\|_{L_t^2 L^3}^2 + \|W_\varepsilon\|_{L_t^4 L^6}^4 + \|W_\varepsilon\|_{L_t^{\frac{2}{1-s}} L^6}^{\frac{2}{1-s}} &\leq 2\mathbb{E}_0^2 \varepsilon^{2\eta_0 \delta} + (\mathbb{E}_0 \varepsilon^{\frac{9\eta_0}{10} \delta})^{\frac{2}{1-s}} \\ &\leq 2\mathbb{E}_0^2 \varepsilon^{2\eta_0 \delta} + (\mathbb{E}_0 \varepsilon^{\frac{9\eta_0}{10} \delta})^{\frac{4}{1+\frac{\eta_0}{5} \delta}} \leq 1, \end{aligned} \quad (5.89)$$

we get that for all  $t \leq T_\varepsilon$  and  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ ,

$$\begin{aligned} \|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau \\ \leq 2\mathbb{D}_0 \mathbb{E}_0^2 \left[ \varepsilon^{\frac{9\eta_0}{5} \delta} + \left( \mathbb{K}_0 |\ln \varepsilon| \right)^{\frac{5}{2}} \varepsilon^{\frac{\eta_0}{8} \delta} \right] \exp \mathbb{D}_0 \left\{ 1 + \mathbb{K}_0 |\ln \varepsilon| + 2\mathbb{E}_0^2 \varepsilon^{2\eta_0 \delta} + (\mathbb{E}_0 \varepsilon^{\frac{9\eta_0}{10} \delta})^{\frac{2}{1-s}} \right\} \\ \leq 2\mathbb{D}_0 \mathbb{E}_0^2 \left[ \varepsilon^{\frac{9\eta_0}{5} \delta} + \left( \mathbb{K}_0 |\ln \varepsilon| \right)^{\frac{5}{2}} \varepsilon^{\frac{\eta_0}{8} \delta} \right] e^{\mathbb{D}_0 \{ 2 - \mathbb{K}_0 \ln \varepsilon \}}. \end{aligned} \quad (5.90)$$

If we assume that  $\varepsilon_0$  is so small that for any  $\varepsilon \in ]0, \varepsilon_0]$ ,

$$\left( \mathbb{K}_0 |\ln \varepsilon| \right)^{\frac{1}{2}} \leq \varepsilon^{-\frac{\eta_0 \delta}{96}}, \quad (5.91)$$

then if we choose  $\mathbb{K}_0$  according to

$$\mathbb{D}_0 \mathbb{K}_0 \stackrel{def}{=} \frac{\eta_0 \delta}{144}, \quad (5.92)$$

recalling that  $\delta \leq \frac{1}{6}$ , we obtain

$$\begin{aligned} \|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau \\ \leq 4\mathbb{D}_0 \mathbb{E}_0^2 \varepsilon^{\eta_0 \delta} \left[ \frac{1}{8} - \left( \frac{1}{2} + \frac{\eta_0}{10} \delta \right) \frac{1}{96} \right] e^{2\mathbb{D}_0 \varepsilon^{-\mathbb{D}_0 \mathbb{K}_0}} \leq 4\mathbb{D}_0 e^{2\mathbb{D}_0} \mathbb{E}_0^2 \varepsilon^{\eta_0 \delta} \left[ \frac{1}{8} - \left( \frac{1}{2} + \delta \right) \frac{1}{96} \right] \\ \leq \mathbb{F}_0 \varepsilon^{\eta_0 \delta \left( \frac{1}{8} - \frac{1}{72} \right)} = \mathbb{F}_0 \varepsilon^{\frac{\eta_0}{9} \delta}. \end{aligned} \quad (5.93)$$

## 5.4 Conclusion of the proof

First assuming  $v_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\frac{1}{2}+\delta}$ , using (5.93) in the case  $s = \frac{1}{2}$  we obtain that for any  $t \leq T_\varepsilon$  (we refer to (5.59) for the definition of  $T_\varepsilon$ ):

$$\|\delta_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|d_\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \leq \mathbb{F}_0 \varepsilon^{\frac{\eta_0}{9}\delta} \leq \frac{1}{16C} \min(\nu, \nu'),$$

for any  $\varepsilon \in ]0, \varepsilon_0]$  if  $\varepsilon_0$  is small enough. This contradicts the definition of  $T_\varepsilon$ , and allows us to get (by the continuation criterion) that  $T_\varepsilon = T_\varepsilon^* = \infty$ . Then thanks to the propagation of regularity we are able to obtain that for any  $s \in [\frac{1}{2}, \frac{1}{2} + \frac{\eta_0}{10}\delta]$  and  $t \geq 0$ ,

$$\|\delta_\varepsilon(t)\|_{\dot{H}^s}^2 + \|d_\varepsilon(t)\|_{\dot{H}^s}^2 + \nu \int_0^t \|\nabla \delta_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau + \nu' \int_0^t \|\nabla d_\varepsilon(\tau)\|_{\dot{H}^s}^2 d\tau \leq \mathbb{F}_0 \varepsilon^{\frac{\eta_0}{9}\delta},$$

which gives (5.56). If we now assume that  $v_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}-\delta} \cap \dot{H}^{\frac{1}{2}+\delta}$ , then the previous estimates is true for any  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$  (using that  $\delta \leq \frac{1}{6}$ , so that (using once more Proposition 6.1) we get:

$$\|D_\varepsilon - W_\varepsilon\|_{L^2 L^\infty} \leq \mathbb{F}_0 \varepsilon^{\frac{\eta_0}{18}\delta}.$$

Combining this with the second line from (5.88) finally gives (5.57). ■

## 6 Appendix

### 6.1 Notations for general Sobolev and Besov spaces

We refer to [6] for a complete presentation of Sobolev and Besov spaces through the Littlewood-Paley decompositions. We will only recall some basic fundamental results and give more details about refined results involving products of functions depending on  $x = (x_1, x_2, x_3)$  by functions only depending on  $x_h = (x_1, x_2)$ .

Assume  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a smooth function supported in the ball  $[0, \frac{4}{3}]$ , which is equal to 1 in a neighborhood of  $[0, \frac{4}{3}]$  and nonincreasing over  $\mathbb{R}_+$ . Setting  $\varphi(r) = \chi(r/2) - \chi(r)$ , we observe that  $\varphi$  is compactly supported in the interval  $\mathcal{C} = [\frac{3}{4}, \frac{8}{3}]$  and we define the homogeneous dyadic blocks in  $\mathbb{R}^d$  as follows: for all  $j \in \mathbb{Z}$  and any  $u$  depending on  $x = (x_1, \dots, x_d)$ , and recalling that  $\widehat{k(D)u}(\xi) = k(\xi)\widehat{u}(\xi)$ , we define

$$\begin{cases} \dot{\Delta}_j u := \varphi(2^{-j}|D|)u = 2^{jd}m(2^j) * u, & \text{with } m(x) = \mathcal{F}^{-1}(\varphi(|\xi|)), \\ \dot{S}_j u = \sum_{l \leq j-1} \dot{\Delta}_l u = \chi(2^{-j}|D|)u = 2^{jd}w(2^j) * u, & \text{with } w(x) = \mathcal{F}^{-1}(\chi(|\xi|)), \end{cases} \quad (6.94)$$

We can now define the homogeneous Besov norms and spaces:

**Definition 6.1** For  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ , we set

$$\dot{B}_{p,r}^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \text{ with } \lim_{j \rightarrow -\infty} \|\dot{S}_j u\|_{L^p} = 0 \text{ and } \|u\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \|(2^{qs} \|\dot{\Delta}_q u\|_{L^p})_{q \in \mathbb{Z}}\|_{\ell^r} < \infty \right\}.$$

**Remark 6.1** The inhomogeneous Sobolev space injects into its homogeneous counterpart as we have:

$$H^s(\mathbb{R}^d) = L^2(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d) = L^2(\mathbb{R}^d) \cap \dot{B}_{2,2}^s(\mathbb{R}^d).$$

As in [16, 17, 18, 19, 20] we will use spaces that are slight modifications of  $L_t^p \dot{B}_{q,r}^s$ : namely the Chemin-Lerner time-space Besov spaces for which the integration in time is performed before the summation with respect to the frequency decomposition index:

**Definition 6.2** ([6] section 2.6.3) For  $s, t \in \mathbb{R}$  and  $a, b, c \in [1, \infty]$ , we define the following norm

$$\|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} = \left\| \left( 2^{js} \|\dot{\Delta}_j u\|_{L_t^a L^b} \right)_{j \in \mathbb{Z}} \right\|_{l^c(\mathbb{Z})}.$$

The space  $\tilde{L}_t^a \dot{B}_{b,c}^s$  is defined as the set of tempered distributions  $u$  such that  $\lim_{j \rightarrow -\infty} S_j u = 0$  in  $L^a([0, t], L^\infty(\mathbb{R}^d))$  and  $\|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} < \infty$ .

Let us also recall the following proposition:

**Proposition 6.1** For all  $a, b, c \in [1, \infty]$  and  $s \in \mathbb{R}$ :

$$\begin{cases} \text{if } a \leq c, & \forall u \in L_t^a \dot{B}_{b,c}^s, & \|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} \leq \|u\|_{L_t^a \dot{B}_{b,c}^s} \\ \text{if } a \geq c, & \forall u \in \tilde{L}_t^a \dot{B}_{b,c}^s, & \|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} \geq \|u\|_{L_t^a \dot{B}_{b,c}^s}. \end{cases}$$

Let us continue with classical injections:

**Proposition 6.2** ([6] Chapter 2) We have:

$$\begin{cases} \text{For any } p \geq 1, & \dot{B}_{p,1}^0 \hookrightarrow L^p, \\ \text{For any } p \in [2, \infty[, & \dot{B}_{p,2}^0 \hookrightarrow L^p, \\ \text{For any } p \in [1, 2], & \dot{B}_{p,p}^0 \hookrightarrow L^p. \end{cases}$$

**Lemma 6.1** ([6] Section 2.11, see also Lemma 5 from [15]) For any  $\alpha, \beta > 0$  there exists a constant  $C_{\alpha,\beta} > 0$  such that for any  $u \in \dot{H}^{s-\alpha}(\mathbb{R}^d) \cap \dot{H}^{s+\beta}(\mathbb{R}^d)$ , then  $u \in \dot{B}_{2,1}^s(\mathbb{R}^d)$  and:

$$\|u\|_{\dot{B}_{2,1}^s} \leq C_{\alpha,\beta} \|u\|_{\dot{H}^{s-\alpha}}^{\frac{\beta}{\alpha+\beta}} \|u\|_{\dot{H}^{s+\beta}}^{\frac{\alpha}{\alpha+\beta}}. \quad (6.95)$$

Let us end this section with a consequence of the previous result involving anisotropic Lebesgue spaces, that we define for  $a, b \in [1, \infty]$  as follows:

$$\|f\|_{L_{h,v}^{a,b}} \stackrel{def}{=} \left\| \|f(x_h, \cdot)\|_{L^b(\mathbb{R}_v)} \right\|_{L^a(\mathbb{R}_h^2)}. \quad (6.96)$$

**Lemma 6.2** For any  $\alpha, \beta > 0$ , there exists a constant  $C_{\alpha,\beta}$  such that for any  $c \in \dot{H}^{1-\alpha}(\mathbb{R}^3) \cap \dot{H}^{1+\beta}(\mathbb{R}^3)$ , then  $c \in L_{h,v}^{\infty,2}(\mathbb{R}^3)$  and:

$$\|c\|_{L_{h,v}^{\infty,2}(\mathbb{R}^3)} \leq C_{\alpha,\beta} \|c\|_{\dot{H}^{1-\alpha}(\mathbb{R}^3)}^{\frac{\beta}{\alpha+\beta}} \|c\|_{\dot{H}^{1+\beta}(\mathbb{R}^3)}^{\frac{\alpha}{\alpha+\beta}}.$$

**Proof:** for any function  $c$

$$\|c\|_{L_{h,v}^{\infty,2}(\mathbb{R}^3)} \stackrel{def}{=} \sup_{x_h \in \mathbb{R}^2} \left( \int_{\mathbb{R}} |c(x_h, x_3)|^2 dx_3 \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}} \left( \sup_{x_h \in \mathbb{R}^2} |c(x_h, x_3)| \right)^2 dx_3 \right)^{\frac{1}{2}}$$

Thanks to the Bernstein lemma and to Lemma 6.1, for any  $\alpha, \beta > 0$  we have:

$$\begin{aligned} \sup_{x_h \in \mathbb{R}^2} |c(x_h, x_3)| &= \|c(\cdot, x_3)\|_{L^\infty(\mathbb{R}^2)} \leq \|c(\cdot, x_3)\|_{\dot{B}_{\infty,1}^0(\mathbb{R}^2)} \leq C \|c(\cdot, x_3)\|_{\dot{B}_{2,1}^1(\mathbb{R}^2)} \\ &\leq C_{\alpha,\beta} \|c(\cdot, x_3)\|_{\dot{H}^{1-\alpha}(\mathbb{R}^2)}^{\frac{\beta}{\alpha+\beta}} \|c(\cdot, x_3)\|_{\dot{H}^{1+\beta}(\mathbb{R}^2)}^{\frac{\alpha}{\alpha+\beta}}. \end{aligned} \quad (6.97)$$

This implies that

$$\begin{aligned} \|c\|_{L_{h,v}^{\infty,2}(\mathbb{R}^3)} &\leq C_{\alpha,\beta} \left( \int_{\mathbb{R}} \| |D_h|^{1-\alpha} c(\cdot, x_3) \|_{L^2(\mathbb{R}^2)}^{\frac{2\beta}{\alpha+\beta}} \cdot \| |D_h|^{1-\beta} c(\cdot, x_3) \|_{L^2(\mathbb{R}^2)}^{\frac{2\alpha}{\alpha+\beta}} dx_3 \right)^{\frac{1}{2}} \\ &\leq C_{\alpha,\beta} \|c\|_{\dot{H}^{1-\alpha}(\mathbb{R}^3)}^{\frac{\beta}{\alpha+\beta}} \|c\|_{\dot{H}^{1+\beta}(\mathbb{R}^3)}^{\frac{\alpha}{\alpha+\beta}}. \blacksquare \end{aligned} \quad (6.98)$$

## 6.2 Product laws

### 6.2.1 General product laws

We refer to [6] for the classical product laws and to [22, 23, 24, 18] for the 2D-3D version (we also refer to [19, 20]) for a 1D-3D version.

**Proposition 6.3** *There exists a constant  $C > 0$  such that for any  $s, t < \frac{3}{2}$  with  $s + t > 0$  and any  $u \in \dot{H}^s(\mathbb{R}^3)$ ,  $v \in \dot{H}^t(\mathbb{R}^3)$ , then  $uv \in \dot{H}^{s+t-\frac{3}{2}}(\mathbb{R}^3)$  and we have:*

$$\|uv\|_{\dot{H}^{s+t-\frac{3}{2}}(\mathbb{R}^3)} \leq C \|u\|_{\dot{H}^s(\mathbb{R}^3)} \|v\|_{\dot{H}^t(\mathbb{R}^3)}.$$

**Proposition 6.4** *There exists a constant  $C > 0$  such that for any  $s, t < 1$  with  $s + t > 0$  and any  $u \in \dot{H}^s(\mathbb{R}^2)$ ,  $v \in \dot{H}^t(\mathbb{R}^3)$ , then  $uv \in \dot{H}^{s+t-1}(\mathbb{R}^3)$  and we have:*

$$\|uv\|_{\dot{H}^{s+t-1}(\mathbb{R}^3)} \leq C \|u\|_{\dot{H}^s(\mathbb{R}^2)} \|v\|_{\dot{H}^t(\mathbb{R}^3)}.$$

**Remark 6.2** *When applying these propositions to estimate a product of the form  $u \cdot \nabla v$  with  $\operatorname{div} u = 0$ , the second condition on the regularity exponents can be relaxed into  $s + t > -\frac{3}{2}$  for the classical law and into  $s + t > -1$  for the 2D-3D law.*

### 6.2.2 Estimates for the external force terms

In this section, we prove the following estimates for the external force terms in System (5.52), namely

$$\mathbb{P}(\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b} + c_\varepsilon \cdot \nabla c_\varepsilon)$$

**Proposition 6.5** Assume  $\delta \in ]0, \frac{1}{4}]$ ,  $\tilde{u}_0, \tilde{b}_0 \in H^\delta$  and  $c_{0,\varepsilon} \in H^{\frac{1}{2}+\delta}$  (inhomogeneous spaces). Let  $(\tilde{u}, \tilde{b})$  and  $c_\varepsilon$  respectively denote the corresponding unique global strong solutions of Systems  $(2D - MHD^3)$  and  $(3D - M_\varepsilon)$ . There exists a constant  $C > 0$  (depending on  $\delta, \nu', \|\tilde{u}_0\|_{L^2}$ ) such that for any  $(\sigma, \sigma') \in [0, \frac{1}{2} + \delta] \times [0, \delta]$ :

$$\begin{cases} \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1 \dot{H}^\sigma} \leq C \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}^2, \\ \|\tilde{b} \cdot \nabla c_\varepsilon\|_{L^{\frac{4}{3}} \dot{H}^{\sigma'}} + \|c_\varepsilon \cdot \nabla \tilde{b}\|_{L^{\frac{4}{3}} \dot{H}^{\sigma'}} \leq C \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}. \end{cases} \quad (6.99)$$

**Proof:** the first term in  $(6.99)_2$  is easily bounded thanks to Proposition 6.4 with  $(s, t) = (\frac{1}{2}, \frac{1}{2} + \sigma')$  (provided that  $\delta < \frac{1}{2}$ ) and interpolation. For any  $t \geq 0$ , we have:

$$\|\tilde{b} \cdot \nabla c_\varepsilon\|_{L_t^{\frac{4}{3}} \dot{H}^{\sigma'}(\mathbb{R}^3)} \leq C \|\tilde{b}\|_{L_t^4 \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \|\nabla c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{1}{2}+\sigma'}(\mathbb{R}^3)} \leq \|\tilde{b}\|_{L_t^\infty L^2}^{\frac{1}{2}} \|\tilde{b}\|_{L_t^2 \dot{H}^1}^{\frac{1}{2}} \|\nabla c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{1}{2}+\sigma'}}.$$

Thanks to Theorems 2.1 and 2.2, we end-up with

$$\|\tilde{b} \cdot \nabla c_\varepsilon\|_{L_t^{\frac{4}{3}} \dot{H}^{\sigma'}} \leq C \|\tilde{b}_0\|_{L^2} \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\sigma'}} \leq C \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}.$$

We could deal with  $(6.99)_1$  using classical product laws from Proposition 6.3 but due to the condition  $s, t < \frac{3}{2}$  we would end-up at best with (for some  $m > 0$ )

$$\|c_\varepsilon \cdot \nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq \|c_\varepsilon\|_{\dot{H}^{\frac{3}{2}-m\delta}} \|\nabla c_\varepsilon\|_{\dot{H}^{\frac{1}{2}+(1+m)\delta}},$$

which requires extra-regularity from  $c_{0,\varepsilon}$ . To avoid this problem we simply do as in [16]: for any  $\sigma \in [0, \frac{1}{2} + \delta]$

$$\begin{aligned} \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{\dot{H}^\sigma} &= \|\operatorname{div}(c_\varepsilon \otimes c_\varepsilon)\|_{\dot{H}^\sigma} \leq 2 \|T_{c_\varepsilon} c_\varepsilon\|_{\dot{H}^{1+\sigma}} + \|R(c_\varepsilon, c_\varepsilon)\|_{\dot{H}^{1+\sigma}} \\ &\leq C (2 \|c_\varepsilon\|_{L^\infty} + \|c_\varepsilon\|_{\dot{B}_{\infty,\infty}^0}) \|c_\varepsilon\|_{\dot{H}^{1+\sigma}}. \end{aligned} \quad (6.100)$$

Thanks to the classical Bernstein lemma and to Lemma 6.1, we have

$$2 \|c_\varepsilon\|_{L^\infty} + \|c_\varepsilon\|_{\dot{B}_{\infty,\infty}^0} \leq 3 \|c_\varepsilon\|_{\dot{B}_{\infty,1}^0} \leq C \|c_\varepsilon\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \leq C \|c_\varepsilon\|_{\dot{H}^{\frac{3}{2}-\delta}}^{\frac{1}{2}} \|c_\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}}^{\frac{1}{2}},$$

so that we obtain for any  $t \geq 0$ :

$$\begin{aligned} \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L_t^1 \dot{H}^\sigma} &\leq C \|c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{3}{2}-\delta}}^{\frac{1}{2}} \|c_\varepsilon\|_{L_t^2 \dot{H}^{\frac{3}{2}+\delta}}^{\frac{1}{2}} \|c_\varepsilon\|_{L_t^2 \dot{H}^{1+\sigma}} \\ &\leq C \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}-\delta}}^{\frac{1}{2}} \|c_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\delta}}^{\frac{1}{2}} \|c_{0,\varepsilon}\|_{\dot{H}^\sigma} \leq C \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}^2, \end{aligned} \quad (6.101)$$

which gives  $(6.99)_1$ . The last term will require most of the work. The following decomposition holds:

$$c_\varepsilon \cdot \nabla \tilde{b} = T_{c_\varepsilon} \nabla \tilde{b} + T_{\nabla \tilde{b}} c_\varepsilon + R(c_\varepsilon, \nabla \tilde{b}),$$

where we have defined

$$\begin{cases} T_{c_\varepsilon} \nabla \tilde{b} \stackrel{def}{=} \sum_{q \in \mathbb{Z}} \dot{S}_{q-j_0} c_\varepsilon \cdot \dot{\Delta}_q^h \nabla \tilde{b}, \\ T_{\nabla \tilde{b}} c_\varepsilon \stackrel{def}{=} \sum_{q \in \mathbb{Z}} \dot{S}_{q-j_0}^h \nabla \tilde{b} \cdot \dot{\Delta}_q c_\varepsilon, \\ R(c_\varepsilon, \nabla \tilde{b}) \stackrel{def}{=} \sum_{q \in \mathbb{Z}} \sum_{\alpha=-j_0}^{j_0} \dot{\Delta}_q c_\varepsilon \cdot \nabla \dot{\Delta}_{q+\alpha}^h \tilde{b}, \end{cases} \quad (6.102)$$

and where  $\dot{\Delta}_q, \dot{S}_q$  refer to the classical dyadic truncation operators in  $\mathbb{R}^3$  (introduced in the previous subsection) and  $\dot{\Delta}_q^h, \dot{S}_q^h$  refer to their counterpart in  $\mathbb{R}^2$  acting on the horizontal variables  $x_h = (x_1, x_2)$ .

**Lemma 6.3** *If  $j_0 \geq 4$ , for any  $q \in \mathbb{Z}$ , we have*

$$\begin{cases} \text{supp} \left( \mathcal{F}(\dot{S}_{q-j_0} c_\varepsilon \cdot \dot{\Delta}_q^h \nabla \tilde{b}) \right) \cup \text{supp} \left( \mathcal{F}(\dot{S}_{q-j_0}^h \nabla \tilde{b} \cdot \dot{\Delta}_q c_\varepsilon) \right) \subset 2^q \mathcal{C}(0, \frac{1}{12}, \frac{10}{3}), \\ \text{supp} \left( \mathcal{F} \left( \sum_{\alpha=-j_0}^{j_0} \dot{\Delta}_q c_\varepsilon \cdot \nabla \dot{\Delta}_{q+\alpha}^h \tilde{b} \right) \right) \subset 2^q B(0, 46). \end{cases} \quad (6.103)$$

**Proof of the lemma :** we emphasize that contrary to the classical decomposition, there is now a slight asymmetry between the previous paraproduct terms due the 2D-3D product which will require us to fix  $j_0$  large enough in order to use the classical methods. For  $f = f(x)$  and  $\tilde{g} = \tilde{g}(x_h)$ , we formally have (denoting the same way the Fourier transform in  $\mathbb{R}^3$  and  $\mathbb{R}_h^2$ ):

$$\mathcal{F}(f\tilde{g})(\xi) = \int_{\mathbb{R}^2} \hat{f}(\xi_h - \eta_h, \xi_3) \widehat{\tilde{g}}(\eta_h) d\eta_h. \quad (6.104)$$

Thanks to this, assuming that  $\text{supp} \widehat{\tilde{g}} \subset \mathbb{R}^2$  and  $\text{supp} \hat{f} \subset \mathbb{R}^3$  are compact sets, and if we define compact sets  $K_h \subset \mathbb{R}^2$  and  $K_v \subset \mathbb{R}$  such that  $\text{supp} \hat{f} \subset K_h \times K_v$ , it easy to see that

$$\text{supp} \mathcal{F}(f\tilde{g}) \subset \left( K_h + \text{supp} \widehat{\tilde{g}} \right) \times K_v. \quad (6.105)$$

Thanks to this, we prove (6.103)<sub>3</sub> easily. In the first paraproduct from (6.102), as for any  $q \in \mathbb{Z}$ , we have

$$\begin{cases} \text{supp} \mathcal{F}(\dot{S}_{q-j_0} c_\varepsilon) \subset 2^{q-j_0} B(0, \frac{4}{3}) \subset 2^q B_h(0, \frac{4}{3} 2^{-j_0}) \times 2^q [-\frac{4}{3} 2^{-j_0}, \frac{4}{3} 2^{-j_0}], \\ \text{supp} \mathcal{F}(\dot{\Delta}_q^h \nabla \tilde{b}) \subset 2^q \mathcal{C}_h(0, \frac{3}{4}, \frac{8}{3}), \end{cases}$$

so that, thanks to (6.105), if  $j_0 \geq 1$ , we get  $\text{supp} \mathcal{F}(\dot{S}_{q-j_0} c_\varepsilon \cdot \dot{\Delta}_q^h \nabla \tilde{b}) \subset 2^q \mathcal{C}(0, \frac{1}{12}, \frac{10}{3})$ .

Let us now turn to the last term: writing (6.104) for  $\mathcal{F}(\dot{S}_{q-j_0}^h \nabla \tilde{b} \cdot \dot{\Delta}_q c_\varepsilon)$ , for the integrated product to be non zero we need that:

$$\begin{cases} |\eta_h| \leq 2^{q-j_0} \frac{4}{3}, \\ 2^q \frac{3}{4} \leq \sqrt{|\xi_h - \eta_h|^2 + \xi_3^2} \leq 2^q \frac{8}{3}. \end{cases}$$

This implies that if  $j_0 \geq 4$ :

$$\begin{aligned} |\xi|^2 &\geq |\xi_h|^2 + \xi_3^2 \geq |\xi_h - \eta_h|^2 - 2|\xi_h - \eta_h| \cdot |\eta_h| + |\eta_h|^2 + \xi_3^2 \\ &\geq 2^{2q} \left( \frac{9}{16} - \frac{2^{6-j_0}}{9} \right) \geq 2^{2q} \frac{17}{144} > 2^{2q} \frac{1}{9}. \end{aligned} \quad (6.106)$$

So, when  $j_0 \geq 4$  we finally obtain that  $\text{supp} \mathcal{F}(\dot{S}_{q-j_0}^h \nabla \tilde{b} \cdot \dot{\Delta}_q c_\varepsilon) \subset 2^q \mathcal{C}(0, \frac{1}{3}, \frac{11}{4})$ , which ends the proof of the lemma  $\blacksquare$ .

We can now focus on estimating the terms from (6.102). Fixing  $j_0 = 4$  and thanks to the previous lemma we can use Lemma 2.47 from [6] and write that for any  $\sigma' \in [0, \delta]$ :

$$\|T_{\nabla \tilde{b}} c_\varepsilon\|_{\dot{H}^{\sigma'}} \leq \left\| (2^{q\sigma'} \|\dot{S}_{q-4}^h \nabla \tilde{b} \cdot \dot{\Delta}_q c_\varepsilon\|_{L^2})_{q \in \mathbb{Z}} \right\|_{l^2(\mathbb{Z})}. \quad (6.107)$$

For any  $q \in \mathbb{Z}$ , thanks to the Bernstein lemma and convolutions estimates (see (6.94)), we have

$$\|\dot{S}_{q-4}^h \nabla \tilde{b} \cdot \dot{\Delta}_q c_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq \|\dot{S}_{q-4}^h \nabla \tilde{b}\|_{L^\infty(\mathbb{R}^2)} \|\dot{\Delta}_q c_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C 2^q \|\tilde{b}\|_{L^\infty(\mathbb{R}^2)} \|\dot{\Delta}_q c_\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

Multiplying by  $2^{q\sigma'}$  and summing in  $l^2(\mathbb{Z})$ , we obtain that thanks to (6.107) and Lemma 6.1,

$$\begin{aligned} \|T_{\nabla \tilde{b}} c_\varepsilon\|_{\dot{H}^{\sigma'}} &\leq C \|\tilde{b}\|_{L^\infty(\mathbb{R}^2)} \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}} \leq C \|\tilde{b}\|_{\dot{B}_{2,1}^1(\mathbb{R}^2)} \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}} \\ &\leq C \|\tilde{b}\|_{\dot{H}^{1-\sigma'}(\mathbb{R}^2)}^{\frac{1}{2}} \|\tilde{b}\|_{\dot{H}^{1+\sigma'}(\mathbb{R}^2)}^{\frac{1}{2}} \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}}, \end{aligned} \quad (6.108)$$

As  $\tilde{b}_0 \in H^\delta(\mathbb{R}^2)$ , thanks to (2.21) we know that  $\|\tilde{b}\|_{\dot{H}^{1+\sigma'}} \in L^2$ . Thanks to (2.20) and by interpolation we have  $\|\tilde{b}\|_{\dot{H}^{1-\sigma'}} \in L^{\frac{2}{1-\sigma'}}$ . Thanks to Proposition 5.1,  $\|c_\varepsilon\|_{\dot{H}^{1+\sigma'}} \in L^p$  for any  $p \in [2, \frac{4}{1-2(\delta-\sigma')}]$ , so taking in particular  $p = \frac{4}{1+\sigma'}$ , we get:

$$\|T_{\nabla \tilde{b}} c_\varepsilon\|_{L^{\frac{4}{3}} \dot{H}^{\sigma'}} \leq C \|\tilde{b}\|_{L^{\frac{2}{1-\sigma'}} \dot{H}^{1-\sigma'}(\mathbb{R}^2)}^{\frac{1}{2}} \|\tilde{b}\|_{L^2 \dot{H}^{1+\sigma'}(\mathbb{R}^2)}^{\frac{1}{2}} \|c_\varepsilon\|_{L^{\frac{4}{1+\sigma'}} \dot{H}^{1+\sigma'}}. \quad (6.109)$$

Thanks to Theorems 2.1, 2.2, and Proposition 5.1, we finally obtain:

$$\|T_{\nabla \tilde{b}} c_\varepsilon\|_{L^{\frac{4}{3}} \dot{H}^{\sigma'}} \leq C \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}. \quad (6.110)$$

As  $\operatorname{div} c_\varepsilon = 0$ , the remainder term is classically rewritten as follows:

$$R(c_\varepsilon, \nabla \tilde{b}) \stackrel{def}{=} \sum_{i=1}^3 \partial_i \left( \sum_{q \in \mathbb{Z}} \sum_{\alpha=-4}^4 \dot{\Delta}_q c_\varepsilon^i \cdot \dot{\Delta}_{q+\alpha}^h \tilde{b} \right),$$

so that

$$\|R(c_\varepsilon, \nabla \tilde{b})\|_{\dot{H}^{\sigma'}} \leq C \left\| \sum_{i=1}^3 \sum_{q \in \mathbb{Z}} \sum_{\alpha=-4}^4 \dot{\Delta}_q c_\varepsilon^i \cdot \dot{\Delta}_{q+\alpha}^h \tilde{b} \right\|_{\dot{H}^{1+\sigma'}}.$$

Denoting as  $(c_q(c_\varepsilon))_{q \in \mathbb{Z}}$  a positive sequence of  $l^2$ -norm equal to 1, and for some fixed  $N_0$  only depending on the radius from (6.103)<sub>3</sub>, we have for any  $q' \in \mathbb{Z}$ ,

$$\begin{aligned} \left\| \dot{\Delta}_{q'} \left( \sum_{i=1}^3 \sum_{q \in \mathbb{Z}} \sum_{\alpha=-4}^4 \dot{\Delta}_q c_\varepsilon^i \cdot \dot{\Delta}_{q+\alpha}^h \tilde{b} \right) \right\|_{L^2} &\leq C \sum_{q \geq q' - N_0} \sum_{\alpha=-4}^4 \|\dot{\Delta}_q c_\varepsilon\|_{L^2(\mathbb{R}^3)} \|\dot{\Delta}_{q+\alpha}^h \tilde{b}\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C \sum_{q \geq q' - N_0} 2^{-q(1+\sigma')} c_q(c_\varepsilon) \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}} \|\tilde{b}\|_{L^\infty(\mathbb{R}^2)}. \end{aligned} \quad (6.111)$$

This entails

$$2^{q'(1+\sigma')} \left\| \dot{\Delta}_{q'} \left( \sum_{i=1}^3 \sum_{q \in \mathbb{Z}} \sum_{\alpha=-4}^4 \dot{\Delta}_q c_\varepsilon^i \cdot \dot{\Delta}_{q+\alpha}^h \tilde{b} \right) \right\|_{L^2} \leq C (a * c(c_\varepsilon))_{q'} \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}} \|\tilde{b}\|_{L^\infty},$$

where we define the  $l^1$  sequence  $(a_q)_{q \in \mathbb{Z}}$  as follows: for any  $q \in \mathbb{Z}$ :

$$a_q = \begin{cases} 2^{q(1+\sigma')} & \text{if } q \leq N_0, \\ 0 & \text{else.} \end{cases}$$

Taking the  $l^2$ -norm, we obtain as previously that:

$$\|R(c_\varepsilon, \nabla \tilde{b})\|_{L^{\frac{4}{3}} \dot{H}^{\sigma'}} \leq C \|\tilde{b}\|_{L^{\frac{2}{1-\sigma'}} \dot{H}^{1-\sigma'}(\mathbb{R}^2)}^{\frac{1}{2}} \|\tilde{b}\|_{L^2 \dot{H}^{1+\sigma'}(\mathbb{R}^2)}^{\frac{1}{2}} \|c_\varepsilon\|_{L^{\frac{4}{1+\sigma'}} \dot{H}^{1+\sigma'}} \leq C \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}. \quad (6.112)$$

Turning to the last term, using first Lemma 6.2 and then convolutions from (6.94), we have that for any  $q \in \mathbb{Z}$ ,

$$\begin{aligned} \|\dot{S}_{q-4}c_\varepsilon \cdot \dot{\Delta}_q^h \nabla \tilde{b}\|_{L^2(\mathbb{R}^3)} &\leq \|\dot{S}_{q-4}c_\varepsilon\|_{L_{h,v}^{\infty,2}(\mathbb{R}^3)} \|\dot{\Delta}_q^h \nabla \tilde{b}\|_{L^2(\mathbb{R}^2)} \\ &\leq C_{\sigma'} \|\dot{S}_{q-4}c_\varepsilon\|_{\dot{H}^{1-\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} \|\dot{S}_{q-4}c_\varepsilon\|_{\dot{H}^{1+\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} 2^q \|\dot{\Delta}_q^h \tilde{b}\|_{L^2(\mathbb{R}^2)} \\ &\leq C_{\sigma'} \|c_\varepsilon\|_{\dot{H}^{1-\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} 2^q \|\dot{\Delta}_q^h \tilde{b}\|_{L^2(\mathbb{R}^2)}. \end{aligned} \quad (6.113)$$

Thanks to (6.103)<sub>1</sub> we conclude that:

$$\|T_{c_\varepsilon} \nabla \tilde{b}\|_{\dot{H}^{\sigma'}} \leq C_{\sigma'} \|c_\varepsilon\|_{\dot{H}^{1-\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} \|c_\varepsilon\|_{\dot{H}^{1+\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} \|\tilde{b}\|_{\dot{H}^{1+\sigma'}(\mathbb{R}^2)}.$$

In addition to the fact that  $\|c_\varepsilon\|_{\dot{H}^{1+\sigma'}} \in L_t^p$  for any  $p \in [2, \frac{4}{1-2(\delta-\sigma')}]$ , thanks once more to Proposition 5.1 we know that when  $\delta \leq \frac{1}{6}$ ,  $\|c_\varepsilon\|_{\dot{H}^{1-\sigma'}} \in L_t^p$  for any  $p \in [\frac{2}{1-\sigma'}, \frac{4}{1-2(\delta+\sigma')}]$  which allows us to obtain that for some  $C = C_{\nu', \delta, \|\tilde{u}_0\|_{L^2}} > 0$ :

$$\|T_{c_\varepsilon} \nabla \tilde{b}\|_{L_t^{\frac{4}{3}} \dot{H}^{\sigma'}} \leq C_{\sigma'} \|c_\varepsilon\|_{L_t^4 \dot{H}^{1-\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} \|c_\varepsilon\|_{L_t^4 \dot{H}^{1+\sigma'}(\mathbb{R}^3)}^{\frac{1}{2}} \|\tilde{b}\|_{L_t^2 \dot{H}^{1+\sigma'}(\mathbb{R}^2)} \leq C \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}} \|\tilde{b}_0\|_{H^\delta}.$$

Gathering the previous estimates with (6.110) and (6.112) ends the proof of (6.99)<sub>2</sub>. ■

## 6.3 Strichartz estimates

### 6.3.1 Isotropic estimates

Consider the following system (external force terms  $F_{ext}, G_{ext}$  depend on  $(t, x)$ )

$$\begin{cases} \partial_t f - \nu \Delta f + \frac{1}{\varepsilon} \mathbb{P}(f \wedge e_3) = F_{ext} + G_{ext}, \\ f|_{t=0} = f_{0,\varepsilon}. \end{cases} \quad (6.114)$$

The Strichartz estimates we use in this article are slight extensions of the ones from [18]:

**Proposition 6.6** 1. (Case  $G_{ext} = 0$ ) For any  $d \in \mathbb{R}$ ,  $r \geq 2$ ,  $q \geq 1$ ,  $\theta \in [0, 1]$ , and  $p \in [1, \frac{2}{\theta(1-\frac{2}{r})}]$ , there exists a constant  $C = C_{p,\theta,r} > 0$  such that for any divergence-free vectorfield  $f_{0,\varepsilon}$  and  $F_{ext}$ , the solution  $f$  of (6.114) satisfies:

$$\| |D|^d f \|_{\tilde{L}^p \dot{B}_{r,q}^0} \leq \frac{C_{p,\theta,r}}{\nu^{\frac{1}{p}-\frac{\theta}{2}(1-\frac{2}{r})}} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \left( \|f_{0,\varepsilon}\|_{\dot{B}_{2,q}^{\sigma_1}} + \|F_{ext}\|_{L^1 \dot{B}_{2,q}^{\sigma_1}} \right), \quad (6.115)$$

with  $\sigma_1 = d + \frac{3}{2} - \frac{3}{r} - \frac{2}{p} + \theta(1 - \frac{2}{r})$ .

2. (Case  $(f_{0,\varepsilon}, F_{ext}) = (0, 0)$ ) For any  $d \in \mathbb{R}$ ,  $k \in [1, 2]$ ,  $r \geq 2$ ,  $q \geq 1$ ,  $\theta \in [0, 1]$ , and  $p \in [2, \frac{2}{\theta(1-\frac{2}{r})}]$ , there exists a constant  $C = C_{p,\theta,r,k} > 0$  such that for any divergence-free vectorfield  $G_{ext}$ , we have:

$$\| |D|^d f \|_{\tilde{L}^p \dot{B}_{r,q}^0} \leq \frac{C_{p,\theta,r,k}}{\nu^{1-\frac{1}{k}+\frac{1}{p}-\frac{\theta}{2}(1-\frac{2}{r})}} \varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})} \|G_{ext}\|_{\tilde{L}^k \dot{B}_{2,q}^{\sigma_2}}, \quad (6.116)$$

where  $\sigma_2 = d - \frac{1}{2} + \frac{2}{k} - \frac{3}{r} - \frac{2}{p} + \theta(1 - \frac{2}{r})$ .

The first point is easily proved from Proposition 10 from [18] (where we only needed to consider the homogeneous case) and the Duhamel formula. The second point is an adaptation (in the case of the rotating fluids) of Proposition 6 from [18]. In particular, as the dispersion estimates are a little better for the rotating fluids compared to the primitive system (see Section 5.2 from [17] and Section 4.3.2 from [18]) this explains why, compared to Proposition 6 from [18],  $\theta$  is upgraded into  $2\theta$ .

**Remark 6.3** 1. *Let us emphasize that as in the cases  $\nu = \nu'$  of Primitive or Strongly stratified Boussinesq systems, these estimates hold for any  $\varepsilon > 0$  (the threshold in  $\varepsilon$  will be set in the bootstrap argument and in Proposition 5.2).*

2. *Of course, when the coefficients satisfy the (more restrictive) conditions from the second point, we can write that (using Proposition 6.1 and the fact that  $k \leq 2$  in the last term):*

$$\| |D|^d f \|_{\tilde{L}^p \dot{B}_{r,q}^0} \leq \frac{\varepsilon^{\frac{\theta}{2}(1-\frac{2}{r})}}{\nu^{\frac{1}{p}-\frac{\theta}{2}(1-\frac{2}{r})}} \left( C_{p,\theta,r} \left( \|f_{0,\varepsilon}\|_{\dot{B}_{2,q}^{\sigma_1}} + \|F_{ext}\|_{L^1 \dot{B}_{2,q}^{\sigma_1}} \right) + \frac{C_{p,\theta,r,k}}{\nu^{1-\frac{1}{k}}} \|G_{ext}\|_{L^k \dot{B}_{2,q}^{\sigma_1+\frac{2}{k}-2}} \right). \quad (6.117)$$

*For the sake of simplicity, we will sometimes use the previous proposition on the velocity  $v_\varepsilon$  splitting it into several parts, thanks to the superposition principle.*

### 6.3.2 Anisotropic estimates

The anisotropic Lebesgue spaces recalled previously appear in the following anisotropic Strichartz estimates that we present in two versions (depending on the regularity of the external force) as in the previous section:

**Proposition 6.7** 1. *(Case  $G_{ext} = 0$ ) For any  $d \in \mathbb{R}$ ,  $m > 2$ ,  $\theta \in ]0, 1]$ ,  $p \in [1, \frac{4}{\theta(1-\frac{2}{m})}]$  there exists a constant  $C = C_{p,\theta,m}$  such that for any divergence-free vectorfields  $f_{0,\varepsilon}$  and  $F_{ext}$ , the solution  $f$  of (6.114) satisfies:*

$$\| |D|^d f \|_{L^p L_{h,v}^{m,2}} \leq \frac{C_{p,\theta,m}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{2}{m})}} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{m})} \left( \|f_{0,\varepsilon}\|_{\dot{B}_{2,1}^{\sigma_3}} + \|F_{ext}\|_{L^1 \dot{B}_{2,1}^{\sigma_3}} \right), \quad (6.118)$$

*with  $\sigma_3 = d + 1 - \frac{2}{m} - \frac{2}{p} + \frac{\theta}{2}(1 - \frac{2}{m})$ .*

2. *(Case  $(f_{0,\varepsilon}, F_{ext}) = (0, 0)$ ) For any  $d \in \mathbb{R}$ ,  $k \in ]1, 2]$ ,  $m > 2$ ,  $\theta \in ]0, 1]$ ,  $p \in [2, \frac{4}{\theta(1-\frac{2}{m})}]$ , there exists a constant  $C = C_{p,\theta,m,k} > 0$  such that for any divergence-free vectorfield  $G_{ext}$ , we have:*

$$\| |D|^d f \|_{L^p L_{h,v}^{m,2}} \leq \frac{C_{p,\theta,m,k}}{\nu^{1-\frac{1}{k}+\frac{1}{p}-\frac{\theta}{4}(1-\frac{2}{m})}} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{m})} \|G_{ext}\|_{\tilde{L}^k \dot{B}_{2,1}^{\sigma_4}}, \quad (6.119)$$

*with  $\sigma_4 = d + \frac{2}{k} - 1 - \frac{2}{m} - \frac{2}{p} + \frac{\theta}{2}(1 - \frac{2}{m})$ .*

**Proof:** In [18] we only considered the homogeneous case. The cases for  $F_{ext}$  ( $k = 1$ ) and  $G_{ext}$  ( $k \in ]1, 2]$ ) are treated adapting our arguments from [18] (see the proof of Proposition 17 therein).

**Remark 6.4** *As previously, when the coefficients satisfy the assumptions from the second point, we can write that:*

$$\| |D|^d f \|_{L^p L_{h,v}^{m,2}} \leq \frac{\varepsilon^{\frac{\theta}{4}(1-\frac{2}{m})}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{2}{m})}} \left( C_{p,\theta,m} \left( \|f_{0,\varepsilon}\|_{\dot{B}_{2,1}^{\sigma_3}} + \|F_{ext}\|_{L^1 \dot{B}_{2,1}^{\sigma_3}} \right) + \frac{C_{p,\theta,m,k}}{\nu^{1-\frac{1}{k}}} \|G_{ext}\|_{L_t^k \dot{B}_{2,1}^{\sigma_3+\frac{2}{k}-2}} \right). \quad (6.120)$$

### 6.3.3 Proof of Proposition 5.2

We prove in this section the adapted isotropic Strichartz estimates for System (5.52).

Taking advantage of Propositions 6.1, 6.2 and 6.5, we can rewrite Proposition 6.6 for System (5.52) with  $(q, k) = (2, \frac{4}{3})$  and for fixed regularity: for any  $d \in \mathbb{R}$ ,  $r \geq 2$ ,  $\theta \in [0, 1]$ , and  $p \in [2, \frac{2}{\theta(1-\frac{2}{r})}]$  such that

$$d + \frac{3}{2} - \frac{3}{r} - \frac{2}{p} + \theta(1 - \frac{2}{r}) = \frac{1}{2} + \delta, \quad (6.121)$$

there exists a constant  $C = C_{p, \theta, r} > 0$  such that the solution of (5.52) satisfies:

$$\begin{aligned} \| |D|^d W_\varepsilon \|_{L^p L^r} &\leq \| |D|^d W_\varepsilon \|_{L^p \dot{B}_{r,2}^0} \leq \| |D|^d W_\varepsilon \|_{\tilde{L}^p \dot{B}_{r,2}^0} \\ &\leq \frac{C_{p, \theta, r}}{\nu^{\frac{1}{p} - \frac{\theta}{2}(1 - \frac{2}{r})}} \varepsilon^{\frac{\theta}{2}(1 - \frac{2}{r})} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L_t^1 \dot{H}^{\frac{1}{2} + \delta}} + \frac{1}{\nu^{\frac{1}{4}}} \|\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b}\|_{L^{\frac{4}{3}} \dot{H}^\delta} \right) \\ &\leq C_{\nu, p, \theta, r} \varepsilon^{\frac{\theta}{2}(1 - \frac{2}{r})} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^2 + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right). \end{aligned} \quad (6.122)$$

- Taking  $(d, p, r) = (1, 2, 3)$ , we obtain that (6.121) is satisfied if and only if  $\theta = 3\delta$ , (6.122) then turns into

$$\|\nabla W_\varepsilon\|_{L^2 L^3} \leq C_{\nu, \nu', \mathbb{C}_0, \delta} \varepsilon^{\frac{\delta}{2}} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^2 + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right),$$

together with the following conditions

$$\begin{cases} \theta \in ]0, 1[ & \iff \delta \leq \frac{1}{3}, \\ p \in [2, \frac{2}{\theta(1-\frac{2}{r})}] & \iff \delta < 1. \end{cases} \quad (6.123)$$

Thanks to the definitions of  $\gamma, \mathbb{K}_0$  (see (5.92)) and the assumptions from Theorem 5.2,

$$\|\nabla W_\varepsilon\|_{L^2 L^3} \leq C_{\nu, \nu', \mathbb{C}_0, \delta} \varepsilon^{\frac{\delta}{2}} \left( \mathbb{C}_0 \varepsilon^{-\gamma} + \left( \frac{\eta_0 \delta}{144 \mathbb{D}_0} |\ln \varepsilon| \right)^{\frac{1}{2}} + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right),$$

and if  $\varepsilon \in ]0, \varepsilon_0]$  with  $\varepsilon_0$  depending on  $\mathbb{C}_0, \mathbb{D}_0, \delta, \eta_0$  so small that (we recall that  $\mathbb{D}_0$  is a constant featured in (5.87)) for any  $\varepsilon \in ]0, \varepsilon_0]$ ,

$$\left( \frac{\eta_0 \delta}{144 \mathbb{D}_0} |\ln \varepsilon| \right)^{\frac{1}{2}} \leq \min \left( \varepsilon^{-\frac{\eta_0 \delta}{96}}, \varepsilon^{-\gamma} \right), \quad (6.124)$$

we finally get that:

$$\|\nabla W_\varepsilon\|_{L^2 L^3} \leq C_{\nu, \nu', \mathbb{C}_0, \delta} \varepsilon^{\frac{\delta}{2} - \gamma} = C_{\nu, \nu', \mathbb{C}_0, \delta} \varepsilon^{\eta_0 \delta}, \quad (6.125)$$

- Choosing  $(d, p, r) = (0, \frac{2}{1-s}, 6)$ , (6.121) is satisfied if and only if  $\theta = \frac{3}{2}(\frac{1}{2} + \delta - s)$ , and (6.122) becomes:

$$\begin{aligned} \|W_\varepsilon\|_{L^{\frac{2}{1-s}} L^6} &\leq C_{\nu, \nu', \delta, s} \varepsilon^{\frac{1}{2}(\frac{1}{2} + \delta - s)} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^2 + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right) \\ &\leq C_{\nu, \nu', \delta, s} \varepsilon^{(1 - \frac{\eta_0}{10}) \frac{\delta}{2}} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^2 + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right). \end{aligned} \quad (6.126)$$

with the corresponding conditions (as we have  $\eta_0 \leq \frac{1}{2}$ )

$$\begin{cases} \theta \in ]0, 1[ \text{ (for any } s \in [\frac{1}{2} - \frac{\eta_0}{10} \delta, \frac{1}{2} + \frac{\eta_0}{10} \delta]) & \iff (1 + \frac{\eta_0}{10}) \delta \leq \frac{2}{3}, \\ p \in [2, \frac{2}{\theta(1-\frac{2}{r})}] & \iff \delta < \frac{1}{2}. \end{cases} \quad (6.127)$$

If  $\varepsilon \leq \varepsilon_0$ , we obtain for any  $s \in [\frac{1}{2} - \frac{\eta_0}{10} \delta, \frac{1}{2} + \frac{\eta_0}{10} \delta]$ ,

$$\|W_\varepsilon\|_{L^{\frac{2}{1-s}} L^6} \leq C_{\nu, \nu', \mathbb{C}_0, \delta} \varepsilon^{(1 - \frac{\eta_0}{10}) \frac{\delta}{2} - \gamma} = C_{\nu, \nu', \delta, \mathbb{C}_0} \varepsilon^{\frac{19}{20} \eta_0 \delta} \leq C_{\nu, \nu', \delta, \mathbb{C}_0} \varepsilon^{\frac{9\eta_0}{10} \delta}.$$

- The previous case when replacing  $s$  by  $\frac{1}{2}$  gives (when  $\delta < \frac{1}{2}$ ):

$$\|W_\varepsilon\|_{L^4 L^6} \leq C_{\nu, \nu', \mathbb{C}_0, \delta} \varepsilon^{\frac{\delta}{2}} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^2 + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right) \leq C_{\nu, \nu', \delta, \mathbb{C}_0} \varepsilon^{\eta_0 \delta} \quad (6.128)$$

- Choosing  $(d, p, r) = (1, \frac{2}{1-s(1-(\alpha_0+1)\delta)}, \frac{3}{\frac{3}{2}-\alpha_0\delta})$ , (6.121) is satisfied if and only if

$$\frac{2}{3}\alpha_0\delta\theta = \frac{1}{2} + (1 - \alpha_0)\delta - s(1 - (1 + \alpha_0)\delta).$$

We recall that  $\alpha_0$  was chosen in (5.79) to ensure that  $\theta \in [0, 1]$ . And the condition on  $p$  is satisfied if, and only if  $(1 - \alpha_0)\delta < \frac{1}{2}$ , which is true as  $\alpha_0$  is larger than 1 and  $\delta \leq \frac{1}{6}$ . This allows us to write that (for some constant  $C_{\nu, \nu', \delta, \eta_0, s}$  bounded when  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$  by some  $C_{\nu, \nu', \delta, \eta_0}$ ):

$$\begin{aligned} \|\nabla W_\varepsilon\|_{L_t^{\frac{1}{1-s(1-(\alpha_0+1)\delta)}} L^{\frac{3}{\frac{3}{2}-\alpha_0\delta}}} &\leq C_{\nu, \nu', \delta, s, \eta_0} \varepsilon^{\frac{\alpha_0\delta\theta}{3}} \left( \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}} + \|c_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^2 + \|\tilde{b}_0\|_{\dot{H}^\delta}^2 \right) \\ &\leq C_{\nu, \nu', \delta, \eta_0, \mathbb{C}_0} \varepsilon^{\frac{\alpha_0\delta\theta}{3} - \gamma}. \end{aligned} \quad (6.129)$$

Thanks to the notations in Theorem 5.2, we can bound from below the previous exponent as follows (as we wish a bound valid for any  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ ):

$$\begin{aligned} \frac{\alpha_0\delta\theta}{3} - \gamma &= \frac{1}{4} + (1 - \alpha_0)\frac{\delta}{2} - s\left(\frac{1}{2} - (1 + \alpha_0)\frac{\delta}{2}\right) - \frac{\delta}{2}(1 - 2\eta_0) \\ &\geq \frac{1}{4} + (1 - \alpha_0)\frac{\delta}{2} - \frac{1}{4}\left(1 + \frac{\eta_0}{5}\delta\right)\left(1 - (1 + \alpha_0)\delta\right) - \frac{\delta}{2}(1 - 2\eta_0) \\ &\geq \frac{\delta}{2} \left( \frac{1}{2} + \frac{\eta_0}{10}(19 + \delta) - \frac{1}{2}\left(1 - \frac{\eta_0}{5}\delta\right)\alpha_0 \right) \geq \frac{\delta}{2(7 + \frac{3}{5}\eta_0\delta)}(-1 + 13\eta_0) \\ &\geq \frac{\delta}{16}(-1 + 13\eta_0) \geq \frac{\eta_0}{16}\delta, \end{aligned} \quad (6.130)$$

where the last estimate is true as  $\eta_0 \geq \frac{1}{12}$ . This finally gives that for any  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ :

$$\|\nabla W_\varepsilon\|_{L_t^{\frac{1}{1-s(1-(\alpha_0+1)\delta)}} L^{\frac{3}{\frac{3}{2}-\alpha_0\delta}}} \leq C_{\nu, \nu', \delta, \eta_0} \varepsilon^{\frac{\eta_0}{16}\delta}.$$

- Taking  $(d, p, r, q) = (0, 2, \infty, 1)$ , and similarly to what we did to obtain (6.122), we have (thanks to Propositions 6.1 and 6.2),

$$\begin{aligned} \|W_\varepsilon\|_{L^2 L^\infty} &\leq \|W_\varepsilon\|_{L^2 \dot{B}_{\infty, 1}^0} \leq \|W_\varepsilon\|_{\tilde{L}^2 \dot{B}_{\infty, 1}^0} \\ &\leq \frac{C_\theta}{\nu^{\frac{1-\theta}{2}}} \varepsilon^{\frac{\theta}{2}} \left( \|v_{0, \varepsilon}\|_{\dot{B}_{2, 1}^{\frac{1}{2} + \theta}} + \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L_t^1 \dot{B}_{2, 1}^{\frac{1}{2} + \theta}} + \frac{1}{\nu^{\frac{1}{4}}} \|\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b}\|_{L^{\frac{4}{3}} \dot{B}_{2, 1}^\theta} \right). \end{aligned} \quad (6.131)$$

Now we can use the arguments from [18]: applying Proposition 6.1 with  $(\alpha, \beta) = (a\theta, b\theta)$ , we have:

$$\|v_{0, \varepsilon}\|_{\dot{B}_{2, 1}^{\frac{1}{2} + \theta}} \leq C_{a, b} \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + (1-a)\theta}}^{\frac{b}{a+b}} \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + (1+b)\theta}}^{\frac{a}{a+b}} \leq C_{a, b} \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \mu\delta}}^{\frac{b}{a+b}} \|v_{0, \varepsilon}\|_{\dot{H}^{\frac{1}{2} + \delta}}^{\frac{a}{a+b}}, \quad (6.132)$$

if we have chosen  $a, b > 0$  and  $\theta \in ]0, 1]$  so that:

$$\begin{cases} (1 - a)\theta = \mu\delta, \\ (1 + b)\theta = \delta. \end{cases}$$

To do this, for some  $b > 0$  (that will be chosen small in a few lines), we simply choose  $\theta = \frac{\delta}{1+b}$  and  $a = 1 - \mu(1+b)$  (we recall that  $\mu \in ]0, 1[$  so  $a$  is positive if  $b > 0$  is small enough). Thanks to the assumption (5.55) we obtain:

$$\|v_{0,\varepsilon}\|_{\dot{B}_{2,1}^{\frac{1}{2}+\theta}} \leq C_{b,\mu} \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\mu\delta}}^{\frac{b}{(1-\mu)(1+b)}} \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\delta}}^{\frac{1-\mu(1+b)}{(1-\mu)(1+b)}} \leq C_{b,\mu} \mathbb{C}_0 \varepsilon^{-\gamma}.$$

Using the previous estimates and Proposition 6.5, we obtain that for some positive constant  $C = C(\nu', \|\tilde{u}_0\|_{L^2}, \delta, b, \mu)$ :

$$\|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1 \dot{B}_{2,1}^{\frac{1}{2}+\theta}} \leq C_{b,\mu} \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1 \dot{H}^{\frac{1}{2}+\mu\delta}}^{\frac{b}{(1-\mu)(1+b)}} \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1 \dot{H}^{\frac{1}{2}+\delta}}^{\frac{1-\mu(1+b)}{(1-\mu)(1+b)}} \leq C \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}^2.$$

Similarly,

$$\|\tilde{b} \cdot \nabla c_\varepsilon\|_{L^{\frac{4}{3}} \dot{B}_{2,1}^\theta} \leq C_{b,\mu} \|\tilde{b} \cdot \nabla c_\varepsilon\|_{L^{\frac{4}{3}} \dot{H}^{\mu\delta}}^{\frac{b}{(1-\mu)(1+b)}} \|\tilde{b} \cdot \nabla c_\varepsilon\|_{L^{\frac{4}{3}} \dot{H}^\delta}^{\frac{1-\mu(1+b)}{(1-\mu)(1+b)}} \leq C \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}},$$

and

$$\|c_\varepsilon \cdot \nabla \tilde{b}\|_{L^{\frac{4}{3}} \dot{B}_{2,1}^\theta} \leq C \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}.$$

Plugging these estimates into (6.132), there exists a constant  $\mathbb{E}_0 = \mathbb{E}_0(\nu, \nu', \mathbb{C}_0, \delta, \eta_0, \mu) > 0$  such that (the last estimate being true because  $\varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0$  has been defined in the proof of the first estimates):

$$\|W_\varepsilon\|_{L^2 L^\infty} \leq C \varepsilon^{\frac{\delta}{2(1+b)}} \left( \mathbb{C}_0 \varepsilon^{-\gamma} + \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}^2 + \|\tilde{b}_0\|_{H^\delta}^2 \right) \leq C \varepsilon^{\frac{\delta}{2(1+b)} - \gamma} = \mathbb{E}_0 \varepsilon^{\frac{\delta}{2}(2\eta_0 + \frac{1}{1+b} - 1)}.$$

As we have:

$$\frac{1}{1+b} - 1 \geq -\eta_0 \iff b \leq \frac{\eta_0}{1-\eta_0},$$

Finally choosing  $b = \frac{\eta_0}{1-\eta_0}$ , we obtain

$$\|W_\varepsilon\|_{L^2 L^\infty} \leq \mathbb{E}_0 \varepsilon^{\frac{\eta_0}{2}\delta}. \quad (6.133)$$

### 6.3.4 Proof of the anisotropic Strichartz estimates for System (5.52)

We prove in this section the last two estimates from Proposition 5.2.

As in the previous section, writing (6.120) for  $k = \frac{4}{3}$  we get that for any  $d \in \mathbb{R}$ ,  $m > 2$ ,  $\theta \in ]0, 1]$ ,  $p \in [2, \frac{4}{\theta(1-\frac{2}{m})}]$ , there exists a constant  $C = C_{p,\theta,m} > 0$  such that:

$$\begin{aligned} & \| |D|^d W_\varepsilon \|_{L^p L_{h,v}^{m,2}} \\ & \leq \frac{C_{p,\theta,m}}{\nu^{\frac{1}{p} - \frac{\theta}{4}(1-\frac{2}{m})}} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{m})} \left( \|v_{0,\varepsilon}\|_{\dot{B}_{2,1}^{\sigma_3}} + \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1 \dot{B}_{2,1}^{\sigma_3}} + \frac{1}{\nu^{\frac{1}{4}}} \|\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b}\|_{L_t^{\frac{4}{3}} \dot{B}_{2,1}^{\sigma_3 - \frac{1}{2}}} \right). \end{aligned} \quad (6.134)$$

**Remark 6.5** Notice that here  $\sigma_3$  is kept as in Proposition 6.7 as we have to deal, as in the end of the previous section, with the third Besov index equal to 1 through the methods from [18].

- Choosing  $(d, p, m) = (0, \frac{2}{1-s}, \infty)$  we obtain that  $\sigma_3 = s + \frac{\theta}{2}$ . Reproducing the arguments leading to (6.132), we define  $a, b > 0$  and  $\theta \in ]0, 1]$  so that:

$$\begin{cases} s + (1-a)\frac{\theta}{2} = \frac{1}{2} + \mu\delta, \\ s + (1+b)\frac{\theta}{2} = \frac{1}{2} + \delta. \end{cases}$$

First consider some small  $b > 0$  (whose value will be precised in the following lines), and simply define (as above, the second term is positive when  $b > 0$  is small enough)

$$\theta = \frac{2}{1+b} \left( \frac{1}{2} + \delta - s \right) \quad \text{and} \quad a = 1 - (1+b) \frac{\frac{1}{2} + \mu\delta - s}{\frac{1}{2} + \delta - s}. \quad (6.135)$$

Then, thanks to Proposition 6.1, we get:

$$\begin{aligned} \|v_{0,\varepsilon}\|_{\dot{B}_{2,1}^{s+\frac{\theta}{2}}} &\leq C_{a,b} \|v_{0,\varepsilon}\|_{\dot{H}^{s+(1-a)\frac{\theta}{2}}}^{\frac{b}{a+b}} \|v_{0,\varepsilon}\|_{s+(1+b)\frac{\theta}{2}}^{\frac{a}{a+b}} \\ &\leq C_{a,b} \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\mu\delta}}^{\frac{b}{a+b}} \|v_{0,\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\delta}}^{\frac{a}{a+b}} \leq C_{b,\mu,\delta,s} \mathbb{C}_0 \varepsilon^{-\gamma}. \end{aligned} \quad (6.136)$$

Similarly, we obtain that:

$$\begin{cases} \|c_\varepsilon \cdot \nabla c_\varepsilon\|_{L^1 \dot{B}_{2,1}^{s+\frac{\theta}{2}}} \leq C_{\nu',\delta,s,\mu,b} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}^2, \\ \|\tilde{b} \cdot \nabla c_\varepsilon + c_\varepsilon \cdot \nabla \tilde{b}\|_{L^{\frac{4}{3}} \dot{B}_{2,1}^{s+\frac{\theta}{2}-\frac{1}{2}}} \leq C_{\nu',\delta,s,\mu,b} \|\tilde{b}_0\|_{H^\delta} \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}. \end{cases}$$

Plugging these estimates and (6.135) into (6.134), thanks to the fact that  $\varepsilon \leq \varepsilon_0$ , we obtain that for all  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ ,

$$\begin{aligned} \|W_\varepsilon\|_{L^{\frac{2}{1-s}} L_{h,v}^{\infty,2}} &\leq C_{\nu,\nu',\mathbb{C}_0,\delta,s,\mu,b} \varepsilon^{\frac{1}{2(1+b)}(\frac{1}{2}+\delta-s)} \left( \mathbb{C}_0 \varepsilon^{-\gamma} + \|c_{0,\varepsilon}\|_{H^{\frac{1}{2}+\delta}}^2 + \|\tilde{b}_0\|_{H^\delta}^2 \right) \\ &\leq C_{\nu,\nu',\mathbb{C}_0,\delta,s,\mu,b} \varepsilon^{\frac{1}{2(1+b)}(\frac{1}{2}+\delta-s)-\gamma} \leq \mathbb{E}_0 \varepsilon^{\frac{\delta}{2}(2\eta_0 + \frac{1}{1+b}(1-\frac{\eta_0}{10})-1)}. \end{aligned} \quad (6.137)$$

As we have

$$2\eta_0 + \frac{1}{1+b} \left(1 - \frac{\eta_0}{10}\right) - 1 \geq \frac{9}{5}\eta_0 \quad \iff \quad b \leq \frac{\eta_0}{10 - 2\eta_0},$$

we can choose  $b = \frac{\eta_0}{10-2\eta_0}$  and finally obtain that for all  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$ ,

$$\|W_\varepsilon\|_{L^{\frac{2}{1-s}} L_{h,v}^{\infty,2}} \leq \mathbb{E}_0 \varepsilon^{\frac{9\eta_0}{10}\delta}.$$

Note that the conditions on  $p, \theta$  are satisfied as we already asked that  $\delta \leq \frac{1}{6}$ .

- For the last estimate, choosing  $(d, p, m) = (1, 2, \frac{2}{1-s})$ , we obtain  $\sigma_3 = s + s\frac{\theta}{2}$ . As previously, for some  $b > 0$  that will be precised later, choosing  $a, \theta$  as follows,

$$\theta = \frac{2}{s+b} \left( \frac{1}{2} + \delta - s \right) \quad \text{and} \quad a = s - (s+b) \frac{\frac{1}{2} + \mu\delta - s}{\frac{1}{2} + \delta - s},$$

ensures that:

$$\begin{cases} s + (s-a)\frac{\theta}{2} = \frac{1}{2} + \mu\delta, \\ s + (s+b)\frac{\theta}{2} = \frac{1}{2} + \delta, \end{cases}$$

which allows us to reproduce the previous arguments and leads for all  $s \in [\frac{1}{2} - \frac{\eta_0}{10}\delta, \frac{1}{2} + \frac{\eta_0}{10}\delta]$  to

$$\begin{aligned} \|\nabla W_\varepsilon\|_{L^2 L_{h,v}^{\frac{2}{1-s},2}} &\leq C_{\nu,\nu',\mathbb{C}_0,\delta,s,\mu,b} \varepsilon^{\frac{s}{2(s+b)}(\frac{1}{2}+\delta-s)-\gamma} \\ &\leq C_{\nu,\nu',\mathbb{C}_0,\delta,s,\mu,b} \varepsilon^{\frac{\delta}{2}(2\eta_0 + \frac{s}{s+b}(1-\frac{\eta_0}{10})-1)} \leq C_{b,c,\nu,\nu',\delta,\mathbb{C}_0} \varepsilon^{\frac{\delta}{2} \left( 2\eta_0 + \frac{\frac{1}{2} - \frac{\eta_0}{10}\delta}{\frac{1}{2} - \frac{\eta_0}{10}\delta + b} (1 - \frac{\eta_0}{10}) - 1 \right)}. \end{aligned} \quad (6.138)$$

Choosing  $b = (\frac{1}{2} - \frac{\eta_0}{10}) \frac{\eta_0}{10-2\eta_0}$  we finally obtain that (the conditions on  $p, \theta$  being satisfied as  $\delta \leq \frac{1}{6}$ .)

$$\|\nabla W_\varepsilon\|_{L^2 L_{h,v}^{\frac{2}{1-s},2}} \leq \mathbb{E}_0 \varepsilon^{\frac{9\eta_0}{10}\delta},$$

which ends the proof of Proposition 5.2. ■

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