



**HAL**  
open science

# Convergence of explicit Runge-Kutta discontinuous Galerkin approximations of the first-order form of Maxwell's equations with low regularity solutions

Alexandre Ern, Jean-Luc Guermond

## ► To cite this version:

Alexandre Ern, Jean-Luc Guermond. Convergence of explicit Runge-Kutta discontinuous Galerkin approximations of the first-order form of Maxwell's equations with low regularity solutions. 2025. ⟨hal-05142005⟩

**HAL Id: hal-05142005**

**<https://hal.science/hal-05142005v1>**

Preprint submitted on 3 Jul 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Convergence of explicit Runge–Kutta discontinuous Galerkin approximations of the first-order form of Maxwell’s equations with low regularity solutions\*

Alexandre Ern<sup>†</sup>      Jean-Luc Guermond<sup>‡</sup>

Draft version July 3, 2025

## Abstract

We establish a convergence result for the approximation of low-regularity solutions to time-dependent PDE systems that have an involution structure similar to Maxwell’s equations and the linear wave equations. The approximation is based on an explicit Runge–Kutta (ERK) time-stepping and the discontinuous Galerkin (dG) method with stabilization (so-called upwind fluxes) in space. The regularity setting only assumes that the exact solution and its first time-derivative are in  $L^\infty(J; H^s(D))$  with a Sobolev regularity index  $s$  in  $(0, \frac{1}{2})$  (here,  $J$  is the time interval and  $D$  the space domain), and that its second time-derivative is in  $L^\infty(J; L^2(D))$ . The two main tools for the convergence analysis are a Ritz projection in space that leverages recent convergence results in operator norm for the dG approximation of the steady form of the PDE, and the  $L^2$ -stability under a standard CFL condition of three-stage, third-order and four-stage, fourth-order ERK schemes. These latter results are known in the literature, but we provide here a somewhat simpler argumentation to prove the  $L^2$ -stability.

**Keywords.** Discontinuous Galerkin, explicit Runge-Kutta method, involution, finite elements, Maxwell’s equations, linear wave equation

**MSC.** 65N25, 35Q61, 65N15, 65N30, 35L05, 35P15

## 1 Introduction

We want to solve time-dependent PDE systems that have an involution structure similar to Maxwell’s equations and the linear wave equations. More precisely, we are interested in approximating in time and space problems of the following type:

$$\partial_t u + A(u) = 0, \quad u(0) = u^0, \quad (1)$$

where the linear operator  $A : D(A) \subset L \rightarrow L$  is unbounded and maximal monotone, and the initial data  $u^0$  is in  $L$ . For the applications we have in mind,  $L$  is a (real or complex) Hilbert space with

---

<sup>†</sup>CERMICS, ENPC, Institut Polytechnique de Paris, 77455 Marne-la-Vallée Cedex 2, France and INRIA Paris, 75589 Paris, France

<sup>‡</sup>Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843, USA.

\*This material is based upon work supported in part by the National Science Foundation grant DMS2110868, the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-18-1-0397, the Army Research Office, under grant number W911NF-19-1-0431, and the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contracts B640889. The support of INRIA and of University Paris-Est is gratefully acknowledged.

inner product  $(\cdot, \cdot)_L$ . Recall that  $A$  being maximal monotone means that  $\Re((A(v), v)_L) \geq 0$  for all  $v \in D(A)$ , and that, for all  $\lambda \in \mathbb{R}_{>0}$  and all  $f \in L$ , there exists  $v \in D(A)$  so that  $\lambda v + A(v) = f$ . Recall also that maximal monotone operators are closed and densely defined, cf. Brezis [6, §7.1]. Recall finally that the Hille–Yosida theorem implies that for all  $u^0 \in D(A)$ , the system (1) has a unique solution in  $C^1([0, +\infty); L) \cap C^0([0, +\infty); D(A))$ .

The class of problems of interest to us in this paper have an additional property that implies that solutions to (1) have some smoothness and can therefore be well approximated. More specifically, we assume that there exists a subspace  $H^s$  dense in  $L$  with compact embedding, i.e.,  $H^s \Subset L$ , so that

$$\text{im}(A) \cap D(A) \subset H^s. \quad (2)$$

In the applications considered herein,  $H^s$  is a fractional-order Sobolev space with smoothness index  $s$  in  $(0, \frac{1}{2})$ , and  $L$  is the space of square integrable functions/vector fields. The compactness of the injection  $H^s \Subset L$  implies that functions in  $\text{im}(A) \cap D(A)$  can be well approximated in  $L$ . Notice in passing that, as  $\text{im}(A)$  is closed, Banach's closed range theorem implies that  $\ker(A^\top)^\perp = \text{im}(A)$  (here, orthogonality is meant in  $L$ ). Hence, if  $u^0 \in \text{im}(A)$ , then  $u(t)$  remains in  $\text{im}(A)$  owing to the identity

$$(u(t), v)_L = (u^0, v)_L - \int_0^t (A(u(\xi)), v)_L d\xi = 0, \quad \forall v \in \ker(A^\top). \quad (3)$$

Then the Hille–Yosida theorem gives  $u(t) \in \text{im}(A) \cap D(A) \subset H^s$ , which, in turn, implies that  $u(t)$  can be well approximated and the quality of the approximation only depends on the initial data. The implication

$$u^0 \in \text{im}(A) \implies u(t) \in \text{im}(A) \quad \forall t > 0, \quad (4)$$

is called involution in Boillat [4], Dafermos [12], and we henceforth adopt this terminology. In the context of Maxwell's equations, the fact that  $u(t) \in \ker(A^\top)^\perp = \text{im}(A)$  is called Gauss's law of electromagnetism.

The problem addressed in this paper is about the time and space approximation of (1). We prove that, when properly stabilized, the discontinuous Galerkin method combined with explicit Runge–Kutta (ERK) schemes is convergent in the  $L^2$ -norm. We do so by using only the smoothness property (2) and the involution property (4) which, when combined, give  $u(t) \in H^s$  for all  $t > 0$ . The two key ingredients of the proof are as follows: (i) ERK methods are  $L^2$ -stable under a standard CFL condition; (ii) the approximation of the steady state problem  $A(v) = f$  with  $f \in \ker(A^\top)^\perp = \text{im}(A)$  using a (properly stabilized) discontinuous Galerkin method converges in operator norm in  $L^2$  with rate  $h^s$ , where  $s$  is the smoothness index in (2). We focus on three-stage, third-order and four-stage, fourth-order ERK schemes. The  $L^2$ -stability of third-order schemes has been proved in Burman et al. [7] and Zhang and Shu [29] (see Remark 4.2 below), and the  $L^2$ -stability of fourth-order schemes has been proved in Xu et al. [28]; see also Sun and Shu [24, 25]. In this paper, we present slightly different proofs that somewhat simplify the argumentation.

The convergence in operator norm of the space approximation of the steady-state problem is proved in [16] for any positive smoothness index  $s$ . This result crucially uses that the discontinuous Galerkin approximation is properly stabilized. For Maxwell's equations, this entails penalizing the jumps of the tangential components of both the magnetic and the electric fields across mesh interfaces and weakly enforcing the boundary conditions. We leverage this result in the convergence analysis herein by introducing the Ritz projection associated with the steady-state problem when forming the time-discrete error equations. This way of proceeding in the convergence analysis of time-dependent PDEs is classical; we refer the reader to Wheeler [27] for the original idea in the context of the heat equation, and we refer to [14, Chaps. 77&78] for examples in the context of time-dependent Friedrichs's systems.

We emphasize that establishing error estimates when the smoothness index  $s$  in (2) is allowed to be smaller than  $\frac{1}{2}$  is a major roadblock in the literature. Considering situations for which  $s \in (0, \frac{1}{2})$  is not an academic exercise as working with heterogeneous materials and non-smooth boundaries systematically leads to smoothness indices that are less than  $\frac{1}{2}$ . The restriction  $s > \frac{1}{2}$  is invoked in most papers dedicated to the analysis of Maxwell's equations, see, in particular, Ciarlet and Zou [10, Thm. 4.2], Fezoui et al. [17, Eq. (16)] and Li [21, Thm. 5.1]. The reader is also referred to Hesthaven and Warburton [18, Thm. 4.2] and Campos Pinto and Sonnendrücker [8] where  $s \geq 2$  is assumed and to Lanteri et al. [20, Sec. 4.2] where  $s \geq 1$  is assumed. The originality of the present paper is to show that, by using the results from [16], it is possible to prove convergence for any  $s > 0$ . Another novel contribution, which is a consequence of using the Ritz projection instead of the  $L^2$ -orthogonal projection to derive the discrete error equation, is an error estimate that does not grow exponentially with respect to the final time. This fact is perhaps not sufficiently emphasized in the literature.

The main convergence results established in the paper are Theorem 5.3 for third-order ERK methods and Theorem 5.7 for fourth-order ERK methods. We emphasize that these results hold for *any* three-stage, third-order and *any* four-stage, fourth-order ERK schemes. The generic smoothness of the solution that is used in the proofs is

$$\|u\|_{L^\infty((0,\infty);H^s)} < \infty, \quad \|\partial_t u\|_{L^\infty((0,\infty);H^s)} < \infty, \quad \|\partial_{tt} u\|_{L^\infty((0,\infty);L)} < \infty. \quad (5)$$

These bounds, which are natural in the context of the Hille–Yosida theorem, can be proved by invoking generic regularity assumptions on the initial data  $u^0$ . The reader is referred to Lemma 2.1 where this result is proved for Maxwell's equations. A similar result is asserted in Lemma 2.3 for the linear wave equation.

The paper is organized as follows. In §2, we show that Maxwell's equations and the linear wave equation have the generic properties invoked in the main convergence results of the paper. We introduce in §3 the discontinuous Galerkin setting and the associated notation. The key properties of the Ritz projection are recalled in Theorem 3.4 in the case of Maxwell's equations and in Theorem 3.8 for the linear wave equations. In §4, we introduce the fully discrete setting. We first rewrite the space semi-discrete problem in a unified setting and then use ERK schemes for the time discretization. Our main convergence results (Theorem 5.3 and Theorem 5.7) are stated and proved in §5. We revisit the proofs of  $L^2$ -stability of the ERK schemes in §6.

## 2 Examples

We show in this section that the generic properties mentioned above hold true for Maxwell's equations and the linear wave equation.

### 2.1 Maxwell's equations

We consider Maxwell's equations in  $\mathbb{R}^3$ . Let  $D \subset \mathbb{R}^d$ ,  $d = 3$ , be an open, bounded, connected polyhedron with Lipschitz boundary. We seek a pair of vector fields  $(\mathbf{H}, \mathbf{E}) : D \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  such that, for all  $t > 0$ ,

$$\partial_t(\mu \mathbf{H}) + \nabla \times \mathbf{E} = \mathbf{0}, \quad \partial_t(\epsilon \mathbf{E}) - \nabla_0 \times \mathbf{H} = \mathbf{0}, \quad \mathbf{H}|_{\partial D} \times \mathbf{n}_D = \mathbf{0}, \quad (6)$$

where  $\mathbf{H}$  is the magnetic field,  $\mathbf{E}$  the electric field, and  $\mathbf{n}_D$  the unit outward normal at the boundary  $\partial D$  of  $D$ . Other boundary conditions can be considered as well. The magnetic permeability  $\mu$  and the electrical permittivity are assumed to be time-independent but can be heterogeneous. We

assume that there is a partition of  $D$  into a finite number of disjoint Lipschitz polyhedra such that  $\mu$  and  $\epsilon$  are piecewise smooth on this partition.

Maxwell's equations fit the framework of the Hille–Yosida theorem by setting

$$u := (\mathbf{H}, \mathbf{E}), \quad (7a)$$

$$L := \mathbf{L}^2(D) \times \mathbf{L}^2(D), \quad ((\mathbf{H}, \mathbf{E}), (\mathbf{h}, \mathbf{e}))_L := (\mu \mathbf{H}, \mathbf{h})_{L^2(D)} + (\epsilon \mathbf{E}, \mathbf{e})_{L^2(D)}, \quad (7b)$$

$$A(u) := \left(\frac{1}{\mu} \nabla \times \mathbf{E}, -\frac{1}{\epsilon} \nabla_0 \times \mathbf{H}\right) \quad \text{with} \quad D(A) = \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D). \quad (7c)$$

Moreover, a key property of (6) is that that  $\mu \mathbf{H}$  is in the image of the curl operator (i.e.,  $\mu \mathbf{H} \in \text{im } \nabla \times$ ) and  $\epsilon \mathbf{E}$  is in the image of the curl operator acting on vector fields with zero tangential trace at the boundary (i.e.,  $\epsilon \mathbf{E} \in \text{im } \nabla_0 \times$ ). Since  $\text{im } \nabla \times = (\ker \nabla_0 \times)^\perp$  and  $\text{im } \nabla_0 \times = (\ker \nabla \times)^\perp$  (here, orthogonality is meant in  $\mathbf{L}^2(D)$ ), the involution property of the solutions to (6) can be reformulated as follows:

$$\begin{aligned} (\mu \mathbf{H}^0, \epsilon \mathbf{E}^0) \in (\ker \nabla_0 \times)^\perp \times (\ker \nabla \times)^\perp &\implies \\ (\mu \mathbf{H}(t), \epsilon \mathbf{E}(t)) \in (\ker \nabla_0 \times)^\perp \times (\ker \nabla \times)^\perp &\quad \forall t > 0, \end{aligned} \quad (8)$$

which are Gauss's laws of magnetism and electrostatics. In addition to implying Gauss's laws, the involution property (8) implies that the solutions to (6) have some smoothness. More precisely, setting

$$\mathbf{X}_{\mu 0} := \{\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \mu \mathbf{h} \in (\ker \nabla_0 \times)^\perp\}, \quad (9a)$$

$$\mathbf{X}_\epsilon := \{\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D) \mid \epsilon \mathbf{e} \in (\ker \nabla \times)^\perp\}, \quad (9b)$$

it is well established in the literature that there exists  $s > 0$  so that

$$\mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon \subset \mathbf{H}^s(D) \times \mathbf{H}^s(D). \quad (10)$$

This is a well-known consequence of the elliptic regularity theory; see e.g., Bonito et al. [5], Jochmann [19]. In general, we have  $s \in (0, \frac{1}{2})$ , which is the case of interest in this paper. When  $\mu$  and  $\epsilon$  are constant or smooth, the smoothness index  $s$  is equal to  $\frac{1}{2}$  in Lipschitz domains, see Costabel [11, Thm. 2], Birman and Solomyak [3, Thm. 3.1], Weber [26], the smoothness index  $s$  is in the interval  $(\frac{1}{2}, 1]$  if  $D$  is a Lipschitz polyhedron (see Amrouche et al. [2, Prop. 3.7]), and  $s = 1$  if  $D$  is convex (see [2, Thm. 2.17]). The reader is also referred to Ciarlet [9, Thm. 16] for situations with heterogeneous properties for which smoothness is established with  $s > \frac{1}{2}$ .

The regularity of the solution to (6) that is invoked in the paper and stated in (5) takes the following form in the context of Maxwell's equations.

**Lemma 2.1** (Smoothness). *Assume that the initial data satisfies  $(\mathbf{H}^0, \mathbf{E}^0) \in D(A) \cap \text{im}(A)$ , i.e.,  $(\mathbf{H}^0, \mathbf{E}^0) \in \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$ . Assume also that  $(\mathbf{H}^0, \mathbf{E}^0) \in D^2(A)$ , i.e.,  $\frac{1}{\epsilon} \nabla_0 \times \mathbf{H}^0 \in \mathbf{H}(\mathbf{curl}; D)$  and  $\frac{1}{\mu} \nabla \times \mathbf{E}^0 \in \mathbf{H}_0(\mathbf{curl}; D)$ . Let  $s > 0$  be the smoothness index so that (10) holds true. Then the solution to (6) is such that, for all  $T > 0$ ,*

$$(\mathbf{H}, \mathbf{E}) \in L^\infty((0, T); \mathbf{H}^s(D) \times \mathbf{H}^s(D)), \quad (11a)$$

$$(\partial_t \mathbf{H}, \partial_t \mathbf{E}) \in L^\infty((0, T); \mathbf{H}^s(D) \times \mathbf{H}^s(D)), \quad (11b)$$

$$(\partial_{tt} \mathbf{H}, \partial_{tt} \mathbf{E}) \in L^\infty((0, T); \mathbf{L}^2(D) \times \mathbf{L}^2(D)). \quad (11c)$$

*Proof.* This is just an application of the Hille–Yosida theorem, but we sketch the proof for completeness. Let  $T > 0$  be a fixed time.

(i) Taking one time derivative of (6), testing the PDEs with the pair  $(\partial_t \mathbf{H}, \partial_t \mathbf{E})$ , and using that  $(\partial_t \mathbf{H})|_{\partial D} \times \mathbf{n}_D = \mathbf{0}$ , gives

$$\frac{d}{dt} \left\{ \|\mu^{\frac{1}{2}} \partial_t \mathbf{H}(t)\|_{L^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \partial_t \mathbf{E}(t)\|_{L^2(D)}^2 \right\} = 0.$$

Since  $(\mathbf{H}^0, \mathbf{E}^0) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$ , this gives

$$\begin{aligned} \|\mu^{-\frac{1}{2}} \nabla \times \mathbf{E}\|_{L^\infty((0,T); L^2(D))}^2 + \|\epsilon^{-\frac{1}{2}} \nabla_0 \times \mathbf{H}\|_{L^\infty((0,T); L^2(D))}^2 = \\ \|\mu^{-\frac{1}{2}} \nabla \times \mathbf{E}^0\|_{L^2(D)}^2 + \|\epsilon^{-\frac{1}{2}} \nabla_0 \times \mathbf{H}^0\|_{L^2(D)}^2 < \infty, \end{aligned}$$

where we used (6) to replace the time derivatives by the curl operators. Moreover, as  $(\mathbf{H}(t), \mathbf{E}(t)) \in (\ker \nabla_0 \times)^\perp \times (\ker \nabla \times)^\perp$  for all  $t \in (0, T)$  owing to (8) and the assumption  $(\mathbf{H}^0, \mathbf{E}^0) \in \text{im}(A)$ , we conclude that both  $\mathbf{H}$  and  $\mathbf{E}$  are in  $L^\infty((0, T); \mathbf{H}^s(D))$ . This establishes (11a).

(ii) Taking two time derivatives of (6), testing with the pair  $(\partial_{tt} \mathbf{H}, \partial_{tt} \mathbf{E})$ , using  $(\partial_{tt} \mathbf{H})|_{\partial D} \times \mathbf{n}_D = \mathbf{0}$ , and  $\mu^{\frac{1}{2}} \partial_{tt} \mathbf{H}(0) = -\mu^{-\frac{1}{2}} \nabla \times (\epsilon^{-1} \nabla_0 \times \mathbf{H}^0)$ ,  $\epsilon^{\frac{1}{2}} \partial_{tt} \mathbf{E}(0) = -\epsilon^{-\frac{1}{2}} \nabla_0 \times (\mu^{-1} \nabla \times \mathbf{E}^0)$ , and reasoning as above gives

$$\begin{aligned} \|\mu^{\frac{1}{2}} \partial_{tt} \mathbf{H}\|_{L^\infty((0,T); L^2(D))}^2 + \|\epsilon^{\frac{1}{2}} \partial_{tt} \mathbf{E}\|_{L^\infty((0,T); L^2(D))}^2 = \\ \|\mu^{-\frac{1}{2}} \nabla \times (\epsilon^{-1} \nabla_0 \times \mathbf{H}^0)\|_{L^2(D)}^2 + \|\epsilon^{-\frac{1}{2}} \nabla_0 \times (\mu^{-1} \nabla \times \mathbf{E}^0)\|_{L^2(D)}^2 < \infty. \end{aligned}$$

As we also have  $\nabla \times \partial_t \mathbf{E} = -\partial_{tt}(\mu \mathbf{H})$  and  $\nabla_0 \times \partial_t \mathbf{H} = \partial_{tt}(\epsilon \mathbf{E})$ , this estimate implies that  $\partial_t \mathbf{H}(t) \in \mathbf{H}_0(\mathbf{curl}; D)$ ,  $\partial_t \mathbf{E}(t) \in \mathbf{H}(\mathbf{curl}; D)$  for all  $t \in (0, T)$ . We conclude that  $(\partial_t \mathbf{H}(t), \partial_t \mathbf{E}(t)) \in \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$  for all  $t \in (0, T)$  since  $(\mu \partial_t \mathbf{H}(t), \epsilon \partial_t \mathbf{E}(t)) \in (\ker \nabla_0 \times)^\perp \times (\ker \nabla \times)^\perp$ . This establishes (11b) and (11c).  $\square$

**Remark 2.2** (Involution). *In view of Lemma 2.1, the involution property of the solution to (6) can be reformulated as  $(\mathbf{H}(0), \mathbf{E}(0)) \in \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon \implies (\mathbf{H}(t), \mathbf{E}(t)) \in \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$ , for all  $t > 0$ .*

## 2.2 Linear wave equation

Let us now consider the linear wave equation in  $\mathbb{R}^d$  with  $d \in \{2, 3\}$ . Assuming again that  $D \subset \mathbb{R}^d$  is an open, bounded, connected polyhedron with Lipschitz boundary, we seek a pair of scalar and vector fields  $(p, \mathbf{v}) : D \rightarrow \mathbb{R} \times \mathbb{R}^d$  such that, for all  $t > 0$ ,

$$\partial_t(\kappa p) + \nabla_0 \cdot \mathbf{v} = 0, \quad \partial_t(\rho \mathbf{v}) + \nabla p = \mathbf{0}, \quad \mathbf{v}|_{\partial D} \cdot \mathbf{n}_D = 0, \quad (12)$$

where  $p$  is the pressure and  $\mathbf{v}$  the velocity. Other boundary conditions can be considered as well. Moreover, the compressibility  $\kappa$  and the density  $\rho$  are time-independent coefficients that can be heterogeneous. We assume that there is a partition of  $D$  into a finite number of disjoint Lipschitz polyhedra such that  $\kappa$  and  $\rho$  are piecewise smooth on this partition.

The linear wave equation fits the framework of the Hille–Yosida theorem by setting

$$u := (p, \mathbf{v}), \quad (13a)$$

$$L := L^2(D) \times L^2(D), \quad ((p, \mathbf{v}), (q, \mathbf{w}))_L := (\kappa p, q)_{L^2(D)} + (\rho \mathbf{v}, \mathbf{w})_{L^2(D)}, \quad (13b)$$

$$A(u) := \left( \frac{1}{\kappa} \nabla_0 \cdot \mathbf{v}, \frac{1}{\rho} \nabla p \right) \quad \text{with} \quad D(A) = H^1(D) \times \mathbf{H}_0(\text{div}; D). \quad (13c)$$

Moreover, since  $\text{im}(\nabla) = \ker(\nabla_0 \cdot)^\perp$  and  $\text{im}(\nabla_0 \cdot) = \ker(\nabla)^\perp$ , the involution property associated with (12) is

$$\begin{aligned} (\kappa p^0, \rho \mathbf{v}^0) \in \ker(\nabla)^\perp \times \ker(\nabla_0 \cdot)^\perp \implies \\ (\kappa p(t), \rho \mathbf{v}(t)) \in \ker(\nabla)^\perp \times \ker(\nabla_0 \cdot)^\perp \quad \forall t > 0. \quad (14) \end{aligned}$$

The involution on the pressure reduces to  $\int_D \kappa(x)p(\mathbf{x}, t) d\mathbf{x} = 0$  for all  $t > 0$  because  $D$  is connected, i.e.,  $\ker(\nabla) = \text{span}\{1\} =: \mathbb{P}_0$ . The involution on the velocity amounts to  $\rho\mathbf{v} \in (\ker \nabla_0 \cdot)^\perp$ , which implies in particular that  $\nabla \times (\rho\mathbf{v}) = \mathbf{0}$ . Let us introduce

$$X_\kappa := \{q \in H^1(D) \mid \kappa q \in \ker(\nabla)^\perp\}, \quad (15a)$$

$$\mathbf{X}_{\rho 0} := \{\mathbf{v} \in \mathbf{H}_0(\text{div}; D) \mid \mathbf{v} \in (\ker \nabla_0 \cdot)^\perp\}. \quad (15b)$$

It is well established in the literature that there exists  $s > 0$  so that

$$\mathbf{X}_{\rho 0} \subset \mathbf{H}^s(D). \quad (16)$$

The smoothness index  $s$  is larger than  $\frac{1}{2}$  if  $\rho$  is constant or smooth. The index  $s$  is in general smaller than  $\frac{1}{2}$  if the mass distribution  $\rho$  is discontinuous. Obviously, we have  $X_\kappa \subset H^1(D)$ . Hence, combining (14) and (16), we infer that the solution to (12) satisfies  $(p(t), \mathbf{v}(t)) \in H^1(D) \times \mathbf{H}^s(D)$  for all  $t > 0$ .

**Lemma 2.3** (Smoothness). *Assume that the initial data for the linear wave equation satisfies  $(p^0, \mathbf{v}^0) \in D(A) \cap \text{im}(A)$ , i.e.,  $(p^0, \mathbf{v}^0) \in X_\kappa \times \mathbf{X}_{\rho 0}$ . Assume also that  $(p^0, \mathbf{v}^0) \in D^2(A)$ , i.e.,  $\frac{1}{\rho} \nabla p^0 \in \mathbf{H}_0(\text{div}; D)$  and  $\frac{1}{\kappa} \nabla \cdot \mathbf{v}^0 \in H^1(D)$ . Let  $s > 0$  be the smoothness index so that (16) holds true. Then the solution to (12) is such that, for all  $T > 0$ ,*

$$(p, \mathbf{v}) \in L^\infty((0, T); H^1(D) \times \mathbf{H}^s(D)), \quad (17a)$$

$$(\partial_t p, \partial_t \mathbf{v}) \in L^\infty((0, T); H^1(D) \times \mathbf{H}^s(D)), \quad (17b)$$

$$(\partial_{tt} p, \partial_{tt} \mathbf{v}) \in L^\infty((0, T); L^2(D) \times L^2(D)). \quad (17c)$$

*Proof.* Proceed as in the proof of Lemma 2.1.  $\square$

**Remark 2.4** (Topology of  $D$ ). *The involution on the velocity implies  $\nabla \times \mathbf{v}(t) = \mathbf{0}$  for all  $t > 0$ , since  $(\ker \nabla_0 \cdot)^\perp = \nabla H^1(D)$ . Notice though that the identity  $\nabla \times \mathbf{v}(t) = \mathbf{0}$  fully describes the involution only if  $D$  is simply connected because  $(\ker \nabla_0 \cdot)^\perp = \ker \nabla \times$  if and only if  $D$  is simply connected. The involution that is independent of the topology of  $D$  is  $\mathbf{v}(t) \in (\ker \nabla_0 \cdot)^\perp$  for all  $t > 0$ .*

### 3 DG setting

We describe in this section the dG setting that is used to approximate (6) in space. Although all the material that follows could be reformulated into one single abstract framework, we refrain to do so to facilitate the reading of the paper. We separately introduce the discrete operators that are specific for Maxwell's equations and the linear wave equation. The material herein is standard and is only recalled for completeness. To be dimensionally consistent, we introduce a length scale,  $\ell_D$ , associated with  $D$ ; it can be, for instance, the diameter of  $D$ . We denote the simulation time by  $T > 0$ .

#### 3.1 Meshes, averages and jumps

Let  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  be a shape-regular family of affine simplicial meshes such that each mesh covers  $D$  exactly and is compatible with the partition of  $D$  associated with the discontinuities of the material properties. The meshes are assumed to be quasi-uniform. The symbol  $K$  denotes a generic mesh cell,  $h_K$  its diameter, and  $\mathbf{n}_K$  its outward unit normal. We define  $\tilde{h}$  as the piecewise constant

function on  $\mathcal{T}_h$  such that  $\tilde{h}|_K = h_K$  for all  $K \in \mathcal{T}_h$ ; we set  $h := \|\tilde{h}\|_{L^\infty(D)}$ . The set of mesh faces,  $\mathcal{F}_h$ , is split into the subset of mesh interfaces, say  $\mathcal{F}_h^\circ$ , and the subset of mesh boundary faces, say  $\mathcal{F}_h^\partial$ . Each interface is shared by two distinct mesh cells which we denote  $K_l, K_r$ . Each boundary face is shared by one mesh cell,  $K_l$ , and the boundary,  $\partial D$ . For every mesh face  $F \in \mathcal{F}_h$ ,  $h_F$  denotes the diameter of  $F$ . Every mesh interface  $F \in \mathcal{F}_h^\circ$  is oriented by the unit normal,  $\mathbf{n}_F$ , pointing from  $K_l$  to  $K_r$ . Every boundary face  $F \in \mathcal{F}_h^\partial$  is oriented by the unit normal  $\mathbf{n}_F := \mathbf{n}_D$ .

The approximation is done using polynomials of degree  $k \geq 0$ . We emphasize that it is legitimate to take  $k = 0$  for the theory developed in this paper. The vector space over  $\mathbb{C}$  composed of  $d$ -variate polynomials of total degree at most  $k$  is denoted  $\mathbb{P}_{k,d}$ . We set  $\mathbf{P}_{k,d} := [\mathbb{P}_{k,d}]^d$  and consider the scalar- and vector-valued broken polynomial spaces

$$P_k^b(\mathcal{T}_h) := \{v_h \in L^2(D) \mid v_h|_K \in \mathbb{P}_{k,d}, \forall K \in \mathcal{T}_h\}, \quad (18a)$$

$$\mathbf{P}_k^b(\mathcal{T}_h) := \{\mathbf{w}_h \in \mathbf{L}^2(D) \mid \mathbf{w}_h|_K \in \mathbf{P}_{k,d}, \forall K \in \mathcal{T}_h\}. \quad (18b)$$

We define the  $L^2$ - and  $\mathbf{L}^2$ -orthogonal projections onto the broken polynomial spaces  $P_k^b(\mathcal{T}_h)$  and  $\mathbf{P}_k^b(\mathcal{T}_h)$  as

$$\Pi_h^b : L^2(D) \rightarrow P_k^b(\mathcal{T}_h), \quad \mathbf{\Pi}_h^b : \mathbf{L}^2(D) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h). \quad (19)$$

For all  $K \in \mathcal{T}_h$ , all  $F \in \mathcal{F}_h$  with  $F \subset \partial K$ , all  $v_h \in P_k^b(\mathcal{T}_h)$ , and all  $\mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ , we define the local trace operators such that  $\gamma_{K,F}^g(v_h)(\mathbf{x}) := v_h|_K(\mathbf{x})$ ,  $\gamma_{K,F}^g(\mathbf{w}_h)(\mathbf{x}) := \mathbf{w}_h|_K(\mathbf{x})$ ,  $\gamma_{K,F}^c(\mathbf{w}_h)(\mathbf{x}) := \mathbf{w}_h|_K(\mathbf{x}) \times \mathbf{n}_F$ ,  $\gamma_{K,F}^d(\mathbf{w}_h)(\mathbf{x}) := \mathbf{w}_h|_K(\mathbf{x}) \cdot \mathbf{n}_F$ . Here, the superscript  $\mathbf{x} \in \{g, c, d\}$  refers to the gradient, curl, and divergence operators, since each trace is naturally associated with the corresponding operator. Then, for all  $F = \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ , we define the jump and average operators such that

$$\llbracket v_h \rrbracket_F^g := \gamma_{K_l,F}^g(v_h) - \gamma_{K_r,F}^g(v_h), \quad \{\!\!\{ v_h \}\!\!\}_F^g := \frac{1}{2}(\gamma_{K_l,F}^g(v_h) + \gamma_{K_r,F}^g(v_h)), \quad (20a)$$

$$\llbracket \mathbf{w}_h \rrbracket_F^{\mathbf{x}} := \gamma_{K_l,F}^{\mathbf{x}}(\mathbf{w}_h) - \gamma_{K_r,F}^{\mathbf{x}}(\mathbf{w}_h), \quad \{\!\!\{ \mathbf{w}_h \}\!\!\}_F^{\mathbf{x}} := \frac{1}{2}(\gamma_{K_l,F}^{\mathbf{x}}(\mathbf{w}_h) + \gamma_{K_r,F}^{\mathbf{x}}(\mathbf{w}_h)). \quad (20b)$$

To allow for more compact expressions, we also set  $\llbracket \mathbf{w}_h \rrbracket_F^{\mathbf{x}} := \{\!\!\{ \mathbf{w}_h \}\!\!\}_F^{\mathbf{x}} := \gamma_{K_l,F}^{\mathbf{x}}(\mathbf{w}_h)$  for all  $F = \partial K_l \cap \partial D \in \mathcal{F}_h^\partial$ .

## 3.2 Maxwell's equations

In this section, we exemplify the dG space semi-discretization for Maxwell's equations. The magnetic permeability and electric permittivity of vacuum are denoted  $\mu_0, \epsilon_0$ , respectively. To simplify the presentation, we assume throughout the paper that the ratios  $\mu_0^{-1}\mu$  and  $\epsilon_0^{-1}\epsilon$  are of order unity, so that these ratios can be hidden in the generic constants used in the error analysis. Setting  $\mathfrak{c} := (\mu_0\epsilon_0)^{-\frac{1}{2}}$ , we introduce the quantity  $\omega := \mathfrak{c}\ell_D^{-1}$  which scales as the reciprocal of a time scale.

### 3.2.1 Discrete bilinear form and involution

We define the jump bilinear forms such that for all  $\mathbf{H}_h, \mathbf{h}_h, \mathbf{E}_h, \mathbf{e}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,

$$s_h^c(\mathbf{H}_h, \mathbf{h}_h) := \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{H}_h \rrbracket_F^c, \llbracket \mathbf{h}_h \rrbracket_F^c)_{L^2(F)}, \quad (21a)$$

$$s_h^{c,\circ}(\mathbf{E}_h, \mathbf{e}_h) := \sum_{F \in \mathcal{F}_h^\circ} (\llbracket \mathbf{E}_h \rrbracket_F^c, \llbracket \mathbf{e}_h \rrbracket_F^c)_{L^2(F)}, \quad (21b)$$

which induce the seminorms  $|\mathbf{h}_h|_J^c := s_h^c(\mathbf{h}_h, \mathbf{h}_h)^{\frac{1}{2}}$  and  $|e_h|_J^{c,\circ} := s_h^{c,\circ}(e_h, e_h)^{\frac{1}{2}}$ . The discrete curl operators

$$\mathbf{C}_{h0} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h), \quad \mathbf{C}_h : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h) \quad (22)$$

are defined as follows: For all  $(\mathbf{h}_h, e_h) \in \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ ,  $\mathbf{C}_{h0}(\mathbf{h}_h)$  and  $\mathbf{C}_h(e_h)$  are the unique members of  $\mathbf{P}_k^b(\mathcal{T}_h)$  such that the following identities hold true for all  $e'_h, \mathbf{h}'_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ :

$$(\mathbf{C}_{h0}(\mathbf{h}_h), e'_h)_{L^2(D)} := (\nabla_h \times \mathbf{h}_h, e'_h)_{L^2(D)} + \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{h}_h \rrbracket_F^c, \{\!\!\{ e'_h \}\!\!\}_F^g)_{L^2(F)}, \quad (23a)$$

$$(\mathbf{C}_h(e_h), \mathbf{h}'_h)_{L^2(D)} := (\nabla_h \times e_h, \mathbf{h}'_h)_{L^2(D)} + \sum_{F \in \mathcal{F}_h^\circ} (\llbracket e_h \rrbracket_F^c, \{\!\!\{ \mathbf{h}'_h \}\!\!\}_F^g)_{L^2(F)}, \quad (23b)$$

where  $\nabla_h \times$  denotes the broken curl operator (evaluated cellwise). The following integration by parts formula holds true:

$$(\mathbf{C}_{h0}(\mathbf{h}_h), e_h)_{L^2(D)} = (\mathbf{h}_h, \mathbf{C}_h(e_h))_{L^2(D)}, \quad \forall (\mathbf{h}_h, e_h) \in \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h). \quad (24a)$$

Setting  $L_h := \mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ , we define the bilinear form  $a_h : L_h \times L_h \rightarrow \mathbb{R}$  such that

$$\begin{aligned} a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, e_h)) := & -(\mathbf{C}_h(\mathbf{E}_h), \mathbf{h}_h)_{L^2(D)} + (\mathbf{C}_{h0}(\mathbf{H}_h), e_h)_{L^2(D)} \\ & + \kappa_H s_h^c(\mathbf{H}_h, \mathbf{h}_h) + \kappa_E s_h^{c,\circ}(\mathbf{E}_h, e_h), \end{aligned} \quad (25)$$

with  $\kappa_H := \mu_0 \omega \ell_D = (\mu_0 / \epsilon_0)^{\frac{1}{2}}$  and  $\kappa_E := \epsilon_0 \omega \ell_D = (\epsilon_0 / \mu_0)^{\frac{1}{2}}$  (notice that  $\kappa_H \kappa_E = 1$ ). The Riesz representative of the bilinear form  $a_h$  is the operator  $A_h : L_h \rightarrow L_h$  such that, for all  $(\mathbf{H}_h, \mathbf{E}_h)$  in  $L_h$ , the following identity holds true, for all  $(\mathbf{h}_h, e_h) \in L_h$ :

$$(A_h(\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, e_h))_{L^2(D) \times L^2(D)} := a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, e_h)). \quad (26)$$

The semi-discrete version of Maxwell's equations (6) consists of seeking the pair  $(\mathbf{H}_h, \mathbf{E}_h) \in C^1([0, T]; L_h)$  so that

$$\partial_t(\epsilon \mathbf{E}_h, \mu \mathbf{H}_h) + A_h(\mathbf{H}_h, \mathbf{E}_h) = \mathbf{0}, \quad (\mathbf{H}_h(0), \mathbf{E}_h(0)) = (\mathbf{H}_h^0, \mathbf{E}_h^0), \quad (27)$$

with initial data  $(\mathbf{H}_h^0, \mathbf{E}_h^0)$  approximating  $(\mathbf{H}^0, \mathbf{E}^0)$ . Assuming that  $(\mathbf{E}_h^0, \mathbf{H}_h^0) \in \text{im}(A_h)$ , the pair  $(\epsilon \mathbf{E}_h, \mu \mathbf{H}_h)$  remains in the image of the operator  $A_h$  as  $t$  grows, i.e.,

$$(\epsilon \mathbf{E}_h(t), \mu \mathbf{H}_h(t)) \in \text{im}(A_h) \quad \forall t > 0. \quad (28)$$

What makes this simple observation nontrivial is that  $\text{im}(A_h)$  is a genuine subset of  $L_h$ . Recall that  $\text{im}(A_h) = \ker(A_h^\top)^\perp$ . Let us now characterize  $\ker(A_h^\top)$ . Let  $\mathbf{P}_k^c(\mathcal{T}_h)$  be the  $\mathbf{H}(\mathbf{curl}; D)$ -conforming finite element space composed of the piecewise Nédélec polynomials of the first family and of degree  $k \geq 0$ , and let  $\mathbf{P}_{k0}^c(\mathcal{T}_h)$  be its subspace composed of the discrete fields with zero tangential trace on  $\partial D$ . We set

$$\mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \{e_h \in \mathbf{P}_k^c(\mathcal{T}_h) \mid \nabla \times e_h = \mathbf{0}\}, \quad (29a)$$

$$\mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \{\mathbf{h}_h \in \mathbf{P}_{k0}^c(\mathcal{T}_h) \mid \nabla_0 \times \mathbf{h}_h = \mathbf{0}\}. \quad (29b)$$

Notice that  $\mathbf{P}_k^c(\mathcal{T}_h)$  is not a subspace of  $\mathbf{P}_k^b(\mathcal{T}_h)$  (actually  $\mathbf{P}_k^c(\mathcal{T}_h) \subsetneq \mathbf{P}_{k+1}^b(\mathcal{T}_h)$ ), but we are going to make use of the following well-known results (see, e.g., [13, Chap. 15&18]):

$$\mathbf{P}_k^b(\mathcal{T}_h) \cap \ker \nabla \times = \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h), \quad (30a)$$

$$\mathbf{P}_k^b(\mathcal{T}_h) \cap \ker \nabla_0 \times = \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h). \quad (30b)$$

The following result is established in [16, Lem. 3.1].

**Lemma 3.1** (Discrete involution). *The following holds true:*

$$\ker(A_h^\top) = \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \times \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h). \quad (31)$$

Lemma 3.1 gives  $\text{im}(A_h) = \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp \times \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$ . Accordingly, we set

$$\mathbf{X}_{\mu 0, h} := \{\mathbf{h}_h \in \mathbf{P}_k^b(\mathcal{T}_h) \mid \mu \mathbf{h}_h \in \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp\}, \quad (32a)$$

$$\mathbf{X}_{\epsilon, h} := \{\mathbf{e}_h \in \mathbf{P}_k^b(\mathcal{T}_h) \mid \epsilon \mathbf{e}_h \in \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp\}. \quad (32b)$$

With this notation, we observe that the solution to (27) satisfies the involution

$$(\mathbf{H}_h(0), \mathbf{E}_h(0)) \in \mathbf{X}_{\mu 0, h} \times \mathbf{X}_{\epsilon, h} \implies (\mathbf{H}_h(t), \mathbf{E}_h(t)) \in \mathbf{X}_{\mu 0, h} \times \mathbf{X}_{\epsilon, h} \quad \forall t > 0. \quad (33)$$

**Remark 3.2** (Quadrangles and hexaedrons). *The identities in (30) do not hold for quadrangles and hexaedrons with the definition of  $\mathbf{P}_k^b(\mathcal{T}_h)$  given in (19). In this case, the dG space  $\mathbf{P}_k^b(\mathcal{T}_h)$  has to be slightly enriched as explained in Perrier [22, §5].*

### 3.2.2 Deflated inf-sup condition and Ritz projection

We equip  $L$  with the norm  $\|(\mathbf{f}, \mathbf{g})\|_L := (\|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \mathbf{g}\|_{\mathbf{L}^2(D)}^2)^{\frac{1}{2}}$ , and we equip  $L_h$  with the mesh-dependent norm

$$\begin{aligned} \|(\mathbf{h}_h, \mathbf{e}_h)\|_{b, h} &:= \omega^{\frac{1}{2}} \|(\mathbf{h}_h, \mathbf{e}_h)\|_L \\ &+ \kappa_H^{\frac{1}{2}} \{ \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_{h0}(\mathbf{h}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{h}_h|_J^c \} + \kappa_E^{\frac{1}{2}} \{ \|\tilde{h}^{\frac{1}{2}} \mathbf{C}_h(\mathbf{e}_h)\|_{\mathbf{L}^2(D)} + |\mathbf{e}_h|_J^{c, \circ} \}. \end{aligned} \quad (34)$$

The following two results established in [15, 16] plays a key role in the paper.

**Lemma 3.3** (Deflated inf-sup condition). *There exists  $c > 0$  so that, for all  $h \in \mathcal{H}$  and for every pair  $(\mathbf{H}_h, \mathbf{E}_h)$  in  $\mathbf{X}_{\mu 0, h} \times \mathbf{X}_{\epsilon, h}$ , the following holds true:*

$$\omega^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{b, h} \lesssim \sup_{(\mathbf{h}_h, \mathbf{e}_h) \in L_h} \frac{|a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h))|}{\|(\mathbf{h}_h, \mathbf{e}_h)\|_L}. \quad (35)$$

**Theorem 3.4** (Ritz projection). *There is  $c$ , so that for all  $h \in \mathcal{H}$  and all  $(\mathbf{H}, \mathbf{E}) \in \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$ , there exists a unique pair  $(\Pi_{0h}(\mathbf{H}), \Pi_h(\mathbf{E})) \in \mathbf{X}_{\mu 0, h} \times \mathbf{X}_{\epsilon, h}$  so that*

$$a_h((\Pi_{0h}(\mathbf{H}), \Pi_h(\mathbf{E})), (\mathbf{h}_h, \mathbf{e}_h)) = a((\mathbf{H}, \mathbf{E}), (\mathbf{h}_h, \mathbf{e}_h)), \quad \forall (\mathbf{h}_h, \mathbf{e}_h) \in L_h, \quad (36a)$$

$$\|(\mathbf{H} - \Pi_{0h}(\mathbf{H}), \mathbf{E} - \Pi_h(\mathbf{E}))\|_L \leq ch^s (\mu_0^{\frac{1}{2}} |\mathbf{H}|_{\mathbf{H}^s(D)} + \epsilon_0^{\frac{1}{2}} |\mathbf{E}|_{\mathbf{H}^s(D)}), \quad (36b)$$

with  $a((\mathbf{H}, \mathbf{E}), (\mathbf{h}_h, \mathbf{e}_h)) := -(\nabla \times \mathbf{E}, \mathbf{h}_h)_{\mathbf{L}^2(D)} + (\nabla_0 \times \mathbf{H}, \mathbf{e}_h)_{\mathbf{L}^2(D)}$ .

**Remark 3.5** (Stabilization vs. centered flux). *Notice that it is essential that the stabilization coefficients  $\kappa_H, \kappa_E$  be nonzero for equality to hold in (31). When the dG numerical flux is centered, i.e.,  $\kappa_H = 0$  and  $\kappa_E = 0$ , we only have  $\text{im}(A_h) \subsetneq \mathbf{P}_{k0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp \times \mathbf{P}_k^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$ . This nevertheless implies that a discrete version of Gauss's laws is satisfied (as observed, e.g., in Campos Pinto and Sonnendrücker [8, §5.4]), but this is not enough for the approximation to be convergent under the stringent smoothness assumptions considered in this paper. More precisely, it is essential that  $\kappa_H > 0$  and  $\kappa_E > 0$  to establish the deflated inf-sup condition (35), which, in turn, gives the estimate (36b). The importance of stabilization is now well recognized in the literature, see e.g., Alvarez et al. [1], Hesthaven and Warburton [18]. In conclusion, the combination of (31) and (35) can be viewed as the discrete counterparts of  $\text{im}(A) \cap D(A) \subset H^s$  only when  $\kappa_H > 0$  and  $\kappa_E > 0$ ; see (2).*

### 3.3 Linear wave equations

In this section, we exemplify the dG space semi-discretization for the linear wave equations. Let  $\kappa_0, \rho_0$  be reference scales for  $\kappa$  and  $\rho$ , respectively. We assume throughout the paper that the ratios  $\kappa_0^{-1}\kappa$  and  $\rho_0^{-1}\rho$  are of order unity. Setting  $\mathfrak{c} := (\kappa_0\rho_0)^{-\frac{1}{2}}$ , we introduce the quantity  $\omega := \mathfrak{c}\ell_D^{-1}$  which scales as the reciprocal of a time scale.

#### 3.3.1 Discrete bilinear form and involution

We define the jump bilinear forms such that, for all  $p_h, q_h \in P_k^b(\mathcal{T}_h)$  and all  $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,

$$s_h^{\mathfrak{g},\circ}(p_h, q_h) := \sum_{F \in \mathcal{F}_h^\circ} (\llbracket p_h \rrbracket_F^{\mathfrak{g}}, \llbracket q_h \rrbracket_F^{\mathfrak{g}})_{L^2(F)}, \quad (37a)$$

$$s_h^{\mathfrak{d}}(\mathbf{v}_h, \mathbf{w}_h) := \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^{\mathfrak{d}}, \llbracket \mathbf{w}_h \rrbracket_F^{\mathfrak{d}})_{L^2(F)}, \quad (37b)$$

which induce the seminorms  $|q_h|_h^{\mathfrak{g},\circ} := s_h^{\mathfrak{g},\circ}(q_h, q_h)^{\frac{1}{2}}$  and  $|\mathbf{v}_h|_h^{\mathfrak{d}} := s_h^{\mathfrak{d}}(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}}$ . The discrete gradient and divergence operators

$$\mathbf{G}_h : P_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^b(\mathcal{T}_h), \quad D_{h0} : P_k^b(\mathcal{T}_h) \rightarrow P_k^b(\mathcal{T}_h) \quad (38)$$

are defined as follows: For all  $(q_h, \mathbf{v}_h) \in P_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ ,  $\mathbf{G}_h(q_h)$  and  $D_{h0}(\mathbf{v}_h)$  are the unique members of  $P_k^b(\mathcal{T}_h)$  and  $P_k^b(\mathcal{T}_h)$  such that the following identities hold true for all  $q'_h \in P_k^b(\mathcal{T}_h)$  and  $\mathbf{v}'_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ :

$$(\mathbf{G}_h(p_h), \mathbf{v}'_h)_{L^2(D)} := (\nabla_h p_h, \mathbf{v}'_h)_{L^2(D)} - \sum_{F \in \mathcal{F}_h^\circ} (\llbracket q_h \rrbracket_F^{\mathfrak{g}}, \llbracket \mathbf{v}'_h \rrbracket_F^{\mathfrak{d}})_{L^2(F)}, \quad (39a)$$

$$(D_{h0}(\mathbf{v}_h), q'_h)_{L^2(D)} := (\nabla_h \cdot \mathbf{v}_h, q'_h)_{L^2(D)} - \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^{\mathfrak{d}}, \llbracket q'_h \rrbracket_F^{\mathfrak{g}})_{L^2(F)}, \quad (39b)$$

where  $\nabla_h$  and  $\nabla_h \cdot$  denote the broken gradient and divergence operators (evaluated cellwise). The following integration by parts formula holds true:

$$(\mathbf{G}_h(q_h), \mathbf{v}_h)_{L^2(D)} = -(q_h, D_{h0}(\mathbf{v}_h))_{L^2(D)}, \quad \forall (q_h, \mathbf{v}_h) \in P_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h). \quad (40a)$$

Setting  $L_h := P_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)$ , we define the bilinear form  $a_h : L_h \times L_h \rightarrow \mathbb{R}$  such that

$$a_h((p_h, \mathbf{u}_h), (q_h, \mathbf{v}_h)) := (D_{h0}(\mathbf{u}_h), q_h)_{L^2(D)} + (\mathbf{G}_h(p_h), \mathbf{v}_h)_{L^2(D)} + \kappa_P s_h^{\mathfrak{g},\circ}(p_h, q_h) + \kappa_V s_h^{\mathfrak{d}}(\mathbf{u}_h, \mathbf{v}_h), \quad (41)$$

with  $\kappa_P := \kappa_0 \omega \ell_D = (\kappa_0 / \rho_0)^{\frac{1}{2}}$  and  $\kappa_V := \rho_0 \omega \ell_D = (\rho_0 / \kappa_0)^{\frac{1}{2}}$  (notice that  $\kappa_P \kappa_V = 1$ ). The Riesz representative of the bilinear form  $a_h$  is the operator  $A_h : L_h \rightarrow L_h$  such that the following identity holds true for all pairs  $(p_h, \mathbf{u}_h), (q_h, \mathbf{v}_h) \in L_h$ :

$$(A_h(p_h, \mathbf{u}_h), (q_h, \mathbf{v}_h))_{L^2(D) \times L^2(D)} := a_h((p_h, \mathbf{u}_h), (q_h, \mathbf{v}_h)). \quad (42)$$

The semi-discrete version of the linear wave equations (12) consists of seeking the pair  $(p_h, \mathbf{u}_h) \in C^1([0, T]; L_h)$  so that

$$\partial_t(\kappa p_h, \rho \mathbf{u}_h) + A_h(p_h, \mathbf{u}_h) = 0, \quad (p_h(0), \mathbf{u}_h(0)) = (p_h^0, \mathbf{u}_h^0), \quad (43)$$

with initial data  $(p_h^0, \mathbf{u}_h^0)$  approximating  $(p^0, \mathbf{u}^0)$ . Assuming that  $(p_h^0, \mathbf{u}_h^0) \in \text{im}(A_h)$ , the pair  $(\kappa p_h, \rho \mathbf{u}_h)$  remains in the image of the operator  $A_h$  as  $t$  grows, i.e.,

$$(\kappa p_h(t), \rho \mathbf{u}_h(t)) \in \text{im}(A_h) \quad \forall t > 0. \quad (44)$$

We now characterize  $\ker(A_h^\top) = \text{im}(A_h)^\perp$ . We first observe that  $\{q_h \in P^b(\mathcal{T}_h) \mid \nabla q_h = \mathbf{0}\} = \mathbb{P}_0$ . Let  $\mathbf{P}_{k0}^d(\mathcal{T}_h)$  be the  $\mathbf{H}_0(\text{div}; D)$ -conforming space composed of the piecewise Raviart–Thomas finite elements of degree  $k \geq 0$  with zero normal trace on  $\partial D$ . We set

$$\mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{P}_{k0}^d(\mathcal{T}_h) \mid \nabla_0 \cdot \mathbf{v}_h = 0\}. \quad (45)$$

Notice that  $\mathbf{P}_{k0}^d(\mathcal{T}_h)$  is not a subspace of  $\mathbf{P}_k^b(\mathcal{T}_h)$  (actually  $\mathbf{P}_{k0}^d(\mathcal{T}_h) \subsetneq \mathbf{P}_{k+1}^b(\mathcal{T}_h)$ ), but the following well-known result holds (see, e.g., [13, Lem. 14.9&Thm. 18.10]):

$$\mathbf{P}_k^b(\mathcal{T}_h) \cap \ker \nabla_0 = \mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h). \quad (46)$$

**Lemma 3.6** (Discrete involution). *The following holds true:*

$$\ker(A_h^\top) = \mathbb{P}_0 \times \mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h). \quad (47)$$

Lemma 3.6 gives  $\text{im}(A_h) = \mathbb{P}_0^\perp \times \mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h)^\perp$ . Accordingly, we set

$$X_{\kappa, h} := \{q_h \in P_k^b(\mathcal{T}_h) \mid \int_D \kappa q_h \, dx = 0\}, \quad (48a)$$

$$\mathbf{X}_{\rho 0, h} := \{\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h) \mid \rho \mathbf{v}_h \in \mathbf{P}_{k0}^d(\text{div} = 0; \mathcal{T}_h)^\perp\}. \quad (48b)$$

With this notation, we observe that the solution to (43) satisfies the involution

$$(p_h(0), \mathbf{u}_h(0)) \in X_{\kappa, h} \times \mathbf{X}_{\rho 0, h} \implies (p_h(t), \mathbf{u}_h(t)) \in X_{\kappa, h} \times \mathbf{X}_{\rho 0, h} \quad \forall t > 0. \quad (49)$$

### 3.3.2 Deflated inf-sup condition and Ritz projection

We equip  $L$  with the norm  $\|(f, \mathbf{g})\|_L := (\|\kappa^{\frac{1}{2}} f\|_{L^2(D)}^2 + \|\rho^{\frac{1}{2}} \mathbf{v}\|_{L^2(D)}^2)^{\frac{1}{2}}$  and we equip  $L_h$  with the mesh-dependent norm

$$\begin{aligned} \|(p_h, \mathbf{u}_h)\|_{b, h} &:= \omega^{\frac{1}{2}} \|(p_h, \mathbf{u}_h)\|_L \\ &+ \kappa_P^{\frac{1}{2}} \{ \|\tilde{h}^{\frac{1}{2}} \mathbf{G}_h(p_h)\|_{L^2(D)} + |p_h|_J^{\text{g}, \circ} \} + \kappa_V^{\frac{1}{2}} \{ \|\tilde{h}^{\frac{1}{2}} D_{h0}(\mathbf{u}_h)\|_{L^2(D)} + |\mathbf{u}_h|_J^{\text{d}} \}. \end{aligned} \quad (50)$$

The following two results established in [15, 16] plays a key role in the paper.

**Lemma 3.7** (Deflated inf-sup condition). *There exists  $c > 0$  so that, for all  $h \in \mathcal{H}$  and for every pair  $(p_h, \mathbf{u}_h)$  in  $X_{\kappa, h} \times \mathbf{X}_{\rho 0, h}$ , the following holds true:*

$$\omega^{\frac{1}{2}} \|(p_h, \mathbf{u}_h)\|_{b, h} \lesssim \sup_{(q_h, \mathbf{v}_h) \in L_h} \frac{|a_h((p_h, \mathbf{u}_h), (q_h, \mathbf{v}_h))|}{\|(q_h, \mathbf{v}_h)\|_L}. \quad (51)$$

**Theorem 3.8** (Ritz projection). *There is  $c$  so that, for all  $h \in \mathcal{H}$  and all  $(p, \mathbf{u}) \in X_\kappa \times \mathbf{X}_{\rho 0}$ , there exists a unique pair  $(\Pi_h(p), \Pi_{0h}(\mathbf{u})) \in X_{\kappa, h} \times \mathbf{X}_{\rho 0, h}$  so that*

$$a_h((\Pi_h(p), \Pi_{0h}(\mathbf{u})), (q_h, \mathbf{v}_h)) = a((p, \mathbf{u}), (q_h, \mathbf{v}_h)), \quad \forall (q_h, \mathbf{v}_h) \in L_h, \quad (52a)$$

$$\|(p - \Pi_h(p), \mathbf{u} - \Pi_{0h}(\mathbf{u}))\|_L \leq ch^s (\kappa_0^{\frac{1}{2}} |p|_{H^s(D)} + \rho_0^{\frac{1}{2}} |\mathbf{u}|_{\mathbf{H}^s(D)}), \quad (52b)$$

with  $a((p, \mathbf{u}), (q_h, \mathbf{v}_h)) := (\nabla \cdot \mathbf{u}, q_h)_{L^2(D)} + (\nabla p, \mathbf{v}_h)_{L^2(D)}$ .

**Remark 3.9** (Smoothness index). *The deflated inf-sup condition and the error estimate for the Ritz projection are proved in [15] under the assumption  $s > \frac{1}{2}$ , but the duality argument from [16] can be easily adapted to prove the deflated inf-sup condition and the error estimate for the Ritz projection in the present setting where the coefficients  $\kappa$  and  $\rho$  are discontinuous and the smoothness index  $s$  is in  $(0, \frac{1}{2})$ . Moreover, as in Remark 3.5, we emphasize that the combination of (47) and (51) can be viewed as the discrete counterparts of  $\text{im}(A) \cap D(A) \subset H^s$  only when  $\kappa_p > 0$  and  $\kappa_v > 0$ . The importance of stabilization is now well recognized in the literature, see e.g., Stanglmeier et al. [23].*

## 4 Fully discrete setting

In this section, we first rewrite the space semi-discrete problems derived in the previous section in a unified setting and then use ERK schemes for its time discretization.

### 4.1 Unified setting for the space semi-discrete problem

Since the proposed method can be used for solving a large class of problems having the same structure as Maxwell's equations and the linear wave equations, we now adopt the abstract notation introduced in §1. This will facilitate the stability and convergence analysis reported below. Recall that we want to solve the time-dependent problem (1) where the linear operator  $A : D(A) \subset L \rightarrow L$  is unbounded and maximal monotone, and  $L$  is a Hilbert space equipped with the inner product  $(\cdot, \cdot)_L$ . To allow for a bit more generality, we consider here complex linear spaces. Recall also that we assume that there exists a subspace  $H^s$  dense in  $L$  with compact embedding, i.e.,  $H^s \Subset L$ , so that

$$X := \text{im}(A) \cap D(A) \subset H^s. \quad (53)$$

We denote by  $|v|_{H^s}$  any semi-norm on  $H^s$  such that  $\|v\|_L + |v|_{H^s}$  is a norm equivalent to  $\|v\|_{H^s}$ . Recalling Lemma 2.1 and Lemma 2.3, the above property means that under natural assumptions on the initial data  $u^0$ , the solution to (1) satisfies the following smoothness property:

$$\|u\|_{L^\infty((0, \infty); H^s)} < \infty, \quad \|\partial_t u\|_{L^\infty((0, \infty); H^s)} < \infty, \quad \|\partial_{tt} u\|_{L^\infty((0, \infty); L)} < \infty. \quad (54)$$

We now approximate ‘‘in space’’ the problem (1) by introducing a sequence  $(L_h)_{h \in \mathcal{H}}$  of finite-dimensional subspaces of  $L$  equipped with the inner product  $(\cdot, \cdot)_L$ . We then consider the ODE system

$$\partial_t u_h + A_h(u_h) = 0, \quad u_h(0) = u_h^0, \quad (55)$$

where  $A_h : L_h \rightarrow L_h$  is a linear operator approximating  $A$ . Let  $\Pi_h^b : L \rightarrow L_h$  denote the  $L$ -orthogonal projection. We formalize the approximation properties of  $\Pi_h^b$  and  $A_h$  by assuming that there exists a linear operator  $\Pi_h : X \rightarrow L_h$  (which we call Ritz projection) and a constant  $c > 0$  such that, for all  $u \in X$  and all  $h \in \mathcal{H}$ ,

$$(A_h \circ \Pi_h)(u) = (\Pi_h^b \circ A)(u), \quad (56a)$$

$$\|\Pi_h(u) - u\|_L \leq ch^s |u|_{H^s}, \quad (56b)$$

$$\|\Pi_h^b(u) - u\|_L \leq ch^s |u|_{H^s}. \quad (56c)$$

We finish by saying a few words on the dissipation properties of  $A_h$ . Just like the operator  $A$  satisfies  $\Re((A(v), v)_L) \geq 0$  for all  $v \in D(A)$ , we assume that  $A_h + A_h^H$  is non-negative (where  $A_h^H$  denotes the Hermitian transpose of  $A_h$ ). More precisely, we set

$$s_h(v_h, w_h) := \left(\frac{1}{2}(A_h + A_h^H)(v_h), w_h\right)_L \quad \forall v_h, w_h \in L_h, \quad (57)$$

and assume that the Hermitian sesquilinear form  $s(\cdot, \cdot)$  is non-negative. We denote

$$|v_h|_S := (s_h(v_h, v_h))^{\frac{1}{2}}, \quad (v_h, w_h)_S := s_h(v_h, w_h), \quad \forall v_h, w_h \in L_h. \quad (58)$$

Recalling that  $(v_h, A_h(w_h))_L = \overline{(A_h(w_h), v_h)_L}$ , the above definitions imply that

$$\Re((A_h(v_h), v_h)_L) = |v_h|_S^2, \quad \forall v_h \in L_h. \quad (59)$$

Notice that (57) can be viewed as an integration by parts formula since

$$(A_h(v_h), w_h)_L = -(v_h, A_h(w_h))_L + 2s_h(v_h, w_h) \quad \forall v_h, w_h \in L_h. \quad (60)$$

One important question regarding ERK schemes is their  $L^2$ -stability under the standard CFL condition  $\tau C(A_h) \leq \sigma$ , where  $C(A_h)$  is a constant depending on the stability properties of the discrete operator  $A_h$ . To be able to estimate  $C(A_h)$ , we assume that there exist two constants  $c_L$  and  $c_S$  so that the following holds true for all  $h \in \mathcal{H}$  and all  $v_h \in L_h$ :

$$\|A_h(v_h)\|_L \leq c_L h^{-1} \|v_h\|_L, \quad |v_h|_S^2 \leq c_S h^{-1} \|v_h\|_L^2. \quad (61)$$

In the dG context, these two inequalities are established by invoking discrete inverse and trace inequalities, and using that the underlying operator  $A$  is a first-order differential operator.

## 4.2 ERK schemes

For the time discretization of the ODE system (55), we consider three-stage, third-order and four-stage, fourth-order ERK schemes, which we call ERK(3,3) and ERK(4,4). Given  $u_h^{n,0} := u_h^n$ , where  $u_h^n$  is the approximation of  $u$  at time  $t^n$ , and given a time step  $\tau$ , the computation of the update  $u_h^{n+1}$  by the ERK(3,3) scheme is done in three stages as follows:

$$u_h^{n,1} = u_h^{n,0} - \tau A_h u_h^{n,0}, \quad (62a)$$

$$u_h^{n,2} = u_h^{n,1} - \frac{1}{2} \tau A_h (u_h^{n,1} - u_h^{n,0}), \quad (62b)$$

$$u_h^{n+1} := u_h^{n,3} = u_h^{n,2} - \frac{1}{3} \tau A_h (u_h^{n,2} - u_h^{n,1}). \quad (62c)$$

The computation of the update  $u_h^{n+1}$  by the ERK(4,4) scheme is done in four stages as follows:

$$u_h^{n,1} = u_h^{n,0} - \tau A_h u_h^{n,0}, \quad (63a)$$

$$u_h^{n,2} = u_h^{n,1} - \frac{1}{2} \tau A_h (u_h^{n,1} - u_h^{n,0}), \quad (63b)$$

$$u_h^{n,3} = u_h^{n,2} - \frac{1}{3} \tau A_h (u_h^{n,2} - u_h^{n,1}), \quad (63c)$$

$$u_h^{n+1} := u_h^{n,4} = u_h^{n,3} - \frac{1}{4} \tau A_h (u_h^{n,3} - u_h^{n,2}). \quad (63d)$$

Let  $P_l(z) := \sum_{m \in \{0:l\}} \frac{1}{m!} z^m$  be the Taylor polynomial approximating the exponential function at order  $l \in \mathbb{N}$ . A straightforward computation shows that (62) amounts to  $u_h^{n+1} = P_3(-\tau A_h)(u_h^n)$  and that (63) amounts to  $u_h^{n+1} = P_4(-\tau A_h)(u_h^n)$ .

Equations (62) and (63) each provide one possible realization of ERK(3,3) and ERK(4,4) schemes. More generally, ERK schemes are specified by means of their Butcher array. However, in the absence of external forcing and for linear evolution PDEs, as in (55), every ERK(3,3) scheme and every ERK(4,4) scheme eventually reduces, after elimination of the intermediate stages, to the identities  $u_h^{n+1} = P_3(-\tau A_h)(u_h^n)$  and  $u_h^{n+1} = P_4(-\tau A_h)(u_h^n)$ , respectively. Therefore, to establish  $L^2$ -stability, it is sufficient to consider the schemes (62) and (63) as they simplify some of the forthcoming algebra.

**Proposition 4.1** ( $L^2$ -stability). *There exists  $\sigma_0 > 0$  so that, for all  $h \in \mathcal{H}$  and all  $\tau$  satisfying the following standard (hyperbolic) CFL condition:*

$$\max(c_L, c_S)\tau h^{-1} \leq \sigma_0, \quad (64)$$

we have

$$\|P_3(-\tau A_h)\|_{\mathcal{L}(L_h)} \leq 1, \quad \|P_4(-\tau A_h)^2\|_{\mathcal{L}(L_h)} \leq 1. \quad (65)$$

Consequently, if the sequence  $(u_h^n)_{n \in \mathbb{N}}$  is generated by the ERK(3,3) scheme, we have  $\|u_h^n\|_L \leq \|u_h^0\|_L$ , and if the sequence  $(u_h^n)_{n \in \mathbb{N}}$  is generated by the ERK(4,4) scheme, we have  $\|u_h^n\|_L \leq \max(\|u_h^0\|_L, \|u_h^1\|_L)$ .

**Remark 4.2** (Literature). *The  $L^2$ -stability of ERK(3,3) schemes is established in Burman et al. [7] and Zhang and Shu [29] under the standard hyperbolic CFL condition of the form (64). In particular, the  $L^2$ -stability of ERK(3,3) schemes in the form  $\|u_h^{n+1}\|_L \leq \|u_h^n\|_L$  readily follows from Burman et al. [7, Lem. 4.2], and the estimate in Step (iii) of the proof of Lemma 4.3 therein with  $\alpha_h^n = \beta_h^n = \gamma_h^n = 0$  and  $\Lambda_h^n = 0$ . Concerning the  $L^2$ -stability of the ERK(4,4) schemes under (64), we refer the reader to Xu et al. [28], see also Sun and Shu [24, 25]. For completeness, we provide somewhat simpler proofs of these statements in Section 6.*

## 5 Convergence

Our objective in this section is to prove convergence under the following minimal regularity assumptions:

$$\|\partial_{tt}u\|_{L^\infty((0,\infty);L)} < \infty, \quad \|\partial_t u\|_{L^\infty((0,\infty);H^s)} < \infty, \quad \|u\|_{L^\infty((0,\infty);H^s)} < \infty, \quad (66)$$

where  $s > 0$ . In what follows, the symbol  $C$  denotes a generic positive constant whose value can change at each occurrence provided it is independent of the mesh and time-step sizes.

### 5.1 Preliminaries

Let  $u_h^0 \in L_h$  be a reasonable approximation of  $u^0$  which we recall is assumed to belong to the space  $X$  defined in (53). For instance, setting  $u_h^0 := \Pi_h^b(u_0)$ , we have

$$\|u^0 - u_h^0\|_L \leq Ch^s |u^0|_{H^s}. \quad (67)$$

To establish the consistency of the approximating scheme we are going to use the following key identity satisfied by the exact solution, which is a direct consequence of (56a) and (1):

$$\Pi_h^b(\partial_t u(t)) = -A_h(\Pi_h(u(t))). \quad (68)$$

### 5.2 ERK(3,3)

We start by introducing the discrete errors

$$e_h^n := e_h^{n,0} := u_h^{n,0} - \Pi_h(u(t^n)), \quad (69a)$$

$$e_h^{n,1} := u_h^{n,1} - \Pi_h(u(t^n) + \tau \partial_t u(t^n)), \quad (69b)$$

$$e_h^{n,2} := u_h^{n,2} - \Pi_h(u(t^n) + \tau \partial_t u(t^n)), \quad (69c)$$

$$e_h^{n+1} := e_h^{n,3} := u_h^{n,3} - \Pi_h(u(t^{n+1})). \quad (69d)$$

**Remark 5.1** ( $e_h^{n,2}$ ). *If full regularity were assumed or if an  $L^\infty(L^2)$ -bound on the third-order time-derivative were available so that we would have  $\partial_{tt}u \in L^\infty(X)$ , it would then be more convenient to set  $e_h^{n,2} := u_h^{n,2} - \Pi_h(u(t^n) + \tau\partial_t u(t^n) + \frac{1}{2}\tau^2\partial_{tt}u(t^n))$ , but comparing  $u_h^{n,2}$  to  $u(t^n) + \tau\partial_t u(t^n)$  is sufficient for our purpose since we just want to establish convergence by invoking as little regularity as possible.*

**Lemma 5.2** (Error equation). *The error equation for the ERK(3,3) scheme is*

$$e_h^{n+1} = P_3(-\tau A_h)(e_h^n) + \tau \sum_{m \in \{0:2\}} Q_{3,m}(-\tau A_h)(R_h^{n,m}), \quad (70)$$

where  $Q_{3,0}(z) := 1 - \frac{1}{2}z + \frac{1}{6}z^2$ ,  $Q_{3,1}(z) := 1 - \frac{1}{3}z$ ,  $Q_{3,2}(z) := 1$ , and

$$R_h^{n,0} := (\Pi_h^b - \Pi_h)(\partial_t u(t^n)), \quad (71a)$$

$$R_h^{n,1} := \tau \Pi_h^b(\partial_{tt}u(t^n)), \quad (71b)$$

$$R_h^{n,2} := \tau^{-1} \Pi_h(u(t^{n+1}) - u(t^n) - \tau\partial_t u(t^n)). \quad (71c)$$

*Proof.* Using the identity (68) satisfied by the exact solution, we write the following identities (tautologies):

$$\begin{aligned} \Pi_h(u(t^n) + \tau\partial_t u(t^n)) &= \Pi_h(u(t^n)) - \tau A_h(\Pi_h(u(t^n))) - \tau R_h^{n,0}, \\ \Pi_h(u(t^n) + \tau\partial_t u(t^n)) &= \Pi_h(u(t^n) + \tau\partial_t u(t^n)) - \tau A_h(\Pi_h(\tau\partial_t u(t^n))) - \tau R_h^{n,1}, \\ \Pi_h(u(t^{n+1})) &= \Pi_h(u(t^n) + \tau\partial_t u(t^n)) - \tau R_h^{n,2}. \end{aligned}$$

Subtracting these relations from (62), we infer that the following discrete error equations hold true:

$$\begin{aligned} e_h^{n,1} &= e_h^{n,0} - \tau A_h e_h^{n,0} + \tau R_h^{n,0}, \\ e_h^{n,2} &= e_h^{n,1} - \frac{1}{2}\tau A_h(e_h^{n,1} - e_h^{n,0}) + \tau R_h^{n,1}, \\ e_h^{n,3} &= e_h^{n,2} - \frac{1}{3}\tau A_h(e_h^{n,2} - e_h^{n,1}) + \tau R_h^{n,2}. \end{aligned}$$

We arrive at the assertion by eliminating the intermediate stages and using  $e_h^n := e_h^{n,0}$ ,  $e_h^{n+1} := e_h^{n,3}$ .  $\square$

**Theorem 5.3** (Convergence). *Assume that the initial data  $u_0$  is such that the solution to (1) satisfies the generic regularity assumption (66) and that the discrete initial condition is chosen so that (67) holds true. Let  $(u_h^n)_{n \in \mathbb{N}}$  be the time-discrete sequence produced by the ERK(3,3) scheme. There exists  $\sigma_0$  and a constant  $C$  so that for all  $h \in \mathcal{H}$ , all  $\tau$  satisfying the standard CFL condition (64), and all  $m \geq 1$ , the following error estimate holds:*

$$\|u(t^m) - u_h^m\|_L \leq C \left\{ h^s |u|_{L^\infty((0,t^m);H^s)} + t^m \left\{ \tau \|\partial_{tt}u\|_{L^\infty((0,t^m);L)} + h^s \|\partial_t u\|_{L^\infty((0,t^m);H^s)} \right\} \right\}. \quad (72)$$

*Proof.* Starting from (70) and invoking the triangle inequality gives

$$\|e_h^{n+1}\|_L \leq \|P_3(-\tau A_h)(e_h^n)\|_L + \tau \sum_{m \in \{0:2\}} \|Q_{3,m}(-\tau A_h)(R_h^{n,m})\|_L.$$

We bound the first term on the right-hand side by invoking (65) from Proposition 4.1, and the second one by using the CFL condition (64) and the first bound in (61). This gives

$$\|e_h^{n+1}\|_L \leq \|e_h^n\|_L + C\tau \sum_{m \in \{0:2\}} \|R_h^{n,m}\|_L.$$

It remains to bound the three residuals on the right-hand side. We have

$$\|R_h^{n,0}\|_L \leq \|(I - \Pi_h)(\partial_t u(t^n))\|_L \leq Ch^s |\partial_t u(t^n)|_{H^s},$$

and

$$\|R_h^{n,1}\|_L \leq \tau \|\partial_{tt} u(t^n)\|_L.$$

Moreover, since  $u(t^{n+1}) - u(t^n) - \tau \partial_t u(t^n) = \int_{t^n}^{t^{n+1}} (t^{n+1} - \xi) \partial_t u(\xi) d\xi$ , we infer that

$$\|R_h^{n,2}\|_L \leq \tau \|\partial_{tt} u\|_{L^\infty((t^n, t^{n+1}); L)}.$$

Putting the above bounds together, summing over  $n \in \{1:m\}$  for all  $m \geq 1$ , and observing that  $\sum_{n \in \{1:m\}} \tau = t^m$  implies that

$$\|e_h^m\|_L \leq \|e_h^0\|_L + Ct^m \left\{ \tau \|\partial_{tt} u\|_{L^\infty((0, t^m); L)} + h^s |\partial_t u|_{L^\infty((0, t^m); H^s)} \right\}.$$

Recalling the assumption (67) on the initial condition and invoking the triangle inequality together with the approximation property (56b) of  $\Pi_h$  proves the claim.  $\square$

**Remark 5.4** (Convergence order). *The convergence order from Theorem 5.3 is  $O(\tau + h^s)$ . Notice that we do not exploit here the higher-order truncation error in time of ERK(3,3). No superfluous or unrealistic smoothness is invoked. The pivotal property of this scheme is its  $L^2$ -stability as established in Proposition 4.1. It is also critical to invoke the Ritz projection and its properties listed in (56a)-(56b).*

### 5.3 ERK(4,4)

We start by introducing the discrete errors

$$e_h^n := e_h^{n,0} := u_h^{n,0} - \Pi_h(u(t^n)), \quad (73a)$$

$$e_h^{n,1} := u_h^{n,1} - \Pi_h(u(t^n) + \tau \partial_t u(t^n)), \quad (73b)$$

$$e_h^{n,2} := u_h^{n,2} - \Pi_h(u(t^n) + \tau \partial_t u(t^n)), \quad (73c)$$

$$e_h^{n,3} := u_h^{n,3} - \Pi_h(u(t^n) + \tau \partial_t u(t^n)), \quad (73d)$$

$$e_h^{n+1} := e_h^{n,4} := u_h^{n,4} - \Pi_h(u(t^{n+1})). \quad (73e)$$

**Remark 5.5** ( $e_h^{n,2}, e_h^{n,3}$ ). *If full regularity were assumed or if an  $L^\infty(L^2)$ -bound on the third- and fourth-order time-derivatives were available so that we would have  $\partial_{tt} u \in L^\infty(X)$  and  $\partial_{ttt} u \in L^\infty(X)$ , it would be more convenient to set  $e_h^{n,2}$  as in Remark 5.1 and  $e_h^{n,3} := u_h^{n,2} - \Pi_h(u(t^n) + \tau \partial_t u(t^n) + \frac{1}{2} \tau^2 \partial_{tt} u(t^n) + \frac{1}{6} \tau^3 \partial_{ttt} u(t^n))$ .*

**Lemma 5.6** (Error equation). *The error equation for the ERK(4,4) scheme is*

$$e_h^{n+1} = P_4(-\tau A_h)(e_h^n) + \tau \sum_{m \in \{0:3\}} Q_{4,m}(-\tau A_h)(R_h^{n,m}), \quad (74)$$

where  $Q_{4,0}(s) := 1 - \frac{1}{2}s + \frac{1}{6}s^2 - \frac{1}{24}s^3$ ,  $Q_{4,1}(s) := 1 - \frac{1}{3}s + \frac{1}{12}s^2$ ,  $Q_{4,2}(s) := 1 - \frac{1}{4}s$ ,  $Q_{4,3}(s) = 1$ ,  $R_h^{n,0}$  and  $R_h^{n,1}$  defined by (71a) and (71b), respectively,  $R_h^{n,2} = 0$  and  $R_h^{n,3} := \tau^{-1}\Pi_h(u(t^{n+1}) - u(t^n) - \tau\partial_t u(t^n))$ .

*Proof.* Using the same identities (tautologies) as for ERK(3,3) and subtracting these relations to (63), we infer that the following discrete error equations hold true:

$$\begin{aligned} e_h^{n,1} &= e_h^{n,0} - \tau A_h e_h^{n,0} + \tau R_h^{n,0}, \\ e_h^{n,2} &= e_h^{n,1} - \frac{1}{2}\tau A_h (e_h^{n,1} - e_h^{n,0}) + \tau R_h^{n,1}, \\ e_h^{n,3} &= e_h^{n,2} - \frac{1}{3}\tau A_h (e_h^{n,2} - e_h^{n,1}) + \tau R_h^{n,2}, \\ e_h^{n,4} &= e_h^{n,2} - \frac{1}{4}\tau A_h (e_h^{n,3} - e_h^{n,2}) + \tau R_h^{n,3}. \end{aligned}$$

Eliminating the intermediate stages leads to the assertion.  $\square$

**Theorem 5.7** (Convergence). *Assume that the exact solution satisfies the regularity assumption (66) and that the discrete initial condition is chosen so that (67) holds true. Let  $(u_h^n)_{n \in \mathbb{N}}$  be the time-discrete sequence produced by the ERK(4,4) scheme. There exists  $\sigma_0$  and a constant  $C$  so that for all  $h \in \mathcal{H}$ , all  $\tau$  satisfying the standard CFL condition (64), and all  $m \geq 1$ , the following error estimate holds:*

$$\|u(t^m) - u_h^m\|_L \leq C \left\{ h^s |u|_{L^\infty((0,t^m);H^s)} + t^m \left\{ \tau \|\partial_{tt} u\|_{L^\infty((0,t^m);L)} + h^s |\partial_t u|_{L^\infty((0,t^m);H^s)} \right\} \right\}. \quad (75)$$

*Proof.* Considering the error equation (74) on two successive time steps, we infer that

$$\begin{aligned} e_h^{n+2} &= P_4(-\tau A_h)^2 (e_h^n) + \tau \sum_{m \in \{0:3\}} P_4(-\tau A_h) (Q_{4,m}(-\tau A_h) (R_h^{n,m})) \\ &\quad + \tau \sum_{m \in \{0:3\}} P_4(-\tau A_h) (Q_{4,m}(-\tau A_h) (R_h^{n+1,m})). \end{aligned}$$

We now invoke the triangle inequality, the bound (65) from Proposition 4.1, the CFL condition (64) and the first bound in (61) to infer that

$$\|e_h^{n+2}\|_L \leq \|e_h^n\|_L + C\tau \sum_{m \in \{0:3\}} \{ \|R_h^{n,m}\|_L + \|R_h^{n+1,m}\|_L \}.$$

All the residuals can be bounded by reasoning as in the proof of Theorem 5.3. Summing over  $n \in \{1:m\}$  for all  $m \geq 2$ , we infer that

$$\|e_h^m\|_L \leq \|e_h^\mu\|_L + Ct^m \left\{ \tau \|\partial_{tt} u\|_{L^\infty((t^\mu, t^m);L)} + h^s |\partial_t u|_{L^\infty((t^\mu, t^m);H^s)} \right\},$$

with  $\mu := m \bmod 2$ . Altogether, this gives

$$\|e_h^m\|_L \leq \max \{ \|e_h^0\|_L, \|e_h^1\|_L \} + Ct^m \left\{ \tau \|\partial_{tt} u\|_{L^\infty((0,t^m);L)} + h^s |\partial_t u|_{L^\infty((0,t^m);H^s)} \right\}.$$

To bound  $\|e_h^1\|_L$ , we use (74) for  $n = 0$  and invoke the same arguments as above. Finally, recalling the assumption (67) on the initial condition and invoking the triangle inequality together with the approximation property (56b) of  $\Pi_h$  proves the claim.  $\square$

## 6 $L^2$ -stability of ERK(3,3) and ERK(4,4)

In this section, we recall the  $L^2$ -stability argument for explicit Runge–Kutta schemes presented in Xu et al. [28] and simplify some of the argumentation.

Our starting point is the integration by parts formula (60) which allows us to compute quantities of the form  $\Re((A_h^k(u_h), A_h^l(v_h)))_L$  for all  $k > l \geq 0$ .

**Lemma 6.1** (Induction formula). *Let  $k > l \geq 0$  be two natural numbers. Set  $n := \lfloor \frac{k-l}{2} \rfloor$ . Then the following identity holds true for all  $v_h$  in  $L_h$ :*

$$\begin{aligned} \Re((A_h^k(v_h), A_h^l(v_h)))_L &= \sum_{m \in \{1:n\}} (-1)^{m+1} 2 \Re((A_h^{k-m}(v_h), A_h^{l+m-1}(v_h)))_S \\ &\quad + (-1)^n \begin{cases} \|A_h^{l+n}(v_h)\|_L^2 & \text{if } k-l = 2n, \\ |A_h^{l+n}(v_h)|_S^2 & \text{if } k-l = 2n+1. \end{cases} \end{aligned} \quad (76)$$

*Proof.* By induction using the integration by parts formula (60).  $\square$

**Lemma 6.2** (Quadratic identity). *Let  $s \geq 1$  and consider  $(s+1)$  real numbers  $(a_k)_{k \in \{0:s\}}$  defining the polynomial  $\mathcal{P}(z) := \sum_{k \in \{0:s\}} a_k z^k$ . The following holds true for all  $v_h \in L_h$ :*

$$\begin{aligned} \|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 &= \|v_h\|_L^2 + \sum_{k \in \{1:s\}} a_k^L \tau^{2k} \|A_h^k(v_h)\|_L^2 \\ &\quad + \sum_{k,l \in \{0:s-1\}} a_{kl}^S \tau^{k+l+1} \Re((A_h^k(v_h), A_h^l(v_h)))_S, \end{aligned} \quad (77)$$

with the coefficients  $(a_k^L)_{k \in \{1:s\}}$  and  $(a_{kl}^S)_{k,l \in \{0:s-1\}}$  such that  $a_{kl}^S = a_{lk}^S$  and

$$a_k^L = a_k^2 + 2 \sum_{m=1}^{\min(k,s-k)} (-1)^m a_{k+m} a_{k-m}, \quad \forall k \in \{1:s\}, \quad (78a)$$

$$a_{kl}^S = 2 \sum_{m=1}^{\min(l+1,s-k)} (-1)^{k+l+m} a_{k+m} a_{l-m+1}, \quad \forall k \in \{0:s-1\}, \forall l \in \{0:k\}. \quad (78b)$$

*Proof.* A direct calculation shows that

$$\begin{aligned} &\|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 \\ &= \|v_h\|_L^2 + \sum_{k \in \{1:s\}} a_k^2 \tau^{2k} \|A_h^k(v_h)\|_L^2 \\ &\quad + \sum_{k \in \{0:s\}} \sum_{l \in \{0:k-1\}} 2a_k a_l (-1)^{k+l} \tau^{k+l} \Re((A_h^k(v_h), A_h^l(v_h)))_L \\ &= \|v_h\|_L^2 + \sum_{k \in \{1:s\}} a_k^2 \tau^{2k} \|A_h^k(v_h)\|_L^2 \\ &\quad + \sum_{k=0}^s \sum_{l=0}^{k-1} 2a_k a_l (-1)^{k+l} \tau^{k+l} \left( \sum_{m=1}^n (-1)^{m+1} 2 \Re((A_h^{k-m}(v_h), A_h^{l+m-1}(v_h)))_S \right. \\ &\quad \left. + (-1)^n \begin{cases} \|A_h^{l+n}(v_h)\|_L^2 & \text{if } k-l = 2n \\ |A_h^{l+n}(v_h)|_S^2 & \text{if } k-l = 2n+1 \end{cases} \right), \end{aligned}$$

where we used (76) in the last step. The assertion follows from straightforward manipulations on the indices and by using the definitions of  $a_k^L$  and  $a_{kl}^S$ .  $\square$

The arrays  $a^L$  and  $a^S$  for ERK(2,2), ERK(3,3), ERK(4,4) are shown below:

$$\text{ERK}(2,2) \quad a^L = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}, \quad a^S = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \quad (79a)$$

$$\text{ERK}(3,3) \quad a^L = \begin{pmatrix} 0 \\ -\frac{1}{12} \\ \frac{1}{36} \end{pmatrix}, \quad a^S = \begin{pmatrix} -2 & 1 & -\frac{1}{3} \\ 1 & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \end{pmatrix}, \quad (79b)$$

$$\text{ERK}(4,4) \quad a^L = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{72} \\ \frac{1}{576} \end{pmatrix}, \quad a^S = \begin{pmatrix} -2 & 1 & -\frac{1}{3} & -\frac{1}{12} \\ 1 & -\frac{2}{3} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{12} & \frac{1}{24} \\ \frac{1}{12} & -\frac{1}{12} & \frac{1}{24} & -\frac{1}{72} \end{pmatrix}. \quad (79c)$$

**Lemma 6.3** ( $L^2$ -stability). *Assume that there exists an integer  $r \geq 1$  so that (i) the  $(r-1) \times (r-1)$  matrix with entries  $(a_{kl}^S)_{k,l \in \{0:r-1\}}$  is negative definite; (ii)  $a_r^L < 0$  and  $a_l^L = 0$  for all  $l \in \{1:r-1\}$ . Assume that the discrete operator  $A_h$  satisfies the estimates (61). There exists  $\sigma_0 > 0$  so that, if the standard CFL condition (64) holds true, then  $\|\mathcal{P}(-\tau A_h)\|_{\mathcal{L}(L_h)} \leq 1$ .*

*Proof.* Let  $v_h \in L_h$ . For all  $k \in \{r+1:s\}$ , invoking the CFL condition (64) yields

$$\tau^k \|A_h^k(v_h)\|_L \leq (c_L \tau h^{-1})^{k-r} \tau^r \|A_h^r(v_h)\|_L \leq \sigma_0^{k-r} \tau^r \|A_h^r(v_h)\|_L.$$

For all  $k \in \{r:s-1\}$  and all  $l \in \{0:r-1\}$ , we have

$$\begin{aligned} \tau^{k+l+1} |(A_h^k(v_h), A_h^l(v_h))_S| &\leq \tau^{k+l+1} |A_h^k(v_h)|_S |A_h^l(v_h)|_S \\ &\leq \tau^{k+l+1} (c_S h^{-1})^{\frac{1}{2}} (c_L h^{-1})^{k-r} \|A_h^r(v_h)\|_L |A_h^l(v_h)|_S \\ &\leq \tau^{r+l+\frac{1}{2}} (c_S \tau h^{-1})^{\frac{1}{2}} (c_S \tau h^{-1})^{k-r} \|A_h^r(v_h)\|_L |A_h^l(v_h)|_S \\ &\leq \frac{1}{2} \sigma_0^{2(k-r)} \tau^{2r} \|A_h^r(v_h)\|_L^2 + \frac{1}{2} \sigma_0 \tau^{2l+1} |A_h^l(v_h)|_S^2. \end{aligned}$$

Since  $(A_h^k(v_h), A_h^l(v_h))_S$  and  $(A_h^l(v_h), A_h^k(v_h))_S$  have the same module, the same bound holds for all  $l \in \{r:s-1\}$  and all  $k \in \{0:r-1\}$ . Finally, for all  $k \in \{r:s-1\}$  and all  $l \in \{r:s-1\}$ , we have

$$\begin{aligned} \tau^{k+l+1} |(A_h^k(v_h), A_h^l(v_h))_S| &\leq \tau^{2r} (c_S \tau h^{-1}) (c_L \tau h^{-1})^{k+l-2r} \|A_h^r(v_h)\|_L^2 \\ &\leq \tau^{2r} \sigma_0^{k+l-2r+1} \|A_h^r(v_h)\|_L^2. \end{aligned}$$

Inserting the above bounds into (77) and since  $a_l^L = 0$  for all  $l \in \{1:r-1\}$ , we infer that

$$\begin{aligned} \|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 &\leq \|v_h\|_L^2 + \tilde{a}_r^L \tau^{2r} \|A_h^r(v_h)\|_L^2 \\ &+ \sum_{k,l=0}^{r-1} a_{kl}^S \tau^{k+l+1} \Re((A_h^k(v_h), A_h^l(v_h))_S) + \sum_{l=0}^{r-1} \left( \sum_{k=r}^{s-1} |a_{kl}^S| \right) \sigma_0 \tau^{2l+1} |A_h^l(v_h)|_S^2, \end{aligned} \quad (80)$$

with

$$\tilde{a}_r^L := a_r^L + \sum_{k=r+1}^s \sigma_0^{2(k-r)} |a_k^L| + \sum_{k=r}^{s-1} \sum_{l=0}^{r-1} \sigma_0^{2(k-r)} |a_{kl}^S| + \sum_{k=r}^{s-1} \sum_{l=r}^{s-1} \sigma_0^{k+l-2r+1} |a_{kl}^S|.$$

Since  $a_r^L < 0$ , the above expression shows that it is possible to choose  $\sigma_0 > 0$  small enough so that  $\tilde{a}_r^L < 0$ . Furthermore, since the submatrix  $(a_{kl}^S)_{k,l \in \{0:r-1\}}$  is negative definite, it is possible to choose  $\sigma_0 > 0$  small enough so that

$$\sum_{k,l=0}^{r-1} a_{kl}^S \tau^{k+l+1} \Re((A_h^k(v_h), A_h^l(v_h))_S) + \sum_{l=0}^{r-1} \left( \sum_{k=r}^{s-1} |a_{kl}^S| \right) \sigma_0 \tau^{2l+1} |A_h^l(v_h)|_S^2 \leq 0.$$

This completes the proof.  $\square$

**Corollary 6.4** ( $L^2$ -stability of ERK(3,3)). *The statement of Proposition 4.1 holds true for ERK(3,3).*

*Proof.* We consider the polynomial  $\mathcal{P}(z) := P_3(z)$ . An inspection of the matrix  $a^S$  in (79b) shows that the assumption of Lemma 6.3 holds true for  $r = 2$ . Indeed, we see that  $a_1^L = 0$  and  $a_2^L = -\frac{1}{2} < 0$ , and the  $2 \times 2$  block

$$\begin{pmatrix} -2 & 1 \\ 1 & -\frac{2}{3} \end{pmatrix}$$

is negative definite.  $\square$

**Corollary 6.5** ( $L^2$ -stability of ERK(4,4)). *The statement of Proposition 4.1 holds true for ERK(4,4).*

*Proof.* We consider first the polynomial  $\mathcal{P}(z) := P_4(z)$ . From (79c), we see that  $a_1^L = 0$ ,  $a_2^L = 0$ , and  $a_3^L = -\frac{1}{72} < 0$ , but the  $3 \times 3$  block  $a^S(0:2, 0:2)$  is not negative definite. But for the scheme ERK(8,4) composed of two steps of the method ERK(4,4), a direct calculation using the polynomial  $\mathcal{P}(z) = P_4(z)^2$  shows that we have  $a_1^L = 0$ ,  $a_2^L = 0$ , and  $a_3^L = -\frac{1}{36} < 0$ , and that the  $4 \times 4$  block  $a^S(0:3, 0:3)$  is proportional to

$$\begin{pmatrix} -144 & 144 & -96 & 48 \\ 144 & -192 & 144 & -78 \\ -96 & 144 & -114 & 65 \\ 48 & -78 & 65 & -37 \end{pmatrix}.$$

This matrix is not negative definite, but the  $3 \times 3$  block  $a^S(0:2, 0:2)$  is, which is sufficient to apply Lemma 6.3 with  $r = 3$ .  $\square$

**Remark 6.6** (Additional dissipation). *A slightly stronger result than the assertion in Lemma 6.3 is that, under the same assumptions, there exists  $\gamma_0 > 0$  so that, if the standard CFL condition (64) holds true, then for all  $v_h \in L_h$ ,*

$$\|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 + \gamma_0 \tau \sum_{k \in \{0:s-1\}} \tau^{2k} |A_h^k(v_h)|_S^2 \leq \|v_h\|_L^2.$$

*This follows from (80) by invoking that the submatrix  $(a_{kl}^S)_{k,l \in \{0:r-1\}}$  is negative definite (so that its eigenvalues are bounded from above away from zero) and that, for all  $k \in \{r:s-1\}$ ,  $\gamma_0 \tau^{2k+1} |A_h^k(v_h)|_S^2 \leq \gamma_0 \sigma_0^{2(k-r)+1} \tau^{2r} \|A_h^r(v_h)\|_L^2$ , which can be bounded by  $-\tilde{a}_r^L \tau^{2r} \|A_h^r(v_h)\|_L^2$  by choosing  $\gamma_0 > 0$  small enough.*

## References

- [1] J. Alvarez, L. Angulo, A. Rubio Bretones, and S. G. Garcia. A spurious-free discontinuous Galerkin time-domain method for the accurate modeling of microwave filters. *IEEE Transactions on Microwave Theory and Techniques (T-MTT)*, 60(8):2359–2369, 2012.

- [2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998.
- [3] M. S. Birman and M. Z. Solomyak.  $L_2$ -theory of the Maxwell operator in arbitrary domains. *Russian Mathematical Surveys*, 42(6):75, 1987.
- [4] G. Boillat. Involutions des systèmes conservatifs. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(17):891–894, 1988. ISSN 0249-6291.
- [5] A. Bonito, J.-L. Guermond, and F. Luddens. Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains. *J. Math. Anal. Appl.*, 408(2):498–512, 2013.
- [6] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, NY, 2011.
- [7] E. Burman, A. Ern, and M. A. Fernández. Explicit Runge-Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems. *SIAM J. Numer. Anal.*, 48(6):2019–2042, 2010.
- [8] M. Campos Pinto and E. Sonnendrücker. Gauss-compatible Galerkin schemes for time-dependent Maxwell equations. *Math. Comp.*, 85(302):2651–2685, 2016.
- [9] P. Ciarlet, Jr. On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields. *Comput. Math. Appl.*, 71(1):85–104, 2016.
- [10] P. Ciarlet, Jr. and J. Zou. Fully discrete finite element approaches for time-dependent Maxwell's equations. *Numer. Math.*, 82(2):193–219, 1999.
- [11] M. Costabel. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Math. Methods Appl. Sci.*, 12(4):365–368, 1990.
- [12] C. M. Dafermos. Quasilinear hyperbolic systems with involutions. *Arch. Rational Mech. Anal.*, 94(4):373–389, 1986.
- [13] A. Ern and J.-L. Guermond. *Finite elements I—Approximation and interpolation*, volume 72 of *Texts in Applied Mathematics*. Springer, Cham, 2021.
- [14] A. Ern and J.-L. Guermond. *Finite elements. III. First-order and time-dependent PDEs*, volume 74 of *Texts in Applied Mathematics*. Springer, Cham, 2021.
- [15] A. Ern and J.-L. Guermond. The discontinuous Galerkin approximation of the grad-div and curl-curl operators in first-order form is involution-preserving and spectrally correct. *SIAM J. Numer. Anal.*, 61(6):2940–2966, 2023.
- [16] A. Ern and J.-L. Guermond. Spectral correctness of the simplicial discontinuous Galerkin approximation of the first-order form of Maxwell's equations with discontinuous coefficients. *SIAM J. Numer. Anal.*, 63(2):661–684, 2025.
- [17] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno. Convergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes. *ESAIM: Modélisation mathématique et analyse numérique*, 39(6):1149–1176, 2005.

- [18] J. S. Hesthaven and T. Warburton. Nodal high-order methods on unstructured grids. I. Time-domain solution of Maxwell's equations. *J. Comput. Phys.*, 181(1):186–221, 2002.
- [19] F. Jochmann. Regularity of weak solutions of Maxwell's equations with mixed boundary-conditions. *Math. Methods Appl. Sci.*, 22(14):1255–1274, 1999.
- [20] S. Lanteri, C. Scheid, and J. Viquerat. Analysis of a generalized dispersive model coupled to a DGTD method with application to nanophotonics. *SIAM J. Sci. Comput.*, 39(3):A831–A859, 2017.
- [21] J. Li. Numerical convergence and physical fidelity analysis for Maxwell's equations in meta-materials. *Comput. Methods Appl. Mech. Engrg.*, 198(37-40):3161–3172, 2009.
- [22] V. Perrier. Discrete de-Rham complex involving a discontinuous finite element space for velocities: the case of periodic straight triangular and Cartesian meshes. *Annales Henri Lebesgue*, 2025. URL <https://inria.hal.science/hal-04564069>.
- [23] M. Stanglmeier, N. C. Nguyen, J. Peraire, and B. Cockburn. An explicit hybridizable discontinuous Galerkin method for the acoustic wave equation. *Comput. Methods Appl. Mech. Engrg.*, 300:748–769, 2016.
- [24] Z. Sun and C.-W. Shu. Stability of the fourth order Runge-Kutta method for time-dependent partial differential equations. *Ann. Math. Sci. Appl.*, 2(2):255–284, 2017.
- [25] Z. Sun and C.-W. Shu. Strong stability of explicit Runge-Kutta time discretizations. *SIAM J. Numer. Anal.*, 57(3):1158–1182, 2019.
- [26] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.*, 2(1):12–25, 1980.
- [27] M. F. Wheeler. A priori  $L_2$  error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal.*, 10:723–759, 1973.
- [28] Y. Xu, Q. Zhang, C.-W. Shu, and H. Wang. The  $L^2$ -norm stability analysis of Runge-Kutta discontinuous Galerkin methods for linear hyperbolic equations. *SIAM J. Numer. Anal.*, 57(4):1574–1601, 2019.
- [29] Q. Zhang and C.-W. Shu. Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws. *SIAM J. Numer. Anal.*, 48(3):1038–1063, 2010.