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GÖPEL VARIETIES

V. BENEDETTI, M. BOLOGNESI, D. FAENZI, L. MANIVEL

ABSTRACT. We show that the Coble hypersurfaces, uniquely characterized by the remarkable property that their singular loci are an abelian surface and a Kummer threefold, respectively, belong to a family of hypersurfaces exhibiting similar behavior, but defined in various types of homogeneous spaces. With the help of Jordan-Vinberg theory, we show how these hypersurfaces can be parametrized by Göpel type varieties inside projectivized representations of complex reflection groups.

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1. INTRODUCTION

The Coble hypersurfaces are characterized by the very unusual properties that their singular loci are an abelian surface for the Coble cubic in \mathbb{P}^8 , and a Kummer threefold for the Coble quartic in \mathbb{P}^7 [Bea03]. Being hypersurfaces, they are easy to handle. The classical Göpel variety parametrizes the Coble quartics one obtains when the Kummer threefold varies, hence the abelian threefold from which it is constructed, hence the genus three curve whose Jacobian is this threefold; it can thus be considered as a moduli space for this kind of objects, endowed with some sort of additional structure [DO88, RSSS13]. For Coble cubics, the analogous object is the classical Burkhardt quartic.

In [BBFM24, BBFM23] we observed that one can construct other interesting objects, such as moduli spaces of vector bundles on curves, as singular loci of hypersurfaces, at the price of extending the class of ambient spaces we are ready to accept, so as to include Grassmannians, quadrics and other homogeneous spaces. Once we are ready for that, we observe that Coble type hypersurfaces do exist in many interesting situations; this is the consequence of natural constructions in terms of *orbital degeneracy loci*, introduced in [BFMT20a, BFMT20b]. These constructions are based on Lie theory and require an action of an algebraic group, typically an irreducible representation. It appears that there exist a series of special representations having rich connections with such classical objects as curves or low genus or low-dimensional abelian

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varieties, endowed with some additional data. These representations were identified in [GSW13], and simultaneously in a series of works by Bhargava and his school (see e.g. [BH16]). Intriguingly, most of them are derived from the exceptional Lie algebras, and one could argue that substantial pieces of classical algebraic geometry are, in a way, shadows of the exceptional Lie groups, observed from the depth of Plato's cave. These representations do not appear in the works of the classics, but they shed a new light on them and open new perspectives.

The aim of this paper is to discuss, for each of these special representations, which Coble type hypersurfaces can be constructed, how they can be parametrized and what kind of moduli space are obtained from these constructions. The main property shared by these special representations, say V acted on by G_0 , is that they are pieces of some cyclic grading of a simple Lie algebra \mathfrak{g} . This allows to put into action the very powerful version of Jordan theory developed in this setting by Kac and Vinberg. The main actor in this theory is the Cartan subspace $\mathfrak{c} \subset V$, which plays the role of the space of diagonal matrices in usual Jordan theory. It is acted on by a complex reflection group $W_{\mathfrak{c}}$, and the resulting hyperplane arrangement contains a wealth of information; by Vinberg's version of Chevalley's restriction theorem, the quotient of \mathfrak{c} by $W_{\mathfrak{c}}$ is isomorphic to the GIT quotient $V//G_0$. As a consequence, any G_0 -equivariant map from V to another representation U restricts to a $W_{\mathfrak{c}}$ -equivariant map from \mathfrak{c} to some $W_{\mathfrak{c}}$ -module $\Theta \subset U$.

The connection with geometry begins when one observes that the centralizer of \mathfrak{c} in G_0 is a finite group of Heisenberg type, similar to the theta-groups associated to polarized abelian varieties. This is of course a strong hint that such an abelian variety could be constructed geometrically, and this is exactly what the theory of orbital degeneracy loci is good for: starting from a point in \mathfrak{c} , we can define sections of several equivariant vector bundles on auxiliary homogeneous spaces Z , bundles whose spaces of global sections are isomorphic to V . Orbital degeneracy loci are then loci where such a section degenerates in some natural way. It was observed in [BH16, GSW13] that various types of abelian varieties can be obtained that way, as well as the Coble hypersurfaces.

In this paper we take one step further by showing that very often, there is a codimension one orbital degeneracy locus of special interest, stratified by successive singular loci that includes the relevant abelian variety; we call such a locus a *Coble type hypersurface*. Sometimes we can even prove that this special singular locus uniquely determines the hypersurface, as in the classical case of the Coble cubics and quartics. According to Jordan-Vinberg theory, we can parametrize these hypersurfaces by restricting to the Cartan subspace; they form a linear system isomorphic to a m -dimensional $W_{\mathfrak{c}}$ -submodule Θ_m of some $S^d \mathfrak{c}^{\vee}$, parametrizing Heisenberg-invariant polynomials. We can then define the *Göpel variety* as the image of the rational map $\gamma : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(\Theta_m)$. In several cases this variety has a nice moduli interpretation. It may even happen that we get several such varieties, when several hypersurfaces of interest can be constructed in different auxiliary spaces.

To summarize, the successive steps we take are as follows.

- Start from a cyclic grading of a simple Lie algebra \mathfrak{g} ; this yields a so called θ -representation V of a group G_0 .
- Identify a Cartan subspace \mathfrak{c} , together with its complex reflection group $W_{\mathfrak{c}}$; its centralizer turns out to be a Heisenberg type subgroup of G_0 .
- Choose an auxiliary G_0 -homogeneous space Z inside which a nonzero $v \in V$ defines a special hypersurface of degree δ ; study the geometry of this hypersurface and its singularity stratification.
- Interpret the equation of this hypersurface, if it is given by a degree d polynomial function on V , as the image of a G_0 -equivariant morphism

$$\Gamma : S^d V \longrightarrow H^0(Z, \mathcal{O}_Z(\delta))$$

- Restrict this morphism to \mathfrak{c} and denote by Θ_m the m -dimensional span of the image, which is naturally endowed with a $W_{\mathfrak{c}}$ -representation structure, to obtain a $W_{\mathfrak{c}}$ -equivariant rational map

$$\gamma : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(\Theta_m).$$

- Whenever possible, describe the image \mathcal{G} of γ : this is our Göpel type variety; finally, provide a modular interpretation.

We apply this program to the series of gradings and representations from the table below. The first column presents an affine Dynkin diagram D of \mathfrak{g} , where the black vertex corresponds to the simple root α_0 that defines the grading, as in [Vin76, section 9]. The second column gives the subalgebra $\mathfrak{g}_0 = Lie(G_0)$ and the \mathfrak{g}_0 -module $V = \mathfrak{g}_1$; the semisimple part of \mathfrak{g}_0 has Dynkin diagram $D_0 = D \setminus \{\alpha_0\}$; the highest weight of V is the sum of the fundamental weights corresponding to the vertices of D_0 connected to α_0 in D (with a coefficient 2 in the third case, because of the double edge); these vertices are located by a cross.

\mathfrak{g}	\mathfrak{g}_0, V	Z	δ	d		§
	$\mathfrak{sl}_5 \times \mathfrak{sl}_5$ $\wedge^2 \mathbb{C}^5 \otimes \mathbb{C}^5$	$\mathbb{P}^4 \times \mathbb{P}^4$	(2, 1)	2	[BH16]	§ 3
	$\mathfrak{sl}_3 \times \mathfrak{sl}_3 \times \mathfrak{sl}_3$ $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$	$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	2	4	Rubik's cubes [BH16]	§ 4
	\mathfrak{so}_{2m} $S^{(2)} \mathbb{C}^{2m}$	$OG(k, 2m)$	2	k	Spin bundles on hyperelliptic curves [DR77, Ram81, Bho84]	§ 5
	\mathfrak{sl}_8 $\wedge^4 \mathbb{C}^8$	$\mathbb{P}^7, \check{\mathbb{P}}^7$ $G(2, 8)$ Fl_8	4 2 2	7 3 4	Coble quartic [RSSS13] Coble quadric [BBFM24]	§ 6.1 § 6.3
	\mathfrak{sl}_9 $\wedge^3 \mathbb{C}^9$	\mathbb{P}^8 $\check{\mathbb{P}}^8$ $G(6, 9)$	3 6 2	4 4 4	Coble cubic dual sextic Coble quadric [BBFM23]	§ 7
	$\mathfrak{sl}_4 \times \mathfrak{so}_{10}$ $\mathbb{C}^4 \otimes \Delta_+$	$G(2, 4)$ Fl_4 \mathbb{P}^3 \mathbb{Q}^8 $OG(2, 10)$	2 2 4 4 2	4 8 12 8 4	quadratic complex [LM25] Coble quadric Kummer surface	§ 8
	\mathfrak{so}_{16} Δ_+	\mathbb{Q}^{14} $OG(2, 16)$	4 2	8 4	Coble quartic Coble quadric	§ 9.7 § 9.6

In the second table below we describe the corresponding little Weyl groups W_c ; the notation G_N refers to the Shephard-Todd classification [ST54]. Their representations Θ_m are always irreducible (of dimension m). They are always Macdonald representations, in the sense that they are generated by certain products of equations of reflection hyperplanes. This is typically the case of the classical Specht modules, which are the irreducible representations of symmetric groups; for instance we get the Specht module $[k, k]$, a representation of the symmetric group S_k of dimension $m = \frac{(2k)!}{k!(k+1)!}$, the k -th Catalan number.

W_c	Θ_m	Göpel	Modular interpretation
G_{16}	Θ_3	$v_2(\mathbb{P}^1)$	genus one curves + degree 5 line bundle
G_{25}	Θ_{12}	projection of $v_4(\mathbb{P}^2)$	genus one curves + two degree 3 line bundles
S_{2k}	$[k, k]$	$(\mathbb{P}^1)^{2k} // PGL_2$	hyperelliptic curves / $2k$ points in \mathbb{P}^1
$W(E_7)$	Θ_{15}, Θ_{21}	$(\mathbb{P}^2)^7 // PGL_3$	plane quartics
G_{32}	Θ_5	Burkhardt quartic	genus two curves + level 3 structure
G_{31}	Θ_5, Θ_9 Θ'_5	Igusa quartic Segre cubic	genus two curves + level 2 structure
$W(E_8)$	Θ_{50}, Θ_{84}		special genus four curves

Relation to previous work. The Göpel variety was defined in [RSSS13] as the moduli space of Coble quartics in \mathbb{P}^7 , parametrized by the so-called Göpel functions [DO88] as a subvariety of \mathbb{P}^{14} ; it is cut out by 35 quadrics and 35 cubics. The moduli space of Coble cubics in \mathbb{P}^8 was computed in [GS15] and shown to coincide with the classical Burkhardt quartic in \mathbb{P}^4 . The moduli space of dual Coble sextics has not been considered and seems computationally challenging. More accessible are the Coble quadrics that we defined in [BBFM24] as quadric sections of the Grassmannian $G(6, 9)$; we show the corresponding Göpel type variety is a projection of the fourth-Veronese of \mathbb{P}^3 from the span of the Burkhardt quartic, and that it is cut out by 300 quadrics (Theorem 7.1). We also recover the classical description of the moduli space of $2k$ -points in \mathbb{P}^1 by considering such a $2k$ -tuple as the base locus of a pencil of quadrics in \mathbb{P}^{2k-1} ; fixing a smooth quadric in the pencil, we define a quadric section of the associated orthogonal Grassmannian $OG(k, 2k)$, whose equation recovers the classical cross-ratio functions [DO88].

The case of 6 points in \mathbb{P}^1 is also related to $SL_4 \times Spin_{10}$; on the side of SL_4 , we recover the classical Segre cubic primal and Castelnuovo-Richmond (or Igusa) quartic; but on the side of $Spin_{10}$ we show that one can also define Coble type hypersurfaces, namely quartic sections of the quadric Q^8 and quadric sections of the orthogonal Grassmannian $OG(2, 10)$.

But the most difficult case we deal with is the half-spin representation of $Spin_{16}$, which is known to be related to a family of special genus four curves. Using spinor bundles, we show how to construct Coble type hypersurfaces as quartic sections of the quadric Q^{14} , and quadric sections of the orthogonal Grassmannian $OG(2, 16)$. We show that the two associated Göpel type varieties, both seven dimensional, span (the projectivization of) two Macdonald representations of the Weyl group $W(E_8)$, of dimension 84 (Proposition 9.16) and 50 (Theorem 9.22), respectively. Conjecturally, these two Göpel type hypersurfaces should contain, inside their singular locus, the two moduli spaces of rank two vector bundles on the genus four curve, with fixed determinant of even or odd degree. For the even moduli space of bundles on a genus four curve with no vanishing theta-null, the existence of such a hypersurface was proved in [OP99]; the case we address here is a disjoint, degenerate one. Also we expect the (smooth) odd moduli space to coincide with the singular locus of our Coble type hypersurface in $OG(2, 16)$; it would be interesting to prove some statement of this type for a general curve of genus four. For curves of genus two and three such statement we established in [BBFM24, BFMT20b].

We find it remarkable that a unified perspective can be given on all these moduli problems, through Coble type hypersurfaces that are geometrically meaningful. In quite a few cases the precise geometry of these hypersurfaces is still to be understood; their successive singular strata are often interesting varieties, but some of them have not been identified yet. We also note that similar Coble type phenomena have been observed in related, but different situations: for example, the moduli space of 8 points in \mathbb{P}^1 embeds in \mathbb{P}^{13} as the singular locus of a uniquely defined cubic [HMSV12]. It would be very nice to discover other relevant examples.

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2. PRELIMINARIES

A grading of the Lie algebra \mathfrak{g} of a semisimple simply connected group G is a decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. In this paper the index i runs over \mathbb{Z} or $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Vinberg developed in [Vin76] the structure theory of $\mathbb{Z}/m\mathbb{Z}$ -graded Lie algebras. Such a grading is always given by an automorphism θ of \mathfrak{g} , such that $\theta^m = \text{id}$. Vinberg studied the action of G^θ (the subgroup of θ -invariant elements in G), whose Lie algebra is \mathfrak{g}_0 , on a graded piece, say \mathfrak{g}_1 ; this is often called a θ -representation of a θ -group, the latter being either G^θ or its universal cover G_0 . He showed that there exist maximal abelian subspaces $\mathfrak{c} \subset \mathfrak{g}_1$ consisting of semisimple elements; they are called Cartan subspaces and are all conjugate under G^θ ; moreover they parametrize closed G^θ -orbits in \mathfrak{g}_1 . The rank of a graded Lie algebra is $\dim(\mathfrak{c})$. A Cartan subspace \mathfrak{c} is acted on by the *little Weyl group* $W_{\mathfrak{c}} := N_{G^\theta}(\mathfrak{c})/Z_{G^\theta}(\mathfrak{c})$, which is a finite reflection group (here N stands for normalizer, and Z for centralizer). This extends the definition of the usual Weyl group of a Cartan subalgebra in a semisimple Lie algebra. He then proved a Chevalley type restriction theorem relating G^θ -invariants in \mathfrak{g}_1 and $W_{\mathfrak{c}}$ -invariants in \mathfrak{c} via restriction to the Cartan subspace:

$$\mathbb{C}[\mathfrak{g}_1]^{G^\theta} \simeq \mathbb{C}[\mathfrak{c}]^{W_{\mathfrak{c}}}.$$

We will see many examples of graded Lie algebras throughout the paper.

Complex reflection groups $W_{\mathfrak{c}}$ acting on a vector space \mathfrak{c} were completely classified in the famous work of Shephard and Todd [ST54]. Among their representations, one can distinguish the so-called Macdonald representations, defined as follows. Consider a system $S := \{c_1, \dots, c_k\} \subset \mathfrak{c}^\vee$ of non-trivial linear forms, defining reflection hyperplanes of reflections $s_1, \dots, s_k \in W_{\mathfrak{c}}$. We can construct a $W_{\mathfrak{c}}$ -representation inside $\mathbb{C}[\mathfrak{c}^\vee]_k$ as the linear span

$$\langle w \cdot (c_1 c_2 \cdots c_k) \rangle_{w \in W_{\mathfrak{c}}} \subset \mathbb{C}[\mathfrak{c}]_k.$$

When it is irreducible, we call it a *Macdonald representation*. Macdonald proved that irreducibility always holds when $W_{\mathfrak{c}}$ is a genuine Weyl group and S is any subsystem of the root system [Mac72]. This generalizes the construction of irreducible modules of symmetric groups as Specht modules.

In all the examples we will meet in the sequel, the Cartan subspace will be the space of invariants in \mathfrak{g}_1 of a finite Heisenberg subgroup H of G_0 , the universal cover of G^θ . Conversely, H can be obtained as the subgroup of G_0 acting trivially on \mathfrak{c} . Finite Heisenberg groups are defined as central extensions

$$1 \rightarrow \mathbb{Z}_k \rightarrow H \rightarrow K \rightarrow 1,$$

where K is a finite abelian group [MNN91]. Moreover, it is required that the commutator map

$$K \times K \rightarrow \mathbb{Z}_k, \quad (x, y) \mapsto \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1},$$

where \tilde{x}, \tilde{y} are lifts of x, y to H , is a non-degenerate skew-symmetric pairing.

3. A TOY CASE

We start with the case of $\text{SL}_5 \times \wedge^2 \text{SL}_5$, or quintuples of skew-symmetric matrices of order five. This is classically related to quintic elliptic curves [BH16, section 4.4]. From our point of view, the corresponding Göpel variety will just be a conic; but treating this case in some detail will allow to introduce in a simple setting the approach we will apply to more interesting, and also more involved cases.

From the Jordan-Vinberg perspective, this example comes from a \mathbb{Z}_5 -grading of \mathfrak{e}_8 , namely

$$\mathfrak{e}_8 = \mathfrak{sl}(A_5) \times \mathfrak{sl}(B_5) \oplus (A_5 \otimes \wedge^2 B_5) \oplus (\wedge^2 A_5 \otimes B_5^\vee) \oplus (\wedge^2 A_5^\vee \otimes B_5) \oplus (A_5^\vee \otimes \wedge^2 B_5^\vee).$$

So $G_0 = \text{SL}(A_5) \times \text{SL}(B_5)$ acts on $\mathfrak{g}_1 = \wedge^2 A_5 \otimes B_5$. The rank, i.e. the dimension of a Cartan subspace, is only two in this case, and the corresponding little Weyl group is the complex reflection group G_{16} [Vin76], of order 600, with invariants of degrees 20 and 30; this group is a product of \mathbb{Z}_5 by the binary icosahedral group [ST54].

3.1. A Cartan subspace. Let a_1, \dots, a_5 be a basis of A_5 , and b_1, \dots, b_5 be a basis of B_5 . We will use the notation $b_{ij} = b_i \wedge b_j \in \wedge^2 B_5$.

Lemma 3.1. *The subspace $\mathfrak{c} \subset A_5 \otimes \wedge^2 B_5$ generated by*

$$\begin{aligned} h_1 &= a_1 \otimes b_{12} + a_2 \otimes b_{23} + a_3 \otimes b_{34} + a_4 \otimes b_{45} + a_5 \otimes b_{51}, \\ h_2 &= a_1 \otimes b_{35} + a_2 \otimes b_{41} + a_3 \otimes b_{52} + a_4 \otimes b_{13} + a_5 \otimes b_{24}, \end{aligned}$$

is a Cartan subspace of \mathfrak{g}_1 .

Proof. We need to prove that h_1, h_2 are semisimple elements of \mathfrak{e}_8 , and commute. For the latter claim, observe that the Lie bracket $\wedge^2 \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ must be proportional, by G_0 -equivariance, to the morphism

$$\wedge^2(A_5 \otimes \wedge^2 B_5) \rightarrow \wedge^2 A_5 \otimes B_5^\vee$$

deduced from the map $S^2(\wedge^2 B_5) \rightarrow \wedge^4 B_5 \simeq B_5^\vee$, where the latter identification is defined by the choice of a generator of $\wedge^5 B_5$, say $b_1 \wedge \dots \wedge b_5$. Explicitly, the Lie bracket of $a_p \otimes b_{ij}$ with $a_q \otimes b_{kl}$ is $a_{pq} \otimes b_{ijkl}$, where b_{ijkl} identifies with ϵb_m^\vee if $(ijklm)$ is a permutation of (12345) of signature ϵ , and zero otherwise. Then the claim that $[h_1, h_2] = 0$ follows from the observation that if $a_p \otimes b_{p,p+1}$ is one of the terms of h_1 , and $a_q \otimes b_{q+2,q-1}$ one of the terms of h_2 , then either $p = q$ of the pairs $(p, p+1)$ and $(q+2, q-1)$ have non-empty intersection.

In order to prove that h_1 is semisimple, we let $x_p^1 = a_p \otimes b_{p,p+1}$, and we observe that the subalgebra \mathfrak{s} of \mathfrak{e}_8 generated by these five elements of \mathfrak{g}_1 is isomorphic to \mathfrak{sl}_5 with its standard \mathbb{Z}_5 -grading: in the canonical basis e_1, \dots, e_5 of \mathbb{C}^5 , $(\mathfrak{sl}_5)_k$ is generated by matrices of the form $e_p^\vee \otimes e_{p+k}$ (with the additional condition that the trace vanishes when $k = 0$). Explicitly, the isomorphism $\theta : \mathfrak{sl}_5 \simeq \mathfrak{s}$ is obtained by sending $e_p^\vee \otimes e_{p+1}$ to $a_{2p-1} \otimes b_{2p-1,2p}$. In particular,

$$\theta^{-1}(h_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is semisimple; so h_1 is semisimple as well. The case of h_2 is completely similar, and the claim follows. \square

3.2. The Heisenberg group. This is the stabilizer H of \mathfrak{c} inside $G_0 = \mathrm{SL}(A_5) \times \mathrm{SL}(B_5)$. We can describe it explicitly.

Lemma 3.2. *The Heisenberg group is a non-trivial central extension*

$$1 \rightarrow \mathbb{Z}_5 \rightarrow H \rightarrow (\mathbb{Z}_5)^2 \rightarrow 1.$$

Proof. We can identify H with its image in $\mathrm{SL}(B_5)$, whose induced action on $\wedge^2 B_5$ has to preserve the two five-planes $I_1 = \langle b_{12}, b_{23}, b_{34}, b_{45}, b_{51} \rangle$ and $I_2 = \langle b_{35}, b_{41}, b_{52}, b_{13}, b_{24} \rangle$, and act on these two spaces in the same way (in their two given basis). Note that if $g \in \mathrm{SL}(B_5)$ preserves I_1 and I_2 , it also has to preserve the varieties D_1 and D_2 of decomposable tensors in $\mathbb{P}(I_1)$ and $\mathbb{P}(I_2)$.

A straightforward computation shows that D_1 and D_2 are two pentagons, D_1 being the union of the five lines joining $b_{i-1,i}$ to $b_{i,i+1}$, while D_2 is the union of the five lines joining $b_{i-2,i}$ to $b_{i,i+2}$. As a consequence, $g \in \mathrm{SL}(B_5)$ preserving D_1 and D_2 must also preserve their singular loci, which are the $\langle b_{i,i+1} \rangle$'s and $\langle b_{i,i+2} \rangle$'s, respectively. Then it also has to preserve the intersections of the corresponding lines in B_5 , which are the $\langle b_i \rangle$'s. So we must have $g(b_i) = \beta_i b_{\sigma(i)}$ for some permutation σ and some scalars β_i . Since $\sigma(i)$ and $\sigma(i) + 1$ must differ by one, the only possibilities are that σ is a cycle ($\sigma(i) = i + s \pmod{5}$), or a reflection ($\sigma(i) = s - i \pmod{5}$); but the latter case is excluded by the fact that g should satisfy $\det(g) = 1$. So we may suppose that $g(a_i) = \alpha_i a_{i+s}$, $g(b_i) = \beta_i b_{i+s}$, with the additional conditions that for any i , $\alpha_i \beta_i \beta_{i+1} = \alpha_i \beta_{i+2} \beta_{i+4} = 1$. So the α_i 's are determined by the β_i 's, which are subject to the condition that $\beta_i \beta_{i+1} = \beta_{i+2} \beta_{i+4}$ for all i . One checks that this is equivalent to

$$\beta_3 = \beta_1^{-1} \beta_2^2, \quad \beta_4 = \beta_1^{-2} \beta_2^3, \quad \beta_5 = \beta_1^2 \beta_2^{-1},$$

for β_1, β_2 any fifth roots of unity (hence $\beta_i^{-1} \beta_{i+1}$ is independent of i). Notice that when $\beta_1 = \beta_2$ and $\sigma = \mathrm{id}$, we get $g = \beta_1 \mathrm{id}$. We conclude that the morphism sending $g \mapsto (\sigma, \beta_1^{-1} \beta_2)$ yields the extension

$$1 \rightarrow \mathbb{Z}_5 \rightarrow H \rightarrow (\mathbb{Z}_5)^2 \rightarrow 1,$$

in agreement with Appendix 10. \square

Corollary 3.3. *The space of Heisenberg invariants in \mathfrak{g}_1 is exactly \mathfrak{c} .*

3.3. A Coble type hypersurface. A generic tensor v in $A_5^\vee \otimes \wedge^2 B_5^\vee$ defines several interesting geometric objects as follows. If U_m is an m -dimensional vector space, we denote by \mathcal{U}_k the tautological rank k vector bundle on a Grassmannian $G(k, U_m)$.

- (1) A genus one curve E , as the codimension five section of $G(2, B_5)$ cut out by the image of A_5 in $\wedge^2 B_5^\vee$ (see [BH16, Theorem 4.14] for a modular interpretation).
- (2) A surface S inside $\mathbb{P}(A_5) \times \mathbb{P}(B_5^\vee)$, cut out by a section of the rank six bundle $A_1^\vee \boxtimes \wedge^2 B_4^\vee$.
- (3) A hypersurface Q defined as the vanishing locus of a global section of $\mathcal{O}(1, 2)$ on $\mathbb{P}(A_5) \times \mathbb{P}(B_5^\vee)$. The singular locus of the hypersurface Q is the surface S .
- (4) A Fano sixfold X inside $G(2, A_5) \times G(3, B_5)$, obtained as the zero locus of a section of $A_2^\vee \boxtimes \wedge^2 B_3^\vee$. The two projections p_1 and p_2 are birational maps. The exceptional locus of p_1 is the singular fourfold Y of lines incident to E , whose normalisation is a \mathbb{P}^3 -bundle over E . On the other hand, p_2 is the blow-up of the degeneracy locus of the morphism defined as the composition $\wedge^2 B_3 \hookrightarrow \wedge^2 B_5 \rightarrow A_5^\vee$.

Proposition 3.4. *The Coble-type hypersurface Q is the unique hypersurface of bidegree $(2, 1)$ which is singular along the ruled surface S .*

Proof. A point (ℓ, h) of $\mathbb{P}(A_5) \times \mathbb{P}(B_5^\vee)$ is in S if and only if the two-form $v(\ell)$ vanishes on the hyperplane $P_h \subset B_5$ defined by h . This implies that $v(\ell)$ has rank two, hence defines a point of E , and then the hyperplane defined by h has to contain its kernel. So S is a ruled surface over E .

Since S is the zero locus of the bundle $\mathcal{E} = A_1^\vee \boxtimes \wedge^2 B_4^\vee$ on $\mathbb{P}(A_5) \times \mathbb{P}(B_5^\vee)$, its ideal sheaf is resolved by the usual Koszul complex. The following generalization is probably well-known.

Lemma 3.5. *If Z is the zero locus of a section of a vector bundle \mathcal{E} or rank e over some smooth variety X , and has the expected codimension, then the k -th power of its ideal sheaf admits a resolution*

$$0 \rightarrow S_{k1^{e-1}} \mathcal{E}^\vee \rightarrow \dots \rightarrow S_{k1} \mathcal{E}^\vee \rightarrow S_k \mathcal{E}^\vee \rightarrow \mathcal{J}_Z^k \rightarrow 0.$$

Proof of the Lemma. Suppose that $Z = Z(s)$ for some section $s \in H^0(X, \mathcal{E})$. Consider inside the projective bundle $\mathbb{P}_X(\mathcal{E})$ the locus \tilde{X} of pairs $(x, [e])$ where $s(x) \in [e]$. On the one hand, the projection to X is an isomorphism outside Z , whose preimage is $F = \mathbb{P}_Z(\mathcal{E})$. By the universal property of blowups, we conclude that $\tilde{X} \simeq Bl_Z(X)$, the blowup of X along Z , and that F is the exceptional divisor. On the other hand, there is on $\mathbb{P}_X(\mathcal{E})$ a tautological sequence $0 \rightarrow \mathcal{O}_{\mathcal{E}}(-1) \rightarrow \pi^* \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$, where π denotes the projection. In particular s defines a section \tilde{s} of \mathcal{Q} that vanishes exactly along \tilde{X} . Hence a Koszul resolution, that we can twist by $\mathcal{O}_{\mathcal{E}}(k)$ to get

$$0 \rightarrow \wedge^{e-1} \mathcal{Q}^\vee \otimes \mathcal{O}_{\mathcal{E}}(k) \rightarrow \dots \rightarrow \mathcal{Q}^\vee \otimes \mathcal{O}_{\mathcal{E}}(k) \rightarrow \mathcal{O}_{\mathcal{E}}(k) \rightarrow \mathcal{O}_{\mathcal{E}}(k)|_{\tilde{X}} \rightarrow 0.$$

By Bott's theorem, $\pi_*(\wedge^p \mathcal{Q}^\vee \otimes \mathcal{O}_{\mathcal{E}}(k)) = S_{k1^p} \mathcal{E}^\vee$, and the higher direct images vanish. Moreover, $\mathcal{O}_{\mathcal{E}}(1)|_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-F)$, hence $\pi_*(\mathcal{O}_{\mathcal{E}}(1)|_{\tilde{X}}) = \pi_* \mathcal{O}_{\tilde{X}}(-kF) = \mathcal{J}_Z^k$. So the push-forward of the (twisted) Koszul complex on $\mathbb{P}_X(\mathcal{E})$ yields the desired complex. \square

In our case, the resulting resolution $\mathcal{F}_\bullet \rightarrow \mathcal{J}_S^2 \rightarrow 0$ has the following terms:

$$\begin{aligned} \mathcal{F}_0 &= S^2 \mathcal{E}^\vee &= \mathcal{O}(-2) \boxtimes (\det(\mathcal{B}_4) \oplus S_{22} \mathcal{B}_4), \\ \mathcal{F}_1 &= S_{21} \mathcal{E}^\vee &= \mathcal{O}(-3) \boxtimes \mathcal{F}_1, \\ \mathcal{F}_2 &= S_{211} \mathcal{E}^\vee &= \mathcal{O}(-4) \boxtimes \mathcal{F}_2, \\ \mathcal{F}_3 &= S_{2111} \mathcal{E}^\vee &= \mathcal{O}(-5) \boxtimes \mathcal{F}_3, \\ \mathcal{F}_4 &= S_{21111} \mathcal{E}^\vee &= \mathcal{O}(-6) \boxtimes \mathcal{F}_4, \\ \mathcal{F}_5 &= S_{211111} \mathcal{E}^\vee &= \mathcal{O}(-7) \boxtimes \wedge^2 \mathcal{B}_4 \otimes \det(\mathcal{B}_4)^3. \end{aligned}$$

After twisting by $\mathcal{O}(2, 1)$, all these terms are clearly acyclic, except possibly the first and last one. Using Bott-Borel-Weil, we check that the first term has $h^0 = 1$, while the last term has $h^6 = 1$, all the other cohomology groups being zero. Hence

$$h^0(\mathbb{P}(A_5) \times \mathbb{P}(B_5^\vee), \mathcal{J}_S^2(2, 1)) = \mathbb{C},$$

which implies our claim. \square

3.4. Equation of the Coble-type hypersurface. The morphism associating to $v \in A_5^\vee \otimes \wedge^2 B_5^\vee$ the Coble-type hypersurface Q is induced by the G_0 -equivariant morphism

$$\Gamma : S^2(A_5^\vee \otimes \wedge^2 B_5^\vee) \longrightarrow S^2 A_5^\vee \otimes B_5$$

Restricting to the Cartan subspace \mathfrak{c} we get a quadratic map

$$\gamma : \mathbb{P}(\mathfrak{c}) \longrightarrow \mathbb{P}(\Theta_3) \subset \mathbb{P}(S^2 A_5^\vee \otimes B_5)$$

that we can write down explicitly. Indeed, let $v = v_1 h_1 + v_2 h_2$ (see Lemma 3.1); its contraction with $x = (x_1, \dots, x_5) \in A_5$ is

$$\omega = x_1(v_1 b_{12} + v_2 b_{13}) + x_2(v_1 b_{23} + v_2 b_{24}) + x_3(v_1 b_{34} + v_2 b_{35}) + x_4(v_1 b_{45} + v_2 b_{41}) + x_5(v_1 b_{51} + v_2 b_{52}).$$

For a linear form $y \in B_5^\vee$, the point $([x], [y])$ belongs to the hypersurface Q when ω is decomposable modulo y , which we can express as the condition that $\omega \wedge \omega = 0$ modulo y , or equivalently $\omega \wedge \omega \wedge y = 0$. We deduce that an equation of Q is $v_1^2 Q_1 - v_1 v_2 Q_2 - v_2^2 Q_3 = 0$, with

$$\begin{aligned} Q_1 &= x_2 x_4 y_1 + x_3 x_5 y_2 + x_4 x_1 y_3 + x_5 x_2 y_4 + x_1 x_3 y_5, \\ Q_2 &= x_3 x_5 y_1 + x_4 x_1 y_2 + x_5 x_2 y_3 + x_1 x_3 y_4 + x_2 x_4 y_5, \\ Q_3 &= x_2 x_3 y_1 + x_3 x_4 y_2 + x_4 x_5 y_3 + x_5 x_1 y_4 + x_1 x_2 y_5. \end{aligned}$$

In particular the image of γ just a conic: this is our Göpel variety for this toy example. Note that the action of G_{16} on Θ_3 descends to the usual action of the icosahedral group on $\mathbb{P}^2 \simeq \mathbb{P}(\Theta_3)$.

4. RUBIK'S CUBES

For a more interesting case, consider three three-dimensional vector spaces A_1, A_2, A_3 , and the action of $\mathrm{SL}(A_1) \times \mathrm{SL}(A_2) \times \mathrm{SL}(A_3)$ on the 27-dimensional tensor product $A_1 \otimes A_2 \otimes A_3$. This was studied in [BH16] under the name of $3 \times 3 \times 3$ Rubik's cubes, in connection with genus one curves (over an arbitrary field of characteristic $\neq 2, 3$). Indeed, one of their main results is that there exists a canonical bijection between "non-degenerate" orbits in $A_1 \otimes A_2 \otimes A_3$, and isomorphism classes of genus one curves, decorated in a certain way [BH16, Theorem 3.1]. Over the complex numbers, the stability problem for these orbits was discussed in [Ng95]. We would like to add a chapter to this beautiful story.

4.1. A Cartan subspace and the Hesse pencil. The fact that $A_1 \otimes A_2 \otimes A_3$ has especially nice properties is related to the fact that it is a theta-representation, being a graded piece of the \mathbb{Z}_3 -grading

$$\mathfrak{e}_6 = \mathfrak{sl}(A_1) \times \mathfrak{sl}(A_2) \times \mathfrak{sl}(A_3) \oplus (A_1 \otimes A_2 \otimes A_3) \oplus (A_1 \otimes A_2 \otimes A_3)^\vee.$$

According to [Vin76], a Cartan subspace \mathfrak{c} has rank three and the associated little Weyl group $W_{\mathfrak{c}}$ is the complex reflection group denoted G_{25} in the Shephard-Todd classification. Its order is 648, and the degrees of a basis of generators of the invariant ring are 6, 9, 12. There are 12 reflection hyperplanes. Moreover, G_{25} is a degree three extension of the Hessian group, the automorphism group of the Hesse pencil of plane cubics with equation

$$\lambda(u^3 + v^3 + w^3) - \mu uvw = 0.$$

For the classical properties of the Hesse pencil we refer to [AD09].

Consider the three dimensional subspace \mathfrak{c} of $A_1 \otimes A_2 \otimes A_3$ generated by

$$\begin{aligned} h_1 &= a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3, \\ h_2 &= a_1 \otimes b_2 \otimes c_3 + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_2, \\ h_3 &= a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_1 \otimes c_3 + a_3 \otimes b_2 \otimes c_1, \end{aligned}$$

where $a_1, a_2, a_3 \in A_1$, $b_1, b_2, b_3 \in A_2$, $c_1, c_2, c_3 \in A_3$.

Lemma 4.1. *The subspace \mathfrak{c} is a Cartan subspace.*

Proof. We need to prove that h_1, h_2, h_3 are semisimple and commute. That they commute easily follows from the fact the Lie bracket of \mathfrak{e}_6 , restricted to $A_1 \otimes A_2 \otimes A_3$, must be a non-zero multiple of the morphism

$$\begin{aligned} \wedge^2(A_1 \otimes A_2 \otimes A_3) &\longrightarrow A_1^\vee \otimes A_2^\vee \otimes A_3^\vee, \\ (a_1 \otimes a_2 \otimes a_3) \wedge (a'_1 \otimes a'_2 \otimes a'_3) &\mapsto (a_1 \wedge a'_1) \otimes (a_2 \wedge a'_2) \otimes (a_3 \wedge a'_3), \end{aligned}$$

where we have chosen volume forms on A_1, A_2, A_3 in order to identify $\wedge^2 A_i$ with A_i^\vee . Now, the fact that h_1 and h_2 commute is a direct consequence of the combinatorial fact that any pure tensor in h_1 and any pure tensor in h_2 have a common factor.

Let us prove that h_1 is semisimple. Let $X_i = a_i \otimes b_i \otimes c_i$ and $Y_i = a_i^\vee \otimes b_i^\vee \otimes c_i^\vee$. Let also Z_1 be the element of $\mathfrak{sl}(A_1) \times \mathfrak{sl}(A_2) \times \mathfrak{sl}(A_3)$ acting diagonally on the basis $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ as $\text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$; and similarly for Z_2 and Z_3 , so that $Z_1 + Z_2 + Z_3 = 0$. We have commutation relations

$$\begin{aligned} [Z_i, X_i] &= 2X_i, & [Z_i, X_j] &= -X_j, & [X_i, Y_i] &= Z_i, \\ [Z_i, Y_i] &= -2Y_i, & [Z_i, Y_j] &= Y_j, & [X_i, X_j] &= Y_k, \end{aligned}$$

where $i \neq j$ and $(i, j, k) = (1, 2, 3)$ up to cyclic order. We deduce that the elements X_i, Y_j, Z_k of \mathfrak{e}_6 generate a Lie subalgebra isomorphic with \mathfrak{sl}_3 , with an explicit isomorphism sending X_i to $e_j^\vee \otimes e_k \in \mathfrak{sl}_3$ and Y_i to $e_k^\vee \otimes e_j \in \mathfrak{sl}_3$. In particular

$$h_1 = X_1 + X_2 + X_3 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

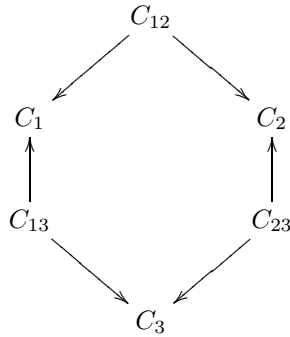
is semisimple, and of course h_2 and h_3 as well. □

Consider the centralizer $Z^+(\mathfrak{c})$ of \mathfrak{c} in $\text{SL}(A_1) \times \text{SL}(A_2) \times \text{SL}(A_3)$. We claim this is a group of order 243. Indeed, let τ be the diagonal element of SL_3 with eigenvalues $1, \zeta, \zeta^2$ for some fixed primitive cubic root of unity ζ . Let σ be the transformation that sends a_i, b_i, c_i to $a_{i+1}, b_{i+1}, c_{i+1}$, where i is considered modulo three. Then $Z^+(\mathfrak{c})$ is generated by σ and the diagonal transformations of type $(\tau, \tau, \tau), (\text{id}, \tau, \tau^2)$, up to permutation of the order, and $(\zeta_1 \text{id}, \zeta_2 \text{id}, \zeta_3 \text{id})$ for $\zeta_1, \zeta_2, \zeta_3$ cubic roots of unity such that $\zeta_1 \zeta_2 \zeta_3 = 1$. The latter transformations act trivially on $A_1 \otimes A_2 \otimes A_3$ and we deduce an exact sequence

$$1 \longrightarrow \mathbb{Z}_3^2 \longrightarrow Z^+(\mathfrak{c}) \longrightarrow Z(\mathfrak{c}) \longrightarrow 1.$$

Here $Z(\mathfrak{c})$ is the image of $Z^+(\mathfrak{c})$ inside $\text{SL}(A_1 \otimes A_2 \otimes A_3)$; this is an abelian group of order 27, while $Z^+(\mathfrak{c})$ is a (non-abelian) Heisenberg group (see 10).

4.2. Six genus one curves. That a generic tensor $t \in A_1 \otimes A_2 \otimes A_3$ defines a genus one curve is quite straightforward: we can see t as a morphism from A_3^\vee to $A_1 \otimes A_2$; when it is injective, its image is a three-dimensional space of linear forms on $A_1^\vee \otimes A_2^\vee$, that cut out $\mathbb{P}(A_1^\vee) \times \mathbb{P}(A_2^\vee) \simeq \mathbb{P}^2 \times \mathbb{P}^2$, generically, along a smooth curve C_{12} with trivial canonical bundle. Alternatively, we get over $\mathbb{P}(A_3^\vee)$ a morphism from $A_1^\vee \otimes \mathcal{O}(-1)$ to $A_2 \otimes \mathcal{O}$ that degenerates along a cubic curve C_3 . This gives six different ways to define the same abstract genus one curve C ; we denote these copies of C by $C_i \subset \mathbb{P}(A_i^\vee)$ for $1 \leq i \leq 3$ and $C_{ij} \subset \mathbb{P}(A_i^\vee) \times \mathbb{P}(A_j^\vee)$ for $1 \leq i < j \leq 3$. But we also get natural morphisms between these six models of C : typically, C_1 is defined by the condition that $t(x, \bullet, \bullet)$ is degenerate; while C_{12} is defined by the condition that $t(x, y, \bullet) = 0$, which of course implies that x belongs to C_1 and y to C_2 . Hence a diagram



where all the arrows are isomorphisms.

Now suppose that $t = uh_1 + vh_2 + wh_3$ belongs to \mathfrak{c} . Let us set coordinates (x_1, x_2, x_3) on A_1 , as well as (y_1, y_2, y_3) on A_2 and (z_1, z_2, z_3) on A_3 . Since the elliptic curve C_1 in $\mathbb{P}(A_1^\vee)$ is defined by the condition

that $t(x, \bullet, \bullet)$ is degenerate, its equation is

$$0 = \det \begin{pmatrix} ux_1 & vx_2 & wx_3 \\ wx_2 & ux_3 & vx_1 \\ vx_3 & wx_1 & ux_2 \end{pmatrix} = x_1 x_2 x_3 (u^3 + v^3 + w^3) - (x_1^3 + x_2^3 + x_3^3) u v w.$$

So we directly recover the Hesse pencil!

Note that C_1 has 6 marked points, which are typically distinct, namely $[u, v, w]$ and its permutations, plus of course the 9 base points of the pencil. The equations of C_2 and C_3 are exactly the same, once we replace (x_1, x_2, x_3) by (y_1, y_2, y_3) and (z_1, z_2, z_3) , respectively.

The correspondence p_{12} from C_1 to C_2 is defined by associating to $[x]$ such that $t(x, \bullet, \bullet)$ is degenerate, the unique $[y]$ such that $t(x, y, \bullet) = 0$.

Lemma 4.2. p_{12} is the translation of the curve C by

$$\tau = [u, v, w] - [0, 1, -1] = [v, w, u] - [1, -1, 0] = [w, u, v] - [1, 0, -1].$$

Proof. A straightforward computation shows that

$$y = (u^2 x_2 x_3 - v w x_1^2, v^2 x_1 x_3 - u w x_2^2, w^2 x_1 x_2 - u v x_3^2).$$

This is the restriction to C_1 of a standard plane Cremona transformation, with three base points that belong to the curve, namely $[u, \epsilon v, \epsilon^2 w]$ for $\epsilon^3 = 1$. One can check that

$$\det \begin{pmatrix} u & v & w \\ x_1 & x_2 & x_3 \\ y_1 & y_3 & y_2 \end{pmatrix} = 0,$$

which means that the correspondence p_{12} is obtained by composing the symmetry of C_1 with respect to the point $[u, v, w]$, chosen as the origin of the group law, with the exchange of the last two coordinates. Note that the latter involution is also a symmetry, with respect to $[0, 1, -1]$, one of the base points of the Hesse pencil. So p_{12} , being the composition of two symmetries, is finally a translation. \square

The correspondences p_{23} and p_{31} are given exactly by the same formulas. Their inverses p_{32} and p_{13} , as well as p_{21} are obtained by exchanging for example v and w . Note in particular that the composition $p_{31} \circ p_{23} \circ p_{12}$ is not the identity, as one could have naively expected, but a translation by 3τ , as already noticed by Cayley. See [AD09, Sect. 5] for a modern treatment.

4.3. A quadric hypersurface of Coble type. We will be interested in a different geometric construction, based on Cayley's hyperdeterminant [GKZ08, Chapter 14]. This is a quartic polynomial in eight variables, that provides an equation of the dual hypersurface to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$. In particular, it provides an equivariant map

$$\text{Ca} : S^4(U_1 \otimes U_2 \otimes U_3) \longrightarrow \det(U_1)^2 \otimes \det(U_2)^2 \otimes \det(U_3)^2,$$

if the three vector spaces U_1, U_2, U_3 are two-dimensional.

In order to be able to use Cayley's hyperdeterminant in our three-dimensional setting, we pass to a relative model. Denote by Q_1, Q_2, Q_3 the rank two quotient vector bundles, respectively on $\mathbb{P}(A_1), \mathbb{P}(A_2), \mathbb{P}(A_3)$. The tensor $t \in A_1 \otimes A_2 \otimes A_3$ defines a global section of the exterior product $Q_1 \boxtimes Q_2 \boxtimes Q_3$. Each fiber of this bundle is a copy of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, whose $GL_2 \times GL_2 \times GL_2$ -orbits are well-known: apart from the trivial orbit closures, one has completely decomposable tensors (the cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$), partially decomposable tensors (three copies of $\mathbb{P}^1 \times \mathbb{P}^3$), and then a quartic hypersurface whose equation is Cayley's hyperdeterminant. Correspondingly, the locus in $\mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$ where the image of v in $Q_1 \boxtimes Q_2 \boxtimes Q_3$ becomes completely decomposable, is in general a smooth surface Σ . The locus where this image falls inside the invariant quartic is a hypersurface \mathfrak{H} of multidegree $(2, 2, 2)$, given by the Cayley hyperdeterminant seen as a vector bundle morphism

$$S^4(Q_1 \boxtimes Q_2 \boxtimes Q_3) \longrightarrow \det(Q_1)^2 \boxtimes \det(Q_2)^2 \boxtimes \det(Q_3)^2 = \mathcal{O}(2, 2, 2).$$

In other words, \mathfrak{H} is a quadric section of the triple Segre product $\mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$, embedded in \mathbb{P}^{26} . This hypersurface \mathfrak{H} and the surface Σ are instances of what we call *orbital degeneracy loci*, about which we refer to [BFMT20a, BFMT20b].

Proposition 4.3. *The locus Σ is an abelian surface. More precisely Σ embeds into $\mathbb{P}(A_1^\vee) \times \mathbb{P}(A_2^\vee) \times \mathbb{P}(A_3^\vee)$ as the surface of colinear triples in $C_1 \times C_2 \times C_3$. In particular, $\Sigma \simeq C \times C$.*

Proof. By definition, over a point (L_1, L_2, L_3) of Σ , there exist vectors q_1, q_2, q_3 such that

$$t = q_1 \otimes q_2 \otimes q_3 \quad \text{mod} \quad L_1 \otimes V_2 \otimes V_3 + V_1 \otimes L_2 \otimes V_3 + V_1 \otimes V_2 \otimes L_3.$$

Note that for t generic, q_i cannot belong to L_i , as follows from a dimension count. If we contract t by a linear form ϕ_1 that vanishes on q_1 and L_1 , we get a tensor in $L_2 \otimes V_3 + V_1 \otimes V_2 \otimes L_3 \subset V_2 \otimes V_3$, which cannot be of maximal rank. This exactly means that $[\phi_1]$ belongs to $C_1 \subset \mathbb{P}(A_1^\vee)$. In fact, another dimension count shows that this tensor cannot be of rank one, so that we can decompose it as

$$t(\phi_1, \bullet, \bullet) = \ell_2 \otimes a_{13} + a_{12} \otimes \ell_3$$

for some generators ℓ_2 of L_2 and ℓ_3 of L_3 , and some vectors $a_{13} \in V_3$ and $a_{12} \in V_2$. Similarly we can find $\phi_2 \in \langle q_2, L_2 \rangle$ and $\phi_3 \in \langle q_3, L_3 \rangle$ so that we can decompose

$$t(\bullet, \phi_2, \bullet) = \ell_1 \otimes a_{23} + a_{21} \otimes \ell_3, \quad t(\bullet, \bullet, \phi_3) = \ell_1 \otimes a_{32} + a_{31} \otimes \ell_2.$$

Let $\psi_{ij} \in A_j^\vee$ be a nonzero linear form, orthogonal to ℓ_j and a_{ij} . Then

$$t(\phi_1, \psi_{12}, \bullet) = t(\phi_1, \bullet, \psi_{13}) = t(\psi_{21}, \phi_2, \bullet) = t(\bullet, \phi_2, \psi_{23}) = t(\psi_{31}, \bullet, \phi_3) = t(\bullet, \psi_{32}, \phi_3) = 0.$$

In particular ψ_{ij} defines a point in C_j , and by Lemma 4.2 and the remark following its proof we have in $Jac^3(C)$ (with some abuse of notations) the relations

$$\tau = \psi_{12} - \phi_1 = \phi_1 - \psi_{13} = \phi_2 - \psi_{21} = \psi_{23} - \phi_2 = \psi_{31} - \phi_3 = \phi_3 - \psi_{32}.$$

Since ℓ_1 is orthogonal to ϕ_1, ψ_{21} and ψ_{31} , the latter define three colinear points in $\mathbb{P}(A_1)$, so their sum in $Jac^3(C)$ is the hyperplane class h_1 of C_1 . We thus get the relations

$$h_1 = \phi_1 + \psi_{21} + \psi_{31}, \quad h_2 = \phi_2 + \psi_{12} + \psi_{32}, \quad h_3 = \phi_3 + \psi_{13} + \psi_{23},$$

where h_i is the hyperplane class of C_i . Summing them up and using the previous relations, we get

$$h_1 + h_2 + h_3 = 3(\phi_1 + \phi_2 + \phi_3).$$

Since the left hand side is constant, we deduce that $\phi_1 + \phi_2 + \phi_3$ is also constant in $Jac^3(C)$, which is exactly our claim. \square

Remark. The hypersurface \mathfrak{H} is, by construction, a quadratic section of $\mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$ that is singular along the abelian surface Σ . We expect \mathfrak{H} to be uniquely characterized by this property, although its singular locus should be strictly bigger.

4.4. The Göpel variety. When t belongs to \mathfrak{c} , we can compute explicitly the equation of \mathfrak{H} . Indeed, on a open subset of $\mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \mathbb{P}(A_3)$, the quotient bundles Q_1, Q_2, Q_3 are obtained by modding out by relations of type $a_3 = x_1 a_1 + x_2 a_2$, $b_3 = y_1 b_1 + y_2 b_2$, $c_3 = z_1 a_1 + z_2 a_2$, which allows to express the section of $Q_1 \otimes Q_2 \otimes Q_3$ defined by t as

$$\begin{aligned} \bar{t} = & u(1 + x_1 y_1 z_1) a_1 \otimes b_1 \otimes c_1 + (u x_1 y_1 z_2 + v y_1 + w x_1) a_1 \otimes b_1 \otimes c_2 + \\ & + (u x_1 y_2 z_1 + v x_1 + w z_1) a_1 \otimes b_2 \otimes c_1 + (u x_2 y_1 z_1 + v z_1 + w y_1) a_2 \otimes b_1 \otimes c_1 + \\ & + (u x_1 y_2 z_2 + v y_2 + w z_2) a_1 \otimes b_2 \otimes c_2 + (u x_2 y_1 z_2 + v z_2 + w x_2) a_2 \otimes b_1 \otimes c_2 + \\ & + (u x_2 y_2 z_1 + v x_2 + w y_2) a_2 \otimes b_2 \otimes c_1 + u(1 + x_2 y_2 z_2) a_2 \otimes b_2 \otimes c_2. \end{aligned}$$

Now we need to take the hyperdeterminant of this expression. Recall from [GKZ08, Chapter 14, Proposition 1.7] that for a $2 \times 2 \times 2$ -tensor $s = \sum_{i,j,k=1,2} s_{ijk} a_i \otimes b_j \otimes c_k$, the hyperdeterminant is

$$Det(s) = \sum s_{111}^2 s_{222}^2 - 2 \sum s_{111} s_{222} s_{112} s_{221} + 4 \sum s_{111} s_{122} s_{212} s_{221},$$

where each sum is obtained by permuting 1 and 2 on the first, second and third indices (so that the first sum has 4 terms, the second one has 6, the last one has 2). After plugging in the coefficients of \bar{t} , and homogeneizing, we deduce the following expression:

$$Q(u, v, w) = \sum_{p+q+r=4} u^p v^q w^r Q_{p,q,r},$$

$$\begin{aligned}
Q_{4,0,0} &= x_1^2 y_1^2 z_1^2 + x_2^2 y_2^2 z_2^2 + x_3^2 y_3^2 z_3^2 - 2x_1 x_2 y_1 y_2 z_1 z_2 - 2x_1 x_3 y_1 y_3 z_1 z_3 - 2x_2 x_3 y_2 y_3 z_2 z_3, \\
Q_{0,4,0} &= x_1^2 y_3^2 z_2^2 + x_2^2 y_1^2 z_3^2 + x_3^2 y_2^2 z_1^2 - 2x_1 x_2 y_1 y_3 z_2 z_3 - 2x_1 x_3 y_2 y_3 z_1 z_2 - 2x_2 x_3 y_1 y_2 z_1 z_3, \\
Q_{0,0,4} &= x_1^2 y_2^2 z_3^2 + x_2^2 y_3^2 z_1^2 + x_3^2 y_1^2 z_2^2 - 2x_1 x_2 y_2 y_3 z_1 z_3 - 2x_1 x_3 y_1 y_2 z_2 z_3 - 2x_2 x_3 y_1 y_3 z_1 z_2, \\
-\frac{1}{4}Q_{3,1,0} &= -\frac{1}{4}Q_{0,1,3} = x_2 x_3 y_1 y_2 z_1 z_3 + x_1 x_2 y_1 y_3 z_2 z_3 + x_1 x_3 y_2 y_3 z_1 z_2, \\
-\frac{1}{4}Q_{3,0,1} &= -\frac{1}{4}Q_{0,3,1} = x_1 x_2 y_2 y_3 z_1 z_3 + x_1 x_3 y_1 y_2 z_2 z_3 + x_2 x_3 y_1 y_3 z_1 z_2, \\
-\frac{1}{4}Q_{1,3,0} &= -\frac{1}{4}Q_{1,0,3} = x_1 x_2 y_1 y_2 z_1 z_2 + x_1 x_3 y_1 y_3 z_1 z_3 + x_2 x_3 y_2 y_3 z_2 z_3, \\
-\frac{1}{2}Q_{0,2,2} &= x_1 x_2 y_1 y_2 z_3^2 + x_2^2 y_1 y_3 z_1 z_3 + x_1 x_3 y_2^2 z_1 z_3 + x_2 x_3 y_2 y_3 z_1^2 + \\
&\quad + x_2 x_3 y_1^2 z_2 z_3 + x_1^2 y_2 y_3 z_2 z_3 + x_1 x_2 y_3^2 z_1 z_2 + x_3^2 y_1 y_2 z_1 z_2 + x_1 x_3 y_1 y_3 z_2^2, \\
-\frac{1}{2}Q_{2,0,2} &= x_1^2 y_1 y_2 z_1 z_3 + x_1 x_2 y_1 y_3 z_1^2 + x_1 x_2 y_2^2 z_2 z_3 + x_1 x_3 y_1^2 z_1 z_2 + \\
&\quad + x_2^2 y_2 y_3 z_1 z_2 + x_2 x_3 y_1 y_2 z_2^2 + x_1 x_3 y_2 y_3 z_3^2 + x_2 x_3 y_3^2 z_1 z_3 + x_3^2 y_1 y_3 z_2 z_3, \\
-\frac{1}{2}Q_{2,2,0} &= x_1 x_2 y_1^2 z_1 z_3 + x_1 x_3 y_1 y_2 z_1^2 + x_2^2 y_1 y_2 z_2 z_3 + x_1^2 y_1 y_3 z_1 z_2 + \\
&\quad + x_2 x_3 y_2^2 z_1 z_2 + x_1 x_2 y_2 y_3 z_2^2 + x_2 x_3 y_1 y_3 z_3^2 + x_3^2 y_2 y_3 z_1 z_3 + x_1 x_3 y_3^2 z_2 z_3, \\
-\frac{1}{4}Q_{2,1,1} &= x_2^2 y_2^2 z_1 z_3 + x_2 x_3 y_1^2 z_1^2 + x_1^2 y_2 y_3 z_1^2 + x_1^2 y_1^2 z_2 z_3 + \\
&\quad + x_2^2 y_1 y_3 z_2^2 + x_1 x_3 y_2^2 z_2^2 + x_1 x_2 y_3^2 z_3^2 + x_3^2 y_1 y_2 z_3^2 + x_3^2 y_3^2 z_1 z_2, \\
-\frac{1}{4}Q_{1,2,1} &= x_1 x_2 y_2^2 z_1^2 + x_2^2 y_1^2 z_1 z_2 + x_1^2 y_1 y_2 z_2^2 + x_1 x_3 y_1^2 z_3^2 + \\
&\quad + x_2^2 y_2 y_3 z_3^2 + x_1^2 y_3^2 z_1 z_3 + x_3^2 y_1 y_3 z_1^2 + x_3^2 y_2^2 z_2 z_3 + x_2 x_3 y_3^2 z_2^2, \\
-\frac{1}{4}Q_{1,1,2} &= x_2^2 y_1 y_2 z_1^2 + x_1^2 y_2^2 z_1 z_2 + x_1 x_2 y_1^2 z_2^2 + x_1^2 y_1 y_3 z_3^2 + \\
&\quad + x_2 x_3 y_2^2 z_3^2 + x_3^2 y_1^2 z_1 z_3 + x_1 x_3 y_3^2 z_1^2 + x_2^2 y_3^2 z_2 z_3 + x_3^2 y_2 y_3 z_2^2.
\end{aligned}$$

In particular we get independent polynomials, up to the three relations

$$(1) \quad Q_{3,1,0} = Q_{0,1,3}, \quad Q_{0,3,1} = Q_{3,0,1}, \quad Q_{1,3,0} = Q_{1,0,3}.$$

We deduce:

Proposition 4.4. *The span of the Göpel variety is a copy of $\mathbb{P}^{11} = \mathbb{P}(\Theta_{12})$, where Θ_{12} is the Macdonald representation of the Hessian group G_{25} generated by the fourth powers of the 12 lines in the Hessian arrangement. The Göpel variety is the image of the bijective projection of $v_4(\mathbb{P}^2)$ from the plane*

$$\Pi = \mathbb{P}\langle u(v^3 - w^3), v(w^3 - u^3), w(u^3 - v^3) \rangle.$$

Its non-normal locus consists in three pinch points. Its ideal in \mathbb{P}^{11} is generated by 36 quadrics, admitting 165 syzygies (156 linear and 9 quadratic syzygies).

Proof. The fact that the Göpel variety is the image of the projection of $v_4(\mathbb{P}_2)$ from Π is just a reformulation of equations (1). This projection can be explicitly written as the morphism

$$[u, v, w] \mapsto [u^4, v^4, w^4, u^3(v+w), v^3(u+w), w^3(u+v), u^2v^2, u^2w^2, v^2w^2, u^2vw, uv^2w, uvw^2],$$

whose image is stable under the action W_c . Indeed, the 12 lines of the Hessian arrangement are the three axis in \mathbb{P}^2 plus the nine lines of equations $\ell = \xi u + \zeta v + \eta w = 0$ for $\xi^3 = \zeta^3 = \eta^3 = \xi\eta\zeta$. These equations ensure that in ℓ^4 , u^3v and u^3w have the same coefficient, and the claim easily follows.

The Jacobian matrix of the above parametrization fails to have maximal rank at the three coordinate points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1) \in \mathbb{P}^2$, whose images in \mathbb{P}^{11} are:

$$\begin{aligned}
p_1 &= (1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^{11}, \\
p_2 &= (0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^{11}, \\
p_3 &= (0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^{11}.
\end{aligned}$$

These points are singular for the Göpel variety, as one can prove by noting that the Jacobian matrix of a system of generators of the ideal of the Göpel variety in \mathbb{P}^{11} has rank 7 at p_1 , p_2 and p_3 . Using the

explicit equations of secants varieties of Veronese varieties available from [Kan99, LO13], we checked with [GS] that the center of projection Π cuts the secant variety of $v_4(\mathbb{P}^2)$ at the three points

$$\begin{aligned} q_1 &= (0 : 1 : -1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \in \mathbb{P}^{14}, \\ q_2 &= (0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : -1 : 0 : 0) \in \mathbb{P}^{14}, \\ q_3 &= (0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : -1) \in \mathbb{P}^{14}, \end{aligned}$$

where we use coordinates indexed by a basis of monomials of degree 4 in u, v, w in the RevLex monomial order. So the Göpel variety has precisely three singular points, which must then be the pinch points p_1, p_2, p_3 , each of these points having a unique preimage in \mathbb{P}^{14} . This last sentence is justified, since q_1, q_2 and q_3 actually lie on the tangential variety of $v_4(\mathbb{P}^2)$. In conclusion, the restriction of π to $v_4(\mathbb{P}^2)$ is bijective onto its image, which is the Göpel variety. Since the latter is singular, we deduce that it is not normal, for otherwise it would be isomorphic to the smooth variety $v_4(\mathbb{P}^2)$ by Zariski's main theorem.

The last claim is the result of a computation with [GS] (see attached file `gopel_rubik`). \square

Remark. For a generic projection, according to [GS] the ideal of the image should be generated by 33 quadrics admitting 123 linear syzygies (see attached file `gopel_rubik`).

5. HYPERELLIPTIC CURVES AND PENCILS OF QUADRICS

The relationship between hyperelliptic curves and pencils of quadrics is very classical [New68, NR69, DR77]. In this section we observe that this has interesting connections with Vinberg's theory, and we reconsider the situation from our point of view of Coble type hypersurfaces. Our starting point is the classical \mathbb{Z}_2 -grading

$$\mathfrak{sl}_m = \mathfrak{so}_m \oplus S^{(2)}\mathbb{C}^m,$$

where $S^{(2)}\mathbb{C}^m \subset S^2\mathbb{C}^m$ is the space of traceless symmetric matrices. As usual \mathfrak{so}_m is seen as the subalgebra of skew-symmetric matrices in \mathfrak{sl}_m , which means that it kills the quadric Q_1 whose matrix is the identity. A Cartan subspace \mathfrak{c} is then simply the space of diagonal (hence symmetric), traceless matrices, which is also the standard Cartan subalgebra of \mathfrak{sl}_m ; when $m = 2g + 2$, the (Heisenberg) group centralizing \mathfrak{c} is an extension of \mathbb{Z}_2^g by \mathbb{Z}_4 when $g + 1$ is odd and $\mathbb{Z}_2 \times \mathbb{Z}_2$ when $g + 1$ is even (see Appendix 10).

5.1. Coble quadrics in orthogonal Grassmannians. Given a pencil of quadrics $\langle Q_1, Q_2 \rangle$ in \mathbb{P}^{2n-1} , such that Q_1 is non-degenerate, the variety of planes which are isotropic with respect to both Q_1 and Q_2 is a subvariety of the copy of the isotropic Grassmannian $OG(2, 2n)$ defined by Q_1 , described as the zero-locus of the global section of S^2U^\vee defined by an equation of Q_2 , where U denotes the tautological subbundle. If Q_2 is general enough, the latter is the singular locus of the hypersurface in $OG(2, 2n)$ consisting in planes on which the restriction of Q_2 is degenerate. This hypersurface is also defined by the condition that the projection of $S^2(S^2U^\vee)$ to $S_{22}U^\vee = \mathcal{O}(2)$ vanishes. So we get a quadratic covariant in the space of quadratic sections of $OG(2, 2n)$.

The condition for the restriction to be degenerate on a plane $V = \langle e, f \rangle$ is simply

$$Q_2(e, e)Q_2(f, f) - Q_2(e, f)^2 = 0,$$

that we need to express in terms of the Plücker coordinates (p_{ij}) of V , with respect to some fixed basis. For this we may suppose that in the chosen coordinates,

$$Q_1 = x_1^2 + \cdots + x_{2n}^2, \quad Q_2 = \alpha_1 x_1^2 + \cdots + \alpha_{2n} x_{2n}^2,$$

with $\alpha_1, \dots, \alpha_{2n}$ pairwise distinct. The condition then becomes

$$\sum_{i,j} \alpha_i \alpha_j (e_i f_j - e_j f_i)^2 = \sum_{i,j} \alpha_i \alpha_j p_{i,j}^2 = 0.$$

This readily extends to any isotropic Grassmannian $OG(k, m)$, in which the locus of k -planes on which Q_2 degenerates is the quadratic section of equation $C(\alpha) = 0$, where

$$(2) \quad C(\alpha) := \sum_{i_1 < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k} p_{i_1, \dots, i_k}^2.$$

5.2. Square-free monomials. Because of the form of this equation, it will be useful to understand the representation of the symmetric group S_{2n} on the space $SF_{2n,n}$ of homogeneous polynomials of degree n spanned by square-free monomials. More generally, denote by $SF_{m,k}$ the representation of the symmetric group S_m on the space of homogeneous polynomials of degree k spanned by square-free monomials in m variables. Its decomposition into irreducible components is as follows, where we use the traditional notation $[\lambda]$ for the irreducible representation of S_m defined by the partition λ of m [FH91].

Proposition 5.1. *For any $m \geq 2k$, $SF_{m,k}$ is multiplicity free and decomposes as*

$$SF_{m,k} \simeq \bigoplus_{p=0}^k [m-p, p].$$

Proof. First observe that the hook-length formula gives

$$\dim[m-p, p] = \frac{(m-2p+1)m!}{p!(m-p+1)!} = \binom{m+1}{p} - 2\binom{m}{p-1}.$$

We deduce the following sum of dimensions:

$$\sum_{p=0}^k \dim[m-p, p] = \sum_{p=0}^k \left(\binom{m}{p} + \binom{m}{p-1} \right) - 2 \sum_{p=0}^k \binom{m}{p-1} = \binom{m}{k} = \dim SF_{m,k}.$$

So it suffices to check that each representation $[m-p, p]$, for $p \leq k$, embeds inside $SF_{m,k}$. For this we need to recall the description of the irreducible representations of S_m as Specht modules. Typically, to each tableau T of shape $(m-p, p)$, numbered by the integers $1, \dots, m$, we associate the polynomial

$$(3) \quad P_T(x) = (\alpha_{a_1} - \alpha_{b_1}) \cdots (\alpha_{a_p} - \alpha_{b_p}),$$

where a_1, \dots, a_{m-p} are the integers appearing on the first row of T , from left to right, and b_1, \dots, b_p are the integers appearing on the second row. Then the span of the polynomials P_T , for T of shape $(m-p, p)$, inside the space of homogeneous polynomials of degree p , is a copy of the irreducible representation $[m-p, p]$. Of course these polynomials are not linearly independent, but their relations are well-understood. It is proved in [Ful97] that the module of relations is generated by the following ones, where T is any tableau of shape $(m-p, p)$:

- (*skew-symmetry relations*) if S is obtained from T by exchanging two numbers in the same column, then $P_S + P_T = 0$;
- (*Plücker type relations*) given two consecutive columns of length two, let R and S be obtained by fixing the top leftmost entry and permuting cyclically the three other entries, then $P_R + P_S + P_T = 0$;
- (*linear type relations*) given two consecutive columns of length two and one, let R, S be obtained by permuting cyclically their three entries, then again $P_R + P_S + P_T = 0$.

In the latter case, if a, b, c are the three entries, the relation is obtained from the obvious linear relation

$$(\alpha_a - \alpha_b) + (\alpha_b - \alpha_c) + (\alpha_c - \alpha_a) = 0,$$

multiplied by the contribution of the other columns. In the previous case, if a, b, c, d are the four entries, a being fixed, the relation is obtained from the quadratic Plücker relation

$$(\alpha_a - \alpha_b)(\alpha_c - \alpha_d) + (\alpha_a - \alpha_c)(\alpha_d - \alpha_b) + (\alpha_a - \alpha_d)(\alpha_b - \alpha_c) = 0,$$

multiplied again by the contribution of the other columns.

We define a nonzero equivariant map from $[m-p, p]$ to $SF_{m,k}$ as follows. For each tableau T of shape $(m-p, p)$, we let

$$Q_T(\alpha) = P_T(\alpha) \times e_{k-p}(\alpha_{c_1}, \dots, \alpha_{c_{m-2p}}),$$

where $\{c_1, \dots, c_{m-2p}\}$ is the complement of $\{a_1, \dots, a_{m-p}, b_1, \dots, b_p\}$. As usual the elementary symmetric function $e_{k-p}(\alpha_{c_1}, \dots, \alpha_{c_{m-2p}})$ is the sum of all square-free monomials of degree $k-p$ in these variables, so that Q_T belongs to $SF_{m,k}$. By mapping P_T to Q_T , we claim that we define a non trivial equivariant map from $[m-p, p]$ to $SF_{m,k}$. Indeed, this will be clear if we can prove that all the linear relations between the P_T 's are also verified by the Q_T 's. This is obvious for the skew-symmetry relations. This is also clear for the Plücker type relations $P_R + P_S + P_T = 0$, since Q_R, Q_S, Q_T are obtained by multiplying P_R, P_S, P_T by the same polynomial.

The case of linear type relations is slightly different. In this case $P_T = A(\alpha_a - \alpha_b)$ for A defined by the first $p - 1$ columns of T , and then $P_R = A(\alpha_b - \alpha_c)$, $P_S = (\alpha_c - \alpha_a)A$. To obtain Q_R , one needs to multiply P_R by an elementary symmetric function of degree $k - p$ in $m - 2p$ variables including α_c , and we can decompose this symmetric function as $\alpha_c e_{k-p-1} + e_{k-p}$ where e_{k-p-1}, e_{k-p} involve the remaining variables. Similarly, we obtain Q_S by multiplying P_S by $\alpha_a e_{k-p-1} + e_{k-p}$ and P_T by $\alpha_b e_{k-p-1} + e_{k-p}$. Finally, the relation $Q_R + Q_S + Q_T = 0$ follows from the identity

$$(\alpha_a - \alpha_b)\alpha_c + (\alpha_b - \alpha_c)\alpha_a + (\alpha_c - \alpha_a)\alpha_b = 0,$$

and the proof is complete. \square

Now consider the equation $C(\alpha)$ as an equivariant morphism

$$(4) \quad \rho_{m,k} : SF_{m,k} \rightarrow H^0(OG(k, m), \mathcal{O}(2)),$$

sending α_I to p_I^2 , for each k -tuple of integers between 1 and m .

Proposition 5.2. *The image of $\rho_{m,k}$ is the irreducible representation $[m - k, k]$.*

Proof. There is a natural injective equivariant map $\sigma_{m,k-1}$ from $SF_{m,k-1}$ to $SF_{m,k}$, sending the monomial α_J , for each $(k - 1)$ -tuple J of integers between 1 and m , to its product by $\sum_{a \notin J} \alpha_a$. The previous Proposition implies that the cokernel is the irreducible representation $[m - k, k]$. Therefore it suffices to prove that $\rho_{m,k}$ vanishes on the image of $\sigma_{m,k-1}$. In other words, we need to check that for each $(k - 1)$ -tuple J ,

$$\sum_{a \notin J} p_{a,J}^2 = 0.$$

The quadratic relations between the Plücker coordinates on $OG(k, m)$ are given by the Plücker relations for the ordinary Grassmannian $G(k, m)$, plus the relations induced by the isotropy conditions. Note that a space U with Plücker coordinates p_I 's is spanned by vectors obtained by contraction of the tensor $\sum_I p_I e_I$, hence of the form $x = \sum_J x_J \sum_a p_{a,J} e_a$ (up to some sign ambiguities that will soon disappear). The basis of e_a 's being Q_1 -orthonormal, the isotropy of U translates to the condition that

$$\sum_a p_{a,J} p_{a,K} = 0$$

for any $(k - 1)$ -tuples J and K . In particular for $K = J$ we obtain the expected relations (independently of the sign ambiguities!), and the proof is complete. \square

5.3. The hyperelliptic Göpel variety. Denote by $\mathfrak{h} \simeq \mathbb{A}^m$ the Cartan algebra of diagonal matrices, and let $\mathfrak{c} \subset \mathfrak{h}$ be the Cartan subspace of traceless diagonal matrices. From the preceding section, we can define a natural rational map

$$(5) \quad C_{m,k} : \mathfrak{h} \dashrightarrow \mathbb{P}([m - k, k]),$$

by the degree k polynomials P_T from (3).

Lemma 5.3. *The base locus of $C_{m,k}$ is the union of $\binom{m}{k-1}$ linear spaces of dimension k , defined by the condition that at least $m - k + 1$ coordinates are equal.*

Proof. Easy and left to the reader. \square

Lemma 5.4. *$C_{m,k}$ factors through the projection to \mathfrak{c} from the identity matrix.*

Proof. Obvious since $P_T(\alpha)$ only depends on the differences $\alpha_i - \alpha_j$. \square

Now consider the action of PGL_2 on \mathfrak{h} defined by the projective collineations

$$(6) \quad \alpha_i \mapsto \frac{a\alpha_i + b}{c\alpha_i + d}, \quad \text{for } ad - bc \neq 0.$$

Lemma 5.5. *$C_{m,k}$ is $PGL(2)$ -invariant if and only if $m = 2k$.*

Proof. This immediately follows from the relation:

$$\frac{a\alpha_i + b}{c\alpha_i + d} - \frac{a\alpha_j + b}{c\alpha_j + d} = \frac{(ad - bc)(\alpha_i - \alpha_j)}{(c\alpha_i + d)(c\alpha_j + d)}.$$

Indeed, we see that for $m = 2k$, each polynomial P_T is multiplied by the same factor $(ad - bc)^k / \prod_p (c\alpha_p + d)$; while for $m \neq 2k$, the multiplication factor clearly depends on T . \square

Definition 5.6. *The hyperelliptic Göpel variety is the image of the map $C_{2k,k}$ in $\mathbb{P}([k, k])$.*

Proposition 5.7. *The hyperelliptic Göpel variety is the GIT quotient $(\mathbb{P}^1)^{2k} // PGL(2)$, with respect to the democratic polarization.*

Proof. This follows from the classical fact (Kempe's theorem) that the coordinate ring of $(\mathbb{P}^1)^{2k} // PGL(2)$ is generated by degree one invariants, which are parametrized by $[k, k]$ (see e.g. [HMSV09, section 2]). \square

As a consequence of [HMSV09, Theorem 1.1], the hyperelliptic Göpel variety $C_{2k,k}$ is cut out by quadrics for any $k > 3$.

In our interpretation, $C_{2k,k}$ sends a pencil of diagonal quadrics to a Coble type hypersurface, which is a quadratic section of $OG(k, 2k)$. But the latter is a union of two spinor varieties, on which $\mathcal{O}(2) = L^{\otimes 4}$ if L denotes the spinor line bundle, i.e. the positive bundle generating its Picard group. This means that our Coble hypersurfaces are in fact *quartic sections* of the spinor varieties. Let us discuss the first values of k in greater detail.

5.4. Genus two. The case $k = 3$ is special because of the exceptional isomorphism between Spin_6 and SL_4 . Recall that this follows for example from the fact that the six-dimensional representation $\wedge^2 \mathbb{C}^4$ of SL_4 admits an invariant quadric, which is just the cone over the Grassmannian $G(2, 4)$. This induces an identification

$$S^{(2)}\mathbb{C}^6 \simeq S_{22}\mathbb{C}^4.$$

Geometrically, this identifies a pencil of quadrics in \mathbb{P}^5 with a fixed smooth member, with a quadratic line complex in $G(2, 4)$. How should we interpret our standard Cartan subspace in $S^{(2)}\mathbb{C}^6$? Given a basis e_1, e_2, e_3, e_4 of \mathbb{C}^4 , an orthonormal basis of $\wedge^2 \mathbb{C}^4$ is given by the six tensors $e_{12} + e_{34}$, $i(e_{12} - e_{34})$, $e_{13} - e_{24}$, $i(e_{13} + e_{24})$, $e_{14} + e_{23}$, $i(e_{14} - e_{23})$, and the space of traceless diagonal quadrics is generated by

$$\begin{aligned} &(e_1 \wedge e_2)^2 + (e_3 \wedge e_4)^2, & (e_1 \wedge e_3)^2 + (e_2 \wedge e_4)^2, & (e_1 \wedge e_4)^2 + (e_2 \wedge e_3)^2, \\ &(e_1 \wedge e_2)(e_3 \wedge e_4), & (e_1 \wedge e_3)(e_2 \wedge e_4), & (e_1 \wedge e_4)(e_2 \wedge e_3), \end{aligned}$$

where the last three quadrics obey the usual Plücker relation. In other words:

Lemma 5.8. *The space of quadrics of the form*

$$\begin{aligned} v = & x((e_1 \wedge e_2)^2 + (e_3 \wedge e_4)^2) + y((e_1 \wedge e_3)^2 + (e_2 \wedge e_4)^2) + z((e_1 \wedge e_4)^2 + (e_2 \wedge e_3)^2) + \\ & + p(e_1 \wedge e_2)(e_3 \wedge e_4) - q(e_1 \wedge e_3)(e_2 \wedge e_4) + r(e_1 \wedge e_4)(e_2 \wedge e_3), \end{aligned}$$

with $p + q + r = 0$, is a Cartan subspace \mathfrak{c} of $S_{22}\mathbb{C}^4$.

The two half-spin modules of Spin_6 have dimension four, and can be identified with the natural representation of SL_4 and its dual. This means that the associated degeneracy loci will be hypersurfaces in \mathbb{P}^3 and $\check{\mathbb{P}}^3$, of degree two with respect to the spinor embedding, which means they are quartic surfaces. Moreover they have degree three with respect to v , hence they are given by some equivariant maps

$$\Gamma : S^3(S_{22}V_4) \longrightarrow S^4V_4 \otimes (\det V_4)^2, \quad \Gamma^\vee : S^3(S_{22}V_4) \longrightarrow S^4V_4^\vee \otimes (\det V_4)^4.$$

One can check that such equivariant morphisms are unique, up to scalar. We could then write them down in tensorial form, and restrict them to \mathfrak{c} to get degree three rational maps

$$\gamma : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(S^4V_4), \quad \gamma^\vee : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(S^4V_4^\vee).$$

In this case it is in fact quicker to compute directly the degeneracy loci of the two conic bundles defined in section 5.1. Let $u = \frac{p-q}{2}$, $t = \frac{q-r}{2}$, $w = \frac{r-p}{2}$, so that again $u + t + w = 0$, one obtains $\gamma(v)$ in terms of our coordinates (a, b, c, d) on V_4 , as

$$x(y^2 + z^2 - w^2)(a^2b^2 + c^2d^2) + y(x^2 + z^2 - t^2)(a^2c^2 + b^2d^2) + z(x^2 + y^2 - u^2)(a^2d^2 + b^2c^2) +$$

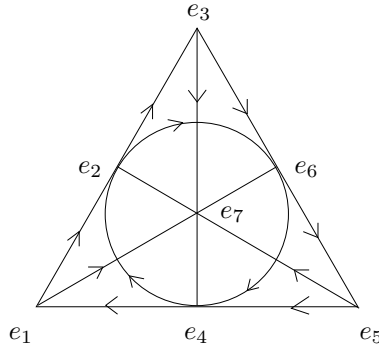
$$+xyz(a^4 + b^4 + c^4 + d^4) - 2(x^2w + y^2t + z^2u + utw)abcd.$$

So we recover classical formulas with five coefficients given by certain cubic forms in five variables. For this case the relevant representation of S_6 is $[3, 3]$, whose dimension is indeed 5. We get a rational map $\gamma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$, whose image is the Segre cubic [Dol12]. If we think of six points in \mathbb{P}^1 as defining a genus 2 curve, the map γ was expressed by the classics in terms of the θ -constants of the curve. We will come back to Kummer surfaces later on, in connection with spinors in ten dimensions.

5.5. Genus three. The case $k = 4$ is special because of triality, which exchanges the three eight-dimensional irreducible representations of Spin_8 . Correspondingly, the three closed orbits in their projectivizations, namely $OG(4, 8)_+$, $OG(4, 8)_-$ and the quadric \mathbb{Q}_6 , are projectively isomorphic. In particular, our special hypersurface in this case will be a quartic section of a six-dimensional quadric.

A nice way to make these isomorphisms concrete is to use the algebra \mathbb{O} of complexified octonions. If $v \in \mathbb{O}$ is a non-zero isotropic vector, then $R_v = \mathbb{O}v$ and $L_v = v\mathbb{O}$ are four dimensional isotropic spaces. Moreover $R_v \cap L_v = \mathbb{C}v$, which shows that R_v and L_v belong to different families of maximal isotropic spaces; by convention, we can decide that R_v defines a point of $OG(4, 8)_+$, and L_v a point of $OG(4, 8)_-$. Then the map $v \mapsto R_v$ (resp. $v \mapsto L_v$) defines an equivariant isomorphism between \mathbb{Q}^6 and $OG(4, 8)_+$ (resp. $OG(4, 8)_-$).

In order to be more specific, we may use the standard basis $e_0 = 1, e_1, \dots, e_7$ of the Cayley algebra, whose multiplication table is encoded in the Cayley plane, suitably oriented.



The multiplication table of the octonions

For a generic isotropic vector $v = v_0e_0 + v_1e_1 + \dots + v_7e_7$, with $|v|^2 = v_0^2 + v_1^2 + \dots + v_7^2 = 0$, R_v is generated by the four vectors v, e_1v, e_2v, e_3v . In particular, its Plücker coordinates p_{i_1, \dots, i_4} are the 70 maximal minors of the 4×8 matrix

$$\begin{pmatrix} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ -v_1 & v_0 & -v_3 & v_2 & v_5 & -v_4 & v_7 & -v_6 \\ -v_2 & v_3 & v_0 & -v_1 & v_6 & -v_7 & -v_4 & v_5 \\ -v_3 & -v_2 & v_1 & v_0 & v_7 & v_6 & -v_5 & -v_4 \end{pmatrix}$$

We know that these Plücker coordinates can be expressed as degree two polynomials in v_0, \dots, v_7 . This means that when $|v|^2 = 0$, we can extract a common quadratic term from the degree four maximal minors of the previous matrix. A few typical computations yield

$$\begin{aligned} p_{0123} &= (v_0^2 + v_1^2 + v_2^2 + v_3^2)^2, \\ p_{4567} &= (v_4^2 + v_5^2 + v_6^2 + v_7^2)^2, \\ p_{1236} &= (v_0^2 + v_1^2 + v_2^2 + v_3^2)(v_1v_5 + v_2v_6 + v_3v_7 - v_0v_4), \\ p_{1246} &= (v_4^2 + v_5^2 + v_6^2 + v_7^2)(v_0v_1 - v_2v_3) - (v_0^2 + v_1^2 + v_2^2 + v_3^2)(v_4v_5 + v_6v_7), \quad \text{etc.} \end{aligned}$$

suggesting that we should extract a quadratic factor $v_0^2 + v_1^2 + v_2^2 + v_3^2 = -v_4^2 - v_5^2 - v_6^2 - v_7^2$ and renormalize the Plücker coordinates to

$$\begin{aligned} p_{0123} &= v_0^2 + v_1^2 + v_2^2 + v_3^2, \\ p_{4567} &= -v_4^2 - v_5^2 - v_6^2 - v_7^2, \\ p_{1236} &= v_1v_5 + v_2v_6 + v_3v_7 - v_0v_4, \\ p_{1246} &= v_2v_3 - v_4v_5 - v_6v_7 - v_0v_1, \quad \text{etc.} \end{aligned}$$

Under this normalization our quadratic section of $OG(4, 8)_+$ becomes a quartic section of \mathbb{Q}^6 that we can write down explicitly. According to [GS] the span of the 70 quartic forms p_{i_1, \dots, i_4}^2 has dimension 35. But the span of the Göpel map in this case turns out to be a vector space Θ_{14} with the following basis:

$$\begin{aligned} A_1 &= v_0v_1v_2v_3 - v_4v_5v_6v_7 & A_8 &= (v_4^2 + v_5^2 + v_6^2 + v_7^2)^2 \\ A_2 &= v_0v_1v_4v_5 - v_2v_3v_6v_7 & A_9 &= (v_2^2 + v_3^2 + v_6^2 + v_7^2)^2 \\ A_3 &= v_0v_1v_6v_7 - v_2v_3v_4v_5 & A_{10} &= (v_2^2 + v_3^2 + v_4^2 + v_5^2)^2 \\ A_4 &= v_0v_2v_4v_6 - v_1v_3v_5v_7 & A_{11} &= (v_1^2 + v_3^2 + v_5^2 + v_7^2)^2 \\ A_5 &= v_0v_2v_5v_7 - v_1v_3v_4v_6 & A_{12} &= (v_1^2 + v_3^2 + v_4^2 + v_6^2)^2 \\ A_6 &= v_0v_3v_4v_7 - v_1v_2v_5v_6 & A_{13} &= (v_1^2 + v_2^2 + v_5^2 + v_6^2)^2 \\ A_7 &= v_0v_3v_5v_6 - v_1v_2v_4v_7 & A_{14} &= (v_1^2 + v_2^2 + v_4^2 + v_7^2)^2 \end{aligned}$$

Note that both septuples are in natural bijection with lines in the Fano plane $\mathbb{P}_{\mathbb{F}_2}^2$!

5.6. Uniqueness. The locus M in $G(k, 2n)$ parametrizing k -planes that are isotropic with respect to a generic pencil of quadratic forms can be described as a singular locus as before: choose a non-degenerate form Q_1 in the pencil $\langle Q_1, Q_2 \rangle$, and consider M as a subvariety of the corresponding orthogonal Grassmannian $OG(k, 2n)$. The locus of k -planes on which Q_2 degenerates is a quadric hypersurface H in $OG(k, 2n)$, and in general M is its k -th singular locus, in the sense that the equation of H vanishes to order k along M . We would like to prove that this property characterizes H uniquely among quadric sections of $OG(k, 2n)$, which would follow from the statement that

$$H^0(OG(k, 2n), \mathcal{J}_M^k(2)) = \mathbb{C}.$$

Note that M is obtained as the zero-locus of a section of $E = S^2U^\vee$ defined by Q_2 , so the resolution of \mathcal{J}_M is given by the usual Koszul complex. The resolution of the powers of this ideal is given by Lemma 3.5 and will allow us to prove:

Proposition 5.9. *Uniqueness holds, i.e. $H^0(OG(k, 2n), \mathcal{J}_M^k(2)) = \mathbb{C}$, as soon as one of the following conditions holds:*

- (1) $\frac{k(k+5)}{2} \leq 2n$,
- (2) $k = 2, 3, 4, 5$ and $n \geq k + 2$;
- (3) $k = 2, 3$ and $n = k + 1$.

The proof is postponed to Appendix 11. We expect this uniqueness statement to hold in full generality, but we have been unable to control the combinatorics of the Bott-Borel-Weil theorem beyond the conditions just stated.

Remark 5.10. In fact, for each fixed $1 \leq k \leq n$, we have a filtration

$$M = D_0^k(Q_2) \subset D_1^k(Q_2) \subset \dots \subset D_{k-1}^k(Q_2) = H \subset D_k^k(Q_2) = OG(k, 2n),$$

by the degeneracy loci

$$D_i^k(Q_2) := \{[U] \in OG(k, 2n) \mid \text{rank}(Q_2|_U) \leq i\}.$$

This filtration has the following properties:

- $D_{i+1}^k(Q_2)$ is the singular locus of $D_i^k(Q_2)$, and $D_0^k(Q_2)$ is smooth;

- $D_0^n(Q_2) = \emptyset$, while $D_4^n(Q_2)$ can be identified with the quotient of the moduli space $\mathrm{SU}_2(C, \mathcal{O}_C)$ (of rank 2 semistable bundles of trivial determinant over a hyperelliptic curve C of genus $n - 1$) by the hyperelliptic involution; $D_3^n(Q_2)$ is the image in this quotient of the fixed locus of the hyperelliptic involution, $D_2^n(Q_2)$ is the Kummer variety of the Jacobian of C , and $D_1^n(Q_2)$ is the set of two-torsion points inside the Kummer. Moreover $D_0^{n-1}(Q_2)$ identifies with the Jacobian of C ([DR77];
- $D_0^k(Q_2)$ for $1 \leq k \leq n - 2$ can be identified with moduli spaces of involution invariant Spin bundles on C ([Ram81]); in particular $D_0^{n-2}(Q_2)$ is isomorphic to the moduli space $\mathrm{SU}_C(2, \mathcal{O}_C(p))$ for $p \in C$;
- $D_i^n(Q_2)$ has also been described, for any i , in terms of moduli spaces of (involution invariant) Spin bundles on C ([Bho84]).

6. GENUS THREE CURVES AND FOUR-FORMS IN EIGHT DIMENSIONS

The geometry of non-hyperelliptic genus three curves is governed by the Lie algebra \mathfrak{e}_7 of the exceptional group E_7 , as shown in [GSW13, RSSS13, BBFM24]. More precisely, given a general element $h \in \mathfrak{e}_7$ one can construct a general plane quartic (i.e. a general non-hyperelliptic genus three curve) with a level-two structure and a marked flex. Notice that there are 24 flexes on a general plane quartic. There are essentially two ways to recover the curve, that we are going to illustrate. Recall that, by the Chevalley Theorem, if $\mathfrak{c} \subset \mathfrak{e}_7$ is a Cartan subalgebra of \mathfrak{e}_7 and W the associated Weyl group, then the GIT quotient $\mathfrak{e}_7 // E_7$ is isomorphic to the quotient \mathfrak{c}/W .

The group W acts by Cremona transformations on the space $(\mathbb{P}^2)^7 // \mathrm{PGL}_3$ of configurations of 7 points in \mathbb{P}^2 . Indeed, this action is generated by consecutive transpositions of the 7 points and the standard Cremona involution ([DO88]). Moreover one can construct a generically finite degree 24 W -equivariant rational map

$$\delta : \mathbb{P}(\mathfrak{c}) \rightarrow (\mathbb{P}^2)^7 // \mathrm{PGL}_3.$$

Let $0 \neq h \in \mathfrak{c}$ be a general element, and let $\delta([h]) = (p_1, \dots, p_7)$. Then the blow-up of \mathbb{P}^2 at the seven points p_1, \dots, p_7 is a del Pezzo surface of degree two. The anticanonical divisor maps this surface onto \mathbb{P}^2 via a $2 : 1$ map branched along a plane quartic, i.e. a genus three curve C . The anticanonical linear system is given by cubics passing through the seven points; among these, there are 24 cuspidal cubics which correspond to the 24 flexes of C . The element $[h]$ corresponds to the seven points plus the choice of one cuspidal cubic passing through them. The level-two structure, on the other hand, comes from the action of the Weyl group: W acts transitively on the set of level-two structures of the Jacobian of the curve C .

6.1. Vinberg theory and the Coble quartic from [RSSS13]. The second way to recover the curve is via its Jacobian. For this, one needs to decompose \mathfrak{e}_7 via the \mathbb{Z}_2 -grading

$$\mathfrak{e}_7 = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{sl}_8 \oplus \wedge^4 V_8,$$

where V_8 is an eight dimensional complex vector space. Vinberg's theory ([Vin76]) shows that in this case one can choose a Cartan subalgebra \mathfrak{c} contained inside $\wedge^4 V_8 \subset \mathfrak{e}_7$; in this case it is also a Cartan subspace of the latter, and the little Weyl group coincides with the genuine Weyl group of E_7 . Moreover, Chevalley's restriction theorem (in its classical and graded versions) yields the identifications

$$\mathfrak{e}_7 // E_7 \simeq \wedge^4 V_8 // \mathrm{SL}_8 \simeq \mathfrak{c}/W.$$

Having this in mind, consider on $\mathbb{P}(V_8) \cong \mathbb{P}^7$ the homogeneous bundle $\wedge^4 \mathcal{Q}$, where \mathcal{Q} is the rank seven quotient bundle on $\mathbb{P}(V_8)$. By the Bott-Borel-Weil Theorem $H^0(\mathbb{P}(V_8), \wedge^4 \mathcal{Q}) = \wedge^4 V_8$. Notice that the fiber of $\wedge^4 \mathcal{Q}$ is naturally isomorphic to $\wedge^4 \mathbb{C}^7$; this representation of GL_7 admits a finite number of orbits (it is a parabolic representation). Let us denote by Y_1 (respectively Y_4) the unique codimension one (resp. four) orbit closure ([KW13]). Then, for $v \in \wedge^4 V_8 \cong H^0(\mathbb{P}(V_8), \wedge^4 \mathcal{Q})$, one can define the following orbital degeneracy loci (ODL):

$$D_{Y_1}(v) := \{x \in \mathbb{P}(V_8) \mid v(x) \in Y_1 \subset \wedge^4 \mathbb{C}^7 \cong (\wedge^4 \mathcal{Q})_x\} \subset \mathbb{P}(V_8),$$

$$D_{Y_4}(v) := \{x \in \mathbb{P}(V_8) \mid v(x) \in Y_4 \subset \wedge^4 \mathbb{C}^7 \cong (\wedge^4 \mathcal{Q})_x\} \subset \mathbb{P}(V_8).$$

It turns out ([GSW13]) that $D_{Y_4}(v)$ is a Kummer threefold, and more precisely the quotient of the Jacobian $J(C)$ of the curve C by ± 1 . Moreover, $D_{Y_1}(v)$ is a quartic hypersurface, isomorphic to the moduli space $SU_C(2, \mathcal{O}_C)$ of semistable vector bundles on C of rank 2 and trivial determinant. The hypersurface $\mathcal{C} := D_{Y_1}(v)$ is also characterized by the property that it is the only quartic which is singular along $J(C)/\pm 1$; this was essentially proved by Coble, hence the name of Coble quartic. In representation theoretical terms, the morphism associating to v the Coble hypersurface \mathcal{C} corresponds to a SL_8 -equivariant rational map

$$\Gamma_1 : S^7(\wedge^4 V_8) \rightarrow S^4 V_8.$$

6.2. The Heisenberg group and the Cartan subspace. The group of two torsion points $J(C)[2] \cong (\mathbb{Z}_2)^6$ acts on the linear system $V_8 = H^0(J(C), 2\Theta)$, where Θ is the theta polarization of $J(C)$. The Heisenberg group H is defined as an extension

$$1 \rightarrow \mathbb{Z}_2 \rightarrow H \rightarrow J(C)[2] \rightarrow 1.$$

Let us fix a basis $x_{000}, x_{001}, x_{010}, \dots, x_{111}$ of V_8^\vee given by the Schrödinger coordinates. The Heisenberg group is generated by the six operations:

$$\begin{aligned} x_{i,j,k} &\mapsto x_{i+1,j,k}, & x_{i,j,k} &\mapsto x_{i,j+1,k}, & x_{i,j,k} &\mapsto x_{i,j,k+1}, \\ x_{i,j,k} &\mapsto (-1)^i x_{i,j,k}, & x_{i,j,k} &\mapsto (-1)^j x_{i,j,k}, & x_{i,j,k} &\mapsto (-1)^k x_{i,j,k}. \end{aligned}$$

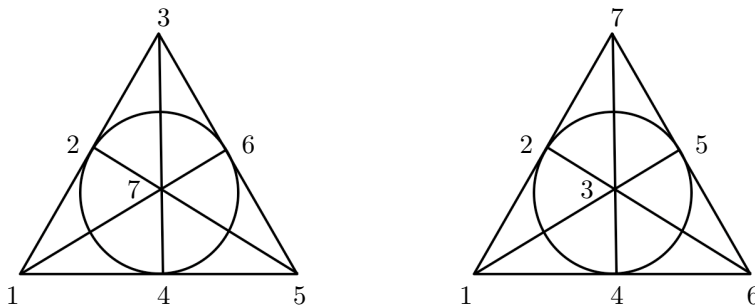
Remark 6.1. The Weyl group W of E_7 can be seen as $N_{GL_8}(H')/H' \times \{\pm 1\}$, where N stands for normalizer, and H' is the extension of H by multiplication by scalars in \mathbb{C}^* .

It turns out that the Cartan subspace $\mathfrak{c} \subset \wedge^4 V_8$ is the Heisenberg-invariant subspace. Let us change notation slightly, renaming $v_0 = x_{000}^\vee, v_1 = x_{001}^\vee, \dots, v_7 = x_{111}^\vee$. A Killing-orthogonal basis of \mathfrak{c} formed by coroots is given by the following four-vectors (this basis appears in [RSSS13], but Cartan subspaces are not mentioned):

$$\begin{aligned} h_1 &= v_{0123} + v_{4567}, & h_2 &= v_{0145} + v_{2367}, \\ h_3 &= v_{0246} + v_{1357}, & h_4 &= v_{0356} + v_{1247}, \\ h_5 &= v_{0257} + v_{1346}, & h_6 &= v_{0347} + v_{1256}, \\ h_7 &= v_{0167} + v_{2345}. \end{aligned}$$

That this is a Cartan subspace can be proved along the same lines as Lemma 4.1: that the h_i commute follows from the definition of the Lie bracket and the fact that for any v_{abcd} in h_i and any v_{efgh} in h_j , the quadruples $abcd$ and $efgh$ have two common indices; that they are semisimple is a consequence of the fact that they are sums of root vectors for pairs of opposite roots.

Remark 6.2. The vectors v_1, \dots, v_7 are in natural bijection with a Fano plane $\mathbb{P}_{\mathbb{F}_2}^2$, whose lines are the triples abc such that v_{0abc} appears in some h_i . So h_1, \dots, h_7 are also in natural bijection with a Fano plane, which is the dual to the previous one. For example v_1 corresponds to the line $h_1 h_2 h_7$ in the dual Fano plane.



The Fano plane and its dual

Consistently with Appendix 10, we get:

Proposition 6.3. *The centralizer H in SL_8 of a Cartan subspace is a finite Heisenberg group, acting on V_8 by a Schrödinger representation.*

Proof. Each h_i can be written in a unique way as the sum of two decomposable tensors, so any element in the centralizer must either fix or permute the two corresponding four planes. This yields a morphism h from H to \mathbb{Z}_2^7 .

Let $K \subset H$ denote the kernel. Any $k \in K$ fixes the 14 four-planes appearing in h_1, \dots, h_7 . Since any line $L_i = \mathbb{C}v_i$ is the intersection of the seven four planes among these fourteen, that contains it, it is also stabilized by k . So k must be diagonal; let us denote its diagonal coefficients by k_0, \dots, k_7 with $k_0 \cdots k_7 = 1$. The condition that k fixes h_1, \dots, h_7 is equivalent to $k_0 k_a k_b k_c = 1$ for any line abc in the Fano plane. Taking the product of these conditions for the three lines intersecting in a yields $(k_0 k_a)^2 = 1$, hence $e_a := k_0 k_a = \pm 1$. Plugging into the previous condition gives $k_0^2 = e_a e_b e_c$ for any line abc , in particular k_0 must be a fourth root of unity. Suppose first that $k_0 = 1$. If we identify points in the Fano plane with nonzero vectors in \mathbb{Z}_2^3 , the condition can be rewritten as $e_{a+b} = e_a e_b$, meaning that e must belong to $\mathrm{Hom}(\mathbb{Z}_2^3, \mathbb{Z}_2)$. We conclude that $K \simeq \mathbb{Z}_4 \times \mathrm{Hom}(\mathbb{Z}_2^3, \mathbb{Z}_2)$.

Now consider some x in the image I of h , that we identify with a sequence x_1, \dots, x_7 with $x_i = -1$ if and only if x exchanges the two four-planes associated to h_i . Consider for example h_1, h_2, h_7 , which correspond to the three lines in the Fano plane that contain 1. If $x_1 = 1$, v_1 must be sent to $\langle v_0, v_1, v_2, v_3 \rangle$; if also $x_2 = 1$, v_1 must be sent to $\langle v_0, v_1, v_4, v_5 \rangle$, hence in fact to $\langle v_0, v_1 \rangle$, and this implies that $x_7 = 1$. Similarly, if $x_1 = -1$, v_1 must be sent to $\langle v_4, v_5, v_6, v_7 \rangle$; if also $x_2 = -1$, v_1 must be sent to $\langle v_2, v_3, v_6, v_7 \rangle$, hence in fact to $\langle v_6, v_7 \rangle$, and this implies that $x_7 = 1$. We conclude that $x_a x_b x_c = 1$ for any line abc in the dual Fano plane. Since there is a canonical identification of \mathbb{Z}_2^m with $\mathrm{Hom}((\mathbb{Z}_2^m)^\vee, \mathbb{Z}_2)$, we conclude that $I \simeq \mathbb{Z}_2^3$. Finally, I can be lifted to a permutation group in $H \subset \mathrm{SL}_8$. Consider for example v_1 ; it corresponds to the line $h_1 h_2 h_7$ in the dual Fano plane, hence to the element x in I defined by $x_1 = x_2 = x_7 = 1$ and $x_3 = x_4 = x_5 = x_6 = -1$. This implies that the planes $\langle v_0, v_1 \rangle, \langle v_2, v_3 \rangle, \langle v_4, v_5 \rangle, \langle v_6, v_7 \rangle$ must be fixed, and it is easy to check that the product of transpositions $(01)(23)(45)(67)$ is a suitable lift. Hence the claim. \square

The restriction of Γ_1 to the Cartan subspace yields a rational map

$$\gamma_1 : \mathbb{P}(\mathfrak{c}) \rightarrow \mathbb{P}(\Theta_{15}) \subset \mathbb{P}(S^4 V_8).$$

It was computed explicitly in [RSSS13] (see also [DO88, IX.7] and [FSM19]), who verified that the span Θ_{15} of its image is the space of Heisenberg invariant degree four polynomials, of dimension fifteen. The closure of the image of γ_1 is the so-called Göpel variety \mathcal{G} . It has degree 175 and is cut-out by 35 cubics and 35 quartics. Moreover, it is a compactification of $(\mathbb{P}^2)^7 // \mathrm{PGL}_3$. Indeed γ_1 , which is generically $24 : 1$, factorizes through δ and a birational map $\eta : (\mathbb{P}^2)^7 // \mathrm{PGL}_3 \rightarrow \mathcal{G}$. The fact that γ_1 is $24 : 1$ translates into the fact that the Coble hypersurface \mathcal{C} only depends on the curve C and not on the choice of the flex point, which is an avatar of Coble's uniqueness result together with the Torelli theorem for $J(C)$. We record:

Corollary 6.4. *The non-hyperelliptic Göpel variety \mathcal{G} in $\mathbb{P}^{14} = \mathbb{P}(\Theta_{15})$ is birational to the moduli space of 7 points on \mathbb{P}^2 .*

Remark 6.5. The Weyl group W acts on $\mathfrak{c}, (\mathbb{P}^2)^7 // \mathrm{PGL}_3$ and Θ_{15} (because it normalizes the Heisenberg group), and all maps γ_1, δ and η are W -equivariant. The morphism γ_1 realizes $\Theta_{15} \subset S^7[\mathfrak{c}^\vee]$ as a Macdonald representation for the action of W . As such a representation, Θ_{15} is linearly generated by the W -orbit of the product of seven orthogonal roots in \mathfrak{c} (here we are identifying $\mathfrak{c} \cong \mathfrak{c}^\vee$ via the Killing form).

We will provide in the next subsection a similar description for the Coble quadric constructed in [BBFM24].

6.3. The Coble quadric and its equations. Consider on $G(2, V_8)$ the homogeneous bundle $\wedge^4 \mathcal{Q}$, where \mathcal{Q} is the rank six quotient bundle on $G(2, V_8)$. By the Borel-Weil Theorem $H^0(G(2, V_8), \wedge^4 \mathcal{Q}) = \wedge^4 V_8$. Notice that the fiber of $\wedge^4 \mathcal{Q}$ is naturally isomorphic to $\wedge^4 \mathbb{C}^6 \cong \wedge^2 \mathbb{C}^6$, i.e. the space of skew-symmetric 6×6 -matrices. Let us denote by Z_1 (respectively Z_6) the orbit closure of matrices with rank at most four (resp. two). Then, for $h \in \wedge^4 V_8$, one can define the following orbital degeneracy loci (ODL):

$$D_{Z_1}(h) := \{x \in G(2, V_8) \mid h(x) \in Z_1 \subset (\wedge^4 \mathcal{Q})_x\} \subset G(2, V_8),$$

$$D_{Z_6}(h) := \{x \in G(2, V_8) \mid h(x) \in Z_6 \subset (\wedge^4 \mathcal{Q})_x\} \subset G(2, V_8).$$

It was shown in [BBFM24] that $D_{Z_6}(h)$ is isomorphic to the moduli space $\mathrm{SU}_C(2, \mathcal{O}_C(p))$ of semistable vector bundles on C of rank 2 and determinant equal to $\mathcal{O}_C(p)$, for some $p \in C$. Moreover, $Q := D_{Z_1}(h)$ is the only quadric hypersurface which is singular along $\mathrm{SU}_C(2, \mathcal{O}_C(p))$, which is why Q was coined the Coble quadric.

In representation theoretical terms, the morphism associating to h the Coble hypersurface Q corresponds to a SL_8 -equivariant rational map

$$\Gamma_2 : S^3(\wedge^4 V_8) \rightarrow S_{22}V_8 = H^0(G(2, V_8), \mathcal{O}(2)).$$

Using Schur's lemma, it is easy to check that there exists a unique such map, up to nonzero scalar.

By restricting Γ_2 to the Cartan subspace, and denoting by $\Theta_{21} \subset S_{22}V_8$ the linear span of the image (which will indeed turn out to be 21-dimensional), we get

$$\gamma_2 : \mathbb{P}(\mathfrak{c}) \rightarrow \mathbb{P}(\Theta_{21}) \subset \mathbb{P}(S_{22}V_8).$$

By analogy with the case of Coble quartics, we will call *Göpel variety* the closure \mathcal{G}_2 of the image of γ_2 . The aim of the rest of this section is to describe γ_2 and \mathcal{G}_2 . Since Γ_2 is SL_8 -equivariant, γ_2 is W -equivariant, and as we will see, also H -invariant. Therefore Θ_{21} is a representation of the group W , and we will show that, similarly to Θ_{15} , it is irreducible, and another one of the so-called Macdonald representations.

Denote by c_1, \dots, c_7 the dual basis of h_1, \dots, h_7 . Let us fix $h = \sum_i c_i h_i \in \mathfrak{c}$. Let us denote by $x_{ij} = x_i \wedge x_j$ the coordinates of $\wedge^2 \mathfrak{h}$. Since $S_{2,2}V_8^\vee \subset S^2(\wedge^2 V_8^\vee)$, we will first interpret $\gamma_2(h)$ as a polynomial $\tilde{\gamma}_2(h) \in S^2(\wedge^2 V_8^\vee)$. Then we will deduce $\gamma_2(h)$ via the projection from $S^2(\wedge^2 V_8^\vee)$ to $S_{22}V_8^\vee$.

Let us define the following elements of $S^2(\wedge^2 V_8^\vee)$:

$$\begin{aligned} Q_{ij|kl|mn|op} &:= x_{ij}^2 + x_{kl}^2 + x_{mn}^2 + x_{op}^2, \\ R_{ij|kl|mn|op} &:= x_{ij}x_{kl} + x_{mn}x_{op}, \\ S_{ijkl} &:= x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}. \end{aligned}$$

We will also denote $c_{ijk} = c_i c_j c_k$.

Lemma 6.6. *The polynomial $\tilde{\gamma}_2(h)$ is the following:*

$$\begin{aligned} &c_{127}Q_{01|23|45|67} + c_1(c_2^2 + c_7^2)R_{01|23|45|67} + c_2(c_1^2 + c_7^2)R_{01|45|23|67} + c_7(c_1^2 + c_2^2)R_{01|67|23|45} \\ &- c_{135}Q_{02|13|46|57} - c_1(c_3^2 + c_5^2)R_{02|13|46|57} + c_3(c_1^2 + c_5^2)R_{02|46|13|57} + c_5(c_1^2 + c_3^2)R_{02|57|13|46} \\ &+ c_{146}Q_{03|12|47|56} + c_1(c_4^2 + c_6^2)R_{03|12|47|56} + c_4(c_1^2 + c_6^2)R_{03|56|12|47} + c_6(c_1^2 + c_4^2)R_{03|47|12|56} \\ &- c_{236}Q_{04|15|26|37} - c_2(c_3^2 + c_6^2)R_{04|15|26|37} - c_3(c_2^2 + c_6^2)R_{04|26|15|37} - c_6(c_2^2 + c_3^2)R_{04|37|15|26} \\ &+ c_{245}Q_{05|14|36|27} + c_2(c_4^2 + c_5^2)R_{05|14|27|36} - c_4(c_2^2 + c_5^2)R_{05|36|14|27} - c_5(c_2^2 + c_4^2)R_{05|27|14|36} \\ &- c_{347}Q_{06|24|35|17} + c_3(c_4^2 + c_7^2)R_{06|24|17|35} + c_4(c_3^2 + c_7^2)R_{06|35|17|24} - c_7(c_3^2 + c_4^2)R_{06|17|24|36} \\ &+ c_{567}Q_{07|25|34|16} + c_5(c_6^2 + c_7^2)R_{07|25|16|34} + c_6(c_5^2 + c_7^2)R_{07|34|16|25} + c_7(c_5^2 + c_6^2)R_{07|16|25|34} \\ &+ c_1^3(S_{0123} + S_{4567}) + c_2^3(S_{0145} + S_{2367}) + c_3^3(S_{0246} + S_{1357}) + c_4^3(S_{0356} + S_{1247}) \\ &+ c_5^3(S_{0257} + S_{1346}) + c_6^3(S_{0347} + S_{1256}) + c_7^3(S_{0167} + S_{2345}). \end{aligned}$$

Proof. We want to construct a nonzero degree three equivariant morphism from $\wedge^4 V_8$ to $S_{2,2}V_8$, and we know that this morphism is unique up to scalar. From the inclusion $S_{2,2}V_8 \subset S^2(\wedge^2 V_8)$ we interpret this as a morphism $S^3(\wedge^4 V_8) \otimes S^2(\wedge^2 V_8^\vee) \rightarrow \det(V_8) \cong \mathbb{C}$. We construct this morphism as the appropriate symmetrization of the morphism $\phi : (\wedge^4 V_8)^{\otimes 3} \otimes (\wedge^2 V_8^\vee)^{\otimes 2} \rightarrow \det(V_8)$ defined as follows: for $w_1, w_2, w_3 \in \wedge^4 V_8$ and $z_1, z_2 \in \wedge^2 V_8^\vee$, let

$$\phi((w_1 \otimes w_2 \otimes w_3) \otimes (z_1 \otimes z_2)) = w_3 \wedge (z_1 \lrcorner w_1) \wedge (z_2 \lrcorner w_2) \in \det(V_8),$$

where \lrcorner is the usual contraction map. The symmetrization of ϕ is SL_8 -equivariant and nonzero. So it must coincide with the unique (up to scalar) morphism $S^3(\wedge^4 V_8) \otimes S^2(\wedge^2 V_8^\vee) \rightarrow \det(V_8)$.

Let us apply the above to some specific cases. So, for instance, if $w_1 = w_2 = w_3 = h_1$, $z_1 = x_{01}$, $z_2 = x_{34}$, we get $\phi((w_1 \otimes w_2 \otimes w_3) \otimes (z_1 \otimes z_2)) = v_{45672301} = v_{01234567}$, which accounts for the term $c_1^3 x_{01} x_{23}$ in $\tilde{\gamma}_2(h)$. If $w_1 = w_2 = h_1$, $w_3 = h_2$, $z_1 = x_{01}$, $z_2 = x_{45}$, we get $\phi((w_1 \otimes w_2 \otimes w_3) \otimes (z_1 \otimes z_2)) = v_{01234567}$, which accounts for the term $c_2 c_1^2 x_{01} x_{45}$ in $\tilde{\gamma}_2(h)$. If $w_1 = h_1$, $w_2 = h_3$, $w_3 = h_5$, $z_1 = x_{02}$, $z_2 = x_{02}$, we

get $\phi((w_1 \otimes w_2 \otimes w_3) \otimes (z_1 \otimes z_2)) = -v_{01234567}$, which accounts for the term $-c_{135}x_{02}^2$ in $\tilde{\gamma}_2(h)$. Modulo permutation of the indices, these are the only non-zero terms that one obtains, and one deduces the expression for $\tilde{\gamma}_2(h)$. \square

Finally, we can explicitly describe the morphism γ_2 . Let us denote

$$\pi_{i|jk|lm}(c) := c_i(c_j^2 + c_k^2 - c_l^2 - c_m^2).$$

Proposition 6.7. *The equation $\gamma_2(h)$ of the Coble quadric is given by*

$$\begin{aligned} & c_{127}Q_{01|23|45|67} + \pi_{1|27|35}(c)R_{01|23|45|67} + \pi_{2|17|36}(c)R_{01|45|23|67} + \pi_{7|12|34}(c)R_{01|67|23|45} \\ & - c_{135}Q_{02|13|46|57}(c) + \pi_{3|15|26}(c)R_{02|46|13|57} + \pi_{5|13|24}(c)R_{02|57|13|47} \\ & + c_{146}Q_{03|12|47|56} + \pi_{1|46|35}(c)R_{03|12|47|56} + \pi_{4|16|25}(c)R_{03|56|12|47} + \pi_{6|14|23}(c)R_{03|47|12|56} \\ & - c_{236}Q_{04|15|26|37} \\ & + c_{245}Q_{05|14|36|27} + \pi_{2|45|36}(c)R_{05|14|27|36} \\ & - c_{347}Q_{06|24|35|17} + \pi_{3|47|26}(c)R_{06|24|17|35} + \pi_{4|37|25}(c)R_{06|35|17|24} \\ & + c_{567}Q_{07|25|34|16} + \pi_{5|67|24}(c)R_{07|25|16|34} + \pi_{6|57|23}(c)R_{07|34|16|25} + \pi_{7|56|34}(c)R_{07|16|25|34}. \end{aligned}$$

Remark 6.8. In the expression above, we have written $\gamma_2(h)$ in terms of a basis. For a more symmetric expression (but with redundant terms, see the proof below) just eliminate from $\tilde{\gamma}_2(h)$ the terms where the S_{ijkl} 's appear.

Proof. The ideal of $G(2, V_8)$ is $\wedge^4 V_8 = \ker(S^2(\wedge^2 V_8) \rightarrow S_{2,2} V_8)$ and it is generated by the Plücker quadrics, which are of the form S_{ijkl} . This implies the relations

$$R_{ij|kl|mn|op} - R_{ik|jl|mo|np} + R_{il|jk|mp|no} = 0.$$

Using these relations, we readily deduce $\tilde{\gamma}_2$ from γ_2 . \square

Proposition 6.9. Θ_{21} has the following properties:

- it is generated by the 28 polynomials $Q_{ij|kl|mn|op}$ and $R_{ij|kl|mn|op}$,
- its dimension is 21,
- it is an H -invariant subspace of $S_{22} V_8$,
- it is a Macdonald irreducible representation of W .

Proof. One checks by hand that the polynomials that generate Θ_{21} are H -invariant. Then the only non-trivial claim is the fact that it is a Macdonald representation. Clearly it is a representation of W inside $S^3 \mathfrak{c}^\vee$, and it contains the monomial c_{127} , which is a product of orthogonal roots. Moreover, one checks that the Macdonald representation generated by c_{127} is 21-dimensional (with [GS] for instance, see attached file `gopel_e7`), so it must coincide with Θ_{21} . \square

There are only two 21-dimensional irreducible representations of W ([Car85]), that we denote V_1 and V_2 . One of them, say V_1 , is simply $\wedge^2 \mathfrak{c}$.

Lemma 6.10. Θ_{21} is isomorphic to V_2 as a W -representation.

Proof. It is sufficient to show that $\Theta_{21} \neq V_1$ as a W -representation. Let $s_i \in W$ be the reflection corresponding to h_i for $i = 1, \dots, 7$, and let $w := s_1 s_2 s_7 \in W$. Thus w sends h_i to $-h_i$ if $i = 1, 2, 7$ and to h_i if $i = 3, 4, 5, 6$. From the explicit basis of Θ_{21} provided above one computes that the trace of the action of w on Θ_{21} is 5, while a direct check shows that the trace of the action of w on $\wedge^2 \mathfrak{h}$ is -3 ; therefore $\Theta_{21} \neq V_1$. \square

Theorem 6.11. *The rational map $\gamma_2 : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathcal{G}_2 \subset \mathbb{P}(\Theta_{21}) \cong \mathbb{P}^{20}$ is birational. Its base locus is given by the W -orbit of $[h_1] \in \mathbb{P}(\mathfrak{c})$, i.e. by the classes of the 63 roots in \mathfrak{c} . The image has 336 singular points, which are the γ_2 -images of 336 lines corresponding to A_2 -subsystems of \mathfrak{e}_7 . Explicitly, these lines are the W -orbit of the line ℓ_0 with equations $c_1 = c_2 + c_3 + c_4 = c_3 - c_5 = c_2 + c_3 + c_6 = c_2 - c_7 = 0$.*

Proof. Proposition 6.7 gives an explicit expression of γ_2 in terms of coordinates. One can plug this in [GS] (see attached file **gopel_e7**) and compute the degree of this rational map, which turns out to be equal to one, and its base locus.

Next, again using [GS], we analyze the subscheme Z of $\mathbb{P}(\mathfrak{c})$ where the Jacobian of γ_2 fails to have maximal rank. It turns out that Z contains ℓ_0 , is of dimension 1 and has degree 336. Since the subscheme Z is W -invariant, it must contain the W -orbit of ℓ_0 , which consists of 336 lines, so this orbit must agree with the top-dimensional part of Z .

On the other hand, ℓ_0 is orthogonal to the roots $\beta_1 = h_2 - h_4 - h_6 + h_7$, $\beta_2 = h_3 - h_4 + h_5 - h_6$, and $\beta_3 = \beta_1 - \beta_2$. Since $\beta_1, \beta_2, \beta_3$ form an A_2 -subsystem of \mathfrak{e}_7 and A_2 subsystems form a W -orbit of 336 elements (see [Osh06]), the top-dimensional part of Z is precisely given by A_2 -subsystems of \mathfrak{e}_7 . Also, [GS] allows to check that ℓ_0 is contracted to a point by γ_2 , so the same holds for each line of the W -orbit of ℓ_0 . One verifies that the resulting points are pairwise distinct. These are singular points of the image of γ_2 , since the exceptional locus of γ_2 is 1-dimensional, by the above analysis of Z . \square

The space $\mathbb{P}(\mathfrak{c})$ parametrizes curves of genus 3 plus a fixed flex point, and the birationality above translates into the fact that Coble quadrics Q , unlike Coble quartics \mathcal{C} , depend on the choice of the flex of the curve C . This is not unexpected since we know that $Q_h := D_{Z_1}(h)$ is isomorphic to $\mathrm{SU}_C(2, \mathcal{O}_C(p))$ for a fixed point $p \in C$; we believe that (C, p) is exactly the genus 3 curve and the flex point corresponding to $[h] \in \mathbb{P}(\mathfrak{c})$. Let us fix now a curve C with its 24 flexes p_1, \dots, p_{24} . Let l_1, \dots, l_{24} be the corresponding points in $\mathbb{P}(\mathfrak{c})$.

Corollary 6.12. $Q_{l_1}, \dots, Q_{l_{24}}$ are not PGL_8 -equivalent inside $G(2, V_8)$.

Proof. If Q_{l_i} and Q_{l_j} are PGL_8 -conjugate then by the birationality of γ_1 we deduce that l_i and l_j are PGL_8 -conjugate inside $\mathbb{P}(\mathfrak{c}) \subset \mathbb{P}(\mathfrak{e}_7)$. However, by the Chevalley Theorem, this is equivalent to the fact that l_i and l_j belong to the same W -orbit, which in turn means that $p_i = p_j$. \square

7. GENUS TWO CURVES AND THREE-FORMS IN NINE DIMENSIONS

This case is induced by the \mathbb{Z}_3 -grading

$$\mathfrak{e}_8 = \mathfrak{sl}_9 \oplus \wedge^3 \mathbb{C}^9 \oplus \wedge^6 \mathbb{C}^9,$$

studied in [VE78]. Its relationship with Coble cubics was first observed in [GSW13], and then considered in greater detail in [GS15, BMT21].

A Cartan subspace \mathfrak{c} of $\wedge^3 V_9$ is given in [VE78]:

$$\begin{aligned} h_1 &= e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6 + e_7 \wedge e_8 \wedge e_9, \\ h_2 &= e_1 \wedge e_4 \wedge e_7 + e_2 \wedge e_5 \wedge e_8 + e_3 \wedge e_6 \wedge e_9, \\ h_3 &= e_1 \wedge e_5 \wedge e_9 + e_2 \wedge e_6 \wedge e_7 + e_3 \wedge e_4 \wedge e_8, \\ h_4 &= e_1 \wedge e_6 \wedge e_8 + e_2 \wedge e_4 \wedge e_9 + e_3 \wedge e_5 \wedge e_7. \end{aligned}$$

Observe that if we denote $e_1 = a_1, e_2 = b_1, e_3 = c_1, e_4 = b_2, e_5 = c_2, e_6 = a_2, e_7 = c_3, e_8 = a_3, e_9 = b_3$, this gives

$$\begin{aligned} h_1 &= a_1 \wedge b_1 \wedge c_1 + a_2 \wedge b_2 \wedge c_2 + a_3 \wedge b_3 \wedge c_3, \\ h_2 &= a_1 \wedge b_2 \wedge c_3 + a_2 \wedge b_3 \wedge c_1 + a_3 \wedge b_1 \wedge c_2, \\ h_3 &= a_1 \wedge b_3 \wedge c_2 + a_2 \wedge b_1 \wedge c_3 + a_3 \wedge b_2 \wedge c_1, \\ h_4 &= a_1 \wedge a_2 \wedge a_3 + b_1 \wedge b_2 \wedge b_3 + c_1 \wedge c_2 \wedge c_3, \end{aligned}$$

suggesting a strong relationship with the case of Rubik's cubes.

The corresponding complex reflection group is G_{32} in the Shephard-Todd classification; it is isomorphic with $\mathbb{Z}_3 \times Sp_4(\mathbb{F}_3)$.

The equation of the Coble cubic in terms of coordinates on the Cartan space was computed in [GS15, (3.9)]; its coefficients are particularly simple quartics, defining a rational map

$$\gamma_1 : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}^4 = \mathbb{P}(\Theta_5).$$

Its image is the Burkhardt quartic, which compactifies the Siegel modular variety $A_2(3)$ parametrizing abelian surfaces with a full level three structure; it is embedded inside the projectivization of a Macdonald representation Θ_5 of G_{32} . Moreover the base locus of γ_1 consists in 40 points and form a single orbit of

G_{32} [GS15, Remark 3.24]. Each of these points is the intersection of three of the 40 reflection hyperplanes, and appear among the "flats" listed in [GS15, Table 1].

The degree of the map is six, and the meaning of the map γ_1 is the following: $\mathbb{P}(\mathfrak{c})/G_{32}$ parametrizes genus two curves with a chosen Weierstrass point (see [GS15, Theorem 3.4]). The morphism γ_1 , which associates to an alternating form a Coble cubic, is the forgetful map that forgets the Weierstrass point. Thus, the Burkhardt quartic can be seen as a *Göpel* variety for genus two curves.

As shown in [BBFM23], one can also define a Coble type quadric in $G(3, V_9)$. In the sequel we give the equation of this quadric and we study the corresponding Göpel map. This is the restriction (and the projectivization) to $\mathbb{P}(\mathfrak{c})$ of the unique SL_9 -equivariant map

$$\Gamma_2 : S^4(\wedge^3 V_9) \longrightarrow H^0(G(3, V_9), \mathcal{O}(2)) \simeq S_{222} V_9^\vee \simeq S_{222222} V_9.$$

Let us denote this restriction, which is a G_{32} -equivariant map, by:

$$\gamma_2 : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathcal{G}_2 \subset \mathbb{P}(\Theta_{30}),$$

where \mathcal{G}_2 is the image of γ_2 and $\Theta_{30} \subset S_{222222} V_9$ is the G_{32} -invariant linear span of \mathcal{G}_2 (which will indeed turn out to be 30-dimensional).

Theorem 7.1. *The space Θ_{30} is the 30-dimensional G_{32} -subrepresentation of $S^4 \mathfrak{c}^\vee$ generated by the orbit of c_1^4 . The map γ_2 is a birational immersive morphism, obtained by composing the 4th Veronese embedding with the projection from*

$$\Theta_5 = \langle c_1 c_2 c_3 c_4, c_1(c_2^3 + c_3^3 + c_4^3), c_2(c_1^3 + c_3^3 - c_4^3), c_3(c_1^3 - c_2^3 + c_4^3), c_4(c_1^3 + c_2^3 - c_3^3) \rangle \simeq \mathbb{P}^4.$$

The ideal of \mathcal{G}_2 is generated by 300 quadrics.

Remark 7.2. The relationship with Rubik's cubes is confirmed by the fact that if we let $c_4 = 0$ in the previous Theorem, we recover the picture of Proposition 4.4.

Remark 7.3. The equation of the Coble quadric, depending on variables c_1, c_2, c_3, c_4 , can be deduced from the quadrics appearing in the following proof and by permutation of the indices.

Proof. Γ_2 can be computed by using the morphism $\phi : S^2(\wedge^3 V_9) \longrightarrow V_9 \otimes \wedge^5 V_9$ defined by sending $(a_1 \wedge a_2 \wedge a_3)(b_1 \wedge b_2 \wedge b_3)$ to

$$\begin{aligned} & a_1 \otimes a_2 \wedge a_3 \wedge b_1 \wedge b_2 \wedge b_3 + a_2 \otimes a_3 \wedge a_1 \wedge b_1 \wedge b_2 \wedge b_3 + a_3 \otimes a_1 \wedge a_2 \wedge b_1 \wedge b_2 \wedge b_3 + \\ & + b_1 \otimes b_2 \wedge b_3 \wedge a_1 \wedge a_2 \wedge a_3 + b_2 \otimes b_3 \wedge b_1 \wedge a_1 \wedge a_2 \wedge a_3 + b_3 \otimes b_1 \wedge b_2 \wedge a_1 \wedge a_2 \wedge a_3. \end{aligned}$$

Then we can compose

$$\begin{aligned} S^4(\wedge^3 V_9) & \hookrightarrow S^2(\wedge^3 V_9) \otimes S^2(\wedge^3 V_9) \xrightarrow{\phi \otimes \phi} (V_9 \otimes \wedge^5 V_9) \otimes (V_9 \otimes \wedge^5 V_9) \xrightarrow{(13)} \\ & \xrightarrow{(13)} (V_9 \otimes \wedge^5 V_9) \otimes (V_9 \otimes \wedge^5 V_9) \longrightarrow \wedge^6 V_9 \otimes \wedge^6 V_9 \longrightarrow S_{222222} V_9, \end{aligned}$$

where the morphism (13) just permutes the first and third terms V_9 . For instance, one gets

$$\begin{aligned} \frac{1}{2}\phi(h_1^2) & = e_1 \otimes (e_{23456} + e_{23789}) + e_2 \otimes (e_{31456} + e_{31789}) + e_3 \otimes (e_{12456} + e_{12789}) + \\ & + e_4 \otimes (e_{56123} + e_{56789}) + e_5 \otimes (e_{64123} + e_{64789}) + e_6 \otimes (e_{45123} + e_{45789}) + \\ & + e_7 \otimes (e_{89123} + e_{89456}) + e_8 \otimes (e_{97123} + e_{97456}) + e_9 \otimes (e_{78123} + e_{78456}), \\ \frac{1}{2}\phi(h_2^2) & = e_1 \otimes (e_{47258} + e_{47369}) + e_2 \otimes (e_{58147} + e_{58369}) + e_3 \otimes (e_{14769} + e_{25869}) + \\ & + e_4 \otimes (e_{71258} + e_{71369}) + e_5 \otimes (e_{82147} + e_{82369}) + e_6 \otimes (e_{93147} + e_{93258}) + \\ & + e_7 \otimes (e_{14258} + e_{14369}) + e_8 \otimes (e_{25147} + e_{25369}) + e_9 \otimes (e_{36147} + e_{36258}), \\ \phi(h_1 h_2) & = e_1 \otimes e_{12347} + e_2 \otimes e_{12358} + e_3 \otimes e_{12369} + e_4 \otimes e_{14567} + e_5 \otimes e_{24568} + \\ & + e_6 \otimes e_{34569} + e_7 \otimes e_{14789} + e_8 \otimes e_{25789} + e_9 \otimes e_{36789}, \\ \phi(h_2 h_3) & = e_1 \otimes e_{14759} + e_2 \otimes e_{25678} + e_3 \otimes e_{34869} + e_4 \otimes e_{13478} + e_5 \otimes e_{12589} + \\ & + e_6 \otimes e_{23769} + e_7 \otimes e_{14267} + e_8 \otimes e_{23458} + e_9 \otimes e_{15369}. \end{aligned}$$

Using this formula we can determine the image by Γ_2 of $c = c_1 h_1 + c_2 h_2 + c_3 h_3 + c_4 h_4$. A direct computation shows that the coefficient of $c_1 c_2 c_3 c_4$ vanishes; since the G_{32} -representation spanned by the orbit of $c_1 c_2 c_3 c_4$ is generated by $c_1 c_2 c_3 c_4, c_1(c_2^3 + c_3^3 + c_4^3), c_2(c_1^3 + c_3^3 - c_4^3), c_3(c_1^3 - c_2^3 + c_4^3), c_4(c_1^3 + c_2^3 - c_3^3)$ (verification made with [GS]), we deduce that the coefficient of each of these elements also vanishes.

Denote by γ_{ijkl} the coefficient of $c_i c_j c_k c_l$. Starting from the above computations, we just need to compose all morphisms to get the explicit expression of $S^4(\wedge^3 V_9) \rightarrow S_{222222} V_9$. We compute directly that

$$\begin{aligned}
\gamma_{1111} &= 6(f_{789}f_{789} + f_{123}f_{123} + f_{456}f_{456}) - 6(f_{789}f_{456} + f_{789}f_{123} + f_{456}f_{123}) + \\
&\quad + 2(-f_{237}f_{189} + f_{238}f_{179} - f_{239}f_{178} + f_{137}f_{289} - f_{138}f_{279} + f_{139}f_{278} - f_{127}f_{389} + \\
&\quad + f_{128}f_{379} - f_{129}f_{378} - f_{234}f_{156} + f_{235}f_{146} - f_{236}f_{145} + f_{134}f_{256} - f_{135}f_{246} + \\
&\quad + f_{136}f_{245} - f_{124}f_{356} + f_{125}f_{346} - f_{126}f_{345} - f_{567}f_{489} + f_{568}f_{479} - f_{569}f_{478} + \\
&\quad + f_{467}f_{589} - f_{468}f_{579} + f_{469}f_{578} - f_{457}f_{689} + f_{458}f_{679} - f_{459}f_{678}), \\
\gamma_{1112} &= 36(-f_{179}f_{346} + f_{178}f_{245} + f_{146}f_{379} - f_{145}f_{278} + f_{479}f_{136} - f_{478}f_{125} + \\
&\quad + f_{289}f_{356} - f_{256}f_{389} - f_{134}f_{679} + f_{124}f_{578} - f_{589}f_{236} + f_{235}f_{689}), \\
\gamma_{1122} &= 24(f_{689}f_{379} - f_{589}f_{278} - f_{568}f_{245} + f_{569}f_{346} + f_{679}f_{389} + f_{479}f_{178} + \\
&\quad + f_{469}f_{356} + f_{467}f_{145} - f_{578}f_{289} + f_{478}f_{179} - f_{458}f_{256} + f_{457}f_{146} + \\
&\quad + f_{239}f_{136} - f_{238}f_{125} + f_{139}f_{236} + f_{137}f_{124} - f_{128}f_{235} + f_{127}f_{134}), \\
\gamma_{1123} &= 48(f_{179}f_{679} + f_{145}f_{458} + f_{278}f_{478} + f_{256}f_{569} + f_{389}f_{589} + f_{346}f_{467} + \\
&\quad + f_{134}f_{139} + f_{125}f_{127} + f_{236}f_{238}).
\end{aligned}$$

By permuting the indices we get a total number of 34 elements in $S^2(\wedge^3 V_9^\vee)$. In order to interpret them as quadrics, one needs to quotient by the elements in $S^2(\wedge^3 V_9^\vee)$ which are contained in the ideal of $G(3, V_9) \subset \mathbb{P}(\wedge^3 V_9)$. This can be done with the help of [GS] (see attached file **construction_gamma_w3C9**) and one finally gets 30 quadric sections of $G(3, V_9)$; since this is also the dimension of the Macdonald representation generated by c_1^4 and since the coefficient of γ_{1111} is different from zero, we deduce that this representation coincides with Θ_{30} . Once γ_2 is written in coordinates, again a computation with [GS] (see attached file **gopel_w3C9**) allows to deduce the remaining statements, by checking that the base locus of γ_2 is empty, that γ_2 has degree one, and that the subscheme defined by the maximal minors of the Jacobian matrix of γ_2 is also empty. \square

Remark 7.4. Recall from [GS15] that \mathfrak{c} parametrizes triples (C, p, ω) , where C is a genus two curve, p is a Weierstrass point in C and ω is a level three structure. Moreover G_{32} acts transitively on level three structures, so that $\mathfrak{h}/G_{32} \cong \wedge^3 V_9 // \text{SL}_9$ parametrizes couples (C, p) . The degree six rational map $\gamma_1 : \mathbb{P}(\mathfrak{c}) \rightarrow \mathbb{P}^4$ whose image is the Burkhardt quartic consists in forgetting the Weierstrass point p ; thus Coble cubics, modulo GL_9 , parametrize curves of genus two. On the other hand, $\gamma_2 : \mathbb{P}(\mathfrak{c}) \rightarrow \mathcal{G}_2 \subset \mathbb{P}(\Theta_{30})$ is birational; thus, modulo GL_9 , Coble quadrics parametrize couples (C, p) , i.e. genus two curves with a chosen Weierstrass point; moreover, there exists a degree six rational map from \mathcal{G}_2 to the Burkhardt quartic which consists in forgetting the Weierstrass point.

8. GENUS TWO CURVES AND FORTUPLES OF SPINORS IN TEN DIMENSIONS

This case is connected to the \mathbb{Z}_4 -grading

$$\mathfrak{e}_8 = \mathfrak{sl}_4 \times \mathfrak{so}_{10} \oplus (\mathbb{C}^4 \otimes \Delta_+) \oplus (\wedge^2 \mathbb{C}^4 \otimes V_{10}) \oplus (\wedge^3 \mathbb{C}^4 \otimes \Delta_-).$$

Orbits in $\mathbb{C}^4 \otimes \Delta_+$ were classified in [dG25] and the geometry of the corresponding sections of the spinor tenfold was studied in [LM25]. One can associate to an element of $\mathbb{C}^4 \otimes \Delta_+$ the following objects: a quadratic line complex (i.e. a quadric section of $G(2, 4)$), a quadric section of the flag manifold $Fl_4 \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$, and a quartic Kummer surface in \mathbb{P}^3 . The first and third constructions already appear in [GSW13]; the second one will be explained below. They can be interpreted in terms of three equivariant morphisms

$$\begin{aligned}
\Gamma_1 &: S^4(\mathbb{C}^4 \otimes \Delta_+) \longrightarrow S_{22}\mathbb{C}^4 \simeq S_{22}(\mathbb{C}^4)^\vee, \\
\Gamma_2 &: S^8(\mathbb{C}^4 \otimes \Delta_+) \longrightarrow S_{422}\mathbb{C}^4 \subset S^4\mathbb{C}^4 \otimes S^4(\mathbb{C}^4)^\vee, \\
\Gamma_3 &: S^{12}(\mathbb{C}^4 \otimes \Delta_+) \longrightarrow S_{444}\mathbb{C}^4 \simeq S^4(\mathbb{C}^4)^\vee.
\end{aligned}$$

A Cartan subspace \mathfrak{c} has a basis

$$\begin{aligned}
p_1 &= a_1 \otimes e_{53} + a_2 \otimes e_{1245} + a_3 \otimes e_{42} + a_4 \otimes e_{31}, \\
p_2 &= a_1 \otimes e_{52} + a_2 \otimes e_{1345} + a_3 \otimes e_{34} + a_4 \otimes e_{12}, \\
p_3 &= a_1 \otimes e_{1234} + a_2 \otimes 1 + a_3 \otimes e_{1235} + a_4 \otimes e_{54}, \\
p_4 &= a_1 \otimes e_{14} + a_2 \otimes e_{23} + a_3 \otimes e_{51} + a_4 \otimes e_{2345},
\end{aligned}$$

where we identify Δ_+ with the even part $\wedge^+ E$ of the exterior algebra of some maximal isotropic subspace E of V_{10} , with basis e_1, \dots, e_5 . Restricting $\Gamma_1, \Gamma_2, \Gamma_3$ to \mathfrak{c} we get rational maps (the first two turn out to be morphisms)

$$\gamma_1 : \mathbb{P}(\mathfrak{c}) \longrightarrow \mathbb{P}(\Theta_5), \quad \gamma_2 : \mathbb{P}(\mathfrak{c}) \longrightarrow \mathbb{P}(\Theta_9), \quad \gamma_3 : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(\Theta'_5),$$

whose images are instances of our Göpel varieties. In fact it is shown in [LM25] that the images of γ_1 is the Castelnuovo-Richmond (or Igusa) quartic, while the image of γ_3 is the Segre cubic primal in \mathbb{P}^4 . Let us therefore focus on Γ_2 and γ_2 .

8.1. A Coble quadric in the flag manifold. Let v be a general element of $\mathbb{C}^4 \otimes \Delta_+$. As we just mentioned (see [GSW13] and [LM25]), one can associate to v

- a Kummer quartic surface $K_v \subset \mathbb{P}^3$,
- a quadratic complex of lines $Q_v \subset G(2, 4)$.

In order to define this quadratic complex, the crucial observation is that $S_{22}\Delta_+$ contains a (unique) Spin_{10} -invariant line (defining the *spinor quadratic complex* from [Kuz18]). Using v we can then pull back this invariant line to a line in $S_{22}\mathbb{C}^4 \simeq H^0(G(2, 4), \mathcal{O}(2))$.

This line defines a quadratic form κ on $\wedge^2 V_4$ (up to scalar) and we can then obtain the Kummer surface K_v in \mathbb{P}^3 as the surface of lines $L \subset V_4$ such that κ is singular on $L \wedge V_4$. Generically, this surface is singular along the sixteen points where the restriction of κ has rank one. Note that the kernel of this restriction is then a hyperplane $L \wedge H$, for some hyperplane H of V_4 containing L . This shows that $\text{Sing}(K_v)$ is naturally embedded into the flag manifold $Fl_4 = Fl(1, 3, V_4)$. And indeed, we can define the dual Kummer surface K_v^\vee in the dual \mathbb{P}^3 , as the set of hyperplanes H of V_4 such that κ is singular on $\wedge^2 V_4$.

That we define Kummer surfaces can be seen by considering the rank two vector bundle \mathcal{E} on Fl_4 with fiber $L \wedge H$. This bundle is endowed with a natural quadratic form induced by κ , and this section of $S^2\mathcal{E}^\vee$ vanishes along a smooth surface A_v . That this surface is an abelian surface follows from two facts: by adjunction, its canonical bundle is trivial; the Hodge number $h^{0,1} = 2$, as follows from a simple computation involving the Koszul complex and the Bott-Borel-Weil theorem. Finally, it is easy to check that the projections of A_v to \mathbb{P}^3 and its dual have degree two over the Kummer surface and its dual, respectively.

One should observe that from this perspective, the abelian surface A_v appears as the singular locus of the hypersurface B_v of Fl_4 where κ degenerates.

Proposition 8.1. *For v generic, the hypersurface B_v is the unique quadric section of Fl_4 that is singular along the abelian surface A_v .*

Proof. We start with the Koszul resolution

$$0 \longrightarrow \det(\mathcal{E})^3 \longrightarrow S^2\mathcal{E} \otimes \det(\mathcal{E}) \longrightarrow S^2\mathcal{E} \longrightarrow \mathcal{J}_{A_v} \longrightarrow 0.$$

Using Lemma 3.5 we get a resolution of $\mathcal{J}_{A_v}^2$, namely

$$0 \longrightarrow S_{211}(S^2\mathcal{E}) \longrightarrow S_{21}(S^2\mathcal{E}) \longrightarrow S^2(S^2\mathcal{E}) \longrightarrow \mathcal{J}_{A_v}^2 \longrightarrow 0.$$

We have $K_2 := S_{211}(S^2\mathcal{E}) = S^2\mathcal{E} \otimes \det(\mathcal{E})^3$, $K_1 := S_{21}(S^2\mathcal{E}) = S^4\mathcal{E} \otimes \det(\mathcal{E}) \oplus S^2\mathcal{E} \otimes \det(\mathcal{E})^2$ and $K_0 = S^2(S^2\mathcal{E}) = S^4\mathcal{E} \oplus \det(\mathcal{E})^2$. Using the Bott-Borel-Weil theorem we compute that the only non zero cohomology groups of their twists by $\mathcal{O}(2) = \det(\mathcal{E})^{-2}$ are

$$H^0(K_0(2)) = H^3(K_2(2)) = \mathbb{C}, \quad H^2(K_0(2)) = S_{22}\mathbb{C}^4, \quad H^2(K_1(2)) = S_{211}\mathbb{C}^4, \quad H^5(K_1(2)) = \wedge^2\mathbb{C}^4.$$

This easily implies that $H^0(\mathcal{J}_{A_v}^2(2)) = H^0(K_0(2)) = \mathbb{C}$, and our claim follows. \square

The equation of B_v as a quadratic function of Q_v must be given by a morphism

$$\Gamma_0 : S^2(S_{22}V_4) \longrightarrow S_{422}V_4.$$

By the Littlewood-Richardson rule, $S_{422}V_4$ has multiplicity one inside $S_{22}V_4 \otimes S_{22}V_4$, in particular there is a unique such equivariant morphism (up to scalar). We will construct it as a composition

$$S^2(S_{22}V_4) \longrightarrow S^2(S^2(\wedge^2 V_4)) \xrightarrow{\ominus} S^2V_4 \otimes S^2(\wedge^3 V_4) \longrightarrow S_{422}V_4 \subset S^2V_4 \otimes S^2V_4^\vee.$$

We define Θ explicitly by sending $((a_1 \wedge a_2)(b_1 \wedge b_2)).((c_1 \wedge c_2)(d_1 \wedge d_2))$ to

$$\begin{aligned} & a_1 c_1 \otimes (b_1 \wedge b_2 \wedge c_2)(a_2 \wedge d_1 \wedge d_2) - a_2 c_1 \otimes (b_1 \wedge b_2 \wedge c_2)(a_1 \wedge d_1 \wedge d_2) \\ & - a_1 c_2 \otimes (b_1 \wedge b_2 \wedge c_1)(a_2 \wedge d_1 \wedge d_2) + a_2 c_2 \otimes (b_1 \wedge b_2 \wedge c_1)(a_1 \wedge d_1 \wedge d_2) \\ & + b_1 c_1 \otimes (a_1 \wedge a_2 \wedge c_2)(b_2 \wedge d_1 \wedge d_2) - b_2 c_1 \otimes (a_1 \wedge a_2 \wedge c_2)(b_1 \wedge d_1 \wedge d_2) \\ & - b_1 c_2 \otimes (a_1 \wedge a_2 \wedge c_1)(b_2 \wedge d_1 \wedge d_2) + b_2 c_2 \otimes (a_1 \wedge a_2 \wedge c_1)(b_1 \wedge d_1 \wedge d_2) \\ & + a_1 d_1 \otimes (b_1 \wedge b_2 \wedge d_2)(a_2 \wedge c_1 \wedge c_2) - a_2 d_1 \otimes (b_1 \wedge b_2 \wedge d_2)(a_1 \wedge c_1 \wedge c_2) \\ & - a_1 d_2 \otimes (b_1 \wedge b_2 \wedge d_1)(a_2 \wedge c_1 \wedge c_2) + a_2 d_2 \otimes (b_1 \wedge b_2 \wedge d_1)(a_1 \wedge c_1 \wedge c_2) \\ & + b_1 d_1 \otimes (a_1 \wedge a_2 \wedge d_2)(b_2 \wedge c_1 \wedge c_2) - b_2 d_1 \otimes (a_1 \wedge a_2 \wedge d_2)(b_1 \wedge c_1 \wedge c_2) \\ & - b_1 d_2 \otimes (a_1 \wedge a_2 \wedge d_1)(b_2 \wedge c_1 \wedge c_2) + b_2 d_2 \otimes (a_1 \wedge a_2 \wedge d_1)(b_1 \wedge c_1 \wedge c_2). \end{aligned}$$

This formula has the required symmetries: it is skew-symmetric in a_1, a_2 , symmetric in a, b , and also symmetric when we exchange a with c and b with d .

As always, we only need to compute the restriction of Γ_0 to a suitable subspace; as in [LM25, Lemma 5.15], we can choose the space \mathfrak{c}_{qc} consisting of quadratic complexes of the form

$$\Theta = P(\pi_{12}^2 - \pi_{34}^2) + Q(\pi_{13}^2 + \pi_{24}^2) + R(\pi_{14}^2 + \pi_{23}^2) + 2S\pi_{12}\pi_{34} + 2T\pi_{14}\pi_{23} - 2U\pi_{13}\pi_{24},$$

with P, Q, R, S, T, U are coefficients such that $S + T + U = 0$ and the π_{ij} 's are Plücker coordinates on $G(2, 4)$ for $1 \leq i < j \leq 4$. (Note that by Lemma 5.8, this is a genuine Cartan subspace!)

Lemma 8.2. *The restriction of Γ_0 to \mathfrak{c}_{qc} is the morphism sending $[P, Q, R, S, T, U]$ to*

$$[QR, PR, PQ, S^2 + Q^2 - R^2, T^2 + P^2 - Q^2, U^2 + R^2 - P^2, UT - PS, TS - QU, SU - RT].$$

Finally, we deduce γ_2 by restriction to \mathfrak{c} , as a composition

$$\gamma_2 : S^8 \mathfrak{c} \longrightarrow S^2 \mathfrak{c}_{qc} \subset S^2(S_{22}V_4) \xrightarrow{\Gamma_0} S_{422}V_4 \subset S^2V_4 \otimes S^2V_4^\vee.$$

We are now ready for an explicit computation, whose result is the following:

Proposition 8.3. *The image of $\gamma_2 : \mathbb{P}(\mathfrak{c}) \longrightarrow \mathbb{P}(\Theta_9) \subset \mathbb{P}(S^2V_4 \otimes S^2V_4^\vee)$ is an isomorphic projection of the second Veronese embedding of the Castelnuovo-Richmond (or Igusa) quartic. Moreover Θ_9 is the G_{31} -representation induced from the 9-dimensional representation $[4, 2]$ of S_6 ; it is again a Macdonald representation.*

8.2. Spinor bundles and orbital degeneracy loci. Recall that each orthogonal Grassmannian $OG(k, 10)$ admits a spinor bundle, whose space of global sections is Δ .

On the quadric \mathbb{Q}^8 , denote by \mathcal{S}_\pm the rank eight spinor bundles, whose spaces of global sections are the half-spin representations Δ_\pm . Recall that the fiber of \mathcal{S}_+ at $x \in \mathbb{Q}^8$ is the quotient $\Delta_+/x\Delta_-$, where $x\Delta_-$ is the image of Δ_- by Clifford multiplication by x .

Given a generic $v \in \mathbb{C}^4 \otimes \Delta_+$, defining a subspace $P_v \subset \Delta_+$, the associated morphism

$$\psi_v^1 : P_v \otimes \mathcal{O}_{\mathbb{Q}^8} \longrightarrow \mathcal{S}_+$$

degenerates along a smooth threefold $D(v)$, and the kernel map is an isomorphism $D(v) \longrightarrow \mathbb{P}^3$. Indeed, ψ_v^1 degenerates at $x \in \mathbb{Q}^8$ when $P_v \cap x\Delta_- \neq 0$. For a nonzero θ in this intersection, $x.\theta = 0$; for v general θ is not a pure spinor and there is a unique such $x = \nu(\theta)$ (up to scalar). So $D(v) = \nu(P_v) \simeq \nu_2\mathbb{P}^3$.

On the other hand, \mathcal{S}_+ is modeled on a half-spin representation of Spin_8 , which is by triality equivalent to the natural representation; in particular it contains a non-degenerate quadric, given at a point $x \in \mathbb{Q}^8$, by the formula

$$q_x(\bar{\theta}) = \langle \nu(\theta), x \rangle,$$

which is easily seen to make sense. Globally, this corresponds to an isomorphism between \mathcal{S}_+ and $\mathcal{S}_+^\vee(1)$. Transplanting the quadric through v , we get a morphism from $\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{Q}^8}$ to $(\mathbb{C}^4)^\vee \otimes \mathcal{O}_{\mathbb{Q}^8}(1)$, whose determinant yields a section of $\mathcal{O}_{\mathbb{Q}^8}(4)$, that is, a quartic section of the quadric. The corresponding morphism

$$\Gamma_1^{10} : S^8(\mathbb{C}^4 \otimes \Delta_+) \longrightarrow S^{(4)}V_{10}$$

is characterized by its restriction to a Cartan subspace \mathfrak{c} ; this can be deduced from the fact that an element $a \in \mathfrak{c}$ induces a map $\tilde{a} : S^2\mathbb{C}^4 \rightarrow V_{10}$ as in [LM25, Lemma 5.7] (where \tilde{a} is denoted γ_a); in this paper, the authors give the explicit expression of the symmetric matrix M of this quadratic form:

$$\begin{aligned} M_{11} &= -2a_2a_3e_2 + 2a_1a_4f_2 - 2a_1a_3e_3 - 2a_2a_4f_3, \\ M_{12} &= (a_1^2 - a_2^2)e_5 + (a_3^2 - a_4^2)f_5, \\ M_{13} &= 2a_3a_4e_1 + (a_1^2 + a_2^2)f_1, \\ M_{14} &= -(a_3^2 + a_4^2)e_4 + 2a_1a_2f_4, \\ M_{22} &= 2a_1a_4e_2 - 2a_2a_3f_2 + 2a_2a_4e_3 + 2a_1a_3f_3, \\ M_{23} &= -2a_1a_2e_4 - (a_3^2 + a_4^2)f_4, \\ M_{24} &= -(a_1^2 + a_2^2)e_1 + 2a_3a_4f_1, \\ M_{33} &= -2a_1a_3e_2 - 2a_2a_4f_2 - 2a_2a_3e_3 - 2a_1a_4f_3, \\ M_{34} &= -(a_3^2 - a_4^2)e_5 + (a_1^2 - a_2^2)f_5, \\ M_{44} &= 2a_2a_4e_2 + 2a_1a_3f_2 + 2a_1a_4e_3 + 2a_2a_3f_3. \end{aligned}$$

Its determinant yields the restriction

$$\gamma_1^{10} : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(S^{(4)}V_{10})$$

whose image is another Göpel type variety.

For $k = 2$, the spinor bundle \mathcal{S}_+ has rank four. The following observation seems new:

Proposition 8.4. *Let s_i be a general section of $\mathcal{S}_+^{\oplus i}$ on $OG(2, 10)$, and Z_i its zero locus.*

- Z_1 is a horospherical variety for Spin_7 ,
- Z_2 is the adjoint variety of the exceptional group G_2 ,
- Z_3 is a smooth conic.

Proof. For each $i \leq 3$, Z_i is smooth of dimension $13 - 4i$ and index $7 - 2i$. This is enough to ensure that Z_1 is a smooth conic. According to [SK77, Proposition 32], Z_2 is acted on by G_2 ; but G_2 has no non-trivial homogeneous space of dimension smaller than five; in dimension five it has only two, the quadric of index 5, and the adjoint variety of index 3, which has therefore to coincide with F .

By [SK77, Proposition 31], Z_1 is acted on by an extension of Spin_7 . Let us denote by $\Delta^+(V_n)$ and $\Delta^-(V_n)$ the two half-spin representations of $\text{Spin}(V_n)$ for n even, and by $\Delta(V_n)$ its unique spin representation for n odd. A general element $\phi \in \Delta^+(V_{10})$ defines a general section s_1 of the spinor bundle \mathcal{S}_+ . For an isotropic plane $P \in OG(2, V_{10})$, $s_1(P)$ vanishes if and only if ϕ belongs to the image of the map

$$P \otimes \Delta^-(V_{10}) \subset V_{10} \otimes \Delta^-(V_{10}) \rightarrow \Delta^+(V_{10}),$$

where the rightmost morphism is defined via Clifford multiplication. If $V_{10} = E \oplus F$ with E and F two transverse isotropic subspaces, we can identify $\Delta^+(V_{10})$ with $\wedge^- F$, the odd part of the exterior algebra of F . Inside $\Delta^+(V_{10})$, spinors that are not pure form a single Spin_{10} -orbit. Let us choose a basis e_1, \dots, e_5 of E , and let f_1, \dots, f_5 be the dual basis of F ; we can assume that $\phi = f_1 \wedge (1 + f_{2345})$. Indeed this is not a pure spinor, but the sum of the two pure spinors f_1 and f_{12345} associated to the maximal isotropic spaces $W := \langle f_1, \dots, f_5 \rangle$ and $W' := \langle f_1, e_2, \dots, e_5 \rangle$. Let $V_1 := W \cap W'$ and $V_8 := (W + W')/V_1$; the latter comes endowed with a quadratic form, induced by the restriction of the quadratic form on V_{10} . Let us also let $U_1 = \langle e_1 \rangle$; we can thus decompose

$$\wedge^2 V_{10} = (V_1 \otimes V_8) \oplus \wedge^2 V_8 \oplus (V_8 \otimes U_1) \oplus (V_1 \otimes U_1).$$

Elements in $OG(2, V_{10})$ of the form $[f_1 \wedge v]$ are zeroes of s_1 , so that the quadric $\mathbb{Q}(V_8/V_1)$ is contained in Z_1 ; elements of the form $[e_1 \wedge v]$ are never zeroes of s_1 and this implies that the linear span of Z_1 is $\mathbb{P}((V_1 \otimes V_8) \oplus \wedge^2 V_8)$. Now notice that ϕ defines a generic element $\psi \in \Delta^+(V_8)$; in coordinates, this is just

$$\psi = 1 + f_{2345} \in \wedge^+(F/\langle f_1 \rangle) \simeq \Delta^+(V_8).$$

In particular ψ is a generic element in $\Delta^+(V_8)$, whose orthogonal (w.r.t. the unique Spin_8 -invariant quadratic form on $\Delta^+(V_8)$) is a $V_7 \subset \Delta^+(V_8)$ endowed with a non-degenerate quadratic form; the copy of Spin_7 in the stabilizer ϕ identifies with $\text{Spin}(V_7)$. We deduce Spin_7 -equivariant identifications $\Delta(V_7) \simeq V_8$ and $\wedge^2 V_8 \simeq \wedge^2 \Delta^+(V_8) \simeq \wedge^2 V_7 \oplus V_7$. Following these identifications, ψ defines a section of the rank 2 spinor bundle \mathcal{S}' on $OG(2, V_8)$. By triality, this bundle can be exchanged with the dual tautological bundle, so

the zero-locus of a general section is isomorphic to $OG(2, V_7)$. Putting all together we deduce that both $\mathbb{Q}(V_8/V_1) \simeq OG(3, V_7)$ and $OG(2, V_7)$ are contained in Z_1 . More than that, consider a maximal weight vector $v_3 \in \Delta(V_7) \simeq V_8$, and a maximal weight vector $v_2 \wedge v_3 \in \wedge^2 V_7 \subset \wedge^2 \Delta(V_7)$, with $v_2 \in \Delta(V_7)$. The fact that $\psi(\langle v_2 \wedge v_3 \rangle) = 0$ means that

$$\exists \widetilde{\psi}_2, \widetilde{\psi}_3 \in \Delta^-(V_8) \quad | \quad \psi = v_2 \cdot \widetilde{\psi}_2 + v_3 \cdot \widetilde{\psi}_3.$$

This implies that $x := [f_1 \wedge v_2 + v_2 \wedge v_3] = [v_2 \wedge (v_3 - f_1)] \in OG(2, V_8)$ is a zero of s_1 since

$$\phi = v_2 \cdot (f_1 \wedge \widetilde{\psi}_2) + (v_3 - f_1) \cdot (f_1 \wedge \widetilde{\psi}_3).$$

Finally, since Z_1 is Spin_7 -invariant, it must contain the Spin_7 -orbit closure of $x \in \mathbb{P}(\Delta(V_7) \oplus \wedge^2 V_7) \subset \mathbb{P}(\wedge^2 V_7)$, i.e. it must contain the horospherical variety denoted by $(B_3, \omega_2, \omega_3)$ in [Pas09]. Both varieties are smooth of the same dimension, and using the Koszul complex for zero loci one shows that $H^0(Z_1, \mathcal{O}_{Z_1}) \cong \mathbb{C}$, i.e. that Z_1 is connected; so the two varieties must coincide. \square

A vector $v \in \mathbb{C}^4 \otimes \Delta_+$ defines a morphism

$$\psi_v^2 : P_v \otimes \mathcal{O}_{OG(2,10)} \longrightarrow \mathcal{S}_+.$$

Let $D_i(v) \subset OG(2, V_{10})$ be the locus where the rank of ψ_v^2 is at most i .

Proposition 8.5. *Let v be general in $\mathbb{C}^4 \otimes \Delta_+$.*

- $D_3(v)$ is a quadric section of $OG(2, V_{10})$, singular along $D_2(v)$.
- $D_2(v)$ admits a resolution $\tilde{D}_2(v)$ which projects to $G(2, 4)$ with generic fiber isomorphic to the adjoint variety of G_2 .
- $D_1(v)$ is a smooth Fano fourfold of index one; it admits a natural projection to \mathbb{P}^3 , which is a conic fibration with discriminant the Kummer quartic surface K_v .

Proof. The first statement follows from the fact that $\det(\mathcal{S}_+) = \mathcal{O}(2)$. The standard resolution $\tilde{D}_2(v)$ of $D_2(v)$ is the set of pairs (L, x) in $G(2, P_v) \times OG(2, V_{10})$ such that $P_v \subset L\Delta_+$. The fiber over L of the projection $\tilde{D}_2(v) \rightarrow G(2, P_v)$ only depends on the orbit of L in $G(2, \Delta_+)$; it is the zero locus of a pair of sections of the spinor bundle, hence by Proposition 8.4 a copy of the adjoint variety of G_2 .

Over $D := D_1(v)$, the rank of ψ_v^2 is everywhere equal to one, so there is an exact sequence

$$0 \rightarrow K \rightarrow \mathbb{C}^4 \otimes \mathcal{O}_D \rightarrow \mathcal{S}_{+|D} \rightarrow \mathcal{C} \rightarrow 0,$$

where the kernel K and the cokernel \mathcal{C} are both vector bundles of rank three. As usual, the normal bundle of D in $OG(2, V_{10})$ is $N_D = \text{Hom}(K, \mathcal{C})$, whose determinant, by the exact sequence above, is $\det(N_D) = \det(K^\vee)^3 \otimes \det(\mathcal{C})^3 = \det(\mathcal{S}_{+|D})^3 = \mathcal{O}_D(6)$. By adjunction, we deduce that $\omega_D = \mathcal{O}_D(-1)$, so D is Fano.

Since K is a corank one subbundle of a trivial rank four bundle, it defines a morphism p from D to \mathbb{P}^3 , the space of hyperplanes. The fiber over $H \in \mathbb{P}^3$ is the zero-locus of a triple of sections of \mathcal{S}_+ , hence a conic in general by Proposition 8.4. This conic only depends on the orbit of H in $G(3, \Delta_+)$, and becomes singular when H degenerates to the codimension one orbit. According to [LM25], this happens exactly when H belongs to the Kummer surface K_v . \square

The equation of the quadric $D_3(v) \subset OG(2, V_{10})$ is given by a morphism

$$\Gamma_2^{10} : S^4(\mathbb{C}^4 \otimes \Delta_+) \rightarrow S_{(2,2)}V_{10}.$$

Restricting to the Cartan subspace \mathfrak{c} , we get a map

$$\gamma_2^{10} : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(S_{(2,2)}V_{10})$$

whose image is another Göpel variety. One can show that γ_1^{10} is the composition of γ_2^{10} with a quadratic map. Moreover, W. de Graaf checked that the morphism $S^4\mathfrak{c} \rightarrow S_{(2,2)}V_{10}$ is injective, so that γ_2^{10} is just a degree four Veronese embedding. This is in sharp contrast with the other cases with discussed, especially since $S^4\mathfrak{c}$ is not an irreducible representation of the complex reflection group G_{31} , but splits as the sum of two submodules of dimension 5 and 30. Nevertheless, the image of γ_2^{10} is a family of very special quadratic sections of $OG(2, 10)$, since they are singular in dimension five, and the singular locus of their singular locus is in general a smooth Fano fourfold, closely related to a Kummer surface.

9. SPECIAL GENUS FOUR CURVES AND SPINORS IN SIXTEEN DIMENSIONS

This case comes from the ultimate model, the \mathbb{Z}_2 -grading

$$\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \Delta,$$

where Δ is a half-spin representation of dimension 128. The symmetric space E_8/Spin_{16} is conjectured to admit a compactification that would deserve to be coined a bioctonionic projective plane. For this group the geometric identification of Coble type hypersurfaces in terms of moduli spaces of vector bundles on curves is yet to be fully uncovered. Even so, it is expected that the hypersurfaces of which we will give a construction in this section should be related to special non-hyperelliptic trigonal curves of genus four (see [GSW13]). Steven Sam and Eric Rains explained to us some unpublished results of theirs that go even further in the direction that we explore in this section.

9.1. The grading. The Lie bracket on $\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \Delta$, since it is compatible with the grading, can be decomposed into three Spin_{16} -equivariant morphisms: the classical bracket on \mathfrak{so}_{16} , a map $\beta_1 : \mathfrak{so}_{16} \otimes \Delta \rightarrow \Delta$ and a map $\beta_2 : \wedge^2 \Delta \rightarrow \mathfrak{so}_{16}$. The fact that these morphisms are Spin_{16} -equivariant identifies them uniquely up to nonzero scalar, because the target is an irreducible representation appearing in the source with multiplicity one.

In particular β_1 must be given by the action of \mathfrak{so}_{16} on Δ . It can be explicitly described in terms of the Clifford action, once we identify \mathfrak{so}_{16} with $\wedge^2 V$. Indeed, there is an equivariant morphism $V \otimes \Delta \rightarrow \Delta_-$, the other half-spin representation. Recall that Δ and Δ_- are constructed by choosing a decomposition $V = E \oplus F$ into maximal isotropic subspaces, and letting $\Delta = \wedge^+ E$ and $\Delta_- = \wedge^- E$. Then $e \in E$ and $f \in F$ act by sending $v \in \Delta$ or Δ_- to the wedge product $e \wedge v$ and the contraction $f \lrcorner v$, respectively. By composition, we readily deduce a morphism $\wedge^2 V \otimes \Delta \rightarrow V \otimes \Delta_- \rightarrow \Delta$, which has to coincide with β_1 , up to nonzero scalar. We can then construct β_2 from β_1 . Indeed, since Δ and \mathfrak{so}_{16} are selfdual, β_1 induces a morphism

$$\wedge^2 \Delta \hookrightarrow \Delta \otimes \Delta \cong \Delta \otimes \Delta^\vee \rightarrow \mathfrak{so}_{16}^\vee \cong \mathfrak{so}_{16},$$

which can be checked to be nonzero and must therefore be proportional to β_2 . An invariant quadratic form on $\Delta = \wedge^+ E$ can be defined by the wedge product followed with the projection to $\wedge^8 E \simeq \mathbb{C}$. We can then give an explicit formula in terms of a basis (a_i) of V and the dual basis (b_i) :

$$(7) \quad \beta_2(h, h') = \sum_{i < j} \langle (a_i \wedge a_j)h, h' \rangle b_i \wedge b_j.$$

9.2. A Cartan subspace. From the general theory [Vin76], we know that Δ contains Cartan subspaces; in this case these are just Cartan subalgebras of \mathfrak{e}_8 , contained in Δ . Here is one, given in terms of a basis e_0, \dots, e_7 of E . We will also denote by f_0, \dots, f_7 the dual basis of F .

Proposition 9.1. *A Killing-orthonormal basis of a Cartan subspace $\mathfrak{c} \subset \Delta$ is given by*

$$\begin{aligned} h_1 &= e_{0123} + e_{4567}, & h_2 &= e_{0145} + e_{2367}, \\ h_3 &= e_{0246} + e_{1357}, & h_4 &= e_{0356} + e_{1247}, \\ h_5 &= e_{0257} + e_{1346}, & h_6 &= e_{0347} + e_{1256}, \\ h_7 &= e_{0167} + e_{2345}, & h_8 &= 1_E + e_{01234567}. \end{aligned}$$

Proof. We need to check that h_1, \dots, h_8 are semisimple and commute. Let \mathfrak{t} denote the standard Cartan subalgebra of \mathfrak{so}_{16} , for which the e_i 's and f_j 's are weight vectors in V . Then \mathfrak{t} is also a Cartan subalgebra of \mathfrak{e}_8 , whose roots are thus partitioned into the roots of \mathfrak{so}_{16} and the weights of Δ . Now, observe that e_{0123} and e_{4567} are in \mathfrak{e}_8 root vectors for opposite roots α and $-\alpha$; the corresponding root spaces generate a subalgebra $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \simeq \mathfrak{sl}_2$. As an element of this \mathfrak{sl}_2 , h_1 is represented by a matrix with zero diagonal coefficients and nonzero off diagonal coefficients. So h_1 is semisimple, and by the same argument h_2, \dots, h_8 are semisimple as well.

Now we check that they commute, by applying formula (7) for the basis (e_i, f_j) , whose dual is the basis (f_i, e_j) . It is clear that $[h_i, h_8] = \beta_2(h_i \wedge h_8) = 0$ for any $i \leq 7$, because since $h_i \in \wedge^4 E$, we always have $(a_p \wedge a_q)h_i \in \wedge^2 E \oplus \wedge^4 E \oplus \wedge^6 E = (\wedge^0 E \oplus \wedge^8 E)^\perp$. For $i, j \leq 7$, the same argument shows that $(a_p \wedge a_q)h_i$ is orthogonal to h_j unless possibly when $a_p \wedge a_q$ is of the form $e_r \wedge f_s$. The action of such an operator on

a monomial e_{abcd} replaces the index s , if present, by r . But two monomials from h_i and h_j always have two indices in common; after the action of $e_r \wedge f_s$ they will still have at least one index in common, and their wedge product will therefore vanish. \square

Remark 9.2. By computing the rank of $ad(h_i)$ one can check that each h_i is a coroot. In order to make an explicit connexion with the root system of E_8 , it might be useful to exhibit an octuple of eight orthogonal roots, to show how we can recover from those the whole root system and the full Weyl group. There is only one such octuple up to the Weyl group action, and one instance is the following:

$$\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & & & & & & 0 & & & & & & & & & 1 \\ \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 2 & 4 & 6 & 5 & 4 & 3 & 2 \\ & & 1 & & & & & & & & & 1 & & & & & & 2 & & & & & & & 3 & & & & \end{array}$$

More generally, we have the following Lemma (see [Osh06, 10.2]).

Lemma 9.3. *Up to the action of the Weyl group, the root system E_8 contains a unique subsystem of type $2A_1, 3A_1$ or $8A_1$, and two distinct root subsystems of type $4A_1$.*

Note that a subsystem of type mA_1 is just given by m pairwise orthogonal roots. In particular any m -tuple of orthogonal roots is equivalent to h_1, \dots, h_m , up to the action of the Weyl group, for $m = 1, 2, 3, 8$ but not for $m = 4$. We will see that h_1, h_2, h_3, h_4 and h_1, h_2, h_4, h_7 belong to different orbits.

Remark 9.4. Since the Weyl group of E_8 acts transitively on roots, h_8^\perp is equivalent to the system orthogonal to a highest root, which is a root system of type E_7 (indeed in this situation $h_8 = \omega_8$, and E_7 is generated by roots $\alpha_1, \dots, \alpha_7$ in Bourbaki's notation). To make this even clearer, notice that $\bigwedge^4 E \subset \Delta$, and recall from Section 6 that $\bigwedge^4 E \cong \bigwedge^4 \mathbb{C}^8$ contains a Cartan subspace of type E_7 . One can compare the elements h_1, \dots, h_7 defined in this section with those defined in section 6 and notice that they are essentially the same.

Proposition 9.5. *The centralizer of a Cartan subspace is a finite Heisenberg subgroup of Spin_{16} , acting on V_{16} by a Schrödinger representation.*

Proof. Each h_i can be decomposed in a unique way as the sum of two pure spinors, and this defines as in the proof of Proposition 6.3, a morphism h from the centralizer H to \mathbb{Z}_2^8 .

Let k belong to the kernel K of h . Then k fixes the sixteen isotropic eight-planes $\langle e_0, \dots, e_7 \rangle, \langle f_0, \dots, f_7 \rangle, \langle e_0, e_a, e_b, e_c, f_p, f_q, f_r, f_s \rangle$ and $\langle e_p, e_q, e_r, e_s, f_0, f_a, f_b, f_c \rangle$, for abc a line in the Fano plane with complement $pqrs$. By considering their intersections, we deduce that k must act diagonally on the e_i 's and f_j 's, and belongs to the corresponding maximal torus in Spin_{16} .

As for the image I , as in Proposition 6.3 we encode an element x in I by a sequence x_1, \dots, x_8 . First note that swapping e_i and f_i for each i defines an element x^{sw} of I such that $x_j^{sw} = -1$ for all $j = 1, \dots, 8$. Multiplying by this element if necessary, we can then suppose that a given x in I verifies $x_8 = 1$. Then reasoning as in the proof of Proposition 6.3 shows that again $x_a x_b x_c = 1$ if abc is a line in the Fano plane, and we conclude the proof in the same way. \square

9.3. The Heisenberg group. The Heisenberg group of (\mathfrak{e}_8, θ) in Proposition 9.5 is an extension

$$1 \rightarrow \mathbb{Z}_2 \rightarrow H \rightarrow (\mathbb{Z}_2)^8 \rightarrow 1$$

acting linearly on V (see also Appendix 10). Let us fix a basis $x_{0000}, x_{0001}, x_{0010}, \dots, x_{0111}$ of E^\vee given by Schrödinger coordinates, and a basis $x_{1000}, x_{1001}, x_{1010}, \dots, x_{1111}$ of F^\vee . The Heisenberg group is explicitly generated by the following eight operations:

$$\begin{array}{llll} x_{i,j,k,l} \mapsto x_{i+1,j,k,l}, & x_{i,j,k,l} \mapsto x_{i,j+1,k,l}, & x_{i,j,k,l} \mapsto x_{i,j,k+1,l}, & x_{i,j,k,l} \mapsto x_{i,j,k,l+1}, \\ x_{i,j,k,l} \mapsto (-1)^i x_{i,j,k,l}, & x_{i,j,k,l} \mapsto (-1)^j x_{i,j,k,l}, & x_{i,j,k,l} \mapsto (-1)^k x_{i,j,k,l}, & x_{i,j,k,l} \mapsto (-1)^l x_{i,j,k,l}. \end{array}$$

Let us change the notation, renaming $e_0 = x_{0000}^\vee, e_1 = x_{0001}^\vee, \dots, e_7 = x_{0111}^\vee$, which is a basis of E , and $f_0 = x_{1000}^\vee, f_1 = x_{1001}^\vee, \dots, f_7 = x_{1111}^\vee$, which is a basis of F . Of course the renumbering must be

coherent with the Fano plane $\mathbb{P}^2(\mathbb{Z}_2)$, the x_{0ijk} being seen as points in $\mathbb{Z}_2^{\oplus 3}$. A direct examination leads to the following observation:

Proposition 9.6. *The space of H -invariants in Δ is the Cartan subspace \mathfrak{c} .*

9.4. Special trigonal genus four curves. A non-hyperelliptic genus 4 curve lies on a unique quadric surface in its canonical embedding in \mathbb{P}^3 , and the locus of curves C where this quadric is singular (i.e., C has a vanishing theta-null) has codimension 1 in the moduli space. This condition also implies that C has a unique g_3^1 , i.e. a unique degree 3 map to \mathbb{P}^1 . If we further impose that this map has a point with non-simple ramification (i.e. triple ramification), the locus loses another dimension, and hence has dimension 7.

Recall the following construction of genus four curves, starting from 8 general points on \mathbb{P}^2 . If we blow them up, we get a degree one del Pezzo surface, which comes with a natural degree 2 map (defined by twice its anticanonical divisor) to a quadric cone in \mathbb{P}^3 . The branch locus is a genus 4 curve C , with a vanishing theta-null. If the only g_3^1 of C has a point of triple ramification, then the corresponding 8 points lie on a cuspidal plane cubic curve ([GSW13, Remark 8.1]) - which is not the case for 8 generic points in \mathbb{P}^2 . Notice also that the 8 points in \mathbb{P}^2 define a pencil of plane cubics, which generically should contain a unique cuspidal plane cubic.

Let \mathfrak{c} be a Cartan subspace in Δ , which is also a Cartan subalgebra of \mathfrak{e}_8 . Thorne has shown how to construct a family of genus four curves parametrized by $\mathbb{P}(\mathfrak{c})/W$, and has given explicit equations of these curves [Tho13, Theorem 1.2 and Theorem 3.8]. It was observed by Steven Sam (personal communication) that those genus four curves have a g_3^1 (i.e. a vanishing theta-null) with a point of triple ramification. This gives a map from $\mathbb{P}(\mathfrak{c})$ (or $\mathbb{P}(\mathfrak{c})/W$) to the moduli space of genus four curves having a g_3^1 with a point of triple ramification.

Conjecture 9.7. *The quotient $\mathbb{P}(\mathfrak{c})/W$ is birational to the subvariety of the moduli space of genus four curves parametrizing curves with a g_3^1 with a point with triple ramification.*

We expect this birational morphism to even extend to a map from \mathfrak{c} to the configuration space of 8 points in \mathbb{P}^2 that lie on a cuspidal cubic in a W -equivariant way (the action of $W = W(E_8)$ on the configuration space of eight points in \mathbb{P}^2 by Cremona transformations was constructed in [DO88]). See also Remark 9.24 for more details.

9.5. Spinor bundles and orbital degeneracy loci. Exactly as in the previous sections we can consider Δ as the space of global sections of a spinor bundle on any orthogonal Grassmannian $OG(k, 16)$. Let us look more in the details the cases $k = 1$ and $k = 2$.

For $k = 1$, we get a spinor bundle \mathcal{S} modeled on a half-spin representation of Spin_{14} . Since the latter admits a degree eight invariant, this induces a map

$$\Gamma_1 : S^8\Delta \longrightarrow H^0(\mathbb{Q}_{14}, S^8\mathcal{S}) \longrightarrow H^0(\mathbb{Q}_{14}, \mathcal{O}(4)) = S^{(4)}\mathbb{C}^{16}.$$

So a generic element v of Δ defines a quartic section Qu_v of the quadric \mathbb{Q}_{14} . We will study this quartic in Section 9.7. We can do even better. As mentioned, the fibers of the spinor bundle \mathcal{S} on the quadric \mathbb{Q}_{14} are naturally isomorphic to the half-spin representation Δ_{14} of Spin_{14} . This is a parabolic representation with a finite number of orbits, classified in [KW11]. Let us focus on those of small codimension. Let us denote by Y_i the unique codimension i orbit closure in Δ_{14} for $i = 1, 5, 10, 14$. We get the following stratification:

$$Y_{14} \subset Y_{10} \subset Y_5 \subset Y_1 \subset \Delta_{14}.$$

Considering the associated orbital degeneracy loci

$$D_{Y_i}(v) = \{x \in \mathbb{Q}_{14} \mid v(x) \in Y_i \subset \Delta_{14}\},$$

we get a stratification

$$D_{Y_{14}}(v) \subset D_{Y_{10}}(v) \subset D_{Y_5}(v) \subset D_{Y_1}(v) = Qu_v \subset \mathbb{Q}_{14},$$

with $D_{Y_i}(v)$ of codimension i for general v . Following the analogy with the other cases presented in this paper, we expect the following conjecture to hold true:

Conjecture 9.8. *Let v be a general element in Δ .*

- i) $D_{Y_{10}}(v)$ is the Kummer fourfold $\text{Jac}(C)/\{\pm \text{id}\}$ of a non-hyperelliptic genus four curve C with a vanishing theta null and unique degree 3 map to \mathbb{P}^1 ;
- ii) $D_{Y_{14}}(v)$ consists of the 256 2-torsion points in the Kummer;
- iii) $D_{Y_5}(v)$ is the moduli space $\text{SU}_C(2, \mathcal{O}_C)$ of semistable rank 2 vector bundles on C with trivial determinant;
- iv) $D_{Y_1}(v) = Qu_v$ is the unique quartic which is singular along $\text{SU}_C(2, \mathcal{O}_C)$.

We provide some evidence for this conjecture by checking that the Hilbert polynomials are the expected ones.

Proposition 9.9. *We have the following:*

- i) The Hilbert polynomial of $D_{Y_5}(v)$ is the same as that of $\text{SU}_C(2, \mathcal{O}_C)$;
- ii) the Hilbert polynomial of $D_{Y_{10}}(v)$ is the same as that of the Kummer fourfold, $8x^4 + 8$;
- iii) $D_{Y_{14}}(v)$ consists of 256 points.

Proof. The three results follow from the fact that we know a locally free resolution of $\mathcal{O}_{D_{Y_\bullet}(v)}$, given in terms of homogeneous bundles on the quadric. Using the Borel-Weil-Bott Theorem and [vLCL92], we can deduce the result.

More precisely, following [BFMT20b], in order to obtain a locally free resolution of $\mathcal{O}_{D_{Y_\bullet}(v)}$ it is sufficient to *relativize* a locally free resolution of \mathcal{O}_{Y_\bullet} . These resolutions for $\bullet = 5, 10, 14$ were obtained in [KW11].

For instance, the resolution of $\mathcal{O}_{D_{Y_5}(v)}$ is

$$0 \rightarrow \mathcal{O}_{\mathbb{Q}_{14}}(-10) \rightarrow \mathcal{E}_{\omega_7}(-7) \rightarrow \mathcal{E}_{\omega_3}(-7) \rightarrow \mathcal{E}_{\omega_3}(-5) \rightarrow \mathcal{E}_{\omega_8}(-4) \rightarrow \mathcal{O}_{\mathbb{Q}_{14}} \rightarrow \mathcal{O}_{D_{Y_5}(v)} \rightarrow 0,$$

where \mathcal{E}_ω is the homogeneous vector bundle of highest weight ω on \mathbb{Q}_{14} , and $\omega_1, \dots, \omega_8$ are the fundamental weights of D_8 with Bourbaki's numeration. Our notation is such that the space of global sections of \mathcal{E}_ω is V_ω , the irreducible representation of Spin_{16} with highest weight ω . More generally, for any $k \geq 0$, $h^0(\mathbb{Q}_{14}, \mathcal{E}_\omega(k)) = \dim V_{\omega+k\omega_1}$ is given by the Weyl dimension formula as a polynomial $P_\omega(k)$. We then get the Hilbert polynomial of $D_{Y_5}(v)$ as

$$H_{D_{Y_5}(v)}(x) = P_0(x) - P_{\omega_8}(x-4) + P_{\omega_3}(x-5) - P_{\omega_3}(x-7) + P_{\omega_7}(x-7) - P_0(x-10).$$

Plugging-in the Weyl dimension formula, we deduce that

$$H_{D_{Y_5}(v)}(x) = 1 + \frac{233x}{70} + \frac{1979x^2}{420} + \frac{29041x^3}{7560} + \frac{31x^4}{15} + \frac{71x^5}{90} + \frac{13x^6}{60} + \frac{103x^7}{2520} + \frac{x^8}{210} + \frac{x^9}{3780}.$$

On the other hand, the Hilbert function, or polynomial, of $\text{SU}_C(2, \mathcal{O}_C)$ is given by the Verlinde formula, namely for genus four [BS93],

$$H_{\text{SU}_C(2, \mathcal{O}_C)}(x) = \left(\frac{x+2}{2}\right)^3 \sum_{i=1}^{x+1} \sin^{-6}\left(\frac{i\pi}{x+2}\right).$$

This is not clearly polynomial in x , so we compute the first ten values for $x = 0, \dots, 9$ and find 1, 16, 136, 800, 3611, 13328, 42048, 117072, 294525, 681472. These values coincide with those obtained for the degree nine polynomial $H_{D_{Y_5}(v)}$, so we can conclude that the two polynomials coincide. \square

Remark 9.10. Following the previous conjecture, Qu_v is a Heisenberg invariant quartic which is singular along $\text{SU}_C(2, \mathcal{O}_C)$. For a genus four curve with no vanishing theta-null, it was proved in [OP99] that the linear system $|2\Theta|$ embeds the moduli space in \mathbb{P}^{15} in such a way that it is contained in the singular locus of a unique Heisenberg-invariant quartic. That such a quartic should exist is suggested by the first values of the Hilbert polynomial: 1, 16, $\binom{16+1}{2}$, $\binom{16+2}{3} - 16$; in particular the minimal degree equations are the sixteen derivatives of the quartic, and the moduli space is not contained in any quadric. Moreover there is a conjectural interpretation of this quartic as a moduli space \mathcal{N}_C of *Galois* Spin_8 -bundles associated to a triple cover $C \rightarrow \mathbb{P}^1$ [OR99].

When C has a vanishing theta-null, \mathcal{N}_C is expected to be a double cover of \mathbb{Q}^{14} branched over a quartic section. Such a conjecture appears in [BGSW14], and is motivated by the similar behavior in genus three. According to Conjecture 9.7, $\mathbb{P}(\mathfrak{h})$ describes an even more degenerate situation, i.e. C has a vanishing theta-null with a triple ramification: in this case as well we expect $\text{SU}_C(2, \mathcal{O}_C)$ to be contained in a quadric

(image of a double cover from \mathcal{N}_C), and also in the singular locus of a unique Heisenberg-invariant quartic (section of the quadric!), i.e. exactly Qu_v .

For $k = 2$, the spinor bundle \mathcal{S} on $OG(2, 16)$ is modeled on a half-spin representation of Spin_{12} , which admits a degree four invariant. We thus get a map

$$\Gamma_2 : S^4\Delta \longrightarrow H^0(OG(2, 16), S^4\mathcal{S}) \longrightarrow H^0(OG(2, 16), \mathcal{O}(2)) = S_{\langle 2,2 \rangle} \mathbb{C}^{16}.$$

So a generic element v of Δ defines a quadric section Q_v of $OG(2, 16)$, which we will study more in detail in Section 9.6. As before, the fibers of the spinor bundle \mathcal{S} on $OG(2, 16)$ are naturally isomorphic to the half-spin representation Δ_{12} of Spin_{12} . This is a parabolic representation with a finite number of orbits, classified in [KW13]. Let us denote by Z_i the unique codimension i orbit closure in Δ_{12} for $i = 1, 7, 16$. We get the following stratification:

$$Z_{16} \subset Z_7 \subset Z_1 \subset \Delta_{12}.$$

Considering the associated orbital degeneracy loci

$$D_{Z_i}(v) = \{x \in OG(2, 16) \mid v(x) \in Z_i \subset \Delta_{12}\},$$

we get a stratification

$$D_{Z_{16}}(v) \subset D_{Z_7}(v) \subset D_{Z_1}(v) = Q_v \subset OG(2, 16),$$

with $D_{Z_i}(v)$ of codimension i for general v . Again, we believe in the following conjecture:

Conjecture 9.11. *Let v be a general element in Δ .*

- i) $D_{Z_{16}}(v)$ is the (smooth) moduli space $\text{SU}_C(2, \mathcal{O}_C(p))$ of semistable rank 2 vector bundles on C with determinant equal to $\mathcal{O}_C(p)$ for a certain point $p \in C$;*
- ii) $D_{Z_1}(v) = Q_v$ is the unique quadric which is singular along $\text{SU}_C(2, \mathcal{O}_C(p))$.*

Remark 9.12. As in the previous remark, it would be interesting to guess, and possibly prove, a similar statement for the general genus four curve. Is the moduli space contained in the singular locus of a special quadric section of the Grassmannian $G(2, 16)$? Recall that for a general genus 3 curve, there is already such a phenomenon [BBFM24]: the moduli space of semistable rank two vector bundles with odd determinant over a curve C of genus three embeds inside $G(2, 8)$, and it is contained in a unique quadric as its singular locus for generic C .

9.6. The Spin_{16} Coble quadric and its equation. We denote by

$$\gamma_2 := \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(\Theta_2) \subset \mathbb{P}(S_{\langle 2,2 \rangle} V)$$

the restriction of Γ_2 to the Cartan subspace \mathfrak{c} of Δ appearing in Proposition 9.1, sending a point to a Heisenberg invariant quadric section of the orthogonal Grassmannian. Here Θ_2 denotes the linear span of the image of γ_2 . The map γ_2 is W -equivariant since Γ_2 is Spin_{16} -equivariant; hence Θ_2 is a representation for W . Our second aim is to identify Θ_2 ; we will see that it is (once again!) a Macdonald representation.

Let us denote by c_1, \dots, c_8 the dual basis of h_1, \dots, h_8 . Moreover, to avoid confusion, we will sometimes denote by $x_0, \dots, x_7, y_0, \dots, y_7$ the dual basis of $e_0, \dots, e_7, f_0, \dots, f_7$.

Lemma 9.13. *The morphism Γ_2 factorizes as*

$$S^4\Delta \hookrightarrow S^2(S^2\Delta) \xrightarrow{S^2\eta} S^2(\bigwedge^4 V) \xrightarrow{\zeta} S_{\langle 2,2 \rangle} V$$

where $\eta : S^2\Delta \rightarrow \bigwedge^4 V$ and $\zeta : S^2(\bigwedge^4 V) \rightarrow S_{\langle 2,2 \rangle} V$ are uniquely defined (up to scalar) by their Spin_{16} -equivariance.

Proof. The uniqueness follows from Schur's lemma and the decomposition of $S^2\Delta$ and $S^2(\bigwedge^4 V)$ into irreducible factors. From the fact that there exists a unique Spin_{16} -equivariant morphism $S^4\Delta \rightarrow S_{\langle 2,2 \rangle} V$ (verified in the same way) we deduce that the composition of the above maps is equal to Γ_2 or the trivial map. As the computations that follow in this section will show, the composition above is not trivial. \square

Note that the morphism ζ can be obtained by first contracting by the quadratic form:

$$(a_1 \wedge a_2 \wedge a_3 \wedge a_4, b_1 \wedge b_2 \wedge b_3 \wedge b_4) \mapsto \langle a_1 \wedge a_2, b_1 \wedge b_2 \rangle (a_3 \wedge a_4) (b_3 \wedge b_4) + \cdots,$$

where \langle, \rangle denotes as usual the autoduality induced by q . This yields a tensor in $S^2(\wedge^2 V)$, that we can project to $S_{22}V$, and then to $S_{(2,2)}V$; the first projection amounts to restricting quadrics on $\mathbb{P}(\wedge^2 V^\vee)$, first to the Grassmannian $G(2, V^\vee)$ (that is, we mod out by the Plücker relations), and then to the orthogonal Grassmannian. In the sequel we will write formulas involving quadrics that will always be considered as defined on the orthogonal Grassmannian.

The restriction from the usual Grassmannian kills a copy of S^2V inside $S^2(\wedge^2 V)$, obtained as the image of the morphism

$$v^2 \mapsto \sum_i (v \wedge e_i)(v \wedge f_i), \quad v \in V.$$

The equivariant complement $S_{(2,2)}V$ embeds in $S_{2,2}V$ as the kernel of the contraction map to S^2V induced by the quadratic form (as well as $S_{2,2}V$ embeds in $S^2(\wedge^2 V)$ as the kernel of the natural morphism to $\wedge^4 V$).

Alternatively, these extra relations can be deduced from the isotropy condition as follows. If a plane P in V has Plücker coordinates given by

$$\sum_{ij} \left(\frac{1}{2} (e_i e_j) e_i \wedge e_j + (e_i f_j) e_i \wedge f_j + \frac{1}{2} (f_i f_j) f_i \wedge f_j \right),$$

we can deduce P itself as being generated by the vectors obtained by contracting this tensor with a generic linear form $\phi = \sum_i (x_i e_i^* + y_i f_i^*)$. This contraction yields

$$\sum_{ij} \left((e_i e_j) x_i e_j + (e_i f_j) (x_i f_j - y_j e_i) + (f_i f_j) y_i f_j \right),$$

which is isotropic for any ϕ if and only if the following relations hold:

$$\begin{aligned} \sum_k \left((e_i e_k) (e_j f_k) + (e_j e_k) (e_i f_k) \right) &= \sum_k \left((f_i f_k) (f_j e_k) + (f_j f_k) (f_i e_k) \right) = 0, \\ \sum_k \left((e_i e_k) (f_j f_k) - (e_i f_k) (e_k f_j) \right) &= 0. \end{aligned}$$

As a consequence, consider the sums

$$I_0 := \sum_i (e_i f_i)^2, \quad I_1 = \sum_{i < j} (e_i f_i) (e_j f_j), \quad I_2 = \sum_{i < j} (e_i e_j) (f_i f_j), \quad I_3 = \sum_{i < j} (e_i f_j) (e_j f_i).$$

From the latter relations for $i = j$, plus the usual Plücker identities, we deduce that

$$(8) \quad I_2 = \frac{1}{2} I_1 + \frac{1}{4} I_0, \quad I_3 = \frac{1}{2} I_1 - \frac{1}{4} I_0.$$

The morphism η . We will first need to understand η and its actions on monomials $h_i h_j$. Notice that each h_i is obtained as the sum of two pure spinors associated to complementary isotropic subspaces. Let us denote by E_i and E'_i these two subspaces. For instance,

$$E_1 = \langle e_0, e_1, e_2, e_3, f_4, f_5, f_6, f_7 \rangle \quad \text{and} \quad E'_1 = \langle f_0, f_1, f_2, f_3, e_4, e_5, e_6, e_7 \rangle,$$

while $E_8 = E$ and $E'_8 = F$. For two distinct indices k, l we denote

$$E_{kl} := E_k \cap E_l, \quad E'_{kl} := E'_k \cap E'_l, \quad E''_{kl} := E'_k \cap E_l, \quad E'''_{kl} := E_k \cap E'_l.$$

Notice that these four subspaces are always four-dimensional. They consist of two pairs of dual spaces (in the sense that they are put in duality by the restriction of the quadratic form), and we will denote them in a such a way that these two pairs are (E_{ab}, E''_{ab}) and (E'_{ab}, E'''_{ab}) . If e_i, e_j, e_k, e_l (resp. f_i, f_j, f_k, f_l or e_i, e_j, f_k, f_l) is a basis of E_{ab}^\bullet with $i < j < k < l$ (resp. $i < j < k < l$ or $i < j, k < l$), let $s_{E_{ab}^\bullet} = e_{ijkl} \in \wedge^4 V$ (resp. f_{ijkl} or $e_{ij} \wedge f_{kl}$). We will denote

$$I_1^k := \{i < j \mid (e_i, e_j \in E_k) \text{ or } (e_i, e_j \in E'_k)\}, \quad I_2^k := \{i < j \mid (i, j) \notin I_1^k\}.$$

Finally, it will be convenient to introduce the notation $\epsilon_k(i, j) := 1$ if $(i, j) \in I_1^k$, $\epsilon_k(i, j) := -1$ otherwise.

Proposition 9.14. *We have the following formulas:*

$$\eta(h_k^2) = \frac{1}{2} \sum_{i < j} \epsilon_k(i, j) e_{ij} \wedge f_{ij}, \quad \eta(h_k h_l) = s_{E_{kl}} + s_{E'_{kl}} + s_{E''_{kl}} + s_{E'''_{kl}}.$$

For instance, we get

$$\eta(h_1 h_2) = e_{01} f_{67} + e_{23} f_{45} + e_{45} f_{23} + e_{67} f_{45}, \quad \eta(h_1 h_8) = e_{0123} + e_{4567} + f_{0123} + f_{4567}.$$

Proof. In terms of a basis (a_i) of V_{16} , with dual basis (b_i) , we have

$$\eta(\delta_1 \delta_2) = \frac{1}{2} \sum_{i < j < k < l} \left(\langle \delta_1, (a_i \wedge a_j \wedge a_k \wedge a_l) \cdot \delta_2 \rangle + \langle \delta_2, (a_i \wedge a_j \wedge a_k \wedge a_l) \cdot \delta_1 \rangle \right) b_i \wedge b_j \wedge b_k \wedge b_l.$$

Using the basis $(e_i, f_j)_{i,j}$ and plugging in δ_1 and δ_2 the elements h_1, \dots, h_8 , one obtains the statement. \square

We can now state our final results for $\gamma_2(h) = \sum_{\alpha} c_{\alpha} P_{\alpha}$, where P_{α} is the quadric on the orthogonal Grassmannian that is the coefficient of the monomial c_{α} . In fact the Q_{α} will not be independent, and it is more satisfactory to write

$$\gamma_2(h) = \sum_{\beta} \kappa_{\beta}(c) Q_{\beta},$$

where the Q_{β} 's are independent and their coefficients $\kappa_{\beta}(c)$ are some quartic polynomials. The following result is proved in Section 12, where one can find the necessary definitions of $Quad$, $F_{ijkl}^{(u)}$, $F_{ij|kl}^{(1u)}$ and $F_{ij|kl}^{(2u)}$, R_a and R'_a .

Proposition 9.15. *The equation $\gamma_2(h)$ of the Coble quadric is given by the sum of 84 terms:*

- 14 terms parametrized by the quadruples $\beta = (ijkl)$ from $Quad$, for which

$$\kappa_{\beta}(c) = c_i c_j c_k c_l, \quad Q_{\beta} = 24 \sum_{u=1}^8 \det(F_{ijkl}^{(u)})^2.$$

- 56 terms parametrized by β consisting of the following data: a pair ij , and two of the three quadruples in $Quad$ that contain it, say $ijkl$ and $ijmn$, but not $ijop$; then

$$\kappa_{\beta}(c) = c_i c_j (c_k^2 + c_l^2 - c_o^2 - c_p^2), \quad Q_{\beta} = -6 \sum_{u=1}^4 \det(F_{ij|kl}^{(1u)}) \cdot \det(F_{ij|kl}^{(2u)}).$$

- 7 terms parametrized by $\beta = a$ a point in the (dual) Fano plane, with

$$\kappa_{\beta}(c) = c_a^4 - c_a^2 \sum_{1 \leq b \neq a \leq 7} c_b^2 + \sum c_u^2 c_v^2, \quad Q_{\beta} = 6R_a - 3I_1 + \frac{3}{2}I_0,$$

where the sum is over the three lines (auw) through a .

- 7 terms again parametrized by $\beta = a$ a point in the (dual) Fano plane, with

$$\kappa_{\beta}(c) = c_a^4 - c_a^2 \sum_{b \neq a} c_b^2 - (c_8^4 - c_8^2 \sum_{b \neq 8} c_b^2), \quad Q_{\beta} = 6R_a + 12R'_a - 5I_1 + \frac{1}{2}I_0 - 2I_2.$$

Proof. The result follows at once from Lemma 12.2, 12.3 and Corollary 12.13. \square

A natural way to interpret γ_2 , since it is equivariant under the Weyl group action, is to understand it as providing an isomorphism (sending each $\kappa_{\beta}(c)$ to Q_{β}) between an invariant space Θ_2 of quartic polynomials on the Cartan subspace, and a space of quadrics on the orthogonal Grassmannian. We have enough information to identify this representation of the Weyl group of E_8 .

Proposition 9.16. *The space Θ_2 is an irreducible Macdonald representation of dimension 84.*

Proof. Since Θ_2 contains some quartics of the form $c_i c_j c_k c_l$ for i, j, k, l distinct, it has to contain a nontrivial Macdonald representation. Since Θ_2 is generated by the 84 quartics κ_{β} , the following Lemma implies that its dimension is in fact equal to 84, the κ_{β} being a basis, and that Θ_2 is equal to this Macdonald representation. \square

Lemma 9.17. *Consider the Macdonald representations of the Weyl group of type E_8 generated by the orbit of some monomial $c_i c_j c_k c_l$ for i, j, k, l pairwise distinct. There are only two such representations and their dimensions are 84 and 210.*

Proof. That there are at most two such representations is a consequence of the fact that there are only two orbits (under the action of the Weyl group) of root subsystems of type $4A_1$ inside E_8 (Lemma 9.3). An explicit computation with [GS] shows that these orbits generate spaces of quartics of respective dimensions 84 and 210 (see attached file `gopel_e8`). \square

Note that the dimension of $S^4 \mathfrak{h}^\vee$ is $330 = 84 + 210 + 36$, and that 36 is the dimension of $S^2 \mathfrak{h}^\vee$, which embeds inside $S^4 \mathfrak{h}^\vee$ by multiplication with the Cartan-Killing form.

The Göpel variety of the quadric. Let us denote by \mathcal{G}_2 the Göpel variety of Spin_{16} , i.e. the image via γ_2 of $\mathbb{P}(\mathfrak{c})$ inside $\mathbb{P}(\Theta_2)$.

Theorem 9.18. *The rational map $\gamma_2 : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathcal{G}_2 \subset \mathbb{P}(\Theta_2) \cong \mathbb{P}^{83}$ is a birational morphism.*

Proof. The computations above gives an explicit expression of γ_2 in terms of coordinates; however, for our computations it was easier to use the expression given by the program in the file `gopel_e8`. One can plug this in [GS] and compute the degree of this rational map, which turns out to be equal to one, and its base locus, which turns out to be empty. \square

9.7. The Spin_{16} Coble quartic. Recall that we have also defined a Coble type hypersurface via the Spin_{16} -equivariant map

$$\Gamma_1 : S^8 \Delta \longrightarrow S^{(4)}V.$$

Let us give some results about the map

$$\gamma_1 := \Gamma_1|_{\mathfrak{c}} : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathbb{P}(\Theta_1) \subset \mathbb{P}(S^{(4)}V),$$

where Θ_1 is defined as the linear span of the image of γ_1 . The map γ_1 is W -equivariant since Γ_1 is Spin_{16} -equivariant, hence Θ_1 is a representation for W . As the previous section has shown, explicitly computing the equations of the Coble quartic in this case is quite challenging. Since we are interested in the algebro-geometric consequences, we just give some structural results, leaving the rest to future mathematicians/programmers.

Lemma 9.19. *The map Γ_1 factorizes through Γ_2 and a morphism $\eta_2 : S^2(S_{(2,2)}V) \rightarrow S^{(4)}V$.*

Proof. The result follows from a computation with [vLCL92]. Indeed, one checks that there exists a unique (up to scalar) equivariant morphism $S^8 \Delta \rightarrow S^{(4)}V$, as well as a unique equivariant morphism $S^2(S_{(2,2)}V) \rightarrow S^{(4)}V$. This implies the claim. \square

As a consequence, it is in principle possible to derive the equation of the Spin_{16} Coble quartic from Proposition 9.15. Indeed the morphism $\eta_2 : S^2(S_{(2,2)}V) \rightarrow S^{(4)}V$ is easy to define in terms of contraction with the invariant quadratic form, by mapping

$$(9) \quad (a_{ij} a_{kl})(a_{mn} a_{op}) \mapsto \langle a_i, a_m \rangle \langle a_k, a_o \rangle a_j a_l a_n a_o + \dots$$

Of course we expect to end up once again with a Macdonald representation. Let us note the following:

Lemma 9.20. *The unique Macdonald W -representation Θ'_1 generated by eight orthogonal roots in E_8 has dimension 50.*

Proof. The uniqueness follows from Lemma 9.3. The dimension statement is a computation with [GS] (see attached file `gopel_e8`). \square

Lemma 9.21. *The space Θ_1 contains the Macdonald W -representation Θ'_1 .*

Proof. We claim that the coefficient of $e_1^4 \in S^{(4)}V$ in $\gamma_1(h)$ is a non-zero multiple of $c_1 c_2 \cdots c_8$; this implies the claim. By formula (9), the only nonzero contributions to e_1^4 are obtained by multiplying two terms among the first 14 quadrics from Proposition 9.15. More precisely, they can only be obtained by multiplying quadrics corresponding to complementary coefficients c_{ijkl} and c_{mnop} , where $\{i, j, k, l, m, n, o, p\} = \{1, \dots, 8\}$. One checks that each such product contributes by a positive integer, and

the result follows in S^4V . This has to remain true in $S^{(4)}V$ since we mod out by products of the invariant quadric $q = \sum_i e_i f_i$, which cannot affect the coefficient of e_1^4 . \square

Theorem 9.22. *We have an isomorphism $\Theta_1 \simeq \Theta'_1$.*

Proof. Let H denote the genus 4 (and level 2) Heisenberg group. By the explicit action of H , described in Section 9.3, one sees that $H \subset SO(16)$ since it preserves the natural quadric hypersurface. The map Γ_1 is $SO(16)$ -equivariant, hence it is also H -equivariant. By Proposition 9.6, \mathfrak{c} is the H -invariant subspace of Δ , hence its image via Γ_1 is contained in the space of Heisenberg invariant quartics on the 14-dimensional smooth quadric. Now, let H' denote the *continuous* Heisenberg group, defined by the extension:

$$1 \rightarrow \mathbb{C}^* \rightarrow H' \rightarrow (\mathbb{Z}_2)^8 \rightarrow 1.$$

Let $N(H')$ denote the normalizer of H' . By [BL04, Ex. 6.14], $N(H')/H' \cong \mathrm{Sp}(2, 8)$. According to [DPvG09], the space of Heisenberg invariant quartics in \mathbb{P}^{15} is a 51 dimensional representation of $\mathrm{Sp}(2, 8)$. Recall also that $\mathrm{Sp}(2, 8)$ contains $O_8^+(\mathbb{Z}_2)$, the stabilizer subgroup of the quadric. By restriction to $O_8^+(\mathbb{Z}_2)$, the 51-dimensional $\mathrm{Sp}(2, 8)$ -representation decomposes into a 50 dimensional representation plus the one dimensional trivial representation of W , generated by the square of the invariant quadric. Of course the latter quartic is killed by restriction to the quadric, and we can identify the 50-dimensional Macdonald representation Θ'_1 with the space of Heisenberg invariant quartic sections of Q , hence with Θ_1 . \square

The Göpel variety of the quartic. Let us denote by \mathcal{G}_1 the *Göpel* variety, i.e. the image via γ_1 of $\mathbb{P}(\mathfrak{c})$ inside $\mathbb{P}(\Theta_1)$.

Theorem 9.23. *The rational map $\gamma_1 : \mathbb{P}(\mathfrak{c}) \dashrightarrow \mathcal{G}_2 \dashrightarrow \mathcal{G}_1 \subset \mathbb{P}(\Theta_1)$ is birational.*

Proof. We used the expression given by our program `göpel_e8` to find the equations of the composition of the morphism γ_1 with the projection $\Theta_1 \rightarrow \Theta'_1$. One can plug this in [GS] and compute the degree of the composition; it turns out to be equal to 1. \square

Again according to [GS15], the base locus of γ_1 has dimension one and degree 8960. Notice that there are 1120 subsystems of type A_2 in the root system E_8 , and that each subsystem defines a line in the projectivized Cartan algebra. We get thus 1120 lines, which are actually the orbit of the line ℓ_0 defined by the homogeneous ideal $I_{\ell_0} = (c_7 - c_4, c_1 + c_4, c_2, c_3, c_5, c_6)$, under the action of the Weyl group. According to [GS15], for each such line ℓ , the ideal of the base locus contains the homogeneous ideal I_ℓ^2 , which has degree 7. However, in the base locus we should get a subscheme of degree 8, rather than 7, supported at each line ℓ . This point remains to be elucidated.

Remark 9.24. As in [RSSS13, Equation 4.4], we conjecture that one can construct a $W(E_8)$ -equivariant birational morphism

$$(10) \quad f : \mathbb{P}(\mathfrak{h}) \cong \mathbb{P}^7 \rightarrow (\mathbb{P}^2)^8 // \mathrm{PGL}(3)$$

defined as

$$(11) \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 \\ d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & d_6^3 & d_7^3 & d_8^3 \end{pmatrix}$$

whose image is the 7 dimensional subvariety given by 8-tuplets lying on a cuspidal cubic ($zx^2 - y^3 = 0$). Then the map given by degree 8 polynomials obtained via the Macdonald representation should be the same as the composition of f with the 49-dimensional linear system given by the Göpel functions $\mathbb{P}^7 \rightarrow \mathbb{P}^{49}$. In the $g = 3$ case the fiber of the corresponding map had degree 24 since in each net there were 24 cuspidal cubics. In this case it seems that the degree should be just one, since for 8-general points on a cuspidal cubic, the pencil of cubics containing the 8 points should contain just one cuspidal cubic.

10. APPENDIX A: HEISENBERG GROUPS

Let us assume we are given a \mathbb{Z}_m -graded simple Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m-1}$, defined by an automorphism θ of \mathfrak{g} generating \mathbb{Z}_m , and a Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$. Let G_0 and G be simply connected groups whose Lie algebras are \mathfrak{g}_0 and \mathfrak{g} ; the image of G_0 in G is the subgroup G^θ of θ -invariant elements of G [Vin76]. We will use the following notation:

- (1) $Z \subset G$ is the center of G , or the kernel of its action on \mathfrak{g} ,
- (2) $Z_0 \subset G_0$ is the center of G_0 , or the kernel of its action on \mathfrak{g}_0 ,
- (3) $Z_0^{\mathfrak{g}} \subset Z_0$ is the kernel of the action of G_0 on \mathfrak{g} ,
- (4) $(Z_0^{\mathfrak{g}})^{\theta} \subset Z$ is the image of $Z_0^{\mathfrak{g}}$ in G ,
- (5) $K_0 \subset Z_0^{\mathfrak{g}}$ is the kernel of its map to (the center Z of) G .

So by definition there is an exact sequence

$$1 \rightarrow K_0 \rightarrow Z_0^{\mathfrak{g}} \rightarrow (Z_0^{\mathfrak{g}})^{\theta} \rightarrow 1.$$

We will use these groups to analyze our main character, which is not the centralizer $Z(\mathfrak{c})$ of \mathfrak{c} in G .

Definition 10.1. The *Heisenberg group* $Z^+(\mathfrak{c})$ of (\mathfrak{g}, θ) is the centralizer of \mathfrak{c} in G_0 .

Let r be the rank of \mathfrak{g} and r_1 the dimension of \mathfrak{c} . The smallest algebraic subalgebra of \mathfrak{g} containing $\mathfrak{c} \subset \mathfrak{g}_1$ is a Lie subalgebra \mathfrak{t} of commuting semisimple elements of dimension $r_1\varphi(m)$, where φ is Euler's function. More precisely, one shows that $\mathfrak{t} = \bigoplus_{(i,m)=1} \mathfrak{t}_i$, where $\mathfrak{t}_i = \mathfrak{t} \cap \mathfrak{g}_i$, $\mathfrak{t}_1 = \mathfrak{c}$, and each term in this decomposition has dimension r_1 . The torus T whose Lie algebra is \mathfrak{t} is also the closure of $\exp(\mathfrak{c})$, where $\exp : \mathfrak{g} \rightarrow G$ is the exponential map. In all our examples we will assume that T is a maximal torus of G , or equivalently that $r = r_1\varphi(m)$ (i.e. that θ -corank(\mathfrak{g}) = 0 as in [BM24]).

Proposition 10.2. *If m is prime and $r = r_1\varphi(m)$, then $Z(\mathfrak{c}) \cong T$ and there is an exact sequence of groups*

$$1 \rightarrow K_0 \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_m)^{r_1} \rightarrow 1.$$

Proof. An element in $Z(\mathfrak{c})$ acts as the identity on $\exp(\mathfrak{c})$, hence also on its closure T . Since T is a Cartan subalgebra it is its own centralizer, hence $Z(\mathfrak{c}) = T$. The image of $Z^+(\mathfrak{c})$ in G is $Z(\mathfrak{c}) \cap G^{\theta}$. Moreover K_0 is contained in $Z^+(\mathfrak{c})$, hence an exact sequence of groups

$$1 \rightarrow K_0 \rightarrow Z^+(\mathfrak{c}) \rightarrow Z(\mathfrak{c}) \cap G^{\theta} \rightarrow 1.$$

So we need to show that $T \cap G^{\theta} = (\mathbb{Z}_m)^{r_1}$. There is a primitive m th-root ζ of unity such that the automorphism θ acts on \mathfrak{t}_i by multiplication by ζ^i . This implies that one can find a basis $\{\alpha_{i,j}\}_{i,j}$ of \mathfrak{t} , where $1 \leq i \leq m-1$ and $1 \leq j \leq r_1$, on which θ acts as follows:

$$\theta(\alpha_{i,j}) = \alpha_{i+1,j} \text{ for } i = 1, \dots, m-2, \quad \theta(\alpha_{m-1,j}) = -\sum_i \alpha_{i,j}.$$

Taking exponentials one deduces the action of θ on $(t_{1,1}, \dots, t_{m-1,1}, \dots, t_{1,r_1}, \dots, t_{m-1,r_1}) \in T$:

$$\theta(t_{1,j}) = (t_{m-1,j})^{-1}, \quad \theta(t_{i,j}) = t_{i-1,j}(t_{m-1,j})^{-1} \text{ for } i = 2, \dots, m-1.$$

It is straightforward to check that $\theta(t_{1,1}, \dots, t_{m-1,r_1}) = (t_{1,1}, \dots, t_{m-1,r_1})$ if and only if $(t_{1,j}, \dots, t_{m-1,j}) = (\zeta^{i_j}, \zeta^{2i_j}, \dots, \zeta^{(m-1)i_j})$ for all j and for integers i_1, \dots, i_{r_1} , so that the set of such θ -invariant elements is isomorphic to $(\mathbb{Z}_m)^{r_1}$. \square

In the previous proof, when m is not prime, one can still say something. We will be interested in only one graded Lie algebra with m not prime, specifically $m = 4$; we will treat this case separately below.

10.0.1. *Explicit cases of interest.* Let us denote by V_n a complex vector space of dimension n , by Δ_{2n} the half-spin representation of the group Spin_{2n} , by A_1, A_2, A_3 three 3-dimensional vector spaces.

$S^{(2)}\mathbb{C}^{2g+2}$: Here $\mathfrak{g} = \mathfrak{sl}_{2g+2}$, $m = 2$, $r_1 = 2g+1$, $\mathfrak{g}_0 = \mathfrak{so}_{2g+2}$, $\mathfrak{g}_1 = S^{(2)}\mathbb{C}^{2g+2}$. The group Z_0 is isomorphic to \mathbb{Z}_4 when $g+1$ is odd and $\mathbb{Z}_2 \times \mathbb{Z}_2$ when $g+1$ is even, and it coincides with $Z_0^{\mathfrak{g}}$. The group $(Z_0^{\mathfrak{g}})^{\theta}$ is given by $\pm \text{id}$, so $K_0 \cong \mathbb{Z}_2$ is the fundamental group of SO_{2g+2} . We deduce two equivalent exact sequences defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow K_0 \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_2)^{2g+1} \rightarrow 1,$$

$$1 \rightarrow Z_0 \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_2)^{2g} \rightarrow 1.$$

$\wedge^4 V_8$: Here $\mathfrak{g} = \mathfrak{e}_7$, $m = 2$, $r_1 = 7$, $\mathfrak{g}_0 = \mathfrak{sl}_8$, $\mathfrak{g}_1 = \wedge^4 V_8$. The group Z_0 is isomorphic to \mathbb{Z}_8 , while $Z_0^{\mathfrak{g}} = 2Z_0 \cong \mathbb{Z}_4$. The group $(Z_0^{\mathfrak{g}})^{\theta} = Z$ is given by $\pm \text{id}$, so $K_0 \cong \mathbb{Z}_2$. We deduce the exact sequence defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow Z_0^{\mathfrak{g}} \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_2)^6 \rightarrow 1.$$

Δ_{16} : Here $\mathfrak{g} = \mathfrak{e}_8$, $m = 2$, $r_1 = 8$, $\mathfrak{g}_0 = \mathfrak{so}_{16}$, $\mathfrak{g}_1 = \Delta_{16}$. The group Z_0 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and it coincides with $Z_0^{\mathfrak{g}}$. The group Z is trivial so $K_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$. We deduce the exact sequence defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow Z_0 \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_2)^8 \rightarrow 1.$$

$\wedge^3 V_9$: Here $\mathfrak{g} = \mathfrak{e}_8$, $m = 3$, $r_1 = 4$, $\mathfrak{g}_0 = \mathfrak{sl}_9$, $\mathfrak{g}_1 = \wedge^3 V_9$. The group Z_0 is isomorphic to \mathbb{Z}_9 , while $Z_0^{\mathfrak{g}} = 3Z_0 \cong \mathbb{Z}_3$. The group Z is trivial so $K_0 = \mathbb{Z}_3$. We deduce the exact sequence defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow Z_0^{\mathfrak{g}} \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_3)^4 \rightarrow 1.$$

$A_1 \otimes A_2 \otimes A_3$: Here $\mathfrak{g} = \mathfrak{e}_6$, $m = 3$, $r_1 = 3$, $\mathfrak{g}_0 = \mathfrak{sl}_3 \times \mathfrak{sl}_3 \times \mathfrak{sl}_3$, $\mathfrak{g}_1 = A_1 \otimes A_2 \otimes A_3$. The group Z_0 is isomorphic to $(\mathbb{Z}_3)^3$, while $Z_0^{\mathfrak{g}} \cong (\mathbb{Z}_3)^2$. Moreover $Z \cong \mathbb{Z}_3$. Recall Cartan's description of the simply connected E_6 as the stabilizer of a cubic form K in 27 variables. This form can be made explicit by choosing three vector spaces A, B, C of dimension 3, letting $V = \text{Hom}(A, B) \oplus \text{Hom}(B, C) \oplus \text{Hom}(C, A)$ and defining $K(\gamma, \beta, \alpha) = \det(\alpha) + \det(\beta) + \det(\gamma) - \text{trace}(\alpha \circ \beta \circ \gamma)$. If $\zeta^3 = 1$, the triple $(\text{id}_A, \zeta \text{id}_B, \zeta^2 \text{id}_C) \in \text{SL}(A) \times \text{SL}(B) \times \text{SL}(C)$ is mapped to ζid_V , and we conclude that $(Z_0^{\mathfrak{g}})^{\theta} = Z$, so $K_0 \simeq \mathbb{Z}_3$. We deduce the exact sequence defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow \mathbb{Z}_3 \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_3)^3 \rightarrow 1,$$

or equivalently (using the exact sequence in which $Z_0^{\mathfrak{g}}$ appears):

$$1 \rightarrow Z_0^{\mathfrak{g}} \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_3)^2 \rightarrow 1.$$

$A_5^{\vee} \otimes \wedge^2 B_5^{\vee}$: Here $\mathfrak{g} = \mathfrak{e}_8$, $m = 5$, $r_1 = 2$, $\mathfrak{g}_0 = \mathfrak{sl}_5 \times \mathfrak{sl}_5$, $\mathfrak{g}_1 = A_5^{\vee} \otimes \wedge^2 B_5^{\vee}$. The group Z_0 is isomorphic to $(\mathbb{Z}_5)^2$, while $Z_0^{\mathfrak{g}} \cong \mathbb{Z}_5$ (which, inside $Z_0 \cong \mathbb{Z}_5^2$, is given by couples (ξ, ξ^2) , where ξ is a 5th root of unity). We have that Z is trivial and $K_0 \cong \mathbb{Z}_5$. We deduce the exact sequence defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow \mathbb{Z}_5 \rightarrow Z^+(\mathfrak{c}) \rightarrow (\mathbb{Z}_5)^2 \rightarrow 1.$$

$V_4 \otimes \Delta_{10}$: Here $\mathfrak{g} = \mathfrak{e}_8$, $m = 4$, $r_1 = 4$, $\mathfrak{g}_0 = \mathfrak{sl}_4 \otimes \mathfrak{so}_{10}$, $\mathfrak{g}_1 = V_4 \otimes \Delta_{10}$. Beware that in this case $m = 4$ is not prime. The group Z_0 is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, while $Z_0^{\mathfrak{g}} = \mathbb{Z}_4$. The group Z is trivial so $K_0 = Z_0^{\mathfrak{g}}$. We deduce the exact sequence defining $Z^+(\mathfrak{c})$:

$$1 \rightarrow Z_0^{\mathfrak{g}} \rightarrow Z^+(\mathfrak{c}) \rightarrow T \cap G^{\theta} \rightarrow 1,$$

where T and G^{θ} are as in Proposition 10.2. The action of θ on a basis $\alpha_{1,j}, \alpha_{2,j}$, $j = 1, 2, 3, 4$ of \mathfrak{t} is given by $\theta(\alpha_{1,j}) = \alpha_{2,j}$, $\theta(\alpha_{2,j}) = -\alpha_{1,j}$. The corresponding action on a θ -invariant torus T is given by the formula: $\theta((t_{1,j}, t_{2,j})) = (t_{2,j}^{-1}, t_{1,j})$, and the θ -fixed elements form a copy of $(\mathbb{Z}_2)^4$.

Remark 10.3. As we have seen above, in all cases, the exact sequence

$$1 \rightarrow Z_0^{\mathfrak{g}} \rightarrow Z^+(\mathfrak{c}) \rightarrow Z^+(\mathfrak{c})/Z_0^{\mathfrak{g}} \rightarrow 1$$

is such that the quotient $Z^+(\mathfrak{c})/Z_0^{\mathfrak{g}}$ is isomorphic to \mathbb{Z}_m^{2g} , for a certain integer g (except when $m = 4$, in which case we have $\mathbb{Z}_{m/2}^{2g}$). Throughout the paper we have seen that we can associate to each of the examples above a family of curves of genus g . Therefore we can identify abstractly the group $Z^+(\mathfrak{c})/Z_0^{\mathfrak{g}}$ with the group of m -torsion (or $m/2$ if $m = 4$) points of the Jacobian of the curve of genus g . In the case $m = 2$ this was already discovered by Thorne (see [Tho13, Theorem 1.3]).

Remark 10.4. It is not clear a priori whether the above definition gives Heisenberg groups in the sense of [MNN91]. In Proposition 10.2 we have shown that $Z^+(\mathfrak{c})$ is indeed an extension of an abelian group by $Z_0^{\mathfrak{g}}$, but $Z_0^{\mathfrak{g}}$ is not always cyclic and, more importantly, we have not shown that this extension is always non-split. When $m = 2$ the fact that we get indeed Heisenberg groups in the sense of [MNN91] is [Tho13, Theorem 1.3] (see also Remark 10.3). In order to check that we get indeed a non-trivial splitting: for the cases of $\wedge^3 V_9$ and $A_5^{\vee} \otimes \wedge^2 B_5^{\vee}$ we refer to [GS15]; for $A_1 \otimes A_2 \otimes A_3$ we refer to the discussion following Lemma 4.1, while we live aside the case $V_4 \otimes \Delta_{10}$ in this paper.

11. APPENDIX B: UNIQUENESS OF HYPERELLIPTIC COBLE QUADRICS

In this Appendix we prove Proposition 5.9. Because of Lemma 3.5, uniqueness follows from the conditions that

$$H^i(OG(k, 2n), S_{k1^i}(S^2U)(2)) = 0 \quad \forall i > 0,$$

and $H^0(OG(k, 2n), S_k(S^2U)(2)) = \mathbb{C}$. There is a classical plethysm formula

$$S_k(S^2U) = \bigoplus_{|\alpha|=k} S_{2\alpha}U,$$

which easily implies the latter assertion; indeed, for $\alpha = (1, \dots, 1)$ we get $S_{2\alpha}U(2) = \mathcal{O}$, and otherwise $S_{2\alpha}U(2)$ has no section by the Borel-Weil theorem. So we focus on the first assertion. We need to prove that for any component $S_\beta U$ of $S_{k1^i}(S^2U)$, with $i > 0$, we have

$$H^i(OG(k, 2n), S_\beta U(2)) = 0.$$

Such cohomology groups are governed by Bott's theorem, which in our situation applies as follows. In terms of the fundamental weights of \mathfrak{so}_{2n} , the highest weight of $S_\beta U(2)$ is

$$\lambda(\beta) = (\beta_{k-1} - \beta_k)\omega_1 + (\beta_{k-2} - \beta_{k-1})\omega_2 + \dots + (\beta_1 - \beta_2)\omega_{k-1} + (2 - \beta_1)\omega_k.$$

Then we need to evaluate $\lambda(\beta) + \rho$ on the positive roots α that have positive coefficient on α_k . There are two types of roots, that express in terms of simple roots either as a sum of distinct roots, or with a coefficient two. So suppose first that the coefficient on α_k is one, and that the first nonzero coefficient is on $\alpha_{k-\ell+1}$ for $1 \leq \ell \leq k$. For $\alpha = \alpha_{k-\ell+1} + \dots + \alpha_k$ we get

$$\langle \lambda(\beta) + \rho, \alpha \rangle = \ell - \beta_\ell + 2.$$

For the same coefficients on the first k roots, the maximal possible root is $\alpha_+ = \alpha_{k-\ell+1} + \dots + \alpha_k + 2\alpha_{k+1} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, for which

$$\langle \lambda(\beta) + \rho, \alpha_+ \rangle = \ell - \beta_\ell + 2(n - k).$$

Between these two extremes, there are a total of $2(n - k)$ roots, covering all the values between $\ell - \beta_\ell + 2$ and $\ell - \beta_\ell + 2(n - k)$.

Now suppose that the coefficient on α_k is two, which means that α is of the form $\alpha_{k-a+1} + \dots + \alpha_{k-b} + 2\alpha_{k-b+1} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, for $1 \leq b < a \leq k$. Then

$$\langle \lambda(\beta) + \rho, \alpha \rangle = 2(n - k + 1) + a + b - \beta_a - \beta_b.$$

So Bott's theorem tells us the following: if $S_\beta U(2)$ is not acyclic, then

- (1) each β_ℓ must be either $\leq \ell + 1$ or $\geq 2(n - k) + \ell + 1$; let ℓ_0 be the maximal integer for which the second possibility holds;
- (2) for any pair of distinct integers a, b between 1 and k , the sum $\beta_a + \beta_b$ must be different from $2(n - k + 1) + a + b$; let m_0 be the number of pairs for which it is bigger.

Then $H^i(OG(k, 2n), S_\beta U(2))$ is nonzero if and only if $i = 2(n - k)\ell_0 + m_0$.

Recall that i is such that $S_\beta U$ is a component of $S_{k1^i}(S^2U)$. Since $S_{k1^i}(S^2U) \subset S_k(S^2U) \otimes \wedge^i(S^2U)$, this implies that $S_\beta U$ is a component of a tensor product $S_{2\alpha}U \otimes S_\gamma U$ for some partition α of size k [Wey03, Proposition 2.3.8 (a)], and some partition γ of size $2i$ obtained by putting together a skew-partition of size i with its mirror along the diagonal, as in [Wey03, Proposition 2.3.9 (a)].

Proposition 11.1. *Suppose that $2n \geq \frac{k(k+5)}{2}$. Then $H^0(OG(k, 2n), \mathcal{J}_M^k(2)) = \mathbb{C}$.*

Proof. Following the discussion and notations above, the condition that $i = 2(n - k)\ell_0 + m_0$ implies $\ell_0 = 0$ since $i < \frac{k(k+1)}{2}$. But this means that $\beta_1 \leq 2$, and then $i = 0$. \square

The combinatorics becomes very messy when k gets bigger compared to n , but we can get more evidence that Proposition 11.1 should hold under much more general hypotheses by making explicit computations for k or n small enough. For example we can show:

Proposition 11.2. *Suppose that $k = 2, 3, 4, 5$ and $n \geq k + 2$. Then $H^0(OG(k, 2n), \mathcal{J}_M^k(2)) = \mathbb{C}$.*

Proof. With [vLCL92] we can compute the plethysms $S_{k,1,\dots,1}S^2U(2)$ explicitly. For instance, when $k = 2$ the resolution takes the following simple form:

$$0 \rightarrow S^2U(-1) \rightarrow S^2U \oplus S^4U(1) \rightarrow \mathcal{O} \oplus S^4U(2) \rightarrow \mathcal{J}_M^2(2) \rightarrow 0.$$

Then we can compute the cohomologies of these bundles by applying the Bott-Borel-Weil theorem, and conclude as before. \square

We can only deal with low values of k because the number of irreducible components of $S_{k,1,\dots,1}S^2U$ grows too fast with k . Just as an example, here is the resolution for $k = 3$:

$$\begin{aligned} 0 \rightarrow S_{2,2}U(-2) \oplus S^4U(-2) \rightarrow S_{4,4}U \oplus S_{3,2}U(-1) \oplus S^2U(-2) \oplus S_{4,1}U(-1) \oplus S_{6,2}U \rightarrow \\ \rightarrow S_{3,3}U \oplus S_{2,1}U(-1) \oplus S_{5,4}U(1) \oplus S_{4,2}U \oplus 2S^3U(-1) \oplus S_{6,3}U(1) \oplus S_{5,1}U \oplus S_{7,2}U(1) \rightarrow \\ \rightarrow S_{5,5}U(2) \oplus U(-1) \oplus S_{4,3}U(1) \oplus 2S_{3,1}U \oplus 2S_{5,2}U(1) \oplus S_{7,3}U(2) \oplus S_{6,1}U(1) \oplus S^7U(1) \rightarrow \\ \rightarrow S_{3,2}U(1) \oplus S^2U \oplus S_{5,3}U(2) \oplus S_{4,1}U(1) \oplus S_{6,2}U(2) \oplus S_{7,1}U(2) \rightarrow \\ \rightarrow \mathcal{O} \oplus S_{4,2}U(2) \oplus S^6U(2) \rightarrow \mathcal{J}_M^3(2) \rightarrow 0. \end{aligned}$$

However, we could also check that:

Proposition 11.3. *Suppose that $k = 2, 3$ and $n = k + 1$. Then $H^0(OG(k, 2n), \mathcal{J}_M^k(2)) = \mathbb{C}$.*

Remark 11.4. For bigger k the same result could possibly be true but we could not check it with our methods since we cannot control cancellations of cohomology groups in long exact sequences.

12. APPENDIX C: COMPUTATIONS FOR THE COBLE QUADRIC IN GENUS FOUR

In this Appendix we provide the details of the computations that lead to Proposition 9.15.

For $h = c_1h_1 + \dots + c_8h_8$, the goal is to compute

$$\Gamma_2(h) = \zeta(\eta(h^2), \eta(h^2)) = \sum_{ijkl} \zeta(\eta(h_ih_j), \eta(h_kh_l)).$$

So we need to calculate the image by ζ of the products $\eta(h_ih_j)\eta(h_kh_l)$. We will start with the case where i, j, k, l are distinct; recall from Lemma 9.3 that there must be two cases up to the Weyl group action. This is immediately visible when we compute that the image of $\eta(h_1h_2)\eta(h_4h_7)$ by ζ vanishes, while that of $\eta(h_1h_2)\eta(h_3h_4)$ is

$$(e_0f_7)^2 + (e_7f_0)^2 + (e_1f_6)^2 + (e_6f_1)^2 + (e_2f_5)^2 + (e_5f_2)^2 + (e_3f_4)^2 + (e_4f_3)^2.$$

A direct inspection leads to the following conclusion:

Lemma 12.1. *There are only 14 quadruples $ijkl$ of distinct integers for which the coefficient of $c_i c_j c_k c_l$ is non zero:*

$$\begin{array}{cccccc} 1234, & 1256, & 1367, & 1457, & 2357, & 2467, & 3456, \\ 5678, & 3478, & 2458, & 2368, & 1468, & 1358, & 1278. \end{array}$$

Denote by *Quad* this collection of quadruples; this is a *Steiner quadruple system* $S(3, 4, 8)$ that already appeared in [Man06] in connection with \mathfrak{e}_8 .

Note that the quadruples from the second series all contain 8, and that suppressing it we get the seven lines of a Fano plane. The quadruples from the first series are their complements. But the integer 8 has in fact no special rôle, and can be replaced by any other integer. A consequence is that any pair ij is contained in exactly three quadruples from *Quad*.

For each quadruple $ijkl$ from *Quad*, the nonzero pairwise intersections of the four subspaces E_{ij}^\bullet with the four subspaces E_{kl}^\bullet are eight subspaces $F_{ijkl}^{(u)}$, $1 \leq u \leq 8$. The resulting decomposition of V is not affected if we permute the indices, for example we obtain the same subspaces if we intersect E_{ik}^\bullet and E_{jl}^\bullet . Moreover each $F_{ijkl}^{(u)}$ has a basis given by two vectors from the basis $e_1, \dots, e_8, f_1, \dots, f_8$; we will denote by $\det(F_{ijkl}^{(u)})$ the wedge product of these two vectors. The order in which we take the wedge of the two vectors is as follows: $e_v \wedge e_w$ and $f_v \wedge f_w$ for $v < w$; $e_v \wedge f_w$ for any v, w . (This does not matter for the next Lemma but will for the next ones.)

Lemma 12.2. *Let $ijkl$ be one of the 14 quadruples from $Quad$. The coefficient of c_{ijkl} is*

$$24 \sum_{u=1}^8 \det(F_{ijkl}^{(u)})^2.$$

Proof. This is a direct computation; the coefficient 24 comes from counting all the permutations of $ijkl$. \square

Now, given a pair ij , denote by $ijkl$, $ijmn$, $ijop$ the three quadruples from $Quad$ that contain it. Let us focus on one of them, say $ijkl$. The eight planes $F_{ijkl}^{(u)}$ are divided in four pairs, each of which generates one of the four subspaces E_{ij} , E'_{ij} , E''_{ij} , E'''_{ij} . Let us denote these four pairs by $(F_{ij|kl}^{(1u)}, F_{ij|kl}^{(2u)})$ for $1 \leq u \leq 4$, where we introduce a bar between ij and kl to stress that, contrary to the previous case, $ijkl$ cannot be permuted.

Lemma 12.3. *For each quadruplet $ijkl$ from $Quad$, the coefficient of $c_i c_j c_k^2$ is the same as that of $c_i c_j c_l^2$, and is equal to*

$$-6 \sum_{u=1}^4 \det(F_{ij|kl}^{(1u)}) \cdot \det(F_{ij|kl}^{(2u)}).$$

Consider the three quadruples $ijkl$, $ijmn$, $ijop$ from $Quad$ that contain ij . The Plücker relations yield the identities

$$\det(F_{ij|kl}^{(1u)}) \cdot \det(F_{ij|kl}^{(2u)}) + \det(F_{ij|mn}^{(1u)}) \cdot \det(F_{ij|mn}^{(2u)}) + \det(F_{ij|op}^{(1u)}) \cdot \det(F_{ij|op}^{(2u)}) = 0.$$

Summing up over u , we deduce that the sum of the coefficient of $c_i c_j c_k^2$ (which is the same as that of $c_i c_j c_l^2$) with that of $c_i c_j c_m^2$ (or $c_i c_j c_n^2$) and that of $c_i c_j c_o^2$ (or $c_i c_j c_p^2$), is zero. As a consequence, we obtain (at most) two independant quadrics for each pair ij , hence a total amount of 56 quadrics in $S_{\langle 2,2 \rangle} V$ that we checked to be independent.

Lemma 12.4. *The coefficient of $c_a^3 c_b$ vanishes for any $a \neq b$.*

Proof. Recall from Lemma 9.3 that the Weyl group acts transitively on pairs of orthogonal roots, so we can suppose that $a = 8$ and $b = 1$. We have $\eta(h_1 h_8) = e_{0123} + e_{4567} + f_{0123} + f_{4567}$. A simple computation gives

$$\zeta\left(\sum_{i < j} e_{ij} f_{ij}, e_{0123}\right) = -2\left((e_{01})(e_{23}) - (e_{02})(e_{13}) + (e_{03})(e_{12})\right),$$

which again is zero by the Plücker identities. The other terms yield the same result, and our second claim follows. \square

In order to compute the remaining coefficients, we will need the following statement:

Lemma 12.5. *Let a, b, c define three distinct lines that are colinear in the dual Fano plane.*

$$(12) \quad \epsilon_a(i, j) \epsilon_b(i, j) \epsilon_c(i, j) = 1,$$

$$(13) \quad \sum_i \epsilon_a(i, j) \epsilon_b(i, k) = -\epsilon_a(j, k) - \epsilon_b(j, k),$$

$$(14) \quad \sum_i \epsilon_a(i, j) \epsilon_a(i, k) = 6\epsilon_a(j, k).$$

Remark 12.6. Notice that the first identity is trivially true if $(a, b, c) = (a, a, 8)$. This suggests considering $(a, a, 8)$ as a sort of line in a finite geometry that would extend the one of the Fano plane.

Our goal will be to express the coefficients c_a^4 and $c_a^2 c_b^2$ in terms of the following fourteen quadrics (where $c \neq 8$):

$$R_c := \sum_{\epsilon_c(i, j)=1} (e_i f_i)(e_j f_j), \quad R'_c := \sum_{\epsilon_c(i, j)=1} (e_i e_j)(f_i f_j).$$

We will also use

$$S_{a,b} = \sum_{\epsilon_a(i, j)=\epsilon_b(i, j)=1} (e_i f_i)(e_j f_j), \quad S'_{a,b} = \sum_{\epsilon_a(i, j)=\epsilon_b(i, j)=1} (e_i e_j)(f_i f_j).$$

The latter has the advantage of involving only four distinct terms each.

Lemma 12.7. *Given $a \neq b$, let d be line that they span in the Fano plane. The quadrics $S_{a,b}$ and $S'_{a,b}$ only depend on d , we denote them S_d and S'_d . Then*

$$R_c = \sum_{c \in d} S_d \quad \text{and} \quad R'_c = \sum_{c \in d} S'_d.$$

Proof. Given a line $d = (abc)$, we can split the remaining lines into three pairs: there are two lines other than d passing through a , respectively b and c . Adding $(0d)$, this gives a partition of $0, \dots, 7$ into four pairs, and one can check they are exactly those pairs (j, k) such that $\epsilon_a(j, k) = \epsilon_b(j, k) = 1$. In particular this does not depend on the choice of the pair (a, b) on the line d .

The second claim can be checked explicitly: starting from the line (123) defined by h_1 , we get the three pairs of lines (27) (passing through 1), (35) (passing through 2), and (46) (passing through 3). These three pairs yields three partitions of $0, \dots, 7$ into four pairs, respectively

$$(01)(23)(45)(67), \quad (02)(13)(46)(57), \quad (03)(12)(47)(56).$$

The union of these pairs is I_1^1 , and this exactly means that $R'_1 = S_1 + S_2 + S_3$. \square

Note that I_0, I_1, I_2, I_3 can be expressed in terms of the R_c, R'_c and S_d since

$$I_1 = \frac{1}{3} \sum_c R_c, \quad I_2 = \frac{1}{3} \sum_c R'_c,$$

and by (8), $I_3 = I_1 - I_2$ and $I_0 = 4I_2 - 2I_1$.

Now we compute

$$\zeta(e_{ij}f_{ij}, e_{ij}f_{ij}) = (e_i f_i)^2 + (e_j f_j)^2 + 2(e_i f_j)(e_j f_i) - 2(e_i e_j)(f_i f_j).$$

By the Plücker identity $(e_i f_j)(e_j f_i) = (e_i f_i)(e_j f_j) - (e_i e_j)(f_i f_j)$, this can be rewritten as

$$(15) \quad \zeta(e_{ij}f_{ij}, e_{ij}f_{ij}) = \left((e_i f_i) + (e_j f_j) \right)^2 - 4(e_i e_j)(f_i f_j),$$

$$(16) \quad \zeta(e_{ij}f_{ij}, e_{ik}f_{ik}) = (e_j f_j)(e_k f_k) \quad \text{for } j \neq k.$$

Lemma 12.8. *The coefficient of c_a^4 is $6R_a - 3I_1 + \frac{3}{2}I_0$.*

Proof. This coefficient is $\zeta(\eta(h_a^2), \eta(h_a^2))$, that we compute, up to a coefficient four, as

$$\sum_{i < j} \sum_{k < l} \epsilon_a(i, j) \epsilon_a(k, l) \zeta(e_{ij}f_{ij}, e_{kl}f_{kl}).$$

A first contribution to this sum is obtained for $(k, l) = (i, j)$, which by (15) and (8) yields $7I_0 + 2I_3 - 2I_2 = 6I_0$.

A second contribution is obtained when (k, l) and (i, j) share exactly one integer. There are four possibilities for this to occur: either $k = i$, or $k = j$, or $l = i$, or $l = j$. We get

$$\begin{aligned} & \sum_{i < j} \sum_{i < l \neq j} \epsilon_a(i, j) \epsilon_a(i, l) \zeta(e_{ij}f_{ij}, e_{il}f_{il}) + \sum_{i < j} \sum_{j < l} \epsilon_a(i, j) \epsilon_a(j, l) \zeta(e_{ij}f_{ij}, e_{jl}f_{jl}) + \\ & \sum_{i < j} \sum_{k < i} \epsilon_a(i, j) \epsilon_a(k, i) \zeta(e_{ij}f_{ij}, e_{ki}f_{ki}) + \sum_{i < j} \sum_{i \neq k < j} \epsilon_a(i, j) \epsilon_a(k, j) \zeta(e_{ij}f_{ij}, e_{kj}f_{kj}). \end{aligned}$$

Using 16 and permuting the indices, this leads to

$$\begin{aligned} & \sum_{l \neq j} (e_j f_j)(e_l f_l) \sum_{i < j, l} \epsilon_a(i, j) \epsilon_a(i, l) + \sum_{l < j} (e_j f_j)(e_l f_l) \sum_{l < i < j} \epsilon_a(i, j) \epsilon_a(l, i) + \\ & + \sum_{l \neq j} (e_j f_j)(e_l f_l) \sum_{i > j, l} \epsilon_a(j, i) \epsilon_a(l, i) + \sum_{l > j} (e_j f_j)(e_l f_l) \sum_{j < i < l} \epsilon_a(j, i) \epsilon_a(i, l). \end{aligned}$$

Splitting the first and third sum and permuting the indices, we can rewrite this as

$$2 \sum_{l < j} (e_j f_j)(e_l f_l) \sum_{i \neq j, l} \epsilon_a(i, j) \epsilon_a(i, l) = 12 \sum_{l < j} \epsilon_a(l, j) (e_j f_j)(e_l f_l),$$

where the last equality follows from (14). Since $2R_a = \sum_{i < j} \epsilon_a(i, j)(e_i f_i)(e_j f_j) + I_1$, this implies the claim. \square

Now we turn to the coefficients of monomials of type $c_a^2 c_b^2$. In the next two Lemmas a and b are two distinct integers between 1 and 7, that we consider as points in the Fano plane. Then they are joined by a unique line, containing a unique extra point that we denote by c .

Lemma 12.9. $\zeta(\eta(h_a h_b)^2) = -2R'_c + 2R_c - 2S_{a,b}$.

Proof. We know that $\eta(h_a h_b) = s_{E_{ab}} + s_{E'_{ab}} + s_{E''_{ab}} + s_{E'''_{ab}}$, where (E_{ab}, E''_{ab}) and (E'_{ab}, E'''_{ab}) are the only nonorthogonal pairs. So we get

$$\zeta(\eta(h_a h_b)^2) = 2\zeta(s_{E_{ab}}, s_{E''_{ab}}) + 2\zeta(s_{E'_{ab}}, s_{E'''_{ab}}).$$

This is equal to

$$-2 \sum_{\epsilon_a(i,j)=\epsilon_b(i,j)=1} e_{ij} f_{ij} + 2 \sum_{\epsilon_a(i,j)=\epsilon_b(i,j)=-1} (e_i f_j)(e_j f_i),$$

which, by the Plücker relations, gives

$$-2 \sum_{\epsilon_a(i,j)=\epsilon_b(i,j)=1} e_{ij} f_{ij} + 2 \sum_{\epsilon_a(i,j)=\epsilon_b(i,j)=-1} (e_{ij} f_{ij}(e_i f_i)(e_j f_j)),$$

which yields the result. \square

Lemma 12.10. $\zeta(\eta(h_a^2), \eta(h_b^2)) = R_c - R_a - R_b - 2R'_c - \frac{1}{4}I_0 + \frac{1}{2}I_1 + I_2$.

Proof. We proceed as in the proof of Lemma 12.8: up to a factor four, we need to compute

$$\sum_{i < j} \sum_{k < l} \epsilon_a(i, j) \epsilon_b(k, l) \zeta(e_{ij} f_{ij}, e_{kl} f_{kl}).$$

There is a first contribution coming from the case where $(k, l) = (i, j)$, which yields

$$\sum_{i < j} \epsilon_c(i, j) \left(((e_i f_i) + (e_j f_j))^2 - 4(e_i e_j)(f_i f_j) \right)$$

because of identity (12). Since $\sum_{i \neq j} \epsilon_c(i, j) = -1$ for any j , this can be rewritten as

$$- \sum_i (e_i f_i)^2 + 2 \sum_{i < j} \epsilon_c(i, j) (e_i f_i)(e_j f_j) - 4 \sum_{i < j} \epsilon_c(i, j) (e_i e_j)(f_i f_j).$$

There is a second contribution coming from the case where (k, l) and (i, j) have exactly one integer in common, which yields, after some rewriting,

$$\sum_{j \neq l} \left(\sum_{i \neq j, l} \epsilon_a(i, j) \epsilon_b(i, l) \right) (e_j f_j)(e_l f_l).$$

Using identity (13), this is also

$$-2 \sum_{j < l} (\epsilon_a(j, l) + \epsilon_b(j, l)) (e_j f_j)(e_l f_l).$$

Dividing by four and using the identities

$$2R_a = \sum_{i < j} \epsilon_a(i, j) (e_i f_i)(e_j f_j) + I_1, \quad 2R'_a = \sum_{i < j} \epsilon_a(i, j) (e_i e_j)(f_i f_j) + I_2,$$

we finally get the result. \square

The previous computation is also valid for $b = 8$ and $c = a$ (see Remark 12.6). We get:

Lemma 12.11. $\zeta(\eta(h_a^2), \eta(h_8^2)) = -2R'_a - \frac{1}{2}I_1 - \frac{1}{4}I_0 + I_2$.

One last computation remains to be done:

Lemma 12.12. $\zeta(\eta(h_a h_8)^2) = -2R'_a$.

Proof. Suppose for example that $a = 1$. We know that $\eta(h_1 h_8) = e_{0123} + e_{4567} + f_{0123} + f_{4567}$, so we need to compute $2\zeta(e_{0123}, f_{0123}) + 2\zeta(e_{4567}, f_{4567})$, which easily yields the result. \square

Denote by $Q_{a,b}$ the coefficient of $c_a^2 c_b^2$.

Corollary 12.13. *For $a \neq b$, let c denote the extra point on the line that joins them in the dual Fano plane; for $b = 8$, let $c = a$. The coefficient of $c_a^2 c_b^2$ is*

$$10R_c - 8S_{a,b} - 2R_a - 2R_b - 12R'_c - \frac{1}{2}I_0 + I_1 + 2I_2.$$

REFERENCES

- [AD09] Michela Artebani and Igor Dolgachev, *The Hesse pencil of plane cubic curves*, Enseign. Math. (2) **55** (2009), no. 3-4, 235–273.
- [BBFM23] Vladimiro Benedetti, Michele Bolognesi, Daniele Faenzi, and Laurent Manivel, *Hecke cycles on moduli of vector bundles and orbital degeneracy loci*, arXiv e-print math.AG/2310.06482, to appear in J. Algebraic Geom., 2023.
- [BBFM24] ———, *The Coble quadric*, Forum Math. Sigma **12** (2024), Paper No. e63, 25.
- [Bea03] Arnaud Beauville, *The Coble hypersurfaces*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 3, 189–194.
- [BFMT20a] Vladimiro Benedetti, Sara Angela Filippini, Laurent Manivel, and Fabio Tanturri, *Orbital degeneracy loci and applications*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **21** (2020), 169–206.
- [BFMT20b] ———, *Orbital degeneracy loci II: Gorenstein orbits*, Int. Math. Res. Not. IMRN (2020), no. 24, 9887–9932.
- [BGSW14] M. Bolognesi, L. Gruson, S. Sam, and J. Weyman, *Working notes on genus 4 curves*, unpublished, 2014.
- [BH16] Manjul Bhargava and Wei Ho, *Coregular spaces and genus one curves*, Camb. J. Math. **4** (2016), no. 1, 1–119.
- [Bho84] Usha N. Bhosle, *Moduli of orthogonal and spin bundles over hyperelliptic curves*, Compositio Math. **51** (1984), no. 1, 15–40. MR 734782
- [BL04] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, second ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004.
- [BM24] Vladimiro Benedetti and Laurent Manivel, *Discriminants of theta-representations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **25** (2024), no. 4, 2125–2148. MR 4873513
- [BMT21] Vladimiro Benedetti, Laurent Manivel, and Fabio Tanturri, *The geometry of the Coble cubic and orbital degeneracy loci*, Math. Ann. **379** (2021), no. 1-2, 415–440.
- [BS93] Aaron Bertram and András Szenes, *Hilbert polynomials of moduli spaces of rank 2. Vector bundles. II*, Topology **32** (1993), no. 3, 599–609.
- [Car85] Roger W. Carter, *Finite groups of lie type : conjugacy classes and complex characters*, Wiley, 1985.
- [dG25] Willem A. de Graaf, *A classification of four-tuples of spinors of a ten dimensional space*, preprint, 2025.
- [DO88] Igor V. Dolgachev and David Ortland, *Point sets in projective spaces and theta functions*, Astérisque (1988), no. 165, 210 pp. (1989).
- [Dol12] Igor V. Dolgachev, *Classical algebraic geometry*, Cambridge University Press, Cambridge, 2012, A modern view.
- [DPvG09] Francesco Dalla Piazza and Bert van Geemen, *Siegel modular forms and finite symplectic groups*, Adv. Theor. Math. Phys. **13** (2009), no. 6, 1771–1814.
- [DR77] Usha N. Desale and Sundararaman Ramanan, *Classification of vector bundles of rank 2 on hyperelliptic curves*, Invent. Math. **38** (1976/77), no. 2, 161–185.
- [FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.
- [FSM19] Eberhard Freitag and Riccardo Salvati Manni, *The Göpel variety*, Exp. Math. **28** (2019), no. 3, 284–291.
- [Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry.
- [GKZ08] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser, 2008, Reprint of the 1994 edition.
- [GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [GS15] Laurent Gruson and Steven V. Sam, *Alternating trilinear forms on a nine-dimensional space and degenerations of (3, 3)-polarized Abelian surfaces*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 3, 755–785.
- [GSW13] Laurent Gruson, Steven V. Sam, and Jerzy Weyman, *Moduli of abelian varieties, Vinberg θ -groups, and free resolutions*, Commutative algebra, Springer, New York, 2013, pp. 419–469.
- [HMSV09] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil, *The equations for the moduli space of n points on the line*, Duke Math. J. **146** (2009), no. 2, 175–226.
- [HMSV12] ———, *The geometry of eight points in projective space: representation theory, Lie theory and dualities*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 6, 1215–1244.
- [Kan99] Vassil Kanev, *Chordal varieties of Veronese varieties and catalecticant matrices*, J. Math. Sci. (New York) **94** (1999), no. 1, 1114–1125, Algebraic geometry, 9.
- [Kuz18] Alexander G. Kuznetsov, *On linear sections of the spinor tenfold. I*, Izv. Ross. Akad. Nauk Ser. Mat. **82** (2018), no. 4, 53–114.

- [KW11] Witold Kraskeiwicz and Jerzy Weyman, *Geometry of orbit closures for the representations associated to gradings of lie algebras of type e_8* , unpublished, 2011.
- [KW13] Witold Kraskiewicz and Jerzy Weyman, *Geometry of orbit closures for the representations associated to gradings of lie algebras of types e_7* , arXiv e-print math.AG/301.0720, 2013.
- [LM25] Yingqi Liu and Laurent Manivel, *On linear sections of the spinor tenfold II*, preprint, 2025.
- [LO13] Joseph M. Landsberg and Giorgio Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 569–606.
- [Mac72] Ian G. Macdonald, *Some irreducible representations of Weyl groups*, Bull. London Math. Soc. **4** (1972), 148–150.
- [Man06] Laurent Manivel, *Configurations of lines and models of Lie algebras*, J. Algebra **304** (2006), no. 1, 457–486.
- [MNN91] David Mumford, Peter Norman, and Madhav Nori, *Tata lectures on theta iii*, Progress in mathematics, n. 97, Birkhäuser Verlag AG, 1991.
- [New68] Peter E. Newstead, *Stable bundles of rank 2 and odd degree over a curve of genus 2*, Topology **7** (1968), 205–215.
- [Ng95] Kok Onn Ng, *The classification of $(3, 3, 3)$ trilinear forms*, J. Reine Angew. Math. **468** (1995), 49–75.
- [NR69] Mudumbai S. Narasimhan and Sundararaman Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. (2) **89** (1969), 14–51.
- [OP99] William Oxbury and Christian Pauly, *Heisenberg invariant quartics and $SU_C(2)$ for a curve of genus four*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 2, 295–319.
- [OR99] William M. Oxbury and Sundararaman Ramanan, *Trigonal curves and Galois $spin(8)$ -bundles*, arXiv e-print math/9907114, 1999.
- [Osh06] Toshio Oshima, *A classification of subsystems of a root system*, arXiv e-print math/0611904, 2006.
- [Pas09] Boris Pasquier, *On some smooth projective two-orbit varieties with Picard number 1*, Math. Ann. **344** (2009), no. 4, 963–987.
- [Ram81] Sundararaman Ramanan, *Orthogonal and spin bundles over hyperelliptic curves*, Proc. Indian Acad. Sci. Math. Sci. **90** (1981), no. 2, 151–166. MR 653952
- [RSSS13] Qingchun Ren, Steven V. Sam, Gus Schrader, and Bernd Sturmfels, *The universal Kummer threefold*, Exp. Math. **22** (2013), no. 3, 327–362.
- [SK77] Mikio Sato and Tatsuo Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.
- [ST54] Geoffrey C. Shephard and John A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.
- [Tho13] Jack A. Thorne, *Vinberg’s representations and arithmetic invariant theory*, Algebra Number Theory **7** (2013), no. 9, 2331–2368.
- [VE78] Èrnest B. Vinberg and Alexander G. Èlašvili, *A classification of the three-vectors of nine-dimensional space*, Trudy Sem. Vektor. Tenzor. Anal. **18** (1978), 197–233.
- [Vin76] Èrnest B. Vinberg, *The Weyl group of a graded Lie algebra.*, Izv. Akad. Nauk SSSR Ser. Mat. (1976), no. no. 3,, 488–526, 709.
- [vLCL92] Marc A.A. van Leeuwen, Arjeh M. Cohen, and Bert Lissner, *LiE, a package for Lie group computations*, Computer Algebra Nederland, Amsterdam, 1992.
- [Wey03] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003.

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