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A TOY MODEL FOR FREQUENCY CASCADE IN THE NONLINEAR SCHRÖDINGER EQUATION

RÉMI CARLES AND ERWAN FAOU

ABSTRACT. We present an elementary approach to observe frequency cascade on forced nonlinear Schrödinger equations. The forcing term (which may also appear as a potential term instead) consists of a constant term, perturbed by a modulated Gaussian well. Algebraic computations provide an explicit frequency cascade when time and space derivatives are discarded from the nonlinear Schrödinger equation. We prove stability results, showing that when derivatives are incorporated in the model, the initial algebraic solution may be little affected, possibly over long time intervals. Numerical simulations are provided, which support the analysis.

1. INTRODUCTION

1.1. Setting. The phenomenon of frequency cascade originates from the theory of turbulence in fluid mechanics [15, 16, 17] and in wave turbulence see e.g. [20, 22, 23]. A paradigmatic description in experimental physics can be viewed as a nonlinear system (typically a wave like system exemplified by the nonlinear Schrödinger equation) forced in a low frequency range, giving rise to some distribution of energy following some power law of the form $|\xi|^{-\nu}$. These power law rules can be explained by many physical arguments involving statistical models, random forcing, kinetic limit, etc.

The mathematical approach to justify the derivation of equations describing turbulent phenomena has been very intense, involving very sophisticated tools and analysis. The general strategy is to derive a domain a validity for a wave kinetic equation possessing power law solution. The analysis can be performed by considering random initial conditions for a dispersive equation set on a large torus [19, 3, 7, 6], leading to resonance analysis and continuous limit arguments. In a parallel approach, random forcing can be considered, [18, 8, 10], as well as model on the whole space [1, 12, 11, 9], with specific initial conditions.

Our goal here is to propose an elementary model, certainly too simple to meet the actual physics described in the above references, for which the frequency cascade phenomenon is rather explicit. We retain the idea of forcing a wave system at low frequency and recovering a power law energy distribution as stationary solution. More precisely, we consider the nonlinear Schrödinger equation on \mathbb{R}^d (or on the torus \mathbb{T}^d , see Remark 1.2), with an external forcing f ,

$$(1.1) \quad i\partial_t u + \varepsilon^2 \Delta u = |u|^{2\sigma} u - f, \quad x \in \mathbb{R}^d,$$

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where $\sigma > 0$, and we ask the following question: from a smooth forcing term f with Fourier transform $\hat{f}(\xi)$ localized around 0, can we obtain singular or quasi-singular solutions u exhibiting power law frequency $\hat{u}(\xi) \sim |\xi|^{-\nu}$ for some ν , at least in some frequency range?

Note that such a phenomenon must be nonlinear, as this cannot hold for the linear equation without the term $|u|^{2\sigma}u$. Moreover, if f is smooth, standard existence results show that the solution to (1.1) remains smooth in general, so a solution cannot exhibit a frequency behavior of the form $|\xi|^{-\nu}$ for *all* $\xi \in \mathbb{R}^d$, but only on some (possibly large) range of frequency. This leads us to the notion of quasi-singular solution, *i.e.* we propose a model where solutions can be of that type.

We consider two cases of forcing term f in (1.1):

- Time independent forcing, $f = f(x)$,
- Time periodic forcing, $f(t, x) = 2Q(x)e^{-3iPt}$, where P is a constant, in the case of a cubic nonlinearity $\sigma = 1$.

We do not consider the presence of other dissipative terms in the general presentation, but we could add a term of the form $i\nu\varepsilon^2\Delta u$ ($\nu > 0$) on the right hand side of (1.1). The stability analysis stated in Theorem 1.6 would remain valid (and could be possibly strengthened thanks to dissipative effects), see also Remark 4.4.

In many situations, the starting point to analyze systems of the form (1.1) is the linear equation, or a specific solution of the stationary Schrödinger equation (like solitons), from which perturbative arguments are developed. Our approach is somehow the opposite, in the sense that we first remove the differential operators from (1.1). We obtain solutions by algebraic means, where singularities or quasi-singularities of Puiseux type $z^{1/p}$ may naturally develop, and then examine their stability when the differential operators are incorporated.

1.2. Cascade cartoon. When f does not depend on time, we consider, for $x \in \mathbb{R}^d$, $\delta \geq 0$, the forcing

$$f(x) = 1 - e^{-\delta}e^{-|x|^2/2}.$$

The algebraic solution U that we consider solves

$$|U|^{2\sigma}U = f.$$

We first remark that U can be taken independent of time, and that f being positive, the above equation has at least one real-valued solution, given by

$$U(x) = f(x)^{\frac{1}{2\sigma+1}} \in \mathbb{R}.$$

We note that since $\sigma > 0$, the power $\frac{1}{2\sigma+1}$ lies in $(0, 1)$. If we assume $\delta = 0$, we can write

$$(1.2) \quad f^{\frac{1}{2\sigma+1}} - 1 = |x|^{\frac{2}{2\sigma+1}}\psi(x),$$

where ψ is smooth everywhere, and rapidly decaying at infinity. We normalize the Fourier transform as

$$\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \psi(x) dx, \quad \psi \in \mathcal{S}(\mathbb{R}^d),$$

so that Plancherel formula reads $\|\hat{\psi}\|_{L^2(\mathbb{R}^d)} = \|\psi\|_{L^2(\mathbb{R}^d)}$. Taking the Fourier transform in (1.2) (as tempered distributions), we can write

$$\mathcal{F}\left(f^{\frac{1}{2\sigma+1}} - 1\right) = \mathcal{F}\left(|x|^{\frac{2}{2\sigma+1}}\right) * \mathcal{F}(\psi).$$

In view of e.g. [14, Theorem 7.1.16], $\mathcal{F}\left(|x|^{\frac{2}{2\sigma+1}}\right)$ is $\left(-d - \frac{2}{2\sigma+1}\right)$ -homogeneous. Since the map $x \mapsto |x|^{\frac{2}{2\sigma+1}}$ is radially symmetric, so is its Fourier transform, and

$$\mathcal{F}\left(|x|^{\frac{2}{2\sigma+1}}\right)(\xi) = c(d, \sigma)|\xi|^{-d - \frac{2}{2\sigma+1}},$$

which enjoys an explicit frequency decay. The idea now is that for $\delta > 0$, a similar cascade remains, up to restricting the range of frequencies, and that the solution thus obtained is sufficiently stable so the phenomenon remains when going back to the nonlinear Schrödinger equation (1.1).

1.3. Properties of the algebraic solution. The rigorous meaning of the comparison symbols used in this article is as follows:

Notation. For $a, b \geq 0$, we write $a \approx b$ if there exists $C > 1$ such that

$$\frac{1}{C}a \leq b \leq Ca.$$

We write $a \ll b$ if there exists $\eta \in (0, 1)$ (small enough) such that

$$a \leq \eta b.$$

Our first result concerns the algebraic solution, denoted by U above: we check that an explicit frequency cascade occurs, on a frequency range depending on the parameter δ .

Theorem 1.1. *Let $d \geq 1$, and for $\delta > 0$, consider*

$$f(x) = 1 - e^{-\delta} e^{-|x|^2/2}.$$

Let $\alpha \in (0, 1)$. Then the Fourier transform of $f^\alpha - 1$ satisfies

$$|\mathcal{F}(f^\alpha - 1)(\xi)| \approx |\xi|^{-2\alpha - d},$$

provided that ξ is restricted to a region of the form

$$0 < \xi_0^2 \leq |\xi|^2 \ll \frac{1}{\delta},$$

for $\xi_0 > 0$ arbitrary.

Remark 1.2. [Spatially periodic case] In the case where x belongs to the circle, $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, the above result can be adapted, by considering

$$\mathbf{f}(x) = 1 - e^{-\delta} e^{\cos x - 1}.$$

Computations are similar to those presented in the proof of Theorem 1.1, up to the introduction of modified Bessel functions of the first kind, and, again, the use of Stirling formula. Detailed computations are cumbersome though, and we leave out this framework here. We simply note that when the radius of the circle becomes unbounded, a natural connection with the model considered on \mathbb{R} appears: for $x \in \mathbb{T}_L = \mathbb{R}/(2\pi L\mathbb{Z})$, we may set

$$\mathbf{f}_L(x) = 1 - e^{-\delta} e^{L^2(\cos(x/L) - 1)},$$

and we check the pointwise convergence

$$\mathbf{f}_L(x) \xrightarrow{L \rightarrow \infty} 1 - e^{-\delta} e^{-x^2/2} = f(x).$$

In the case where the forcing term is periodic in time ($P \neq 0$), we choose a profile Q which may seem surprising at first sight, but which is constructed in order to make a connection with the previous result (see Section 3 for details).

Theorem 1.3. *Let $d \geq 1$, $\sigma = 1$, and $\varepsilon = 0$ in (1.1). Setting $f(t, x) = 2Q(x)e^{-3it}$ with*

$$Q(x) = 4e^{-\frac{3\delta}{2} - \frac{3|x|^2}{4}} - 3e^{-\frac{\delta}{2} - \frac{|x|^2}{4}}, \quad \delta > 0,$$

(1.1) has three solutions of the form $u_k(t, x) = U_k(x)e^{-3it}$ with U_k real-valued, $k = 0, 1, 2$. Two of these three profiles enjoy the same frequency cascade as in Theorem 1.1 (with $\alpha = \frac{1}{2}$), say U_1 and U_2 :

$$\mathcal{F}(U_k)(\xi) \approx |\xi|^{-1-d}, \quad k = 1, 2,$$

provided that ξ is restricted to a region of the form

$$0 < \xi_0^2 \leq |\xi|^2 \ll \frac{1}{\delta},$$

for $\xi_0 > 0$ arbitrary.

1.4. Stability. Once the algebraic solution is constructed, and its frequency cascade is analyzed, we have to go back to the nonlinear Schrödinger equation (1.1). It turns out that the presence of the parameter δ introduced above is crucial in order not to destroy the frequency cascade, when at least the Laplacian term $\varepsilon^2 \Delta$ is present.

For rather general σ and d , we show that if $\varepsilon > 0$ is sufficiently small (compared to $\delta > 0$), then incorporating the Laplacian term from (1.1) into the algebraic equation does not destroy the initial structure, to construct a stationary solution. At this stage, we need a smooth nonlinearity, since we will consider higher order derivatives, hence the assumption that σ is an integer in the next result.

Theorem 1.4. *Let $d \geq 1$. Suppose that*

- *Either $\sigma > 0$ is an integer, $\alpha = \frac{1}{2\sigma+1}$, $f =: 2Q$ is like in Theorem 1.1, $P = 0$, and $u_0 = f^\alpha$,*
- *Or $P = \sigma = 1$, $\alpha = \frac{1}{2}$, Q is like in Theorem 1.3, and $u_0 = U_1$ (see Section 3.2 for the precise definition of U_1).*

Then there exists $K > 0$ such that if $0 < \varepsilon \ll \delta^K \ll 1$, there exists a real-valued solution $u_\varepsilon = u_0 + v$ to

$$(1.3) \quad -\varepsilon^2 \Delta u_\varepsilon - 3P u_\varepsilon + |u_\varepsilon|^{2\sigma} u_\varepsilon = 2Q,$$

such that

$$\sup_{\xi \in \mathbb{R}^d} \left| |\xi|^{2\alpha+d} (\widehat{u}_\varepsilon(\xi) - \widehat{u}_0(\xi)) \right| \lesssim \frac{\varepsilon}{\delta^K} \ll 1.$$

In particular, u_ε enjoys the same frequency cascade as u_0 , as described in Theorems 1.1 and 1.3, respectively.

Remark 1.5 (Focusing case). If instead of (1.1), where the nonlinearity is defocusing, we consider

$$(1.4) \quad i\partial_t u + \varepsilon^2 \Delta u = -|u|^{2\sigma} u + f, \quad x \in \mathbb{R}^d,$$

then Theorems 1.1 and 1.6 (when $u_0 = f^{\frac{1}{2\sigma+1}}$) remain unchanged, as will be clear from the proofs. In the case where f is periodic in time, $f(t, x) = 2Q(x)e^{-3iPt}$, then the only difference in Theorem 1.3 is that $P = 1$ has to be replaced by $P = -1$, and

then, with this modification, the whole of Theorem 1.6 remains valid. The main difference concerns Theorem 1.4, since (1.3) is replaced by (recall that $P = -1$ now, and $\sigma = 1$)

$$(1.5) \quad -\varepsilon^2 \Delta u_\varepsilon + 3u_\varepsilon - |u_\varepsilon|^2 u_\varepsilon = -2Q.$$

Linearizing the left hand side about u_0 , we find the operator $-\varepsilon^2 \Delta + 3 - 3U_k^2$, where for $k = 0, 1, 2$, U_k are the solutions provided by Theorem 1.3. Unlike what happens in the defocusing case, this operator is elliptic provided that U_k^2 remains uniformly smaller than 1. Resuming the proof of Theorem 1.4 then shows for $k = 0, 1, 2$, each U_k^2 takes larger values than 1, and Theorem 1.3 may not be true in the focusing case.

When the nonlinearity is smooth ($\sigma \in \mathbb{N}$) and L^2 -subcritical or critical (like in the statement below), Strichartz estimates make it possible to prove the following dynamical stability result:

Theorem 1.6. *Let $d = 1$ with $\sigma = 1$ or $\sigma = 2$, or $d = 2$ with $\sigma = 1$. Denote by $\mathbf{u}_0 = f^{\frac{1}{2\sigma+1}}$ when f is like in Theorem 1.1, and $\mathbf{u}_0 = U_k(x)e^{-3it}$ ($k = 2, 3$) when $\sigma = 1$ and f is like in Theorem 1.3. Suppose that u solves (1.1) with*

$$(1.6) \quad u|_{t=0} = \mathbf{u}_0|_{t=0} + v_0.$$

If $v_0 \in L^2(\mathbb{R}^d)$ is such that

$$\|v_0\|_{L^2(\mathbb{R}^d)} \ll \varepsilon^{d/2},$$

and if

$$\varepsilon^{2-\frac{d}{2}} \ll \delta^{\frac{d}{4}+1-\frac{1}{2\sigma+1}},$$

then there exist $T > 0$ and a unique solution $u = \mathbf{u}_0 + v$ to (1.1)–(1.6), such that

$$v \in C([0, T]; L^2(\mathbb{R}^d)) \cap L^{\frac{4\sigma+4}{d}}([0, T]; L^{2\sigma+2}(\mathbb{R}^d)).$$

In addition, there exists $C > 0$ such that

$$\|v(t)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{d/2} e^{Ct}, \quad 0 \leq t \leq T.$$

If there exists $\iota, C > 0$ independent of ε such that

$$\|v_0\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{d/2+\iota} \quad \text{and} \quad \varepsilon^{2-\frac{d}{2}-\iota} \ll \delta^{\frac{d}{4}+1-\frac{1}{2\sigma+1}},$$

then there exist $\varepsilon_0 \in (0, 1)$ and c_1 independent of $\varepsilon \in (0, \varepsilon_0)$ such that the above conclusions remain valid with $T = T^\varepsilon = c_1 \log \varepsilon^{-1}$.

Remark 1.7 (Potential driven cascade). If instead of a source term, we set (the same) f as a potential, and consider

$$(1.7) \quad i\partial_t u + \varepsilon^2 \Delta u = |u|^{2\sigma} u - fu, \quad x \in \mathbb{R}^d,$$

then our algebraic solution is still given like in Theorem 1.1, with $\alpha = \frac{1}{2\sigma}$ instead of $\alpha = \frac{1}{2\sigma+1}$. Up to this change for the value of α and minor modifications, Theorem 1.4 (first case), and Theorem 1.6 remain valid. Note that we may also replace f with $f + \omega$ for some $\omega \in \mathbb{R}$: $u_0(t, x) := e^{i\omega t} U(x)$ solves (1.7) in the case $\varepsilon = 0$, where $U = f^{1/(2\sigma)}$ exhibits a frequency cascade, and this solution is stable in the sense of Theorem 1.6.

Remark 1.8 (Fluid mechanics). Developing on the previous remark, we may consider the Madelung transform: $\rho := |u|^2$ and $J = \text{Im}(\bar{u}\nabla u)$ solve the Euler-Korteweg (or quantum Euler) equation

$$\begin{cases} \partial_t \rho + 2\varepsilon^2 \nabla \cdot J = 0, \\ \partial_t J + 2\varepsilon^2 \nabla \left(\frac{J \otimes J}{\rho} \right) + \frac{\sigma}{\sigma+1} \nabla \rho^{\sigma+1} - \rho \nabla f = \frac{\varepsilon^2}{2} \Delta \nabla \rho - 2\varepsilon^2 \nabla \cdot (\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}), \end{cases}$$

see e.g. [4]. The algebraic solution becomes $\rho_0 = f^{1/\sigma}$: if $\sigma > 1$, then we can still invoke Theorem 1.1 to describe a frequency cascade for ρ_0 . On the other hand, there is no quasi-singularity for ρ_0 when $\sigma = 1$.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1. It paves the way for the proof of Theorem 1.3, which is given in Section 3. The stability analysis, leading to Theorems 1.4 and 1.6, is presented in Section 4. Finally, numerical simulations illustrate and complement the analysis in Section 5.

2. TIME INDEPENDENT FORCING

In this section, we consider

$$f(x) = 1 - e^{-\delta} e^{-\beta|x|^2/2}, \quad \beta, \delta > 0.$$

The parameter δ is crucial in the sequel, while β is here mostly to show some form of generality (see also Remark 2.1).

To lighten notations, we introduce $\alpha \in (0, 1)$, and consider the function $f^\alpha - 1$. Of course, α has to be thought of as $\alpha = \frac{1}{2\sigma+1}$. The function $f^\alpha - 1$ is given by a rather explicit power series. Indeed, for $|X| < 1$,

$$(1 - X)^\alpha - 1 = \sum_{n \geq 1} a_n (-X)^n,$$

where the formula for the a_n 's is standard,

$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}.$$

Euler formula reads, for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

so we can rewrite ($z = \alpha - n + 1 \notin \mathbb{Z}$ indeed)

$$a_n = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)} \frac{\sin(\pi(\alpha-n+1))}{\pi} \Gamma(n-\alpha) = (-1)^{n-1} \frac{\sin(\pi\alpha)}{\pi} \frac{\Gamma(\alpha+1)\Gamma(n-\alpha)}{\Gamma(n+1)}.$$

Stirling formula yields

$$\Gamma(z) \underset{z \rightarrow \infty}{\sim} \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}.$$

Therefore

$$\frac{\Gamma(n-\alpha)}{\Gamma(n+1)} \underset{n \rightarrow \infty}{\sim} \frac{c}{n^{\alpha+1}},$$

for some $c > 0$ whose exact value can be computed, but which is left out here. Since we always have $\alpha > 0$, the series is normally converging, and

$$(2.1) \quad \begin{aligned} \mathcal{F}(f^\alpha - 1)(\xi) &= \sum_{n \geq 1} (-1)^n a_n \mathcal{F}\left(e^{-n\delta} e^{-n\beta|x|^2/2}\right)(\xi) \\ &= \sum_{n \geq 1} (-1)^n a_n \frac{1}{(\beta n)^{d/2}} e^{-n\delta} e^{-\frac{|\xi|^2}{2n\beta}}. \end{aligned}$$

In view of the above computations,

$$(-1)^n a_n \underset{n \rightarrow \infty}{\sim} -\frac{\sin(\pi\alpha)}{\pi} \Gamma(\alpha + 1) \frac{c}{n^{\alpha+1}} = \frac{c_0}{n^{\alpha+1}},$$

for some $c_0 < 0$ whose precise value is irrelevant for our purpose.

The sum on n is then compared to the continuous counterpart via an integral. This makes sense when the integrated function is monotonic. In the present case, and after replacing a_n with its leading order asymptotic behavior, the function we consider is

$$g(y) = \frac{1}{y^{\alpha+1+d/2}} e^{-y\frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}},$$

where y is a placeholder for $n\beta$. Indeed, there exist $c_0, C_0 > 0$ such that

$$\frac{c_0}{n^{\alpha+1}} \leq (-1)^n a_n \leq \frac{C_0}{n^{\alpha+1}}, \quad \forall n \geq 1.$$

Let

$$k = \alpha + 1 + d/2, \quad \eta = \frac{\delta}{\beta}, \quad \gamma = \frac{|\xi|^2}{2}.$$

Then

$$g(y) = \frac{1}{y^k} e^{-\eta y - \frac{\gamma}{y}}, \quad g'(y) = \frac{1}{y^k} e^{-\eta y - \frac{\gamma}{y}} \left(-\frac{k}{y} - \eta + \frac{\gamma}{y^2} \right).$$

The sign of g' is given by the second order polynomial in $1/y$,

$$P(X) = \gamma X^2 - kX - \eta.$$

The function g is nonincreasing for

$$\frac{1}{y} = X < \frac{k + \sqrt{\Delta}}{2\gamma}, \quad \Delta = k^2 + 4\eta\gamma.$$

In the range $\frac{\delta}{\beta} |\xi|^2 \ll 1$, $\Delta \approx k^2$, and the above condition reduces to

$$y > \frac{\gamma}{k} = \frac{|\xi|^2}{2\alpha + 2 + d} =: y_0.$$

Therefore, g is increasing on $(0, y_0)$, decreasing on (y_0, ∞) . Since for all $a > 0$, the map $z \mapsto z^a e^{-z}$ is bounded on $[0, \infty)$, the map

$$y \mapsto \left(\frac{\gamma}{y} \right)^a e^{-\frac{\gamma}{y}}$$

is bounded on $(0, \infty)$, independently of $\gamma > 0$. Writing

$$g'(y) = -\frac{\eta}{\gamma^k} \left(\frac{\gamma}{y} \right)^k e^{-\eta y - \frac{\gamma}{y}} - \frac{k}{\gamma^{k+1}} \left(\frac{\gamma}{y} \right)^{k+1} e^{-\eta y - \frac{\gamma}{y}} + \frac{1}{\gamma^{k+1}} \left(\frac{\gamma}{y} \right)^{k+2} e^{-\eta y - \frac{\gamma}{y}},$$

we get a dependence of bound upon ξ of the form

$$|g'(y)| \lesssim \frac{1}{|\xi|^{2k+2}} + \frac{1}{|\xi|^2}.$$

In passing, this shows that g is integrable near the origin. So if we assume that $|\xi| \geq \xi_0 > 0$, $|g'|$ is controlled on $(0, \infty)$ by a constant which depends on ξ_0, α and d (only). Therefore,

- As $g' > 0$ on $(0, y_0)$, there exists c_1, C_1 (independent of $|\xi|^2 \geq \xi_0^2$) such that

$$\begin{aligned} c_1 \int_0^{y_0} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy &\leq \sum_{n, \beta \leq y_0} \frac{(-1)^{n-1}}{(\beta n)^{d/2}} a_n e^{-n\delta} e^{-\frac{|\xi|^2}{2n\beta}} \\ &\leq C_1 \int_0^{y_0} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy, \end{aligned}$$

- As $g' < 0$ on (y_0, ∞) , there exists c_2, C_2 (independent of $|\xi|^2$ since $\|g'\|_{L^\infty}$ is independent of $|\xi|^2 \geq \xi_0^2$) such that

$$\begin{aligned} c_2 \int_{y_0}^{\infty} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy &\leq \sum_{n, \beta > y_0} \frac{(-1)^{n-1}}{(\beta n)^{d/2}} a_n e^{-n\delta} e^{-\frac{|\xi|^2}{2n\beta}} \\ &\leq C_2 \int_{y_0}^{\infty} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy. \end{aligned}$$

Now we set $c_3 = \min(c_1, c_2)$ and $C_3 = \max(C_1, C_2)$, so we have

$$\begin{aligned} c_3 \int_0^{\infty} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy &\leq \sum_{n \geq 1} \frac{(-1)^{n-1}}{(\beta n)^{d/2}} a_n e^{-n\delta} e^{-\frac{|\xi|^2}{2n\beta}} \\ &\leq C_3 \int_0^{\infty} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy. \end{aligned}$$

So we can take $\xi_0 > 0$ arbitrary, and under the constraint

$$|\xi|^2 \ll \frac{\beta}{\delta},$$

we can write

$$\int_0^{\infty} \frac{1}{y^{\alpha+1+d/2}} e^{-y \frac{\delta}{\beta}} e^{-\frac{|\xi|^2}{2y}} dy \approx c\beta^\alpha |\xi|^{-2\alpha-d} \int_0^{\infty} z^{\alpha-1+d/2} e^{-z/2} dz,$$

thus proving the explicit frequency power law, and Theorem 1.1.

Remark 2.1 (Other power for the singularity). To model a cancellation for f of the form $|x|^{2k}$, with $k > 0$ (integer or not), we may consider

$$\left(1 - e^{-\delta} e^{-\beta|x|^2/2}\right)^k.$$

We can then proceed similarly, by simply replacing $\frac{1}{2\sigma+1}$ by $\frac{k}{2\sigma+1}$ for the value α . The case $k < 2\sigma + 1$ requires no modification. If $k > 2\sigma + 1$, and if $\frac{k}{2\sigma+1}$ is not an integer, we write

$$\frac{k}{2\sigma+1} = n + \alpha, \quad \alpha \in (0, 1),$$

and

$$\left(1 - e^{-\delta} e^{-\beta|x|^2/2}\right)^k = \left(1 - e^{-\delta} e^{-\beta|x|^2/2}\right)^n \left(1 - e^{-\delta} e^{-\beta|x|^2/2}\right)^\alpha.$$

The first factor on the right hand side is smooth, and the second is treated like below: only the second factor is responsible for a quasi-singularity.

3. TIME PERIODIC FORCING

3.1. Cardano formula. We now assume that

$$f = f(t, x) = 2Q(x)e^{-3iPt} \quad \text{and} \quad \sigma = 1,$$

where no assumption is made so far on the sign of P . It turns out that the analysis below yields a result compatible with the previous approach ($P = 0$) only for $P > 0$. We somehow proceed in a reversed order compared to the previous section: we seek an algebraic solution to (1.1), and impose that the profile has a similar behavior as in the previous section. This will lead to the definition of Q .

The first step consists in writing the solution u to (1.1) under the (approximate) form $u(t, x) = e^{-3iPt}U(x)$, and removing the Laplacian (we formally let $\varepsilon \rightarrow 0$). If Q is real, then so is U , which therefore solves

$$(3.1) \quad U^3 - 3PU - 2Q = 0.$$

The solutions are given by Cardano formula, we briefly recall the corresponding method. We plug $U = \alpha + \beta \in \mathbb{R}$ into (3.1), and we find

$$\alpha^3 + \beta^3 + (3\alpha\beta - 3P)(\alpha + \beta) - 2Q = 0.$$

We impose

$$\begin{cases} \alpha\beta = P & \Leftrightarrow & \alpha^3\beta^3 = P^3 \\ \alpha^3 + \beta^3 = 2Q \end{cases}$$

and this yields that α^3 and β^3 are the roots of

$$X^2 - 2QX + P^3 = 0 \quad \Leftrightarrow \quad X = Q \pm \sqrt{Q^2 - P^3}.$$

We discuss this formula according to the sign of the discriminant $Q^2 - P^3$, which depends on the value of $Q(x)$.

- $Q^2 > P^3$. Note that necessarily $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. In addition, $\alpha^3 + \beta^3 = 2Q$ and $\alpha + \beta = U$ have the same sign, since

$$\alpha^3 + \beta^3 = (\alpha + \beta) \underbrace{(\alpha^2 - \alpha\beta + \beta^2)}_{\geq 0}.$$

- $Q^2 = P^3$. Then we have $\alpha^3 = \beta^3 = Q$. If we take $\alpha = j^k Q^{\frac{1}{3}}$ where $Q^{\frac{1}{3}}$ denote the real cubic root of the real function Q , $j = e^{\frac{2i\pi}{3}}$, $k = 0, 1, 2$, then
 - If $k = 0$ then $U = 2Q^{\frac{1}{3}}$ has the same sign as Q .
 - If $k = 1$ or $k = 2$ (the same case by noticing $\bar{j} = j^2$) then $U = 2Q^{\frac{1}{3}} \cos(\frac{2\pi}{3}) = -Q^{\frac{1}{3}}$. We obtain the solution

$$Q^2 = P^3, \quad U = -Q^{\frac{1}{3}}.$$

If $P \neq 0$ is constant, this solution cannot exhibit singularities as we have necessarily $P > 0$, $Q(x) = P^{\frac{3}{2}}$, and thus $U \sim -\sqrt{P}$ is smooth near the point x considered.

- $Q^2 < P^3$. Then as $P > 0$ is constant we can write

$$Q = P^{\frac{3}{2}}R, \quad \text{with } R < 1.$$

Then we can take

$$\alpha^3 = Q + i\sqrt{P^3 - Q^2} = P^{\frac{3}{2}} \left(R + i\sqrt{1 - R^2} \right), \quad \text{and } \beta = \bar{\alpha},$$

with the condition $R^2 \leq 1$ and thus $R = \cos(\theta(x))$ for some function

$$\theta(x) = \arccos(R) \in [0, \pi].$$

Let us remark that θ can have jumps in the derivative, but is at least continuous if R is smooth.

Remark 3.1. At this level, we could also define another lifting as $\theta(x) + 2k(x)\pi$ is also solution, with $k(x) \in \mathbb{Z}$. Then we would have jump in the derivative, but these jumps could be compensated, see below.

Note that $\alpha\beta = |\alpha|^2 = P$, and hence $|\alpha| = P^{\frac{1}{2}}$. Then we have

$$\alpha^3 = P^{\frac{3}{2}}(\cos \theta + i|\sin \theta|) = P^{\frac{3}{2}}e^{i\theta},$$

as $\theta \in [0, \pi]$. Note that the limiting cases $\theta = 0$ and $\theta = \pi$ correspond to the previous case $Q^2 = P^3$. Then we can take with $k(x) \in \{0, 1, 2\}$ depending possibly on x ,

$$\alpha = P^{\frac{1}{2}}e^{i\frac{\theta(x)}{3} + i\frac{2k(x)\pi}{3}} = P^{\frac{1}{2}}e^{i\phi(x)}.$$

In the sequel, we assume $P = 1$ to lighten notations, given we have already assumed $P > 0$.

3.2. Derivation of the forcing term. We make the above formal discussion more precise, in order to derive some explicit formula for Q . In the case where $P = 1$ and the discriminant $Q^2 - P^3 < 0$, we have found that

$$\alpha^3 = Q + i\sqrt{1 - Q^2}, \quad \beta = \bar{\alpha}.$$

Let $j = e^{2i\pi/3}$. We have three solutions, and all of them are real-valued,

$$U_k = j^k \left(Q + i\sqrt{1 - Q^2} \right)^{1/3} + j^{-k} \left(Q - i\sqrt{1 - Q^2} \right)^{1/3}, \quad k = 0, 1, 2,$$

where the cubic root is the same for $k = 0, 1$ and 2 , noticing that the imaginary part of $Q + i\sqrt{1 - Q^2}$ is never zero, provided that $Q^2 < 1$ everywhere. More precisely, for $Q(x) \in (0, e^{-\eta}]$ for all $x \in \mathbb{R}^d$, we can write $Q(x) = \cos \theta(x)$ for some $\theta(x) \in [\arccos e^{-\eta}, \frac{\pi}{2}]$. We have

$$\begin{aligned} U_k(x) &= j^k e^{i\theta(x)/3} + j^{-k} e^{-i\theta(x)/3} = 2 \cos \left(\frac{\theta(x)}{3} + \frac{2k\pi}{3} \right) \\ &= 2 \cos \left(\frac{\theta(x)}{3} \right) \cos \left(\frac{2k\pi}{3} \right) - 2 \sin \left(\frac{\theta(x)}{3} \right) \sin \left(\frac{2k\pi}{3} \right), \end{aligned}$$

and thus

$$\begin{aligned} U_0(x) &= 2 \cos\left(\frac{\theta(x)}{3}\right), \\ U_1(x) &= -\cos\left(\frac{\theta(x)}{3}\right) - \sqrt{3} \sin\left(\frac{\theta(x)}{3}\right), \\ U_2(x) &= -\cos\left(\frac{\theta(x)}{3}\right) + \sqrt{3} \sin\left(\frac{\theta(x)}{3}\right). \end{aligned}$$

To fall back into the situation of the case $P = 0$, we assume that

$$\sin\left(\frac{\theta(x)}{3}\right) = \left(1 - e^{-\delta - \frac{|x|^2}{2}}\right)^{\frac{1}{2}}, \quad \cos\left(\frac{\theta(x)}{3}\right) = e^{-\frac{\delta}{2} - \frac{|x|^2}{4}}.$$

For $k = 1$ or 2 , the sine term is a special case of Theorem 1.1. Classical trigonometric formulas (special case of Chebyshev polynomial) yield

$$\begin{aligned} Q(x) &= \cos(\theta(x)) = 4 \cos\left(\frac{\theta(x)}{3}\right)^3 - 3 \cos\left(\frac{\theta(x)}{3}\right) \\ &= 4e^{-\frac{3\delta}{2} - \frac{3|x|^2}{4}} - 3e^{-\frac{\delta}{2} - \frac{|x|^2}{4}}. \end{aligned}$$

Together with Theorem 1.1, this yields Theorem 1.3.

4. STABILITY

The starting point in this section is simply the purely algebraic solution considered in Section 2 or Section 3, U or U_k , $k = 1, 2$. Denote by $\mathbf{u}_0 = u_0 = U(x)$ the solution constructed in Section 2, and $\mathbf{u}_0 = u_0(x)e^{-3it} = U_k(x)e^{-3it}$ ($k = 1, 2$) the solution constructed in Section 3 when $\sigma = 1$, that is, like in Theorem 1.4. We set $\alpha = \frac{1}{2\sigma+1}$ when f is stationary (like in Section 2), and $\alpha = \frac{1}{2}$ when $f = 2Q(x)e^{-3it}$ and the nonlinearity is cubic (like in Section 3).

We start our analysis with rather straightforward estimates for $u_0 - 1$:

Lemma 4.1. *Let $s \geq 0$. We have*

$$\|u_0 - 1\|_{\dot{H}^s} \lesssim \begin{cases} 1 & \text{if } \alpha > s/2 - d/4, \\ \log \frac{1}{\delta} & \text{if } \alpha = s/2 - d/4, \\ \delta^{\alpha - s/2 + d/4} & \text{if } \alpha < s/2 - d/4, \end{cases}$$

and

$$\|\langle \cdot \rangle^s \Delta u_0\|_{L^2} \lesssim \delta^{\alpha - 1 - d/4}, \quad \text{where } \langle x \rangle = \sqrt{1 + |x|^2}.$$

Proof. Recalling (2.1), we write

$$\|u_0 - 1\|_{\dot{H}^s} \lesssim \frac{1}{\beta d/2} \sum_{n \geq 1} \frac{e^{-n\delta}}{n^{1+\alpha+d/2}} \left\| |\xi|^s e^{-\frac{|\xi|^2}{2n\beta}} \right\|_{L^2(\mathbb{R}^d)} \lesssim \beta^{s/2-d/4} \sum_{n \geq 1} \frac{e^{-n\delta}}{n^{1+\alpha-s/2+d/4}}.$$

As $e^{-n\delta} \leq 1$, the conclusion is straightforward when $\alpha > s/2 - d/4$. Otherwise, we split the sum into two, $n \leq 1/\delta$ and $n > \delta$. For $n \leq \delta$, the role of the exponential is negligible, and we write

$$\sum_{n \leq 1/\delta} \frac{e^{-n\delta}}{n^{1+\alpha-s/2+d/4}} \leq \sum_{n \leq 1/\delta} \frac{1}{n^{1+\alpha-s/2+d/4}} \underset{\delta \rightarrow 0}{\sim} \int_1^{1/\delta} \frac{dy}{y^{1+\alpha-s/2+d/4}} \underset{\delta \rightarrow 0}{\sim} \delta^{\alpha-s/2+d/4},$$

with the obvious modification in the case $\alpha = s/2 - d/4$. The sum for $n > 1/\delta$ can be viewed as a Riemann sum,

$$\begin{aligned} \sum_{n>1/\delta} \frac{e^{-n\delta}}{n^{1+\alpha-s/2+d/4}} &= \delta^{\alpha-s/2+d/4} \times \delta \sum_{n\delta>1} \frac{e^{-n\delta}}{(\delta n)^{1+\alpha-s/2+d/4}} \\ &\underset{\delta \rightarrow 0}{\sim} \delta^{\alpha-s/2+d/4} \int_1^\infty \frac{e^{-y}}{y^{1+\alpha-s/2+d/4}} dy. \end{aligned}$$

The general Sobolev estimates follow. To estimate the momenta of Δu_0 , we write

$$\mathcal{F}(\Delta u_0) = \sum_{n \geq 1} (-1)^n a_n \frac{1}{(\beta n)^{d/2}} |\xi|^2 e^{-n\delta} e^{-\frac{|\xi|^2}{2n\beta}},$$

so

$$\| \langle \cdot \rangle^s \Delta u_0 \|_{L^2} \lesssim \sum_{n \geq 1} \frac{|a_n|}{n^{d/2}} e^{-n\delta} \left\| |\xi|^2 e^{-\frac{|\xi|^2}{2n\beta}} \right\|_{H^s}.$$

In view of the estimate (the largest contribution being the L^2 -norm),

$$\left\| |\xi|^2 e^{-\frac{|\xi|^2}{2n\beta}} \right\|_{H^s} \lesssim n^{1+d/4},$$

we can resume the above estimate (setting $s = 2$ there), and the lemma follows. \square

4.1. Construction of a stationary profile.

Time independent forcing. We first consider the case where f does not depend on time, like in Section 2, and linearize (1.3) about the algebraic solution $f^{\frac{1}{2\sigma+1}}$. Again, we write $u_\varepsilon = u_0 + v$, where $u_0 = f^\alpha = f^{\frac{1}{2\sigma+1}}$, with now the condition $v \in H^1(\mathbb{R})$ independent of time. We obtain

$$-\varepsilon^2 \Delta v + (\sigma + 1)|u_0|^{2\sigma} v + \sigma u_0^{\sigma+1} \bar{u}_0^{\sigma-1} \bar{v} = \varepsilon^2 \Delta u_0 + \mathcal{O}(|v|^2).$$

Note that u_0 is real-valued, and so is v , hence

$$-\varepsilon^2 \Delta v + (2\sigma + 1)u_0^{2\sigma} v = -\varepsilon^2 \Delta u_0 + \mathcal{O}(v^2).$$

As $f(x) \geq 1 - e^{-\delta}$ and $0 < \alpha < 1$, we have¹ $f^\alpha \geq 1 - e^{-\alpha\delta} \geq \alpha\delta$, along with $1 \geq u_0 \geq \alpha\delta$. Lemma 4.1 yields the bound

$$(4.1) \quad \|\varepsilon^2 \Delta u_0\|_{\dot{H}^s} \lesssim \varepsilon^2 \beta^{1+s/2-d/4} \delta^{\alpha-1-s/2-d/4}.$$

The operator $-\varepsilon^2 \Delta + (2\sigma + 1)u_0^{2\sigma}$ is elliptic, uniformly in ε ,

$$(4.2) \quad \langle -\varepsilon^2 \Delta v + (2\sigma + 1)u_0^{2\sigma} v, v \rangle \geq (2\sigma + 1)(\alpha\delta)^{\frac{2\sigma}{2\sigma+1}} \|v\|_{L^2}^2.$$

We note that for $s > 0$, $\mathcal{F}H^s(\mathbb{R}^d)$ is the weighted L^2 -space

$$\mathcal{F}H^s(\mathbb{R}^d) = \left\{ \psi \in L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} (1 + |x|^2)^s |\psi(x)|^2 dx < \infty \right\}.$$

Lemma 4.2. *Let $\sigma, \beta > 0$ and $k \in \mathbb{N}$ be given. There exist a constant C and $a = a(k, d, \sigma) > 0$ such that the following holds.*

¹The inequality $x^\alpha - y^\alpha \leq (x - y)^\alpha$ for all $0 \leq y \leq x$ is standard and easy to prove.

- Let $g \in H^k(\mathbb{R}^d)$. Then for all $\varepsilon \in (0, 1]$, $\delta > 0$ and u_0 as above, there exists a unique solution $v \in H^k(\mathbb{R}^d)$ to the equation

$$(4.3) \quad -\varepsilon^2 \Delta v + (2\sigma + 1)u_0^{2\sigma} v = g,$$

and moreover, we have the estimate

$$\|v\|_{H^k} \leq \frac{C}{\delta^a} \|g\|_{H^k}.$$

- If in addition $g \in \mathcal{FH}^k(\mathbb{R}^d)$, then $v \in \mathcal{FH}^k(\mathbb{R}^d)$ and

$$\|\langle \cdot \rangle^k v\|_{L^2} \leq \frac{C}{\delta^a} \|\langle \cdot \rangle^k g\|_{L^2}.$$

Proof. We have

$$(4.4) \quad \begin{aligned} \langle -\varepsilon^2 \Delta v + (2\sigma + 1)u_0^{2\sigma} v, v \rangle &\gtrsim \delta^{\frac{2\sigma}{2\sigma+1}} \|v\|_{L^2}^2 + \varepsilon^2 \|v\|_{H^1}^2 \\ &\gtrsim \min\left(\delta^{\frac{2\sigma}{2\sigma+1}}, \varepsilon^2\right) \|v\|_{H^1}^2. \end{aligned}$$

Thus the Lax-Milgram Theorem guarantees the existence of v in H^1 . The previous bound then shows that

$$\|v\|_{L^2} \leq \frac{C}{\delta^{\frac{2\sigma}{2\sigma+1}}} \|g\|_{L^2}.$$

In view of the construction of u_0 and of Lemma 4.1,

$$\|u_0\|_{\dot{H}^k} \lesssim \delta^{\alpha-k/2-d/4}, \quad \text{and} \quad c\delta \leq u_0 \leq 1.$$

Then we calculate, with $D = \partial_{x_j}$, for $j \in \{1, \dots, d\}$,

$$-\varepsilon^2 \Delta(Dv) + (2\sigma + 1)u_0^{2\sigma}(Dv) = Dg - 2\sigma(2\sigma + 1)u_0^{2\sigma-1}(Du_0)v,$$

from which we deduce

$$\begin{aligned} \delta^{2\sigma} \|Du\|_{L^2} &\leq C \|Dg\|_{L^2} + C \|u_0^{2\sigma-1}(Du_0)\|_{L^\infty} \|v\|_{L^2} \\ &\leq C \|Dg\|_{L^2} + C \delta^{-b} \|g\|_{L^2} \end{aligned}$$

for some constants b and C independent of δ and ε . We then deduce the first part of the result by induction on k .

Suppose that $g \in \mathcal{FH}^k$, and multiply (4.3) by $\langle x \rangle^k$:

$$-\varepsilon^2 \langle x \rangle^k \Delta v + (2\sigma + 1)u_0^{2\sigma} \langle x \rangle^k v = \langle x \rangle^k g.$$

Introducing the commutator

$$[\langle x \rangle^k, \Delta] = \langle x \rangle^k \Delta - \Delta (\langle x \rangle^k \cdot) = -2k \langle x \rangle^{k-2} x \cdot \nabla + \mathcal{O}(\langle x \rangle^{k-2}),$$

the above equation also reads

$$-\varepsilon^2 \Delta (\langle x \rangle^k v) + (2\sigma + 1)u_0^{2\sigma} \langle x \rangle^k v = \langle x \rangle^k g + \varepsilon^2 [\langle x \rangle^k, \Delta]v.$$

Proceed like above, and take the L^2 inner product with $\langle x \rangle^k v$: on the one hand,

$$\left\langle -\varepsilon^2 \Delta (\langle x \rangle^k v) + (2\sigma + 1)u_0^{2\sigma} \langle x \rangle^k v, \langle x \rangle^k v \right\rangle \geq \delta^{\frac{2\sigma}{2\sigma+1}} \|\langle x \rangle^k v\|_{L^2}^2,$$

and on the other hand,

$$\begin{aligned} \left\langle \langle x \rangle^k g + \varepsilon^2 [\langle x \rangle^k, \Delta] v, \langle x \rangle^k v \right\rangle &\leq \| \langle x \rangle^k g \|_{L^2} \| \langle x \rangle^k v \|_{L^2} \\ &\quad - 2k\varepsilon^2 \int_{\mathbb{R}^d} \langle x \rangle^{k-2} x \cdot \nabla v \langle x \rangle^k v \\ &\quad + C \int_{\mathbb{R}^d} \langle x \rangle^{k-2} |v| \langle x \rangle^k |v|. \end{aligned}$$

The second term on the right hand side is integrated by parts,

$$\begin{aligned} -2 \int_{\mathbb{R}^d} \langle x \rangle^{k-2} x \cdot \nabla v \langle x \rangle^k v &= - \int_{\mathbb{R}^d} \langle x \rangle^{2k-2} x \cdot \nabla (v^2) = \int_{\mathbb{R}^d} \nabla \cdot (\langle x \rangle^{2k-2} x) v^2 \\ &= \mathcal{O} \left(\int_{\mathbb{R}^d} \langle x \rangle^{2k-2} v^2 \right). \end{aligned}$$

Cauchy-Schwarz inequality yields

$$\left\langle \langle x \rangle^k g + \varepsilon^2 [\langle x \rangle^k, \Delta] v, \langle x \rangle^k v \right\rangle \leq \| \langle x \rangle^k v \|_{L^2} \left(\| \langle x \rangle^k g \|_{L^2} + \varepsilon^2 \| \langle x \rangle^{k-2} v \|_{L^2} \right).$$

Since we know from the first part of the lemma that

$$\|v\|_{L^2} \lesssim \delta^{-\frac{2\sigma}{2\sigma+1}} \|g\|_{L^2},$$

the end of the lemma follows by induction. \square

We infer the following result, where we do not try to optimize the parameters involved in the statement.

Proposition 4.3. *Let $k > d/2$ and $\sigma > 0$ be integers. There exist $M(k, \sigma, d) > 0$ such that if*

$$\varepsilon \ll \delta^{2M} \ll 1,$$

then there exists $v \in H^k(\mathbb{R}^d) \cap \mathcal{FH}^k(\mathbb{R}^d)$ such that $u_\varepsilon = u_0 + v$ solves

$$(4.5) \quad -\varepsilon^2 \Delta u_\varepsilon + |u_\varepsilon|^{2\sigma} u_\varepsilon = f, \quad f(x) = 1 - e^{-\delta} e^{-|x|^2/2},$$

and

$$\|v\|_{H^k(\mathbb{R}^d)} + \left\| \langle \cdot \rangle^k v \right\|_{L^2(\mathbb{R}^d)} \lesssim \frac{\varepsilon}{\delta^M} \ll \delta^M \ll 1.$$

Proof. Using the decomposition $u_\varepsilon = u_0 + v$, (4.5) reads, in terms of v ,

$$-\varepsilon^2 \Delta v + (2\sigma + 1) u_0^{2\sigma} v = -\varepsilon^2 \Delta u_0 - 2\sigma(2\sigma + 1) v^2 \int_0^1 (1 - \theta) |u_0 + \theta v|^{2\sigma-2} (u_0 + \theta v) d\theta,$$

where we have used Taylor formula applied to the function $f(y) = |y|^{2\sigma} y$, for $y \in \mathbb{R}$. Denote by $G(\varepsilon, u_0, v)$ the above right hand side. We consider the iterative scheme

$$(4.6) \quad -\varepsilon^2 \Delta v_{n+1} + (2\sigma + 1) u_0^{2\sigma} v_{n+1} = G(\varepsilon, u_0, v_n),$$

initialized with $v_0 = 0$. Lemmas 4.1 and 4.2 yield, since $H^k(\mathbb{R}^d)$ is a Banach algebra and σ is an integer,

$$\begin{aligned} \|v_{n+1}\|_{H^k} &\leq \frac{C}{\delta^a} \left(\varepsilon^2 \delta^{\alpha-1-k/2-d/4} + \|v_n\|_{H^k}^2 \left(\|u_0 - 1\|_{H^k}^{2\sigma-1} + \|v_n\|_{H^k}^{2\sigma-1} \right) \right) \\ &\leq \frac{C}{\delta^a} \left(\varepsilon^2 \delta^{\alpha-1-k/2-d/4} + \|v_n\|_{H^k}^2 \left(C \delta^{\alpha-k/2-d/4} + \|v_n\|_{H^k}^{2\sigma-1} \right) \right). \end{aligned}$$

In the above estimate, we have considered $u_0 - 1$ and not u_0 : as $G(\varepsilon, u_0, v_n)$ is a linear combination of terms of the form $u_0^{2\sigma+1-\ell} v_n^\ell$, $\ell \geq 2$, either all the derivatives fall on v_n , and we use the estimate

$$\|u_0^{2\sigma+1-\ell} \partial^\beta v_n^\ell\|_{L^2} \leq \|u_0\|_{L^\infty}^{2\sigma+1-\ell} \|\partial^\beta v_n^\ell\|_{L^2} \leq \|v_n\|_{H^k}^\ell,$$

or at least one derivative affects u_0 , and then u_0 can be replaced by $u_0 - 1$ in the corresponding term(s). We thus have an estimate of the form

$$\|v_{n+1}\|_{H^k} \leq C \left(\frac{\varepsilon^2}{\delta^{\alpha_1}} + \delta^{-\alpha_2} \|v_n\|_{H^k}^2 + \|v_n\|_{H^k}^{2\sigma+1} \right), \quad \alpha_1, \alpha_2 > 0.$$

To simplify the presentation, we may suppose $\alpha_1 = \alpha_2 =: M$. We see that the property $\|v_n\|_{H^k} \lesssim \varepsilon \delta^{-M}$ is propagated by induction, provided that

$$\frac{\varepsilon^2}{\delta^M} \ll \frac{\varepsilon}{\delta^M}, \quad \frac{\varepsilon^2}{\delta^{3M}} \ll \frac{\varepsilon}{\delta^M}, \quad \frac{\varepsilon}{\delta^M} \ll 1.$$

The strongest condition is the middle one, which yields the first condition of the proposition. Up to decreasing the implicit constant $0 < \eta < 1$ in the notation

$$\varepsilon \ll \delta^{2M} \iff \varepsilon \leq \eta \delta^{2M},$$

we infer that the series $\sum \|v_{n+1} - v_n\|_{H^k}$ converges geometrically, that is,

$$v_n \xrightarrow[n \rightarrow \infty]{} v \quad \text{in } H^k(\mathbb{R}^d),$$

and $u_\varepsilon = u_0 + v$ solves (4.5).

Lemma 4.2 also yields

$$\|\langle \cdot \rangle^k v\|_{L^2} \lesssim \frac{1}{\delta^a} \left(\|\varepsilon^2 \langle \cdot \rangle^k \Delta u_0\|_{L^2} + \|\langle \cdot \rangle^k u_0^{2\sigma-1} v^2\|_{L^2} + \|\langle \cdot \rangle^k v^{2\sigma+1}\|_{L^2} \right).$$

In view of the estimate $\|u_0\|_{L^\infty} \leq 1$ and since $\|v\|_{L^\infty} \lesssim \|v\|_{H^k} \lesssim \varepsilon \delta^{-M}$, we infer, invoking Lemma 4.1,

$$\|\langle \cdot \rangle^k v\|_{L^2} \lesssim \frac{\varepsilon^2}{\delta^{\alpha_3}} + (\|v\|_{L^\infty} + \|v\|_{L^\infty}^{2\sigma}) \|\langle \cdot \rangle^k v\|_{L^2} \lesssim \frac{\varepsilon^2}{\delta^{\alpha_3}} + \frac{\varepsilon}{\delta^M} \|\langle \cdot \rangle^k v\|_{L^2},$$

for some $\alpha_3 > 0$, hence, since $\frac{\varepsilon}{\delta^M} \ll \delta^M \ll 1$,

$$\|\langle \cdot \rangle^k v\|_{L^2} \lesssim \frac{\varepsilon^2}{\delta^{\alpha_3}},$$

and the proposition follows. \square

Remark 4.4 (Case with damping). If in (1.1), the term $\varepsilon^2 \Delta$ is replaced with $(1 - i\nu)\varepsilon^2 \Delta$ with $\nu > 0$ so the new term is dissipative, then we can adapt the above approach. Indeed, we obtain in this case (with $u_0 \in \mathbb{R}$)

$$i\nu \varepsilon^2 \Delta v - \varepsilon^2 \Delta v + (\sigma + 1)|u_0|^{2\sigma} v + \sigma|u_0|^{2\sigma} \bar{v} = -(1 - i\nu)\varepsilon^2 \Delta u_0 + \mathcal{O}(|v|^2).$$

Taking the real and imaginary parts by writing $v = p + iq$ linear operator associated with this system is

$$\begin{pmatrix} -\varepsilon^2 \Delta + (2\sigma + 1)u_0^{2\sigma} & -\nu \varepsilon^2 \Delta \\ \nu \varepsilon^2 \Delta & -\varepsilon^2 \Delta + u_0^{2\sigma} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

We can write this system as

$$\begin{pmatrix} A & C \\ -C^T & B \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

where A and B are similar to the previous case (invertible in H^s with inverse norm independent of ε). When we solve the linear system we can write:

$$p = A^{-1}f - A^{-1}Cq,$$

and thus

$$(B + C^T A^{-1}C)q = C^T A^{-1}f + g,$$

and $B + C^T A^{-1}C$, which is now self-adjoint should be invertible for ν small enough. Indeed, the estimate (4.4) when $\varepsilon^2 \ll \delta^{2\sigma}$ implies that

$$\|A^{-1}Cu\|_{\dot{H}^s} \leq C\nu\|u\|_{H^s},$$

uniformly in ε . Hence we have

$$\|CA^{-1}Cu\|_{H^s} \leq C\varepsilon^2\nu\|u\|_{H^{s+2}},$$

and in particular for ν small enough, the operator $B + C^T A^{-1}C$ satisfies the coercive estimate (4.4) as well as the uniform estimate (4.2). We thus deduce that the same result holds for the dissipative case. Cumbersome details are left out.

Time dependent forcing and cubic nonlinearity. In the case where $f = 2Q(x)e^{-3it}$ and $\sigma = 1$ considered in Section 3 (see Theorem 1.3), the equation for v becomes

$$-\varepsilon^2\Delta v - 3Pv + 3U_k^2v = -\varepsilon^2\Delta U_k + \mathcal{O}(v^2).$$

Recall that for $k = 0, 1, 2$, U_k is given by

$$U_k(x) = 2 \cos\left(\frac{\theta(x)}{3} + \frac{2k\pi}{3}\right), \quad \cos\left(\frac{\theta(x)}{3}\right) = e^{-\frac{\delta}{2} - \frac{|x|^2}{4}} \in \left(0, e^{-\frac{\delta}{2}}\right].$$

In particular,

$$\frac{\theta(x)}{3} \in \left[\arccos e^{-\frac{\delta}{2}}, \frac{\pi}{2}\right),$$

and therefore

$$\frac{\theta(x)}{3} + \frac{2k\pi}{3} \in \left[\arccos e^{-\frac{\delta}{2}} + \frac{2k\pi}{3}, (4k+3)\frac{\pi}{6}\right).$$

This shows that $|U_0|$ and $|U_2|$ are arbitrarily close to zero. On the other hand, writing for instance

$$U_1(x) = -\cos\left(\frac{\theta(x)}{3}\right) - \sqrt{3}\sin\left(\frac{\theta(x)}{3}\right) = -e^{-\frac{\delta}{2} - \frac{|x|^2}{4}} - \sqrt{3}\left(1 - e^{-\delta - \frac{|x|^2}{2}}\right)^{1/2},$$

we compute

$$|U_1(x)|^2 \geq e^{-\delta - \frac{|x|^2}{2}} + 3\left(1 - e^{-\delta - \frac{|x|^2}{2}}\right) = 3 - 2e^{-\delta - \frac{|x|^2}{2}} \geq 3 - 2e^{-\delta} > 1,$$

and we see that the operator $-\varepsilon^2\Delta - 3 + 3U_1^2$ is uniformly elliptic in the sense that

$$\langle -\varepsilon^2\Delta v - 3v + 3U_1^2v, v \rangle \geq \varepsilon^2\|\nabla v\|_{L^2}^2 + c(\delta)\|v\|_{L^2}^2,$$

where $c(\delta) > 0$ provided that $\delta > 0$. We can thus resume the proof of Lemma 4.2. When considering derivatives of v , we have like above

$$-\varepsilon^2\Delta(Dv) - v + 3U_1^2Dv = Dg - 6U_1DU_1v,$$

and we can conclude similarly.

End of the proof of Theorem 1.4. The idea now is to use Sobolev embedding at the level of \hat{u}_ε , as opposed to u_ε directly, and write,

$$\sup_{\xi \in \mathbb{R}^d} \left| |\xi|^{d+2\alpha} (\hat{u}_\varepsilon(\xi) - \hat{u}_0(\xi)) \right| \lesssim \left\| |\cdot|^{d+2\alpha} (\hat{u}_\varepsilon - \hat{u}_0) \right\|_{H^s(\mathbb{R}^d)},$$

provided that $s > d/2$, where α corresponds to the frequency cascade for u_0 , as stated in Theorem 1.4. In view of Plancherel formula,

$$\left\| |\cdot|^{d+2\alpha} (\hat{u}_\varepsilon - \hat{u}_0) \right\|_{H^s(\mathbb{R}^d)} = \left\| \langle \cdot \rangle^s \Lambda^{d+2\alpha} (u_\varepsilon - u_0) \right\|_{L^2(\mathbb{R}^d)}, \quad \Lambda = (-\Delta)^{1/2}.$$

The operator involved above has symbol $\langle x \rangle^s |\xi|^{d+2\alpha}$, and Young inequality yields the pointwise estimate

$$\langle x \rangle^s |\xi|^{d+2\alpha} \leq \langle x \rangle^{3s} + |\xi|^{\frac{3}{2}(d+2\alpha)}.$$

Let

$$k = \left\lceil \frac{3}{2}(d+2\alpha) \right\rceil.$$

Then $k > 3d/2$, and we set $s = k/3 > d/2$, so we have the pointwise estimate

$$\langle x \rangle^s |\xi|^{d+2\alpha} \leq \langle x \rangle^k + \langle \xi \rangle^k.$$

Formally, this implies

$$\left\| \langle \cdot \rangle^s \Lambda^{d+2\alpha} (u_\varepsilon - u_0) \right\|_{L^2(\mathbb{R}^d)} \lesssim \left\| \langle \cdot \rangle^k (u_\varepsilon - u_0) \right\|_{L^2(\mathbb{R}^d)} + \|u_\varepsilon - u_0\|_{H^k(\mathbb{R}^d)}.$$

The justification of this inequality is not completely obvious though, and we refer to [13] (see also [2]) for a complete proof, based on pseudodifferential calculus.

Invoking Proposition 4.3 with the above k , we get some $M > 0$ such that if $\varepsilon \ll \delta^{2M} \ll 1$,

$$\left\| \langle \cdot \rangle^k (u_\varepsilon - u_0) \right\|_{L^2(\mathbb{R}^d)} + \|u_\varepsilon - u_0\|_{H^k(\mathbb{R}^d)} \lesssim \frac{\varepsilon}{\delta^M} \ll \delta^M \ll 1,$$

and thus

$$\sup_{\xi \in \mathbb{R}^d} \left| |\xi|^{d+2\alpha} (\hat{u}_\varepsilon(\xi) - \hat{u}_0(\xi)) \right| \lesssim \frac{\varepsilon}{\delta^M} \ll \delta^M \ll 1,$$

so Theorem 1.4 follows with $K = M/2$.

4.2. Conserved energy. Even in the presence of the forcing term f (which we keep considering independent of time now), the equation (1.1) enjoys a Hamiltonian structure. To see that some energy is conserved, one may follow the usual strategy for Schrödinger equations (see e.g. [5, Section 3.1]): multiply (1.1) by $\partial_t \bar{u}$, integrate in space, and consider the real part of the outcome. This yields formally

$$\frac{d}{dt} E_\varepsilon(u(t)) = 0,$$

where, since f does not depend on time, the energy E_ε is defined by

$$E_\varepsilon(w) = \frac{\varepsilon^2}{2} \|\nabla w\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2\sigma+2} \|w\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2} - \operatorname{Re} \int_{\mathbb{R}^d} f \bar{w}.$$

As a matter of fact, this computation is only formal in the framework that we consider, since as pointed out before, we consider perturbations of the algebraic solution u_0 , and u_0 belongs to no other Lebesgue space than $L^\infty(\mathbb{R}^d)$. Writing $u = u_0 + v$, giving a rigorous meaning to a conserved energy for v becomes possible,

at least when σ is an integer, in the spirit of renormalization. For simplicity, we consider the case $\sigma = 1$.

Recall the standard notation

$$\langle g, h \rangle = \operatorname{Re} \int_{\mathbb{R}^d} g \bar{h}.$$

Lemma 4.5. *Let $d \geq 1$, and assume that f is real-valued. Let $u_\varepsilon(x) \in \mathbb{R}$ be such that*

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon^3 = f(x).$$

Let u be the solution to (1.1) with $\sigma = 1$, and let $v = u - u_\varepsilon$. If u is such that $v \in C([0, T]; H^1 \cap L^4)$, then formally,

$$\mathcal{E}(v) = \frac{\varepsilon^2}{2} \|\nabla v\|_{L^2}^2 + \|u_\varepsilon v\|_{L^2}^2 + \int u_\varepsilon^2 (\operatorname{Re} v)^2 + \operatorname{Re} \int |v|^2 v u_\varepsilon + \frac{1}{4} \|v\|_{L^4}^4$$

is independent of $t \in [0, T]$.

Proof. We first note the identity, for $w_1, w_2 \in H^1 \cap L^4$ independent of time,

$$E_\varepsilon(w_1 + w_2) = E_\varepsilon(w_1) + \operatorname{Re} \int \bar{w}_2 (-\varepsilon^2 \Delta w_1 + |w_1|^2 w_1 - f) + \mathcal{E}(w_2),$$

where, in the definition of \mathcal{E} , we have replaced u_ε with w_1 . Substituting w_1 with u_ε and w_2 with v formally yields the conservation of $\mathcal{E}(v)$. Proceeding like in [5, Lemma 6.5.3], that is, multiplying u_ε and v by $e^{-\eta|x|^2}$ then letting $\eta \rightarrow 0$, we infer that if v has the regularity assumed in the statement, then indeed $\mathcal{E}(v)$ is independent of time. \square

Cauchy-Schwarz inequality implies, since u_ε is real-valued,

$$\left| \int u_\varepsilon |v|^2 \operatorname{Re} v \right| \leq \left(\int |v|^4 \right)^{1/2} \left(\int u_\varepsilon^2 (\operatorname{Re} v)^2 \right)^{1/2},$$

hence by Young inequality,

$$\left| \operatorname{Re} \int |v|^2 v \bar{u}_\varepsilon \right| \leq \left(\int |v|^4 \right)^{1/2} \left(\int u_\varepsilon^2 (\operatorname{Re} v)^2 \right)^{1/2} \leq \frac{1}{4} \int |v|^4 + \int u_\varepsilon^2 (\operatorname{Re} v)^2.$$

This readily yields

$$\mathcal{E}(v) \geq \frac{\varepsilon^2}{2} \|\nabla v\|_{L^2}^2 + \int u_\varepsilon^2 |v|^2,$$

which is a rather weak form of coercivity, since the lower bound for u_ε^2 involves a positive power of δ , as

$$u_\varepsilon(x) \geq u_0(x) - \|v\|_\infty \geq (1 - e^{-\delta})^{\frac{1}{2\sigma+1}} - \delta^M \gtrsim \delta^{\frac{1}{2\sigma+1}},$$

where we have used Proposition 4.3 and Sobolev embedding to neglect the contribution of v . It is possible to infer some dynamical stability from such estimates, up to losing some powers of ε or δ . Loosely speaking, if $\mathcal{E}(v)$ is small initially, then $\varepsilon \|\nabla v(t)\|_{L^2}$ and $\delta \|v(t)\|_{L^2}$ remain small as long as $v \in C([0, T]; H^1 \cap L^4)$, which yields some smallness for v in H^1 , provided that $\mathcal{E}(v|_{t=0})$ is sufficiently small in terms of ε and δ . We do not pursue this direction here, which could easily lead to cumbersome statements.

4.3. Stability via Strichartz estimates. When the nonlinearity is smooth and L^2 -subcritical or critical, we can directly get a dynamical stability result thanks to Strichartz estimates. The cases thus covered are:

- cubic or quintic in 1D ($d = 1$ and $\sigma = 1$ or 2),
- cubic in 2D ($d = 2$ and $\sigma = 1$).

We consider perturbations of the form $u = \mathbf{u}_0 + v$: the goal is to prove that v is small in L^2 on some nontrivial time interval. The nonlinearity has to be smooth because \mathbf{u}_0 is in no Lebesgue space apart from L^∞ , so for instance $|\mathbf{u}_0|^2 v$ and $|v|^2 v$ are not estimated in the same space, so writing

$$\left| |\mathbf{u}_0 + v|^{2\sigma}(\mathbf{u}_0 + v) - |\mathbf{u}_0|^{2\sigma}\mathbf{u}_0 \right| \lesssim (|\mathbf{u}_0|^{2\sigma} + |v|^{2\sigma})|v|$$

is not suited to the use of Strichartz estimates in this context. The nonlinearity has to be smooth so $|\mathbf{u}_0 + v|^{2\sigma}(\mathbf{u}_0 + v) - |\mathbf{u}_0|^{2\sigma}\mathbf{u}_0$ can be written as a finite sum of terms like evoked above.

Definition 4.6. *Let $d = 1$ or 2 . A pair (q, r) is **admissible** if $2 \leq r \leq \infty$ if $d = 1$, $2 \leq r < \infty$ if $d = 2$, and*

$$\frac{2}{q} = \delta(r) := d \left(\frac{1}{2} - \frac{1}{r} \right).$$

The following Strichartz estimates easily follow from the usual ones (see e.g. [5, 21]) by considering the rescaling $u(t, \frac{x}{\varepsilon})$:

Lemma 4.7 (Strichartz estimates). *Denote $U_0^\varepsilon(t) = e^{i\varepsilon^2 t \Delta}$.*

(1) *Homogeneous Strichartz estimate. For any admissible pair (q, r) , there exists C_q independent of ε such that*

$$\varepsilon^{2/q} \|U_0^\varepsilon \varphi\|_{L^q(\mathbb{R}; L^r)} \leq C_q \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\mathbb{R}^d).$$

(2) *Inhomogeneous Strichartz estimate. For a time interval I , denote*

$$D_I^\varepsilon(F)(t, x) = \int_{I \cap \{\tau \leq t\}} U_0^\varepsilon(t - \tau) F(\tau, x) d\tau.$$

For all admissible pairs (q_1, r_1) and (q_2, r_2) , and any interval I , there exists $C = C_{r_1, r_2}$ independent of ε and I such that

$$(4.7) \quad \varepsilon^{2/q_1 + 2/q_2} \|D_I^\varepsilon(F)\|_{L^{q_1}(I; L^{r_1})} \leq C \|F\|_{L^{q_2}(I; L^{r_2})},$$

for all $F \in L^{q_2}(I; L^{r_2})$.

First case: $\sigma = 1$, with $d = 1$ or $d = 2$. A function $u = \mathbf{u}_0 + v$ solves (1.1) if and only if v solves

$$(4.8) \quad i\partial_t v + \varepsilon^2 \Delta v = 2|\mathbf{u}_0|^2 v + \mathbf{u}_0^2 \bar{v} + 2\mathbf{u}_0 |v|^2 + \bar{\mathbf{u}}_0 v^2 + |v|^2 v - \varepsilon^2 \Delta \mathbf{u}_0.$$

With initial data v_0 and $t \in I \subset [0, +\infty)$, Duhamel's formula reads

$$v(t) = U_0^\varepsilon(t) v_0 - i D_I^\varepsilon (2|\mathbf{u}_0|^2 v + \mathbf{u}_0^2 \bar{v} + 2\mathbf{u}_0 |v|^2 + \bar{\mathbf{u}}_0 v^2 + |v|^2 v - \varepsilon^2 \Delta \mathbf{u}_0).$$

Local existence in the type of space evoked in Theorem 1.6 follows from a standard fixed point argument on the above Duhamel's formula, involving the same estimates as below. We focus on such estimates and the bootstrap argument, and leave out the local existence part.

Let q_0 such that the pair $(q_0, 4)$ is admissible: $q_0 = \frac{8}{d}$, so $q_0 = 8$ if $d = 1$, $q_0 = 4$ if $d = 2$. Strichartz estimates yield

$$\begin{aligned} \|v\|_{L^\infty(I;L^2)} &\lesssim \|v_0\|_{L^2} + \|\mathbf{u}_0\|^2 v\|_{L^1(I;L^2)} + \|\mathbf{u}_0 v^2\|_{L^1(I;L^2)} \\ &\quad + \varepsilon^{-d/4} \|v^3\|_{L^{q'_0}(I;L^{4/3})} + \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^1(I;L^2)}. \end{aligned}$$

Using Hölder inequality, and the fact that $\|\mathbf{u}_0\|_{L^\infty} \leq 1$,

$$\begin{aligned} \|v\|_{L^\infty(I;L^2)} &\lesssim \|v_0\|_{L^2} + \|v\|_{L^1(I;L^2)} + \|v\|_{L^2(I;L^4)}^2 \\ &\quad + \varepsilon^{-d/4} \|v\|_{L^{3q'_0}(I;L^4)}^3 + |I| \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

As $q_0 > 2$ and $3q'_0 \leq q_0$ (with equality in the L^2 -critical case $d = 2$),

$$\|v\|_{L^2(I;L^4)} \leq |I|^a \|v\|_{L^{q_0}(I;L^4)}, \quad \|v\|_{L^{3q'_0}(I;L^4)} \leq |I|^b \|v\|_{L^{q_0}(I;L^4)},$$

for some $a > 0$ and $b \geq 0$, and so, for $I \subset [0, T]$ with T finite,

$$\|v\|_{L^\infty(I;L^2)} \lesssim \|v_0\|_{L^2} + \|v\|_{L^1(I;L^2)} + \|v\|_{L^{q_0}(I;L^4)}^2 + \varepsilon^{-d/4} \|v\|_{L^{q_0}(I;L^4)}^3 + \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}.$$

We get rid of the second term on the right hand side by Grönwall lemma: for $I = [0, t]$, $0 \leq t \leq T$,

$$(4.9) \quad \begin{aligned} \|v\|_{L^\infty(I;L^2)} &\lesssim e^{Ct} \|v_0\|_{L^2} + e^{Ct} \|v\|_{L^{q_0}(I;L^4)}^2 + \varepsilon^{-d/4} e^{Ct} \|v\|_{L^{q_0}(I;L^4)}^3 \\ &\quad + e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

We note that the constant C can essentially be taken equal to one (slightly above), since there is no constant in the L^2 -estimate, and we have absorbed algebraic terms $|I|^a = t^a$ in the exponential. We then use Strichartz estimates again, but with the $L^{q_0} L^4$ -norm on the left hand side:

$$\begin{aligned} \|v\|_{L^{q_0}(I;L^4)} &\lesssim \varepsilon^{-d/4} \|v_0\|_{L^2} + \varepsilon^{-d/4} \|\mathbf{u}_0\|^2 v\|_{L^1(I;L^2)} + \varepsilon^{-d/4} \|\mathbf{u}_0 v^2\|_{L^1(I;L^2)} \\ &\quad + \varepsilon^{-d/2} \|v^3\|_{L^{q'_0}(I;L^{4/3})} + \varepsilon^{-d/4} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^1(I;L^2)} \\ &\lesssim \varepsilon^{-d/4} \|v_0\|_{L^2} + \varepsilon^{-d/4} \|v\|_{L^1(I;L^2)} + \varepsilon^{-d/4} \|v\|_{L^2(I;L^4)}^2 \\ &\quad + \varepsilon^{-d/2} \|v\|_{L^{3q'_0}(I;L^4)}^3 + \varepsilon^{-d/4} T \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2} \\ &\lesssim \varepsilon^{-d/4} \|v_0\|_{L^2} + \varepsilon^{-d/4} \|v\|_{L^{q_0}(I;L^4)}^2 + \varepsilon^{-d/2} \|v\|_{L^{q_0}(I;L^4)}^3 + \varepsilon^{-d/4} T \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2} \\ &\quad + \varepsilon^{-d/4} \left(e^{Ct} \|v_0\|_{L^2} + e^{Ct} \|v\|_{L^{q_0}(I;L^4)}^2 + \varepsilon^{-d/4} e^{Ct} \|v\|_{L^{q_0}(I;L^4)}^3 + e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2} \right), \end{aligned}$$

where we have estimated $\|v\|_{L^1(I;L^2)}$ by the first step. It yields the largest contribution, and thus

$$(4.10) \quad \begin{aligned} \|v\|_{L^{q_0}(I;L^4)} &\lesssim \varepsilon^{-d/4} e^{Ct} \|v_0\|_{L^2} + \varepsilon^{-d/4} e^{Ct} \|v\|_{L^{q_0}(I;L^4)}^2 \\ &\quad + \varepsilon^{-d/2} e^{Ct} \|v\|_{L^{q_0}(I;L^4)}^3 + \varepsilon^{-d/4} e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

We argue by bootstrap. So long as

$$\|v\|_{L^{q_0}(0,t;L^4)} \leq \eta \varepsilon^{d/4},$$

for some $\eta > 0$ sufficiently small, the terms $\|v\|_{L^{q_0}(0,t;L^4)}$ on the right hand side of (4.10) are absorbed by the left hand side.

One can take $\eta = \varepsilon^\iota$ with $\iota > 0$, and an Ehrenfest time is allowed, like $c \log \frac{1}{\varepsilon}$ (until the exponential becomes too big). More precisely:

$$\begin{aligned} \|v_0\|_{L^2} &\leq \eta \varepsilon^{d/2} \quad \text{or} \quad \varepsilon^{d/2+\iota}, \\ \varepsilon^2 \|\Delta \mathbf{u}_0\|_{L^2} &\leq \eta \varepsilon^{d/2} \quad \text{or} \quad \varepsilon^{d/2+\iota}. \end{aligned}$$

In view of Lemma 4.1, the last condition follows from

$$\varepsilon^{2-d/2} \ll \delta^{1+d/4-\alpha},$$

as $1 + d/4 > \alpha$ under the assumptions of Theorem 1.6.

Remark 4.8. If $\alpha > 1 + d/4$ (see Remark 2.1), no extra smallness assumption is needed.

Once the bootstrap condition is granted, we resume the estimate (4.9),

$$\|v\|_{L^\infty(0,t;L^2)} \lesssim \varepsilon^{d/2} e^{Ct}.$$

Second case: $\sigma = 2$ and $d = 1$. We have

$$\begin{aligned} \|v\|_{L^q L^r} &\lesssim \varepsilon^{-2/q} \|v_0\|_{L^2} + \sum_{j=1}^5 \varepsilon^{-2/q-2/q_j} \|\mathbf{u}_0^{5-j} v^j\|_{L^{q'_j} L^{r'_j}} + \varepsilon^{-2/q} |I| \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2} \\ &\lesssim \varepsilon^{-2/q} \|v_0\|_{L^2} + \sum_{j=1}^5 \varepsilon^{-2/q-2/q_j} \|v^j\|_{L^{q'_j} L^{r'_j}} + \varepsilon^{-2/q} |I| \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}, \end{aligned}$$

where all the above pairs are admissible. We now choose successively $(q, r) = (\infty, 2)$ and $(q, r) = (6, 6)$.

- For $j = 1$, we choose $(q_1, r_1) = (\infty, 2)$.
- For $j = 2$, we choose $(q_2, r_2) = (12, 3)$, and use Hölder inequality

$$\|v^2\|_{L^{12/11} L^{3/2}} \leq \|v\|_{L^{4/3} L^2} \|v\|_{L^6 L^6}.$$

- For $j = 3$, we choose $(q_3, r_3) = (\infty, 2)$, and write

$$\|v^3\|_{L^1 L^2} = \|v\|_{L^3 L^6}^3 \leq |I|^{1/2} \|v\|_{L^6 L^6}^3.$$

- For $j = 4$, we choose $(q_4, r_4) = (12, 3)$, and use Hölder inequality

$$\|v^4\|_{L^{12/11} L^{3/2}} = \|v\|_{L^{11/3} L^6}^4 \leq |I|^{7/66} \|v\|_{L^6 L^6}^4.$$

- For $j = 5$, we choose $(q_5, r_5) = (6, 6)$, and write

$$\|v^5\|_{L^{6/5} L^{6/5}} = \|v\|_{L^6 L^6}^5.$$

We come up with:

$$\begin{aligned} \|v\|_{L^\infty L^2} &\lesssim \|v_0\|_{L^2} + \|v\|_{L^1 L^2} + \varepsilon^{-1/6} \|v\|_{L^6 L^6} |I|^{3/4} \|v\|_{L^\infty L^2} + |I|^{1/2} \|v\|_{L^6 L^6}^3 \\ &\quad + \varepsilon^{-1/6} |I|^{7/66} \|v\|_{L^6 L^6}^4 + \varepsilon^{-1/3} \|v\|_{L^6 L^6}^5 + |I| \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

Grönwall lemma implies

$$\begin{aligned} \|v\|_{L^\infty L^2} &\lesssim e^{Ct} \|v_0\|_{L^2} + \varepsilon^{-1/6} \|v\|_{L^6 L^6} e^{Ct} \|v\|_{L^\infty L^2} + e^{Ct} \|v\|_{L^6 L^6}^3 \\ &\quad + \varepsilon^{-1/6} e^{Ct} \|v\|_{L^6 L^6}^4 + \varepsilon^{-1/3} e^{Ct} \|v\|_{L^6 L^6}^5 + e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

The bootstrap condition is now

$$(4.11) \quad \varepsilon^{-1/6} e^{Ct} \|v\|_{L^6 L^6} \ll 1,$$

which implies that in the above estimate, the second term on the right hand side is absorbed by the left hand side,

$$\begin{aligned} \|v\|_{L^\infty L^2} &\lesssim e^{Ct} \|v_0\|_{L^2} + e^{Ct} \|v\|_{L^6 L^6}^3 + \varepsilon^{-1/6} e^{Ct} \|v\|_{L^6 L^6}^4 \\ &\quad + \varepsilon^{-1/3} e^{Ct} \|v\|_{L^6 L^6}^5 + e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

Applying Strichartz inequality again, changing only (q, r) to consider $(q, r) = (6, 6)$, we infer

$$\begin{aligned} \|v\|_{L^6 L^6} &\lesssim \varepsilon^{-1/3} e^{Ct} \|v_0\|_{L^2} + \varepsilon^{-1/3} e^{Ct} \|v\|_{L^6 L^6}^3 + \varepsilon^{-1/2} e^{Ct} \|v\|_{L^6 L^6}^4 \\ &\quad + \varepsilon^{-2/3} e^{Ct} \|v\|_{L^6 L^6}^5 + \varepsilon^{-1/3} e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}. \end{aligned}$$

Under the bootstrap assumption (4.11), the terms involving $\|v\|_{L^6 L^6}$ on the right hand side are absorbed by the left hand side, and

$$\|v\|_{L^6 L^6} \lesssim \varepsilon^{-1/3} e^{Ct} \|v_0\|_{L^2} + \varepsilon^{-1/3} e^{Ct} \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2}.$$

Therefore, the assumption on v_0 and \mathbf{u}_0 for the argument to be conclusive is, like in the one-dimensional cubic case,

$$\|v_0\|_{L^2} + \|\varepsilon^2 \Delta \mathbf{u}_0\|_{L^2} \ll \sqrt{\varepsilon}.$$

Invoking Lemma 4.1 again, the second condition is satisfied provided that

$$\varepsilon^2 \delta^{\frac{1}{5}-1-\frac{1}{4}} \ll \sqrt{\varepsilon} \iff \varepsilon^{2-\frac{d}{2}} \ll \delta^{\frac{d}{4}+1-\frac{1}{2\sigma+1}}, \quad \text{as } d = 1 \text{ and } \sigma = 2,$$

and we conclude like in the first case $\sigma = 1$, thus proving Theorem 1.6.

5. NUMERICAL SIMULATIONS

We conclude this paper by exhibiting numerically some solutions constructed above. We consider the case $d = 1$ and the cubic nonlinearity. All numerical simulations are done using $N = 2^{14}$ grid points on a interval $[-2\pi, 2\pi]$. In Figure 1 we illustrate Theorem 1.1 by plotting the solution $\hat{u}_0(\xi)$ obtained by computing $f(x)^{\frac{1}{3}}$ and taking the discrete Fourier transform. We observe the expected power law $|\xi|^{-\frac{5}{3}}$ on an interval of larger size when δ gets smaller.

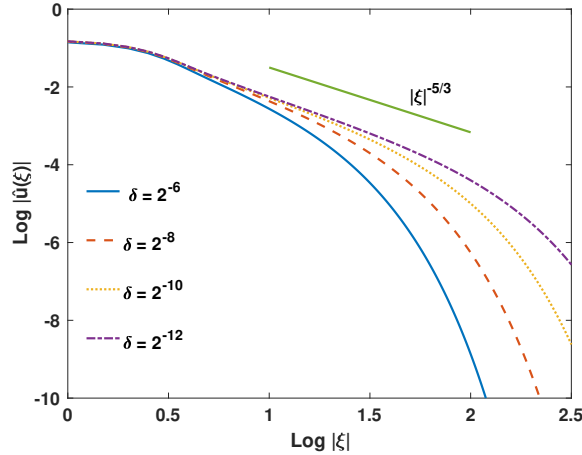


FIGURE 1. $\hat{u}_0(\xi)$ in the case $P = 0$, $d = 1$, $\varepsilon = 0$

In Figure 2, we fix δ and compute the solutions $\hat{u}_\varepsilon(\xi)$ given by Theorem 1.4 for various values of ε . We observe that when ε becomes small enough, the solution indeed converges towards u_0 . The numerical scheme used to calculate u_ε is the implementation of the iterative scheme (4.6). The discretization of the Laplace operator is done by standard finite differences with Dirichlet boundary conditions, which allows a simple inversion of the linearized operator. Note that when ε becomes too large, the solution becomes sensitive to the numerical discretization and the existence of solution is unclear.

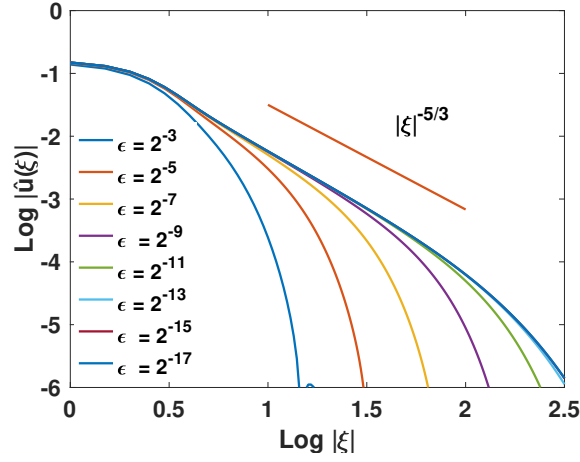


FIGURE 2. $\hat{u}_0(\xi)$ in the case $P = 0$, $d = 1$, $\delta = 10^{-4}$

In Figures 3 and 4 we repeat the same computations, in the case $P = 1$ given by Theorems 1.3 and 1.4, confirming the expected power law $|\xi|^{-2}$.

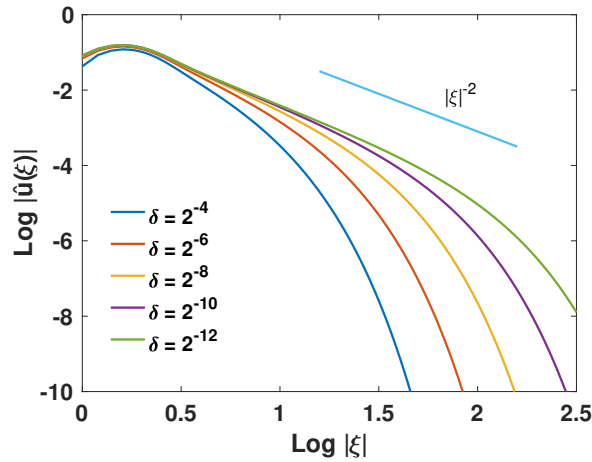


FIGURE 3. $\hat{u}_0(\xi)$ in the case $P = 1$, $d = 1$, $\varepsilon = 0$ and $\delta = 2^{-j}$

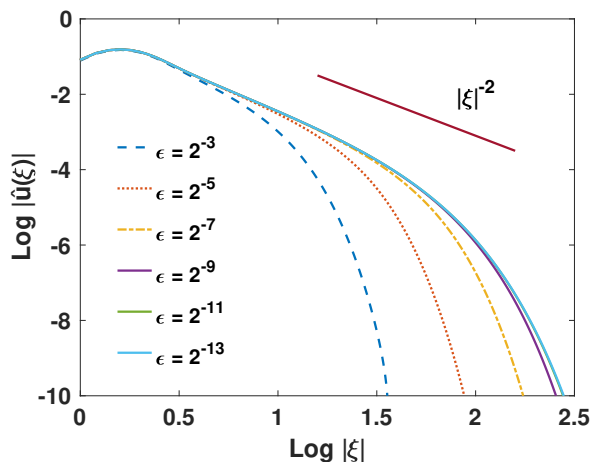


FIGURE 4. $\hat{u}_\varepsilon(\xi)$ in the case $P = 1$, $d = 1$, $\varepsilon = \delta 2^{-j}$ and $\delta = 2^{-10}$

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