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MODERATE DEVIATIONS FOR MANDELBROT'S MULTIPLICATIVE PROCESSES

FUQING GAO AND QUANSHENG LIU

ABSTRACT. Let W be a non-negative random variable with $EW = 1$, and let $\{W_{\mathbf{i}}\}$ be a family of independent copies of W , indexed by all the finite sequences $\mathbf{i} = i_1 \cdots i_n$ of positive integers. For fixed r and n the random multiplicative measure μ_r^n has the density $W_{i_1} \cdots W_{i_1 \cdots i_n}$ with respect to the Lebesgue measure on $[0, 1]$ on each r -adic interval $A_{i_1 \cdots i_n}^r = [\sum_{k=1}^n (i_k - 1)r^{-k}, \sum_{k=1}^n (i_k - 1)r^{-k} + r^{-n})$. If $EW \log W < \log r$, the sequence $\{\mu_r^n\}_n$ converges a.s. weakly to the Mandelbrot measure μ_r^∞ . For each fixed $1 \leq n \leq \infty$, we study moderate deviations for the sequence of random measures $\{\mu_r^n\}_r$ as $r \rightarrow \infty$. We prove that for each $1 \leq n < \infty$, $\{\mu_r^n, r \geq 1\}$ satisfies a moderate deviation principle if $E(e^{\delta W}) < \infty$ for some $\delta > 0$ and that $\{\mu_r^\infty, r \geq 1\}$ satisfies a moderate deviation principle if W is bounded.

1. INTRODUCTION

Let us recall the basic model of the Mandelbrot cascade [11, 12, 7]. Let $\mathbb{N}^* = \{1, 2, \dots\}$ be the set of positive integers, and let

$$\mathbb{U} = \{\emptyset\} \cup \cup_{k=1}^{\infty} (\mathbb{N}^*)^k$$

be the set of all finite sequences $i_1 \cdots i_n$ containing the null sequence \emptyset . Let W be a non-negative random variable with $EW = 1$ and $P(W = 1) \neq 1$, and let $\{W_{\mathbf{i}} : \mathbf{i} \in \mathbb{U}\}$ be independent copies of W . Let \mathcal{F}_0 be the trivial σ -algebra, and for $n > 1$, let \mathcal{F}_{n-1} be the σ -algebra generated by $\{W_{i_1}, \dots, W_{i_1, \dots, i_{n-1}} : 1 \leq i_1 \leq r, \dots, 1 \leq i_{n-1} \leq r\}$. Let λ be the Lebesgue measure on $[0, 1]$. Fix $r \geq 2$, for every $n \geq 1$, let μ_r^n be the random measure on $[0, 1]$ defined by

$$(1.1) \quad \mu_r^n(f) = \int f d\mu_r^n = \sum_{1 \leq i_1, \dots, i_n \leq r} W_{i_1, \dots, i_n} \int_{A_{i_1 \cdots i_n}^r} f d\lambda,$$

for each $f \in \mathcal{L}^1([0, 1], \lambda)$, where $A_{i_1 \cdots i_n}^r = [\sum_{k=1}^n (i_k - 1)r^{-k}, \sum_{k=1}^n (i_k - 1)r^{-k} + r^{-n})$.

For fixed $r \geq 2$, almost surely (a.s.) the sequence of random measures $\{\mu_r^n, n \geq 1\}$ is weakly convergent to the Mandelbrot measure μ_r^∞ for multiplicative cascades, as $n \rightarrow \infty$. The measure μ_r^∞ is not a.s. the null measure if and only if $\log r > EW \log W$ (see Kahane and Peyrière [7]). This measure and its extensions have been studied by many authors, see for example ([7], [9]). Asymptotic properties of the masses $Z_r^\infty := \mu_r^\infty(1)$ as $r \rightarrow \infty$, have been studied in [10]. Asymptotic properties of the measures μ_r^n as $r \rightarrow \infty$, including LLN, FCLT, FLIL and LDP, have been studied in [9].

As a complementarity of [9], here we consider moderate deviations for $(\mu_r^n)_r$ as $r \rightarrow \infty$. We prove that the moderate deviation principle (MDP) holds in the strong (or τ) topology under the assumption:

$$E(\exp\{\theta W\}) < \infty \text{ for some } \theta > 0 \tag{A_1}$$

if $1 \leq n < \infty$, and under the assumption:

$$\bar{w} := \text{ess sup } W < \infty \tag{A_\infty}$$

if $n = \infty$.

For convenience, let us introduce the conception of large deviations(cf.[4]).

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Definition 1.1. Let E be a metric space with metric ρ , and let $\{Y_r : r \geq 1\}$ be a family of E -valued random variables. Let $\lambda(r)$ be a sequence of positive real numbers satisfying $\lambda(r) \rightarrow 0$ as $r \rightarrow \infty$.

(1). A function $I(\cdot) : E \rightarrow [0, +\infty]$ is said to be a rate function if it is lower semicontinuous and it is said to be a good rate function if its level set $\{x \in E : I(x) \leq a\}$ is compact for all $a \geq 0$.

(2). The sequence $\{Y_r : r \geq 1\}$ is said to satisfy a large deviation principle (LDP) with speed $\lambda(r)$ and rate function $I(\cdot)$ if for any closed set F in E

$$\limsup_{r \rightarrow \infty} \lambda(r) \log P(Y_r \in F) \leq - \inf_{x \in F} I(x),$$

and open set G in E

$$\liminf_{r \rightarrow \infty} \lambda(r) \log P(Y_r \in G) \geq - \inf_{x \in G} I(x).$$

(3). The sequence $P(Y_r \in \cdot)$ is exponentially tight if for every $L > 0$, there is a compact set K_L in E such that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \lambda(r) \log P\{Y_r \notin K_L\} \leq -L.$$

Throughout this paper, Let $\{a(r), r \geq 1\}$ be a sequence of positive numbers satisfying

$$\frac{a(r)}{r} \rightarrow 0, \quad \frac{r \log r}{a^2(r)} \rightarrow 0.$$

Let \mathcal{M} denote the set of finite sign measures on $[0, 1]$ and let \mathcal{B} be the Borel σ -field on $[0, 1]$. For $A \in \mathcal{B}$, let p_A be the map $\mathcal{M} \rightarrow \mathbb{R}^+$ defined by $p_A(\nu) = |\nu(A)|$. The strong topology or τ -topology on \mathcal{M} is the coarsest topology which makes continuous the maps p_A , $A \in \mathcal{B}$. It turns \mathcal{M} into a gauge space with the separating family $\mathcal{D} := \{d_A\}_{A \in \mathcal{B}}$ of pseudo-metrics, where $d_A(\nu, \mu) := |\nu(A) - \mu(A)|$ for $\nu, \mu \in \mathcal{M}$. We equip \mathcal{M} with \mathbf{B} , the σ -field generated by p_A , $A \in \mathcal{B}$. We denote by \mathcal{B}_b the set of bounded measurable functions on $[0, 1]$.

The following theorem is the main result in this paper.

Theorem 1.1. (1). For each fixed $1 \leq n < \infty$, if (A_1) holds, then the sequence $\{\frac{r}{a(r)}(\mu_r^n - \lambda), r \geq 1\}$ satisfies the LDP on \mathcal{M} equipped with the strong topology, with speed $\frac{r}{a^2(r)}$ and good rate function I , where

$$(1.2) \quad I(\nu) = \begin{cases} \int_0^1 \frac{1}{2\sigma^2} |\dot{\nu}|^2 d\lambda & \text{if } \dot{\nu} = \frac{d\nu}{d\lambda} \text{ exists} \\ +\infty & \text{otherwise} \end{cases}$$

where $\sigma^2 := \text{Var}(W)$.

(2). If (A_∞) holds, then the sequence $\{\frac{r}{a(r)}(\mu_r^\infty - \lambda), r \geq 1\}$ satisfies the LDP on \mathcal{M} equipped with the strong topology, with speed $\frac{r}{a^2(r)}$ and good rate function $I(\nu)$.

2. EXPONENTIAL APPROXIMATIONS

In this section we will prove the following family of truncated variables

$$(2.1) \quad \mu_r^{n,M}(f) = \sum_{1 \leq i_1, \dots, i_n \leq r} W_{i_1} \widetilde{W}_{i_1 i_2}^M \cdots \widetilde{W}_{i_1 \dots i_n}^M \int_{A_{i_1 \dots i_n}^r} f d\lambda, \quad M \geq 1,$$

is exponentially good approximations of $\{\frac{r}{a(r)}(\mu_r^n - \lambda), r \geq 1\}$, where $\widetilde{W}_{\mathbf{i}}^M = (W_{\mathbf{i}} \wedge M) / E_M$ for $\mathbf{i} \in \mathbb{U}$, $E_M = E(W \wedge M)$. We denote by convention $\mu_r^{1,M} = \mu_r^1$.

Proposition 2.1. Let $\delta > 0$ and $n \geq 2$.

(1). Under Assumption (A_1) ,

$$(2.2) \quad \limsup_{M \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} |\mu_r^n(f) - \mu_r^{n,M}(f)| > \delta \right) = -\infty.$$

(2). Under Assumption (A₁), for every $f \in \mathcal{B}_b$ and $M > 0$,

$$(2.3) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} \left| \mu_r^{n,M}(f) - \mu_r^1(f) \right| > \delta \right) = -\infty.$$

2.1. A recursive relation on μ_r^n . Fix $1 \leq k \leq r$, if the weights $W_{i_1}, \dots, W_{i_1 \dots i_n}$ in (1.1) are replaced by $W_{ki_1}, \dots, W_{ki_1 \dots i_n}$, the corresponding measures will be denoted by $\mu_r^n \circ \Theta_k$ ($1 \leq n < \infty$), and its weak limit (as $n \rightarrow \infty$) by $\mu_r^\infty \circ \Theta_k$. For fixed r and f , the random variables $(\mu_r^\infty \circ \Theta_k)(f)$, $1 \leq k \leq r$, are independent of each other and have the same law as $\mu_r^\infty(f)$, and as a family they are independent of (W_1, \dots, W_r) . For $k = 1, \dots, r$, let τ_k^r be the operator acting on functions from $[0, 1]$ to \mathbb{R} , defined by

$$\tau_k^r f(x) = f\left(\frac{k-1+x}{r}\right), \quad x \in [0, 1].$$

Since $t \in A_{i_1 \dots i_n}^r$ if and only if $r(t - \frac{i_1-1}{r}) \in A_{i_2 \dots i_n}^r$, we have, for $f \in \mathcal{L}^1([0, 1], \lambda)$,

$$(2.4) \quad \mu_r^n(f) = \int f d\mu_r^n = \sum_{k=1}^r W_k \sum_{1 \leq i_2, \dots, i_n \leq r} W_{ki_2} \dots W_{ki_2 \dots i_n} \int_{A_{i_2 \dots i_n}^r} \frac{1}{r} f\left(\frac{s+k-1}{r}\right) ds,$$

so that for each $1 \leq n < \infty$,

$$(2.5) \quad \mu_r^n(f) = \frac{1}{r} \sum_{k=1}^r W_k (\mu_r^{n-1} \circ \Theta_k)(\tau_k^r f),$$

with the convention $\mu_r^0 \circ \Theta_k = \lambda$. Taking the limit as $n \rightarrow \infty$ in (2.5), we see that a.s. for every $f \in \mathcal{C}([0, 1])$,

$$(2.6) \quad \mu_r^\infty(f) = \frac{1}{r} \sum_{k=1}^r W_k (\mu_r^\infty \circ \Theta_k)(\tau_k^r f).$$

2.2. Exponential inequalities.

Lemma 2.1. (Hoeffding's inequality) If $X \in [\alpha, \beta]$, then

$$(2.7) \quad \log E e^{t(X-EX)} \leq \frac{1}{8}(\beta - \alpha)^2 t^2.$$

By independence and Hoeffding's inequality, one can get the following exponential inequality ([9]).

Lemma 2.2. Fix $n \geq 1$ and let U^1, U^2, \dots, U^n be independent and bounded random variables with $E(U^n) = 0$. Let $(U_{i_1 \dots i_n}^n)$ be a family of independent r.v. indexed by (n, i_1, \dots, i_n) , such that for every $1 \leq k \leq n$, $U_{i_1 \dots i_k}^k$ has the same distribution as U^k . Let $\mathbf{a} = (a_{i_1 \dots i_n}^r : 1 \leq i_1 \dots i_n \leq r; r \geq 1)$ be a family of real numbers such that $\|\mathbf{a}\| := \sup_r \max_{1 \leq i_1 \dots i_n \leq r} |a_{i_1 \dots i_n}^r| < \infty$. For $r \geq 1$, set

$$(2.8) \quad \Gamma_r(\mathbf{a}) := r^{-n} \sum_{1 \leq i_1, \dots, i_n \leq r} U_{i_1}^1 \dots U_{i_1 \dots i_n}^n a_{i_1 \dots i_n}^r.$$

Then for any $\delta > 0$,

$$(2.9) \quad P(\Gamma_r(\mathbf{a}) > \delta) \leq \exp\left(-\frac{\delta^2 r^n}{2L^2 n \|\mathbf{a}\|^2}\right)$$

where $L = \max_{1 \leq k \leq n} \{\|U^k\|_\infty\}$.

2.3. Proof of Proposition 2.1.

Lemma 2.3. For any $1 \leq n < \infty$, and any $\delta > 0$,

$$(2.10) \quad \limsup_{M \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log \sup_{\|\mathbf{a}\| \leq 1} P \left(\frac{r}{a(r)} D_r^{n, \mathbf{a}}(M) > \delta \right) = -\infty,$$

where

$$D_r^{n, \mathbf{a}}(M) = r^{-n} \left| \sum_{1 \leq i_1, \dots, i_n \leq r} (W_{i_1 i_2} \cdots W_{i_1 \cdots i_n} - \widetilde{W}_{i_1 i_2}^M \cdots \widetilde{W}_{i_1 \cdots i_n}^M) a_{i_1 \cdots i_n}^r \right|, \quad n \geq 2,$$

and $\mathbf{a} = \{a_{i_1 \cdots i_n}^r : 1 \leq i_1, \dots, i_n \leq r; r \geq 1\}$ be a family of real numbers such that $\|\mathbf{a}\| < \infty$.

Proof. We first consider the case $n = 1$. For any $\alpha > 0$, by Chebyshev inequality, we have

$$\begin{aligned} & P \left(\frac{r}{a(r)} D_r^{n, \mathbf{a}}(M) > \delta \right) \\ & \leq e^{-\delta \alpha r / a^2(r)} E \left(\exp \left\{ \frac{a(r)}{r} \sum_{i=1}^r (W_{i_1} - \widetilde{W}_{i_1}^M) a_{i_1}^r \right\} \right) \\ & \leq e^{-\delta \alpha r / a^2(r)} \left(1 + \frac{\alpha a(r)}{2r^2} E(W_{i_1} - \widetilde{W}_{i_1}^M)^2 (a_{i_1}^r)^2 + o \left(\frac{a^2(r)}{r^2} \right) \right)^r, \end{aligned}$$

which implies

$$\limsup_{M \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log \sup_{\|\mathbf{a}\| \leq 1} P \left(\frac{r}{a(r)} D_r^{n, \mathbf{a}}(M) > \delta \right) \leq -\delta \alpha \rightarrow -\infty \text{ as } \alpha \rightarrow \infty.$$

Now we assume that (2.10) holds for $n - 1$. Since

$$\begin{aligned} W_{i_1} \cdots W_{i_1 \cdots i_n} - \widetilde{W}_{i_1}^M \cdots \widetilde{W}_{i_1 \cdots i_n}^M &= W_{i_1} (W_{i_1 i_2} \cdots W_{i_1 \cdots i_n} - \widetilde{W}_{i_1 i_2}^M \cdots \widetilde{W}_{i_1 \cdots i_n}^M) \\ &\quad + (W_{i_1} - \widetilde{W}_{i_1}^M) \widetilde{W}_{i_1 i_2}^M \cdots \widetilde{W}_{i_1 \cdots i_n}^M, \end{aligned}$$

we get that for each \mathbf{a} with $\|\mathbf{a}\| \leq 1$,

$$(2.11) \quad D_r^{n, \mathbf{a}}(M) \leq \frac{1}{r} \sum_{k=1}^r W_k D_r^{n-1, \mathbf{a}^{(k)}}(M) \circ \Theta_k + \bar{D}_r^n(M),$$

where $\mathbf{a}^{(k)} = \{a_{k i_2 \cdots i_n}^r : 1 \leq i_2, \dots, i_n \leq r; r \geq 1\}$, and

$$\bar{D}_r^n(M) = r^{-1} \left| \sum_{k=1}^r (W_k - \widetilde{W}_k^M) (\bar{Z}_r^{n-1}(M) \circ \Theta_k) \right|.$$

and

$$(2.12) \quad \bar{Z}_r^{n-1}(M) = r^{1-n} \sum_{1 \leq i_1, \dots, i_{n-1} \leq r} \widetilde{W}_{i_1}^M \cdots \widetilde{W}_{i_1 \cdots i_{n-1}}^M.$$

If $\frac{r}{a(r)} D_r^{n-1, \mathbf{a}^{(k)}}(M) \circ \Theta_k \leq \varepsilon$ for all $1 \leq k \leq r$, then $\frac{r}{a(r)} \frac{1}{r} \sum_{k=1}^r W_k D_r^{n-1, \mathbf{a}^{(k)}}(M) \circ \Theta_k \leq \varepsilon Z_r^1$, where $Z_r^1 = \frac{1}{r} \sum_{k=1}^r W_k$. Therefore, for $\delta > 2\varepsilon$,

$$(2.13) \quad \begin{aligned} & P \left(\frac{r}{a(r)} \sum_{k=1}^r W_k D_r^{n-1, \mathbf{a}^{(k)}}(M) \circ \Theta_k > \frac{\delta}{2} \right) \\ & \leq P \left(Z_r^1 > \frac{\delta}{2\varepsilon} \right) + r \sup_{\|\mathbf{a}^{(k)}\| \leq 1, 1 \leq k \leq r} P \left(\frac{r}{a(r)} D_r^{n-1, \mathbf{a}^{(k)}}(M) > \varepsilon \right). \end{aligned}$$

By Chebyshev inequality, we have that for any $\alpha > 0$,

$$\begin{aligned} \frac{r}{a^2(r)} \log P \left(Z_r^1 > \frac{\delta}{2\varepsilon} \right) &\leq -\frac{\alpha\delta}{2\varepsilon} + \frac{r^2}{a^2(r)} \log E(\exp\{a^2(r)\alpha W/r^2\}) \\ &\leq -\frac{\alpha\delta}{2\varepsilon} + \frac{r^2}{a^2(r)} \log \left(1 + \frac{a^2(r)}{r^2} \alpha E(W) + o\left(\frac{a^2(r)}{r^2}\right) \right) \end{aligned}$$

which implies that

$$(2.14) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(Z_r^1 > \frac{\delta}{2\varepsilon} \right) \leq -\frac{\alpha\delta}{2\varepsilon} + \alpha \rightarrow -\infty \text{ as } \varepsilon \rightarrow 0.$$

Therefore, by the induction hypothesis, we have

$$(2.15) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log \sup_{\|\mathbf{a}\| \leq 1} P \left(\frac{r}{a(r)} \sum_{k=1}^r W_k D_r^{n-1, \mathbf{a}^{(k)}}(M) \circ \Theta_k > \frac{\delta}{2} \right) = -\infty.$$

On the other hand, for any $\alpha \in \mathbb{R}$, By conditioning and independence, we have

$$\begin{aligned} &E \left(\exp \left\{ \frac{\alpha a(r)}{r} \sum_{k=1}^r (W_k - \widetilde{W}_k^M) (\tilde{Z}_r^{n-1}(M) \circ \Theta_k) \right\} \right) \\ &= \left(1 + \frac{\alpha a^2(r)}{r^2} E(W_1 - \widetilde{W}_1^M)^2 E(\tilde{Z}_r^{n-1}(M))^2 + o\left(\frac{a^2(r)}{r^2}\right) \right)^r. \end{aligned}$$

Since $\lim_{r \rightarrow \infty} E(\tilde{Z}_r^{n-1}(M))^2 = 1$, we have

$$\lim_{r \rightarrow \infty} \frac{r}{a^2(r)} \log E \left(\exp \left\{ \frac{\alpha a(r)}{r} \sum_{k=1}^r (W_k - \widetilde{W}_k^M) (\tilde{Z}_r^{n-1}(M) \circ \Theta_k) \right\} \right) = \frac{\alpha a^2(r)}{r^2} E(W_1 - \widetilde{W}_1^M)^2.$$

By Gärtner-Ellis's theorem, it is easy to get that

$$(2.16) \quad \lim_{M \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} \bar{D}_r^n(M) > \delta/2 \right) = -\infty.$$

Therefore, by (2.11), (2.15), and (2.16), we complete the proof of (2.10). □

Proof of Proposition 2.1

(1). For any $f \in \mathcal{B}_b$,

$$(2.17) \quad |\mu_r^n(f) - \mu_r^{n, M}(f)| \leq \tilde{D}_r^n(M),$$

where

$$\tilde{D}_r^n(M) = r^{-n} \left| \sum_{1 \leq i_1, \dots, i_n \leq r} W_{i_1} (W_{i_1 i_2} \cdots W_{i_1 \dots i_n} - \widetilde{W}_{i_1 i_2}^M \cdots \widetilde{W}_{i_1 \dots i_n}^M) a_{i_1 \dots i_n}^r \right|,$$

and

$$a_{i_1 \dots i_n}^r = r^n \int_{A_{i_1 \dots i_n}^r} f d\lambda, \quad n \geq 2.$$

Since $\|\mathbf{a}\| \leq 1$ and

$$\tilde{D}_r^n(M) = \frac{1}{r} \sum_{k=1}^r W_k D_r^{n-1, \mathbf{a}^{(k)}}(M) \circ \Theta_k,$$

we have that for any $2 \leq n < \infty$ and any $\delta > 0$,

$$(2.18) \quad \limsup_{M \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} \tilde{D}_r^n(M) > \delta \right) = -\infty$$

which implies (2.2).

(2). By the triangle inequality, it is enough to prove that for every $n \geq 2$, $\delta > 0$ and $f \in \mathcal{B}_b$,

$$(2.19) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} |\mu_r^{n,M}(f) - \mu_r^{n-1,M}(f)| > \delta \right) = -\infty.$$

In fact, for $n \geq 2$,

$$(2.20) \quad \mu_r^{n,M}(f) - \mu_r^{n-1,M}(f) = \frac{1}{r^n} \sum_{1 \leq i_1, \dots, i_n \leq r} W_{i_1} \widetilde{W}_{i_1 i_2}^M \cdots \widetilde{W}_{i_1 \cdots i_{n-1}}^M \left(\widetilde{W}_{i_1 \cdots i_n}^M - 1 \right) a_{i_1 \cdots i_n}^r,$$

where $a_{i_1 \cdots i_n}^r = r^n \int_{A_{i_1 \cdots i_n}^r} f d\lambda$. For $k = 1, \dots, r$, set

$$(2.21) \quad Q_k = P \left(\frac{r}{a(r)} \left| \frac{1}{r^{n-1}} \sum_{1 \leq i_2, \dots, i_n \leq r} \widetilde{W}_{k i_2}^M \cdots \widetilde{W}_{k i_2 \cdots i_{n-1}}^M \left(\widetilde{W}_{k i_2 \cdots i_n}^M - 1 \right) a_{k i_2 \cdots i_n}^r \right| > \varepsilon \right).$$

Then

$$(2.22) \quad P \left(\frac{r}{a(r)} |\mu_r^{n,M}(f) - \mu_r^{n-1,M}(f)| > \delta \right) \leq P(Z_r^1 > \delta/\varepsilon) + \sum_{k=1}^r Q_k.$$

For every $M > 0$, the variables \widetilde{W}_i and $\widetilde{W}_i - 1$ are bounded by some $L_M > 0$, by Lemma 2.2, we have

$$(2.23) \quad Q_k \leq 2 \exp \left(-\frac{a^2(r)\varepsilon^2}{2r^2} r^{n-1} (L_M)^{2-2n} \right)$$

For $n \geq 3$, inequalities (2.22), (2.23) and (2.14) yield

$$(2.24) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} |\mu_r^{n,M}(f) - \mu_r^{n-1,M}(f)| > \delta \right) = -\infty.$$

We get (2.19) for $n \geq 3$.

For the case $n = 2$, the proof is similar to (2.16). In fact, for any $\alpha \in \mathbb{R}$, By conditioning and independence, we have

$$\begin{aligned} & E \left(\exp \left\{ \frac{\alpha a^2(r)}{r} \frac{r}{a(r)} (\mu_r^{n,M}(f) - \mu_r^{n-1,M}(f)) \right\} \right) \\ &= E \left(\exp \left\{ \frac{\alpha a(r)}{r} \sum_{i_2=1}^r \left(\widetilde{W}_{i_1 i_2}^M - 1 \right) \frac{1}{r} \sum_{i_1=1}^r W_{i_1} a_{i_1 i_2}^r \right\} \right) \\ &\leq \left(1 + \frac{\alpha a^2(r)}{r^2} E(\widetilde{W}^M - 1)^2 \sigma^2 + o\left(\frac{a^2(r)}{r^2}\right) \right)^r. \end{aligned}$$

Therefore, by Chebyshev inequality, it is easy to get (2.19). □

3. PROOF OF MAIN RESULT

3.1. The MDP for $(\mu_r^1)_r$ in the strong topology. The MDP for the empirical measures can be obtained by Dawson-Gärtner's theorem (cf. [2] Theorem 3.4) and calculating the Laplace asymptotic integral (cf. [3], [6],[13],[14]). Similar to the empirical measures case, it is easy to get the following The MDP for $(\mu_r^1)_r$.

Theorem 3.1. *If (A_1) holds, then the sequence $\left\{ \frac{r}{a(r)} (\mu_r^1 - \lambda), r \geq 1 \right\}$ satisfies the LDP on \mathcal{M} equipped with the strong topology, with speed $\frac{r}{a^2(r)}$ and good rate function $I(\nu)$.*

Proof. For any $f \in \mathcal{C}_b[0, 1]$, we have

$$\begin{aligned}
\Lambda(f) &:= \lim_{r \rightarrow \infty} \frac{r}{a^2(r)} \log E \left(\exp \left\{ \frac{a^2(r)}{r} \frac{r}{a(r)} (\mu_r^1 - \lambda)(f) \right\} \right) \\
&= \lim_{r \rightarrow \infty} \frac{r}{a^2(r)} \log E \left(\exp \left\{ a(r) \sum_{k=1}^r (W-1) \int_{[(k-1)/r, k/r]} f d\lambda \right\} \right) \\
&= \lim_{r \rightarrow \infty} \frac{r}{a^2(r)} \sum_{k=1}^r \log \left(1 + \frac{\sigma^2 a^2(r)}{2} \left(\int_{[(k-1)/r, k/r]} f d\lambda \right)^2 \right) \\
&= \frac{\sigma^2}{2} \lim_{r \rightarrow \infty} r \sum_{k=1}^r \left(\int_{[(k-1)/r, k/r]} f d\lambda \right)^2 = \frac{\sigma^2}{2} \int_{[0,1]} f^2 d\lambda.
\end{aligned}$$

and

$$\Lambda^*(\nu) := \sup_{f \in \mathcal{C}_b[0,1]} \left\{ \int_{[0,1]} f d\nu - \Lambda(f) \right\} = I(\nu), \quad \nu \in \mathcal{M}.$$

Therefore, by Dawson-Gärtner's theorem, one can get that that $\left\{ \frac{r}{a(r)} (\mu_r^1 - \lambda), r \geq 1 \right\}$ satisfies the LDP on \mathcal{M} equipped with the weak topology, with speed $\frac{r}{a^2(r)}$ and good rate function $I(\nu)$. It is known that I is also good for the τ -topology, that is, the sets $K_L = \{\nu \in \mathcal{M} : I(\nu) \leq L\}$ is also τ -compact. By for any $f \in \mathcal{B}_b$,

$$\Lambda(f) := \lim_{r \rightarrow \infty} \frac{r}{a^2(r)} \log E \left(\exp \left\{ \frac{a^2(r)}{r} \frac{r}{a(r)} (\mu_r^1 - \lambda)(f) \right\} \right) \leq \frac{\sigma^2}{2} \int_{[0,1]} f^2 d\lambda$$

and K_L τ -compact for each $0 < L < \infty$, one can obtained (cf. [1],[4]) that for any closed set F in τ -topology,

$$\lim_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} (\mu_r^1 - \lambda) \in F \right) \leq - \inf_{\nu \in F} I(\nu).$$

Therefore, $\left\{ \frac{r}{a(r)} (\mu_r^1 - \lambda), r \geq 1 \right\}$ satisfies the LDP on \mathcal{M} equipped with the strong topology. \square

3.2. The MDP for $(\mu_r^n)_r$, with $n \geq 2$, in the strong topology. (1). By Proposition 2.1, for every $f \in \mathcal{B}_b$ and every $\delta > 0$, we have

$$(3.1) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} | \mu_r^n(f) - \mu_r^1(f) | > \delta \right) = -\infty,$$

which implies that $\left\{ \frac{r}{a(r)} (\mu_r^n - \lambda), r \geq 1 \right\}$ and $\left\{ \frac{r}{a(r)} (\mu_r^1 - \lambda), r \geq 1 \right\}$ are exponentially equivalent in the strong topology. Since $\left\{ \frac{r}{a(r)} (\mu_r^1 - \lambda), r \geq 1 \right\}$ satisfies the LDP by Theorem 3.1, we conclude that so does $\left\{ \frac{r}{a(r)} (\mu_r^n - \lambda), r \geq 1 \right\}$.

(2). Since $\bar{w} \geq 1$, we have $|W-1| \leq \bar{w}$. For any $f \in \mathcal{B}_b$ with $\|f\|_\infty \leq 1$, by Lemma 2.2, we have

$$(3.2) \quad P(|\mu_r^k(f) - \mu_r^{k-1}(f)| > \delta) \leq 2 \exp \left(-\delta^2 r^k \bar{w}^{-2k} / 2 \right),$$

for each $1 \leq k < \infty$. This yields

$$\begin{aligned}
P(|\mu_r^\infty(f) - \mu_r^1(f)| > \delta) &\leq P \left(\sum_{k=2}^{\infty} |\mu_r^k(f) - \mu_r^{k-1}(f)| > \delta \right) \\
&\leq \sum_{k=2}^{\infty} P \left(|\mu_r^k(f) - \mu_r^{k-1}(f)| > \delta 2^{-k+1} \right) \leq C \exp \left(-\frac{\delta^2 r^2}{8\bar{w}^4} \right)
\end{aligned}$$

for r large enough, where $C > 0$ is a constant independent of r . This gives

$$(3.3) \quad \limsup_{r \rightarrow \infty} \frac{r}{a^2(r)} \log P \left(\frac{r}{a(r)} |\mu_r^\infty(f) - \mu_r^1(f)| > \delta \right) = -\infty,$$

so that $\left\{ \frac{r}{a(r)} (\mu_r^\infty - \lambda), r \geq 1 \right\}$ and $\left\{ \frac{r}{a(r)} (\mu_r^1 - \lambda), r \geq 1 \right\}$ are exponentially equivalent and we end the proof as in (1). □

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