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Boundary output feedback stabilization of a cascade of N heat equations

Hugo Lhachemi, Christophe Prieur, and Emmanuel Trélat ^{*†‡}

Abstract

This paper solves the problem of output feedback stabilization for a cascade of N heat equations that are coupled at the boundary, the input being a scalar boundary control applied to the first heat equation of the cascade, and the scalar output being either a distributed or a pointwise in-domain measurement done on the last equation of the cascade. Two different configurations are studied in details. The first one consists of a cascade of N heat equations with totally disconnected spectra. The second one consists of a cascade of N identical heat equations, inducing eigenvalues of multiplicity N . In both cases, the problem is solved thanks to a spectral analysis and a study of the modal controllability and observability properties. The key point is that the generalized eigenvectors form a Riesz basis of the state space. The stabilization property is established in L^2 and H^1 norms.

1 Introduction

The design of control strategies for partial differential equation (PDE) cascades remains challenging. The case of coupled hyperbolic-parabolic PDE systems has been studied in [5, 9] using backstepping transformations. Control of a heat-wave PDE boundary coupling has been investigated in [27, 16]. Boundary null controllability of two coupled parabolic PDEs has been studied in [3]. Hyperbolic-elliptic couplings have been studied in [6, 17]. Cascades of heat-heat equations have been reported in [24, 11, 20, 21, 15]. The situation becomes even more stringent for cascades of N PDEs. In [1, 2] the authors deal with cascades of N interconnected hyperbolic systems using backstepping-based methods.

To our knowledge, few contributions have addressed the topic of an arbitrary number N of coupled heat equations. The observer design for a system of N heat equations with in-domain and boundary couplings has been studied in [4]. The state-feedback boundary control of N in-domain coupled heat equations, by means of N boundary controls, has been reported in [23]. Finally, the state-feedback boundary control of a chain of N heat-ODE cascades has been studied in [25] for PDEs that cascade into the ODEs through bounded operators.

The present paper reports one of the first contributions regarding the boundary output feedback stabilization of a cascade of N heat equations by means of one scalar boundary control and one

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scalar measurement. We consider the system

$$y_t^j = y_{xx}^j + a_j y^j, \quad 1 \leq j \leq N, \quad (1a)$$

$$y_x^j(t, 0) = y^{j+1}(t, 1), \quad y_x^j(t, 1) = 0, \quad 1 \leq j \leq N-1, \quad (1b)$$

$$y_x^N(t, 0) = u(t), \quad y_x^N(t, 1) = 0, \quad (1c)$$

$$y^j(0, x) = y_0^j(x), \quad 1 \leq j \leq N, \quad (1d)$$

where $a_1, \dots, a_N \in \mathbb{R}$. Here, $u(t)$ stands for the control input at time t , and $y_0^j \in L^2(0, 1)$, $1 \leq j \leq N$, represent the initial conditions. Two different configurations are studied. The first one consists of a cascade of N heat equations with two-by-two distinct eigenvalues. At the opposite, the second one consists of a cascade of N identical heat equations, inducing eigenvalues of multiplicity N . In both cases, the problem is solved by developing spectral reduction techniques (see [18, 7, 14, 22]) thanks to a full characterization of the eigenlements of the underlying operator. A key point of our approach is that the study of the eigenlements is not done for each component of the cascade individually, but directly for the full cascade seen as one single system (see [16]). We show that the generalized eigenvectors of the system form a Riesz basis of the state space, allowing us to study the modal controllability and observability properties of the cascade. Based on these results, output feedback control strategies are designed. The output, that can take the form either of a distributed or of a pointwise measurement, is done solely on the last equation of the cascade (described by y^1 in (1)). We establish stabilization results in L^2 and H^1 norms.

The paper is organized as follows. The configuration of the PDE cascade (1) with two-by-two distinct eigenvalues is studied in Section 2. The case of N identical heat equations, inducing eigenvalues of multiplicity N , is reported in Section 3. Concluding remarks are provided in Section 4.

2 N heat equations with two-by-two distinct eigenvalues

In this section, we make the following standing assumption, meaning that the spectra of the coupled heat equations are totally disconnected.

Assumption 1. *Setting $\lambda_{i,k} = a_i - k^2 \pi^2$ and $\Lambda_i = \{\lambda_{i,k} \mid k \in \mathbb{N}\}$ for all $1 \leq i \leq N$ and $k \in \mathbb{N}$, we assume that $i \neq j \Rightarrow \Lambda_i \cap \Lambda_j = \emptyset$.*

2.1 Spectral properties

We define the real Hilbert space

$$\mathcal{H}^0 = (L^2(0, 1))^N \quad (2a)$$

endowed with its usual inner product

$$\langle (f^j)_{1 \leq j \leq N}, (g^j)_{1 \leq j \leq N} \rangle = \sum_{j=1}^N \int_0^1 f^j(x) g^j(x) dx. \quad (2b)$$

We define the unbounded operator $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}^0$ on

$$D(\mathcal{A}) = \left\{ (f^j)_{1 \leq j \leq N} \in \mathcal{H}^0 \mid (f^j)_{1 \leq j \leq N} \in (H^2(0, 1))^N, \quad (f^N)'(0) = (f^N)'(1) = 0, \right. \\ \left. (f^j)'(0) = f^{j+1}(1), \quad (f^j)'(1) = 0, \quad 1 \leq j \leq N-1 \right\} \quad (3)$$

by $\mathcal{A}(f^j)_{1 \leq j \leq N} = \left((f^j)'' + a_j f^j \right)_{1 \leq j \leq N}$.

Lemma 1. Under Assumption 1, the eigenvalues of \mathcal{A} are given by

$$\lambda_{i,k} = a_i - k^2\pi^2, \quad 1 \leq i \leq N, \quad k \in \mathbb{N}, \quad (4)$$

with associated eigenvectors $\phi_{i,k} = (\phi_{i,k}^j)_{1 \leq j \leq N} \in D(\mathcal{A})$ given by

$$\phi_{i,k}^j(x) = \begin{cases} \frac{(-1)^{k+i-j} \mu_k}{\prod_{j'=j}^{i-1} r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} \cosh(r_{i,k}^j(1-x)) & \text{if } 1 \leq j \leq i-1, \\ \varphi_k(x) \triangleq \mu_k \cos(k\pi x) & \text{if } j = i, \\ 0 & \text{if } i+1 \leq j \leq N, \end{cases} \quad (5)$$

where $\mu_k = \begin{cases} 1 & \text{if } k = 0 \\ \sqrt{2} & \text{if } k \geq 1 \end{cases}$ and $r_{i,k}^j$ is one of the two square roots of $\lambda_{i,k} - a_j$.

Proof. For a given $c \neq 0$, let us first compute the solution of $f'' + af = \lambda f$ such that $f'(1) = 0$ and $f'(0) = c$ when $\lambda - a \neq -k^2\pi^2$ for any $k \in \mathbb{N}$. Denoting by r one of the two square roots of $\lambda - a \neq 0$, we have $f(x) = \alpha e^{rx} + \beta e^{-rx}$ for some $\alpha, \beta \in \mathbb{C}$. The boundary conditions imply that

$$\begin{bmatrix} r & -r \\ re^r & -re^{-r} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}.$$

The square matrix is invertible if and only if $r^2 \sinh r \neq 0$, that is $r \neq ik\pi$ for any $k \in \mathbb{Z}$. This is satisfied based on our assumption that $r^2 = \lambda - a \neq -k^2\pi^2$ for any $k \in \mathbb{N}$. Hence $\alpha = -\frac{c}{2r \sinh r} e^{-r}$ and $\beta = -\frac{c}{2r \sinh r} e^r$, implying that $f(x) = -\frac{c}{r \sinh r} \cosh(r(1-x))$. The proof of the lemma then follows by using an induction argument. \square

In order to establish that the set $\Phi = \{\phi_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ of eigenvectors of \mathcal{A} is a Riesz basis of \mathcal{H}^0 , we first compute the eigenvectors of the adjoint operator \mathcal{A}^* of \mathcal{A} , which is given by $\mathcal{A}^*((g^j)_{1 \leq j \leq N}) = ((g^j)'' + a_j g^j)_{1 \leq j \leq N}$ on the domain

$$D(\mathcal{A}^*) = \left\{ (g^j)_{1 \leq j \leq N} \in \mathcal{H}^0 \mid (g^j)_{1 \leq j \leq N} \in (H^2(0,1))^N, \quad (g^1)'(0) = (g^1)'(1) = 0, \right. \\ \left. (g^j)'(0) = 0, (g^j)'(1) = -g^{j-1}(0), 2 \leq j \leq N \right\}. \quad (6)$$

Lemma 2. Under Assumption 1, the eigenvalues of \mathcal{A}^* are $\lambda_{i,k}$ with associated eigenvectors $\psi_{i,k} = (\psi_{i,k}^j)_{1 \leq j \leq N} \in D(\mathcal{A}^*)$ given by

$$\psi_{i,k}^j(x) = \begin{cases} 0 & \text{if } 1 \leq j \leq i-1, \\ \varphi_k(x) = \mu_k \cos(k\pi x) & \text{if } j = i, \\ \frac{(-1)^{j-i} \mu_k}{\prod_{j'=i+1}^j r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} \cosh(r_{i,k}^j x) & \text{if } i+1 \leq j \leq N. \end{cases} \quad (7)$$

Proof. Given $c \neq 0$, let us compute the solution of $g'' + ag = \lambda g$ such that $g'(0) = 0$ and $g'(1) = c$ when $\lambda - a \neq -k^2\pi^2$ for any $k \in \mathbb{N}$. Denoting by r one of the two square roots of $\lambda - a \neq 0$, we have $g(x) = \alpha e^{rx} + \beta e^{-rx}$ for some $\alpha, \beta \in \mathbb{C}$. The boundary conditions imply that

$$\begin{bmatrix} r & -r \\ re^r & -re^{-r} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}.$$

The square matrix is invertible if and only if $r^2 \sinh r \neq 0$, that is $r \neq ik\pi$ for any $k \in \mathbb{Z}$. This is satisfied based on our assumption that $r^2 = \lambda - a \neq -k^2\pi^2$ for any $k \in \mathbb{N}$. Hence $\alpha = \beta = \frac{c}{2r \sinh r}$, implying that $g(x) = \frac{c}{r \sinh r} \cosh(rx)$. The proof of the lemma then follows by using an induction argument. \square

Lemma 3. $\Phi = \{\phi_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ is a Riesz basis of \mathcal{H}^0 , of dual Riesz basis $\Psi = \{\psi_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$. Moreover \mathcal{A} generates a C_0 -semigroup.

Proof. We start by noting that Φ and Ψ are biorthogonal, i.e., $\langle \phi_{i,k}, \psi_{i',k'} \rangle = \delta_{(i,k),(i',k')}$. Indeed, from their expressions, we have $\langle \phi_{i,k}, \psi_{i,k} \rangle = \int_0^1 \phi_{i,k}^i(x) \psi_{i,k}^i(x) dx = \int_0^1 \varphi_k(x)^2 dx = 1$. If $(i,k) \neq (i',k')$ then $\lambda_{i,k} \neq \lambda_{i',k'}$, hence $\lambda_{i,k} \langle \phi_{i,k}, \psi_{i',k'} \rangle = \langle \mathcal{A} \phi_{i,k}, \psi_{i',k'} \rangle = \langle \phi_{i,k}, \mathcal{A}^* \psi_{i',k'} \rangle = \lambda_{i',k'} \langle \phi_{i,k}, \psi_{i',k'} \rangle$ implying that $\langle \phi_{i,k}, \psi_{i',k'} \rangle = 0$.

Based on [26, Chap. 1, Thm. 9], it suffices to show that $\text{span } \Phi$ and $\text{span } \Psi$ are dense in \mathcal{H}^0 , and that each of Φ and Ψ form a Bessel sequence¹.

Let us show that $\text{span } \Phi$ is dense in \mathcal{H}^0 . Let $f = (f^i)_{1 \leq i \leq N} \in \mathcal{H}^0$ be such that $\langle f, \phi_{i,k} \rangle = 0$ for all integers $1 \leq i \leq N$ and $k \in \mathbb{N}$. Let us prove that $f = 0$. For $i = 1$ we have $\langle f, \phi_{1,k} \rangle = \int_0^1 f^1(x) \varphi_k(x) dx = 0$ for all $k \in \mathbb{N}$. Since $(\varphi_k)_{k \geq 0}$ forms a Hilbert basis of $L^2(0,1)$, we infer that $f^1 = 0$. Assume next we have shown that $f^1 = f^2 = \dots = f^l = 0$ for some $1 \leq l \leq N-1$. Then for $i = l+1$ we have $\langle f, \phi_{l+1,k} \rangle = \sum_{j=1}^{l+1} \int_0^1 f^j(x) \phi_{l+1,k}^j(x) dx = \int_0^1 f^{l+1}(x) \varphi_k(x) dx = 0$ for all $k \in \mathbb{N}$. As before, this implies that $f^{l+1} = 0$. By induction, we infer that $f = 0$, giving the claimed conclusion. A similar argument is employed to show that $\text{span } \Psi$ is dense in \mathcal{H}^0 .

In order to prove that Φ and Ψ are Bessel sequences, we first establish asymptotic estimates of $\phi_{i,k}^j$ for $1 \leq j \leq i-1$ and of $\psi_{i,k}^j$ for $i+1 \leq j \leq N$ as $k \rightarrow +\infty$. All the asymptotic estimates derived below hold for $k \rightarrow +\infty$, uniformly with respect to i, j and $x \in [0,1]$. Recalling that $(r_{i,k}^j)^2 = \lambda_{i,k} - a_j = -\alpha_{i,j} - k^2\pi^2$ with $\alpha_{i,j} = a_j - a_i$, for all sufficiently large integers k we have $r_{i,k}^j = i\sqrt{k^2\pi^2 + \alpha_{i,j}}$ with $\sqrt{k^2\pi^2 + \alpha_{i,j}} \in \mathbb{R}$. Hence

$$r_{i,k}^j = ik\pi \sqrt{1 + \frac{\alpha_{i,j}}{k^2\pi^2}} = i(k\pi + \beta_{i,k}^j) \quad (8a)$$

where

$$\beta_{i,k}^j = \frac{\alpha_{i,j}}{2k\pi} + O(1/k^3) \in \mathbb{R}. \quad (8b)$$

We thus infer that $\sinh(r_{i,k}^j) = i(-1)^k \sin(\beta_{i,k}^j) = i \frac{(-1)^k \alpha_{i,j}}{2k\pi} + O(1/k^3)$. Combining the two latter asymptotic expansions, we have $r_{i,k}^j \sinh(r_{i,k}^j) = \frac{(-1)^{k+1} \alpha_{i,j}}{2} + O(1/k^2)$, yielding

$$\frac{1}{r_{i,k}^j \sinh(r_{i,k}^j)} = \frac{1}{\frac{(-1)^{k+1} \alpha_{i,j}}{2} + O(1/k^2)} = \frac{(-1)^{k+1} 2}{\alpha_{i,j}} + O(1/k^2). \quad (9)$$

By the definition (5) of $\phi_{i,k}^j$ for $1 \leq j \leq i-1$, we deduce from (9) that

$$\prod_{j'=j}^{i-1} \frac{1}{r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} = \prod_{j'=j}^{i-1} \left(\frac{(-1)^{k+1} 2}{\alpha_{i,j'}} + O(1/k^2) \right) = \frac{2^{i-j} (-1)^{(k+1)(i-j)}}{\prod_{j'=j}^{i-1} \alpha_{i,j'}} + O(1/k^2). \quad (10)$$

In view of (8a) we get $\cosh(r_{i,k}^j(1-x)) = (-1)^k \cos(k\pi x) \cos(\beta_{i,k}^j(1-x)) + (-1)^k \sin(k\pi x) \sin(\beta_{i,k}^j(1-x))$. Recalling (8b), we have

$$\cos(\beta_{i,k}^j(1-x)) = 1 + O(1/k^2), \quad \sin(\beta_{i,k}^j(1-x)) = \frac{\alpha_{i,j}}{2k\pi} (1-x) + O(1/k^3), \quad (11)$$

¹A sequence $(\varphi_n)_{n \geq 0}$ of a separable Hilbert space X is said to be a Bessel sequence if $(\langle f, \varphi_n \rangle)_{n \geq 0} \in \ell^2(\mathbb{N})$ for all $f \in X$.

hence

$$\cosh(r_{i,k}^j(1-x)) = (-1)^k \cos(k\pi x) + O(1/k).$$

Since $\mu_k = \sqrt{2}$ for $k \geq 1$, we deduce from (5) (for $1 \leq j \leq i-1$) that

$$\begin{aligned} \phi_{i,k}^j(x) &= \frac{(-1)^{k+i-j} \mu_k}{\prod_{j'=j}^{i-1} r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} \cosh(r_{i,k}^j(1-x)) = \frac{2^{i-j} (-1)^{k(i-j)}}{\prod_{j'=j}^{i-1} \alpha_{i,j'}} \sqrt{2} \cos(k\pi x) + O(1/k) \\ &= A_{i,k}^j \varphi_k(x) + R\phi_{i,k}^j(x) \end{aligned} \quad (12)$$

with $A_{i,k}^j = \frac{2^{i-j} (-1)^{k(i-j)}}{\prod_{j'=j}^{i-1} \alpha_{i,j'}}$ and

$$R\phi_{i,k}^j(x) \triangleq \phi_{i,k}^j(x) - A_{i,k}^j \varphi_k(x) = O(1/k). \quad (13)$$

We are now in a position to prove that Φ is a Bessel sequence. Let $f = (f^j)_{1 \leq j \leq N}$ be fixed. For an arbitrary integer $1 \leq i \leq N$, we infer from (5) (for $i+1 \leq j \leq N$) and (12) that

$$\langle f, \phi_{i,k} \rangle = \sum_{j=1}^{i-1} A_{i,k}^j \int_0^1 f^j \varphi_k dx + \sum_{j=1}^{i-1} \int_0^1 f^j R\phi_{i,k}^j dx + \int_0^1 f^i \varphi_k dx.$$

Since $(\varphi_k)_{k \geq 0}$ is a Hilbert basis of $L^2(0,1)$, we get that $\left(\int_0^1 f^j \varphi_k dx \right)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$ for all $1 \leq j \leq i$. Furthermore, since $|A_{i,k}^j| = \frac{2^{i-j}}{\prod_{j'=j}^{i-1} |\alpha_{i,j'}|}$ is independent of k , we deduce that $\left(A_{i,k}^j \int_0^1 f^j \varphi_k dx \right)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$ for all $1 \leq j \leq i-1$. Finally, since $R\phi_{i,k}^j(x) = O(1/k)$, we infer that

$$\left| \int_0^1 f^j R\phi_{i,k}^j dx \right| \leq \|f^j\|_{L^2} \|R\phi_{i,k}^j\|_{L^2} = O(1/k),$$

hence $\left(\int_0^1 f^j R\phi_{i,k}^j dx \right)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$ for all $1 \leq j \leq i-1$. This shows that $(\langle f, \phi_{i,k} \rangle)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$. This result holds for all $1 \leq i \leq N$, hence $(\langle f, \phi_{i,k} \rangle)_{1 \leq i \leq N, k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, showing that Φ is a Bessel sequence.

It remains to show that Ψ is also a Bessel sequence. Recalling the definition (7) of $\psi_{i,k}^j$ for $i+1 \leq j \leq N$, we infer from (9) that

$$\prod_{j'=i+1}^j \frac{1}{r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} = \prod_{j'=i+1}^j \left(\frac{(-1)^{k+1+2}}{\alpha_{i,j'}} + O(1/k^2) \right) = \frac{2^{j-i} (-1)^{(k+1)(j-i)}}{\prod_{j'=i+1}^j \alpha_{i,j'}} + O(1/k^2).$$

Owing to (8a), we have $\cosh(r_{i,k}^j x) = \cos(k\pi x) \cos(\beta_{i,k}^j x) - \sin(k\pi x) \sin(\beta_{i,k}^j x)$. Recalling (8b), we infer that $\cos(\beta_{i,k}^j x) = 1 + O(1/k^2)$ and $\sin(\beta_{i,k}^j x) = \frac{\alpha_{i,j}}{2k\pi} x + O(1/k^3)$, hence $\cosh(r_{i,k}^j x) = \cos(k\pi x) + O(1/k)$. Since $\mu_k = \sqrt{2}$ for $k \geq 1$, we deduce that

$$\psi_{i,k}^j(x) = \frac{2^{j-i} (-1)^{k(j-i)}}{\prod_{j'=i+1}^j \alpha_{i,j'}} \sqrt{2} \cos(k\pi x) + O(1/k) = A_{i,k}^j \varphi_k(x) + R\psi_{i,k}^j(x)$$

with $A_{i,k}^j = \frac{2^{j-i} (-1)^{k(j-i)}}{\prod_{j'=i+1}^j \alpha_{i,j'}}$ and $R\psi_{i,k}^j(x) \triangleq \psi_{i,k}^j(x) - A_{i,k}^j \varphi_k(x) = O(1/k)$. Following now the same procedure as in the previous paragraph, we infer that Ψ is a Bessel sequence.

Hence Φ is a Riesz basis, of dual Riesz basis Ψ . Using (4), since \mathcal{A} is closed, we infer that \mathcal{A} is a Riesz spectral operator that generates a C_0 -semigroup (see [8]). \square

The previous result is instrumental for studying the solutions in L^2 norm, that is, in the Hilbert space \mathcal{H}^0 . The following result will be useful for studying the solutions in H^1 norm, that is in the Hilbert space

$$\mathcal{H}^1 = (H^1(0, 1))^N \quad (14a)$$

endowed with its natural inner product

$$\langle (f^j)_{1 \leq j \leq N}, (g^j)_{1 \leq j \leq N} \rangle = \sum_{j=1}^N \int_0^1 (f^j(x)g^j(x) + (f^j)'(x)(g^j)'(x)) dx. \quad (14b)$$

Lemma 4. *The set $\tilde{\Phi} = \{\tilde{\phi}_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$, where $\tilde{\phi}_{i,k} = \frac{1}{\sqrt{1+k^2\pi^2}}\phi_{i,k}$, is a Riesz basis of \mathcal{H}^1 .*

Proof. We define $\hat{\Phi} = \{\hat{\phi}_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ where $\hat{\phi}_{i,k} = (\hat{\phi}_{i,k}^j)_{1 \leq j \leq N}$ and

$$\hat{\phi}_{i,k}^j(x) = \begin{cases} A_{i,k}^j \varphi_k(x) & \text{if } 1 \leq j \leq i-1, \\ \varphi_k(x) & \text{if } j = i, \\ 0 & \text{if } i+1 \leq j \leq N. \end{cases}$$

We infer from (12) that $\|\phi_{i,k}^j - \hat{\phi}_{i,k}^j\|_{L^\infty} = O(1/k)$, hence $\sum_{1 \leq i \leq N} \sum_{k \geq 0} \|\phi_{i,k} - \hat{\phi}_{i,k}\|_{\mathcal{H}^0}^2 < \infty$. Since $\hat{\Phi}$ is ω -linearly independent, Bari's theorem (see [10]) implies that $\hat{\Phi}$ is a Riesz basis of \mathcal{H}^0 .

Let Δ be the Neumann Laplacian operator defined by $\Delta f = f''$ on $D(\Delta) = \{f \in H^2(0, 1) \mid f'(0) = f'(1) = 0\}$. Then $\text{id} - \Delta$ is a positive selfadjoint operator, mapping isometrically and surjectively $D(\Delta)$ to $L^2(0, 1)$, and thus the mapping $J = \sqrt{\text{id} - \Delta} : H^1(0, 1) \rightarrow L^2(0, 1)$ is a surjective isometry (note that $\langle Jf_1, Jf_2 \rangle = \langle (1 - \Delta)f_1, f_2 \rangle = \int_0^1 (-f_1'' + f_1)f_2 dx = \int_0^1 (f_1'f_2' + f_1f_2) dx = \langle f_1, f_2 \rangle$). Therefore the mapping

$$\mathfrak{J} : \begin{array}{ccc} \mathcal{H}^1 & \rightarrow & \mathcal{H}^0 \\ (f^j)_{1 \leq j \leq N} & \mapsto & (Jf^j)_{1 \leq j \leq N}. \end{array}$$

is also a surjective isometry. Hence, $\tilde{\Phi} = \mathfrak{J}^{-1}\hat{\Phi} = \{\check{\phi}_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ with $\check{\phi}_{i,k} = \mathfrak{J}^{-1}\hat{\phi}_{i,k}^j$ is a Riesz basis of \mathcal{H}^1 . Noting that $J^{-1}\varphi_k = \frac{1}{\sqrt{1+k^2\pi^2}}\varphi_k$, we have

$$\check{\phi}_{i,k}^j(x) = \begin{cases} \frac{A_{i,k}^j}{\sqrt{1+k^2\pi^2}}\varphi_k(x) & \text{if } 1 \leq j \leq i-1, \\ \frac{1}{\sqrt{1+k^2\pi^2}}\varphi_k(x) & \text{if } j = i, \\ 0 & \text{if } i+1 \leq j \leq N. \end{cases}$$

To conclude the proof, we apply Bari's theorem by comparing $\tilde{\Phi}$ to the Riesz basis $\hat{\Phi}$ of \mathcal{H}^1 . Since $\tilde{\Phi}$ is ω -linearly independent, it remains to show that

$$\begin{aligned} \sum_{1 \leq i \leq N} \sum_{k \in \mathbb{N}} \|\tilde{\phi}_{i,k} - \check{\phi}_{i,k}\|_{\mathcal{H}^1}^2 &= \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq i-1} \sum_{k \in \mathbb{N}} \|\tilde{\phi}_{i,k}^j - \check{\phi}_{i,k}^j\|_{H^1}^2 \\ &= \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq i-1} \sum_{k \in \mathbb{N}} \frac{1}{1+k^2\pi^2} \|R\phi_{i,k}^j\|_{H^1}^2 < \infty \end{aligned} \quad (15)$$

where we have used (12). Again, all the asymptotic estimates derived below hold for $k \rightarrow +\infty$, uniformly with respect to i, j and $x \in [0, 1]$. We know that $R\phi_{i,k}^j(x) = O(1/k)$, hence $\|R\phi_{i,k}^j\|_{L^2}^2 =$

$O(1/k^2)$. Hence it suffices to analyze the asymptotic behavior of $\|(R\phi_{i,k}^j)'\|_{L^2}^2$. For integers $1 \leq j \leq i-1$, using (5), we have

$$(\phi_{i,k}^j)'(x) = \frac{(-1)^{k+i-j+1} \mu_k r_{i,k}^j}{\prod_{j'=j}^{i-1} r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} \sinh(r_{i,k}^j(1-x)). \quad (16)$$

From (8a) we deduce that

$$\begin{aligned} \sinh(r_{i,k}^j(1-x)) &= -i(-1)^k \sin(k\pi x) \cos(\beta_{i,k}^j(1-x)) + i(-1)^k \cos(k\pi x) \sin(\beta_{i,k}^j(1-x)) \\ &= -i(-1)^k \sin(k\pi x) + O(1/k) \end{aligned}$$

where the last equality is inferred from (11). Combining the latter asymptotic behavior with (8a), we have $r_{i,k}^j \sinh(r_{i,k}^j(1-x)) = (-1)^k k\pi \sin(k\pi x) + O(1)$. Using this result and (10), we infer from (16) that

$$(\phi_{i,k}^j)'(x) = A_{i,k}^j \varphi_k'(x) + O(1), \quad (17)$$

hence, in view of (13), $(R\phi_{i,k}^j)'(x) = (\phi_{i,k}^j)'(x) - A_{i,k}^j \varphi_k'(x) = O(1)$. This shows that $\|(R\phi_{i,k}^j)'\|_{L^2}^2 = O(1)$, and thus, recalling that $\|R\phi_{i,k}^j\|_{L^2}^2 = O(1/k^2)$, we have $\|R\phi_{i,k}^j\|_{H^1}^2 = O(1)$. Hence (15) holds true. This completes the proof. \square

2.2 Spectral reduction of the system

We define the function $\varphi(x) = -\frac{1}{2}(1-x)^2$, so that $\varphi(1) = \varphi'(1) = 0$ and $\varphi'(0) = 1$. Making the change of variable

$$\tilde{y}^N(t, x) = y^N(t, x) - \varphi(x)u(t), \quad (18)$$

we infer from (1) that

$$y_t^j = y_{xx}^j + a_j y^j, \quad 1 \leq j \leq N-1 \quad (19a)$$

$$\tilde{y}_t^N = \tilde{y}_{xx}^N + a_N \tilde{y}^N + (\varphi'' + a_N \varphi)u - \varphi \dot{u} \quad (19b)$$

$$y_x^j(t, 0) = \begin{cases} y^{j+1}(t, 1), & 1 \leq j \leq N-2 \\ \tilde{y}^N(t, 1), & j = N-1 \end{cases} \quad (19c)$$

$$y_x^j(t, 1) = 0, \quad 1 \leq j \leq N-1 \quad (19d)$$

$$\tilde{y}_x^N(t, 0) = \tilde{y}_x^N(t, 1) = 0. \quad (19e)$$

Defining the states $\mathcal{X} = (y^j)_{1 \leq j \leq N}$ (original coordinates) and $\tilde{\mathcal{X}} = ((y^j)_{1 \leq j \leq N-1}, \tilde{y}^N)$ (homogeneous coordinates), we infer from (18) that

$$\tilde{\mathcal{X}} = \mathcal{X} + \beta u \quad (20)$$

and (19) is written as

$$\dot{u} = v \quad (21a)$$

$$\frac{d\tilde{\mathcal{X}}}{dt} = \mathcal{A}\tilde{\mathcal{X}} + \alpha u + \beta v \quad (21b)$$

where $\alpha = (0, \dots, 0, \varphi'' + a_N \varphi)$ and $\beta = (0, \dots, 0, -\varphi)$.

We now define

$$x_{i,k} = \langle \mathcal{X}, \psi_{i,k} \rangle, \quad \tilde{x}_{i,k} = \langle \tilde{\mathcal{X}}, \psi_{i,k} \rangle, \quad \alpha_{i,k} = \langle \alpha, \psi_{i,k} \rangle, \quad \beta_{i,k} = \langle \beta, \psi_{i,k} \rangle. \quad (22)$$

By the Riesz basis property established in Lemma 3, we have

$$\tilde{\mathcal{X}}(t) = (y^1(t, \cdot), y^2(t, \cdot), \dots, y^{N-1}(t, \cdot), \tilde{y}^N(t, \cdot)) = \sum_{i=1}^N \sum_{k \in \mathbb{N}} \tilde{x}_{i,k}(t) \phi_{i,k}. \quad (23)$$

The change of variable formula (20) implies that $\tilde{x}_{i,k} = x_{i,k} + \beta_{i,k}u$, while the projection of the system dynamics in homogeneous coordinates (21) gives

$$\dot{\tilde{x}}_{i,k} = \lambda_{i,k} \tilde{x}_{i,k} + \alpha_{i,k}u + \beta_{i,k}v. \quad (24)$$

Combining the two latter equations, we infer that

$$\dot{x}_{i,k} = \lambda_{i,k}x_{i,k} + m_{i,k}u, \quad (25)$$

where $m_{i,k} = \alpha_{i,k} + \lambda_{i,k}\beta_{i,k}$. Since all eigenvalues are simple, the controlled equation (25) (mode $\lambda_{i,k}$) is controllable if and only if $m_{i,k} \neq 0$. The coefficients $m_{i,k}$ are characterized by the following lemma.

Lemma 5. *We have*

$$m_{i,k} = -\psi_{i,k}^N(0) = \begin{cases} -\mu_k & \text{if } i = N, \\ -\frac{(-1)^{N-i}\mu_k}{\prod_{j'=i+1}^N r_{i,k}^{j'} \sinh(r_{i,k}^{j'})} & \text{if } 1 \leq i \leq N-1. \end{cases}$$

In particular, $m_{i,k} \neq 0$ for all $1 \leq i \leq N$ and $k \in \mathbb{N}$.

Proof. Using the definition (22) of $\alpha_{i,k}$ and using integrations by parts, we have

$$\begin{aligned} \alpha_{i,k} &= \int_0^1 (\varphi'' + a_n \varphi) \psi_{i,k}^N dx = [\varphi' \psi_{i,k}^N - \varphi (\psi_{i,k}^N)']_0^1 + \int_0^1 \varphi ((\psi_{i,k}^N)'' + a_n \psi_{i,k}^N) dx \\ &= -\psi_{i,k}^N(0) - \langle \beta, \mathcal{A}^* \psi_{i,k} \rangle = -\psi_{i,k}^N(0) - \lambda_{i,k} \beta_{i,k}, \end{aligned}$$

where we have used that $\varphi(1) = \varphi'(1) = 0$, $\varphi'(0) = 1$, and $(\psi_{i,k}^N)'(0) = 0$. \square

Corollary 1. *Under Assumption 1, each mode $\lambda_{i,k}$ of the PDE cascade (1) is controllable.*

2.3 Output feedback control strategy

Given $c \in L^2(0, 1)$, we consider the following distributed output, measured on the last component y^1 of the PDE cascade (1):

$$z(t) = \int_0^1 c(x) y^1(t, x) dx \quad (26)$$

Our objective is to design an output feedback control strategy for (1) based on the measurement $z(t)$.

Note that the measurement must be done on the last component y^1 of the cascade; indeed a measurement done on an intermediate component of the cascade (say y_j for some $2 \leq i \leq N$) would not allow to reconstruct the dynamics of the last components of the cascade (say $y_{j'}$ for $1 \leq j' \leq j-1$): observability would fail.

Let $\delta > 0$ be the targeted exponential decay rate for the closed-loop system. Using (4), we choose, for any $1 \leq i \leq N$, a sufficiently large integer $n_{0,i} \in \mathbb{N}$ such that $\lambda_{i,n_{0,i}} < -\delta$, implying

that $\lambda_{i,k} \leq \lambda_{i,n_{0,i}} < -\delta$ for all integers $k \geq n_{0,i}$. We now write a finite-dimensional dynamics that captures the $n_{0,i}$ first modes, for $1 \leq i \leq N$, as follows. Defining the state vector

$$\begin{aligned} X_{1,i} &= [\tilde{x}_{i,0} \quad \tilde{x}_{i,1} \quad \dots \quad \tilde{x}_{i,n_{0,i}-1}]^\top, \quad 1 \leq i \leq N, \\ X_1 &= \text{col}(X_{1,1}, X_{1,2}, \dots, X_{1,n}), \end{aligned}$$

we infer from (24) that

$$\dot{X}_{1,i} = A_{1,i}X_{1,i} + B_{1,u,i}u + B_{1,v,i}v, \quad 1 \leq i \leq N,$$

where

$$\begin{aligned} A_{1,i} &= \text{diag}(\lambda_{i,0}, \lambda_{i,1}, \dots, \lambda_{i,n_{0,i}-1}), \quad B_{1,u,i} = [\alpha_{i,0} \quad \alpha_{i,1} \quad \dots \quad \alpha_{i,n_{0,i}-1}]^\top, \\ B_{1,v,i} &= [\beta_{i,0} \quad \beta_{i,1} \quad \dots \quad \beta_{i,n_{0,i}-1}]^\top, \end{aligned}$$

yielding

$$\dot{X}_1 = A_1X_1 + B_{1,u}u + B_{1,v}v \quad (27)$$

where

$$\begin{aligned} A_1 &= \text{diag}(A_{1,1}, A_{1,2}, \dots, A_{1,N}), \quad B_{1,u} = \text{col}(B_{1,u,1}, B_{1,u,2}, \dots, B_{1,u,N}), \\ B_{1,v} &= \text{col}(B_{1,v,1}, B_{1,v,2}, \dots, B_{1,v,N}). \end{aligned}$$

Recalling that $v = \dot{u}$, we define the augmented state vector $X_{1,a} = \text{col}(X_1, u)$, yielding the dynamics

$$\dot{X}_{1,a} = A_{1,a}X_{1,a} + B_{1,a}v \quad (28)$$

where

$$A_{1,a} = \begin{bmatrix} A_1 & B_{1,u} \\ 0 & 0 \end{bmatrix}, \quad B_{1,a} = \begin{bmatrix} B_{1,v} \\ 1 \end{bmatrix}. \quad (29)$$

Under Assumption 1 that ensures that the eigenvalues are all simple, the Hautus test implies that the pair $(A_{1,a}, B_{1,a})$ is controllable if and only if $m_{i,k} = \alpha_{i,k} + \lambda_{i,k}\beta_{i,k} \neq 0$ for all $1 \leq i \leq N$ and all $0 \leq k \leq n_{0,i} - 1$. Lemma 5 implies the following result.

Lemma 6. *Under Assumption 1, the pair $(A_{1,a}, B_{1,a})$ is controllable.*

Remark 1. *At this step, it is possible to design a state-feedback control strategy, as follows. By the above lemma, there exists a matrix K such that $A_{1,a} - B_{1,a}K$ has a spectral abscissa less than $-\delta < 0$, and we set the feedback control $v = -KX_{1,a}$. The stability of the resulting closed-loop system (1) is established in \mathcal{H}^0 norm, as defined by (2), thanks to the Lyapunov functional*

$$V = X_{1,a}^\top P X_{1,a} + \gamma \sum_{i=1}^N \sum_{k \geq n_{0,i}} \tilde{x}_{i,k}^2,$$

where $P \succ 0$ is the solution of the Lyapunov equation $(A_{1,a} - B_{1,a}K)^\top P + P(A_{1,a} - B_{1,a}K) + 2\delta P + I = 0$ and $\gamma > 0$ is chosen small enough so that $\dot{V} + 2\delta V \leq 0$. For classical solutions, the stability can also be established in \mathcal{H}^1 norm, as defined by (14), thanks to the Lyapunov functional $V = X_{1,a}^\top P X_{1,a} + \gamma \sum_{i=1}^N \sum_{k \geq n_{0,i}} (1 + k^2) \tilde{x}_{i,k}^2$.

In order to define our output feedback control strategy, using (23), we infer from (26) that

$$z(t) = \sum_{i=1}^N \sum_{k \in \mathbb{N}} c_{i,k} \tilde{x}_{i,k}(t) = C_1 X_1(t) + \sum_{i=1}^N \sum_{k \geq n_{0,i}} c_{i,k} \tilde{x}_{i,k}(t) \quad (30)$$

where

$$c_{i,k} = \int_0^1 c(x) \phi_{i,k}^1(x) dx \quad (31)$$

and $C_{1,i} = [c_{i,0} \ c_{i,1} \ \dots \ c_{i,n_{0,i}-1}]$ and $C_1 = [C_{1,1} \ C_{1,2} \ \dots \ C_{1,N}]$. Since the eigenvalues are simple, the Hautus test implies that the pair (A_1, C_1) is observable if and only if $c_{i,k} \neq 0$ for all $1 \leq i \leq N$ and $0 \leq k \leq n_{0,i} - 1$. Using the definition (31) of $c_{i,k}$ and the expression of $\phi_{i,k}$ given by (5), we infer the following result.

Lemma 7. *Under Assumption 1, the pair (A_1, C_1) is observable if and only if*

$$\int_0^1 c(x) \cos(k\pi x) dx \neq 0 \quad \text{if } i = 1, \quad (32a)$$

$$\int_0^1 c(x) \cosh(r_{i,k}^1(1-x)) dx \neq 0 \quad \text{if } i \geq 2, \quad (32b)$$

for all $1 \leq i \leq N$ and $0 \leq k \leq n_{0,i} - 1$. Recall that $r_{i,k}^1$ is one of the two square roots of $\lambda_{i,k} - a_1$.

We are now in a position to define our output feedback control strategy. Let $n_i \geq n_{0,i}$ be integers to be chosen large enough, later on. We define the following controller dynamics (see [19]):

$$\dot{u} = v \quad (33a)$$

$$\dot{\hat{X}}_1 = A_1 \hat{X}_1 + B_{1,u} u + B_{1,v} v - L \left(C_1 \hat{X}_1 + \sum_{i=1}^N \sum_{k=n_{0,i}}^{n_i} c_{i,k} \hat{x}_{i,k} - z \right) \quad (33b)$$

$$\dot{\hat{x}}_{i,k} = \lambda_{i,k} \hat{x}_{i,k} + \alpha_{i,k} u + \beta_{i,k} v, \quad 1 \leq i \leq N, \quad n_{0,i} \leq k \leq n_i \quad (33c)$$

$$v = -K_x \hat{X}_1 - k_u u \quad (33d)$$

where, setting $n_0 = \sum_{i=1}^N n_{0,i}$, $k_u \in \mathbb{R}$ and $K_x \in \mathbb{R}^{1 \times n_0}$ are the feedback gains while $L \in \mathbb{R}^{n_0}$ is the observer gain.

Theorem 1. *Let $a_1, \dots, a_N \in \mathbb{R}$ be such that Assumption 1 holds and let $c \in L^2(0,1)$. Let $\delta > 0$ be arbitrary. For any $1 \leq i \leq N$, let $n_{0,i} \in \mathbb{N}$ be such that $\lambda_{i,n_{0,i}} < -\delta$. Assuming that the observability assumption (32) holds, let $K = [K_x \ k_u] \in \mathbb{R}^{1 \times (n_0+1)}$ and $L \in \mathbb{R}^{n_0}$ be such that $A_{1,a} - B_{1,a}K$ and $A_1 - LC_1$ are Hurwitz with spectral abscissa less than $-\delta$.*

Then, for large enough integers $n_i \geq n_{0,i}$, $1 \leq i \leq N$, there exists $C > 0$ such that, for all initial conditions $y_0^j \in L^2(0,1)$, $1 \leq j \leq N$, $u(0) = u_0 \in \mathbb{R}$, $\hat{X}_1(0) \in \mathbb{R}^{n_0}$, and $\hat{x}_{i,k}(0) \in \mathbb{R}$, the solutions of the closed-loop system consisting of the PDE cascade (1), the output (26) and the controller (33) satisfy

$$\begin{aligned} & \left\| (y^j(t, \cdot))_{1 \leq j \leq N} \right\|_Y + |u(t)| + \|\hat{X}_1(t)\| + \sum_{i=1}^N \sum_{k=n_{0,i}}^{n_i} |\tilde{x}_{i,k}(t)| \\ & \leq C e^{-\delta t} \left(\left\| (y_0^j)_{1 \leq j \leq N} \right\|_Y + |u_0| + \|\hat{X}_1(0)\| + \sum_{i=1}^N \sum_{k=n_{0,i}}^{n_i} |\tilde{x}_{i,k}(0)| \right) \quad (34) \end{aligned}$$

for every $t \geq 0$, with $Y = \mathcal{H}^0$ defined by (2).

If the initial condition is such that $y_0^j \in H^2(0, 1)$ with $(y^j)'(0) = y^{j+1}(1)$ and $(y^j)'(1) = 0$ for all $1 \leq j \leq N-1$ and $(y^N)'(0) = u(0)$ and $(y^N)'(1) = 0$, the conclusion of the theorem holds with $Y = \mathcal{H}^1$ defined by (14).

Proof. Define $e_{i,k} = \tilde{x}_{i,k} - \hat{x}_{i,k}$ and

$$\begin{aligned} E_{1,i} &= [e_{i,0} \ e_{i,1} \ \dots \ e_{i,n_{0,i}-1}], \quad E_1 = \text{col}(E_{1,1}, E_{1,2}, \dots, E_{1,N}), \\ \hat{X}_{2,i} &= [\hat{x}_{i,n_{0,i}} \ \hat{x}_{i,n_{0,i}+1} \ \dots \ \hat{x}_{i,n_i}], \quad \hat{X}_2 = \text{col}(\hat{X}_{2,1}, \hat{X}_{2,2}, \dots, \hat{X}_{2,N}), \\ E_{2,i} &= [(n_{0,i}+1)^\kappa e_{i,n_{0,i}} \ \dots \ (n_i+1)^\kappa e_{i,n_i}], \quad E_2 = \text{col}(E_{2,1}, E_{2,2}, \dots, E_{2,N}), \\ \hat{X}_{1,a} &= \text{col}(\hat{X}^1, u), \quad X = \text{col}(\hat{X}_{1,a}, E_1, \hat{X}_2, E_2). \end{aligned}$$

In this proof we set $\kappa = 0$. Different values will be set later on in the proofs of Theorems 2 and 3. Combining (24), (27), (28), (30) and (33), we obtain

$$\dot{X} = FX + \mathcal{L} \sum_{i=1}^N \zeta_i \quad (35)$$

where the measurement residues are defined by

$$\zeta_i = \sum_{k \geq n_i+1} c_{i,k} \tilde{x}_{i,k}, \quad 1 \leq i \leq N, \quad (36)$$

and

$$F = \begin{bmatrix} A_{1,a} - B_{1,a}K & L_a C_1 & 0 & L_a C_2 \\ 0 & A_1 - LC_1 & 0 & -LC_2 \\ [0 \ B_{2,u}] - B_{2,v}K & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}, \quad \mathcal{L} = \text{col}(L_a, -L, 0, 0),$$

where

$$\begin{aligned} K &= [K_x \ k_u], \quad L_a = \text{col}(L, 0), \\ A_{2,i} &= \text{diag}(\lambda_{i,n_{0,i}}, \lambda_{i,n_{0,i}+1}, \dots, \lambda_{i,n_i}), \quad A_2 = \text{diag}(A_{2,1}, A_{2,2}, \dots, A_{2,N}), \\ B_{2,u,i} &= [\alpha_{i,n_{0,i}} \ \alpha_{i,n_{0,i}+1} \ \dots \ \alpha_{i,n_i}]^\top, \\ B_{2,u} &= \text{col}(B_{2,u,1}, B_{2,u,2}, \dots, B_{2,u,N}), \quad B_{2,v,i} = [\beta_{i,n_{0,i}} \ \beta_{i,n_{0,i}+1} \ \dots \ \beta_{i,n_i}]^\top, \\ B_{2,v} &= \text{col}(B_{2,v,1}, B_{2,v,2}, \dots, B_{2,v,N}), \\ C_{2,i} &= \left[\frac{c_{i,n_{0,i}}}{(n_{0,i}+1)^\kappa} \ \frac{c_{i,n_{0,i}+1}}{(n_{0,i}+2)^\kappa} \ \dots \ \frac{c_{i,n_i}}{(n_i+1)^\kappa} \right], \quad C_2 = [C_{2,1} \ C_{2,2} \ \dots \ C_{2,N}]. \end{aligned}$$

We recall that $\kappa = 0$, which ensures that $\|C_{2,i}\| = O(1)$ as $n_i \rightarrow +\infty$. Finally the input u and its time derivative v are

$$u = E\hat{X}_{1,a} = \tilde{E}X, \quad v = -K\hat{X}_{1,a} = -\tilde{K}X, \quad (37)$$

where $E = [0 \ \dots \ 0 \ 1]$, $\tilde{E} = [E \ 0 \ 0 \ 0]$, and $\tilde{K} = [K \ 0 \ 0 \ 0]$.

To establish the claimed stability estimate (34) in \mathcal{H}^0 norm, we define the Lyapunov functional candidate

$$V = X^\top PX + \sum_{i=1}^N \sum_{k \geq n_i+1} \tilde{x}_{i,k}^2 \quad (38)$$

for some matrix $P \succ 0$. The equivalence between (38) and the (squared) \mathcal{H}^0 norm holds true by Lemma 3. In case of classical solutions, establishing the stability estimate (34) in \mathcal{H}^1 norm is obtained thanks to the Lyapunov functional candidate

$$V = X^\top P X + \sum_{i=1}^N \sum_{k \geq n_i+1} (1+k^2) \tilde{x}_{i,k}^2 \quad (39)$$

for some matrix $P \succ 0$. The equivalence between (39) and the \mathcal{H}^1 norm holds true by Lemma 4. Indeed, using (23), we have

$$\tilde{\mathcal{X}}(t) = \sum_{i=1}^N \sum_{k \geq 0} \sqrt{1+k^2\pi^2} \tilde{x}_{i,k}(t) \frac{1}{\sqrt{1+k^2\pi^2}} \phi_{i,k}. \quad (40)$$

This series converges *a priori* in \mathcal{H}^0 norm. But, based on Lemma 3, the series also converges in \mathcal{H}^1 norm for classical solutions because $\tilde{\mathcal{X}}(t, \cdot) \in D(\mathcal{A})$. Using again Lemma 3, we infer that $\|\tilde{\mathcal{X}}(t, \cdot)\|_{\mathcal{H}^1}^2$ is equivalent to $\sum_{i=1}^N \sum_{k \geq 0} (1+k^2) \tilde{x}_{i,k}(t)^2$. This justifies the definition (39) for V .

Establishing the stability in \mathcal{H}^0 or in \mathcal{H}^1 norm now follows a similar reasoning, and we focus on the \mathcal{H}^1 norm. The computation of the time derivative of V defined by (39) along the solutions of (24) and (35), combined with the use of Young's inequality, gives

$$\begin{aligned} \dot{V} + 2\delta V &= \tilde{X}^\top \begin{bmatrix} F^\top P + PF + 2\delta P & P\mathcal{L} & \dots & P\mathcal{L} \\ \mathcal{L}^\top P & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^\top P & 0 & \dots & 0 \end{bmatrix} \tilde{X} \\ &+ 2 \sum_{i=1}^N \sum_{k \geq n_i+1} (1+k^2)(\lambda_{i,k} + \delta) \tilde{x}_{i,k}^2 + 2 \sum_{i=1}^N \sum_{k \geq n_i+1} (1+k^2) \tilde{x}_{i,k} (\alpha_{i,k} u + \beta_{i,k} v) \\ &\leq \tilde{X}^\top \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} & \dots & P\mathcal{L} \\ \mathcal{L}^\top P & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^\top P & 0 & \dots & 0 \end{bmatrix} \tilde{X} + 2 \sum_{i=1}^N \sum_{k \geq n_i+1} (1+k^2) \left(\lambda_{i,k} + \frac{1+k^2}{\epsilon} + \delta \right) \tilde{x}_{i,k}^2 \end{aligned}$$

for any $\epsilon > 0$, with $\tilde{X} = \text{col}(X, \zeta_1, \zeta_2, \dots, \zeta_N)$ and

$$\Theta_{1,1} = F^\top P + PF + 2\delta P + \epsilon \left(\sum_{l=1}^N S_{\alpha_l, n_l} \tilde{E}^\top \tilde{E} + \sum_{l=1}^N S_{\beta_l, n_l} \tilde{K}^\top \tilde{K} \right)$$

where we have used (37) and we have defined $S_{\alpha_i, n_i} = \sum_{k \geq n_i+1} \alpha_{i,k}^2 < \infty$ and $S_{\beta_i, n_i} = \sum_{k \geq n_i+1} \beta_{i,k}^2 < \infty$. Using (36) and the Cauchy-Schwarz inequality, we infer that

$$\zeta_i^2 \leq \underbrace{\sum_{k \geq n_i+1} c_{i,k}^2}_{=S_{\zeta_i, n_i} < \infty} \sum_{k \geq n_i+1} \tilde{x}_{i,k}^2.$$

Combining the two latter estimates, we get

$$\dot{V} + 2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \sum_{i=1}^N \sum_{k \geq n_i+1} (1+k^2) \Gamma_{i,k} \tilde{x}_{i,k}^2 \quad (41)$$

where

$$\Theta_1 = \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} & \dots & P\mathcal{L} \\ \mathcal{L}^\top P & -\eta_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}^\top P & 0 & \dots & -\eta_N \end{bmatrix}, \quad (42a)$$

$$\Gamma_{i,k} = 2 \left(\lambda_{i,k} + \frac{1+k^2}{\epsilon} + \delta \right) + \frac{\eta_i S_{\zeta_i, n_i}}{1+k^2} = 2 \left(-k^2 \left(\pi^2 - \frac{1}{\epsilon} \right) + a_i + \delta + \frac{1}{\epsilon} \right) + \frac{\eta_i S_{\zeta_i, n_i}}{1+k^2} \quad (42b)$$

for any $\epsilon, \eta_1, \dots, \eta_N > 0$. In particular, for all $\epsilon > 1/\pi^2$, we have $\Gamma_{i,k} \leq \Gamma_{i, n_i+1}$ for all $k \geq n_i + 1$. We thus infer that

$$\dot{V} + 2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \sum_{1 \leq i \leq N} \Gamma_{i, n_i+1} \sum_{k \geq n_i+1} (1+k^2) \tilde{x}_{i,k}^2$$

for all $\epsilon > 1/\pi^2$ and all $\eta_1, \dots, \eta_N > 0$. Hence $\dot{V} + 2\delta V \leq 0$, giving the claimed stability estimate (34), provided that there exist integers $n_i \geq n_{0,i} + 1$, real numbers $\epsilon > 1/\pi^2$ and $\eta_1, \dots, \eta_N > 0$, and a matrix $P \succ 0$ such that

$$\Theta_1 \preceq 0, \quad \Gamma_{i, n_i+1} \leq 0, \quad 1 \leq i \leq N.$$

Let us prove the feasibility of these constraints. Recall that F is Hurwitz with spectral abscissa less than $-\delta$. Note also that $\|C_2\| = O(1)$, $\|B_{2,u}\| = O(1)$, and $\|B_{2,v}\| = O(1)$ as $(n_i)_{1 \leq i \leq N} \rightarrow +\infty$. Hence, the application of the lemma in Appendix of [13] shows that the solution $P \succ 0$ to $F^\top P + PF + 2\delta P = -I$ is such that $\|P\| = O(1)$ as $(n_i)_{1 \leq i \leq N} \rightarrow +\infty$. We now arbitrarily fix the constant $\epsilon > 1/\pi^2$ and we define

$$\eta_i = \begin{cases} \frac{1}{\sqrt{S_{\zeta_i, n_i}}} & \text{if } S_{\zeta_i, n_i} \neq 0 \\ n_i & \text{otherwise} \end{cases}, \quad 1 \leq i \leq N.$$

We then infer that, for all $1 \leq i \leq N$, $\Gamma_{i, n_i+1} \rightarrow -\infty$ as $n_i \rightarrow +\infty$. Moreover, the Schur complement shows that $\Theta_1 \preceq 0$ for n_1, n_2, \dots, n_N selected sufficiently large. The last claim uses the facts that $\|\tilde{E}\| = 1$, $\|\tilde{K}\| = \|K\|$, and $\|\mathcal{L}\| = \sqrt{2}\|L\|$ are constants not depending on n_i while $\|P\| = O(1)$ as $(n_i)_{1 \leq i \leq N} \rightarrow +\infty$ and that $S_{\alpha_i, n_i}, S_{\beta_i, n_i} \rightarrow 0$ as $n_i \rightarrow +\infty$. The proof is complete. \square

2.4 Extension to pointwise measurements

2.4.1 Pointwise measurement of y^1

In this section, we briefly show how to adapt our previous result to the case of a pointwise measurement of y^1 of the form:

$$z(t) = y^1(t, \xi_p) \quad (43)$$

for some $\xi_p \in [0, 1]$. Using (23), the series expansion (30) still holds true but with $c_{i,k} = \phi_{i,k}^1(\xi_p)$. Similarly to Lemma 7, and in view of the expression of $\phi_{i,k}^1$ provided by (5), the pair (A_1, C_1) is observable if and only if:

$$\cos(k\pi\xi_p) \neq 0, \quad 1 \leq k \leq N_{0,1} - 1, \quad (44a)$$

$$\cosh(r_{i,k}^1(1 - \xi_p)) \neq 0, \quad 2 \leq i \leq N, 0 \leq k \leq n_{0,i} - 1, \quad (44b)$$

where $r_{i,k}^1$ is one of the two square roots of $\lambda_{i,k} - a_1$.

Theorem 2. *Considering the pointwise measurement (43) for some $\xi_p \in [0, 1]$, replacing the observability condition (32) by (44), the same statement as Theorem 1 holds true for the Hilbert space $Y = \mathcal{H}^0$ defined by (2).*

Proof. The proof follows the one of Theorem 1 except for the estimation of the measurement residues ζ_i defined by (36). Recall that for the pointwise measurement (43) we have defined $c_{i,k} = \phi_{i,k}^1(\xi_p)$. In view of (5) in the case $i = 1$ and (12) in the case $2 \leq i \leq N$, we have $c_{i,k} = O(1)$ as $k \rightarrow +\infty$. Consequently we set $\kappa = 1$ so that we have $\|C_{2,i}\| = O(1)$ as $n_i \rightarrow +\infty$, hence $\|C_2\| = O(1)$ as $(n_i)_{1 \leq i \leq N} \rightarrow +\infty$. Moreover we have

$$\zeta_i^2 \leq \underbrace{\sum_{k \geq n_i+1} \frac{c_{i,k}^2}{1+k^2}}_{=S_{\zeta_i, n_i} < \infty} \sum_{k \geq n_i+1} (1+k^2) \tilde{x}_{i,k}^2.$$

We thus infer that (41) holds with

$$\Gamma_{i,k} = 2 \left(\lambda_{i,k} + \frac{1+k^2}{\epsilon} + \delta \right) + \eta_i S_{\zeta_i, n_i} = 2 \left(-k^2 \left(\pi^2 - \frac{1}{\epsilon} \right) + a_i + \delta + \frac{1}{\epsilon} \right) + \eta_i S_{\zeta_i, n_i}$$

while Θ_1 is still given by (42a). The end of the proof follows the arguments of the proof of Theorem 1. \square

2.4.2 Pointwise measurement of y_x^1

The case of a pointwise measurement of y_x^1 of the form:

$$z(t) = y_x^1(t, \xi_p), \quad (45)$$

for some $\xi_p \in [0, 1]$, is more stringent. This is due to the fact that the system consisting of the PDE cascade (1) and the output (45) is not observable. Indeed, consider the initial condition $y_0^1(x) = 1$ and $y_0^i(x) = 0$ for $2 \leq i \leq N$. We infer that the solution of (1) is $y^1(t, x) = e^{a_1 t}$ and $y^i(t, x) = 0$ for $2 \leq i \leq N$. In particular, (45) gives an identically zero measurement $z = 0$ for a nontrivial state trajectory. This remark highlights the fact that if $a_1 \geq 0$, which corresponds to the case of a cascade (1) that is not open-loop exponentially stable since $\lambda_{1,0} = a_1 \geq 0$, then the pointwise measurement of y_x cannot be used to design an output feedback control strategy for stabilizing the PDE cascade (1).

However, the approach developed in this paper still presents some merits in the case of the pointwise measurement (45) if we assume moreover that the last component of the PDE cascade (1), described by y^1 , is open-loop exponentially stable. Indeed, assume that $a_1 < 0$ and let us choose the targeted exponential decay rate for the closed-loop system trajectories as $\delta \in (0, -a_1)$. Then, following the construction reported in the previous subsection, no mode $\lambda_{1,k}$ has to be included in the construction of the matrices to build the finite-dimensional dynamics (27). This is achieved by setting $N_{0,1} = 0$. Then, using (23), the series expansion (30) still holds true but with $c_{i,k} = (\phi_{i,k}^1)'(\xi_p)$. Now, similarly to Lemma 7 and in view of the expression of $\phi_{i,k}^1$ provided by (5), the pair (A_1, C_1) is observable if and only if:

$$\sinh(r_{i,k}^1(1 - \xi_p)) \neq 0, \quad 2 \leq i \leq N, \quad 0 \leq k \leq n_{0,i} - 1, \quad (46)$$

where $r_{i,k}^1$ is one of the two square roots of $\lambda_{i,k} - a_1$. Note that Assumption 1 implies that $r_{i,k}^1 \neq 0$ for $i \geq 2$.

Remark 2. *The fact that the mode $\lambda_{1,0} = a_1$ is never observable for the pointwise measurement (45) is reflected by the fact that $\phi_{1,0}^1(x) = 1$, hence $c_{1,0} = (\phi_{1,0}^1)'(\xi_p) = 0$.*

Theorem 3. *In the setting of Theorem 2, we further assume that $a_1 < 0$ and we choose $\delta \in (0, -a_1)$, we set $N_{0,1} = 0$, the observability assumption (44) is replaced by (46), the system output (43) is replaced by the Neumann measurement (45). Then the statement of Theorem 2 holds true.*

Remark 3. *The constraint $\delta \in (0, -a_1)$ reflects the fact that, in the context of an output feedback, the exponential stability of the closed-loop system, in the sense of (34), is associated with an exponential decay rate δ that cannot be faster than the slowest unobservable mode $\lambda_{1,0} = a_1 < 0$.*

Proof. The only difference with the proof of Theorem 2 is in the estimation of the measurement residues ζ_i defined by (36). For the pointwise measurement (45) we have defined $c_{i,k} = (\phi_{i,k}^1)'(\xi_p)$. Based on (5) in the case $i = 1$ and (17) in the case $2 \leq i \leq N$, we note that $c_{i,k} = O(1)$ as $k \rightarrow +\infty$. Thus, we set $\kappa = 7/4$ so that we have $\|C_{2,i}\| = O(1)$ as $n_i \rightarrow +\infty$, hence $\|C_2\| = O(1)$ as $(n_i)_{1 \leq i \leq N} \rightarrow +\infty$. We also have

$$\zeta_i^2 \leq \underbrace{\sum_{k \geq n_i+1} \frac{c_{i,k}^2}{(1+k^2)^{7/4}}}_{=S_{\zeta_i, n_i} < \infty} \sum_{k \geq n_i+1} (1+k^2)^{7/4} \tilde{x}_{i,k}^2.$$

Therefore, we infer that (41) holds with

$$\begin{aligned} \Gamma_{i,k} &= 2 \left(\lambda_{i,k} + \frac{1+k^2}{\epsilon} + \delta \right) + (1+k^2)^{3/4} \eta_i S_{\zeta_i, n_i} \\ &= 2 \left(-k^2 \left(\pi^2 - \frac{1}{\epsilon} \right) + a_i + \delta + \frac{1}{\epsilon} \right) + (1+k^2)^{3/4} \eta_i S_{\zeta_i, n_i} \end{aligned}$$

while Θ_1 is still given by (42a). The proof then follows the same arguments as before. \square

3 Cascade of N identical heat equations

In this section, we address the case of N identical heat equations and we make the following standing assumption:

Assumption 2. $a \triangleq a_1 = a_2 = \dots = a_N$.

3.1 Spectral properties

In the Hilbert space \mathcal{H}^0 defined by (2), we consider the unbounded operator $\mathcal{A}((f^j)_{1 \leq j \leq N}) = ((f^j)'' + a f^j)_{1 \leq j \leq N}$ on the domain defined by (3). We start by studying the generalized eigenelements of \mathcal{A} .

Lemma 8. *Let $k \in \mathbb{N}$, $\gamma \in \mathbb{R}$, and $g_0, g_1 \in L^2(0,1)$ be such that $\int_0^1 \cos(k\pi s) g_1(s) ds \neq 0$. Then, the functions $f : [0,1] \rightarrow \mathbb{R}$ and real numbers $\nu \in \mathbb{R}$ that satisfy $f'' + k^2 \pi^2 f = g_0 + \nu g_1$, $f'(0) = \gamma$, $f'(1) = 0$, are given by*

$$\nu = -\frac{\gamma + \int_0^1 g_0(s) ds}{\int_0^1 g_1(s) ds}, \quad f(x) = c + \gamma x + \int_0^x (x-s)(g_0(s) + \nu g_1(s)) ds, \quad (47)$$

if $k = 0$, and by

$$\begin{aligned} \nu &= -\frac{\gamma + \int_0^1 \cos(k\pi s) g_0(s) ds}{\int_0^1 \cos(k\pi s) g_1(s) ds}, \\ f(x) &= c\sqrt{2} \cos(k\pi x) + \frac{\gamma}{k\pi} \sin(k\pi x) + \frac{1}{k\pi} \int_0^x \sin(k\pi(x-s))(g_0(s) + \nu g_1(s)) ds, \end{aligned}$$

if $k \geq 1$, where $c \in \mathbb{R}$ is an arbitrary constant that corresponds to the solutions to the homogeneous ODE (47), i.e., when $g_0 = g_1 = 0$ and $\gamma = 0$.

Lemma 9. We define a sequence of functions as follows. For $k = 0$, we first set

$$\sigma_0^1(x) = 1. \quad (48a)$$

Then, we iteratively define, for integers $n \geq 2$,

$$\sigma_0^n(x) = \sigma_0^{n-1}(1)x + \sum_{i=1}^{n-1} \nu_0^i \int_0^x (x-s) \sigma_0^{n-i}(s) ds. \quad (48b)$$

where

$$\nu_0^{n-1} = -\sigma_0^{n-1}(1) - \sum_{i=1}^{n-2} \nu_0^i \int_0^1 \sigma_0^{n-i}(s) ds. \quad (48c)$$

For positive integers $k \geq 1$, we first set

$$\sigma_k^1(x) = \sqrt{2} \cos(k\pi x). \quad (49a)$$

Then, we iteratively define, for integers $n \geq 2$,

$$\sigma_k^n(x) = \frac{\sigma_k^{n-1}(1)}{k\pi} \sin(k\pi x) + \frac{1}{k\pi} \sum_{i=1}^{n-1} \nu_k^i \int_0^x \sin(k\pi(x-s)) \sigma_k^{n-i}(s) ds. \quad (49b)$$

where

$$\nu_k^{n-1} = -\sqrt{2} \left(\sigma_k^{n-1}(1) + \sum_{i=1}^{n-2} \nu_k^i \int_0^1 \cos(k\pi s) \sigma_k^{n-i}(s) ds \right). \quad (49c)$$

Then we have, for $n = 1$ and for all $k \in \mathbb{N}$,

$$(\sigma_k^1)'' + k^2 \pi^2 \sigma_k^1 = 0, \quad (\sigma_k^1)'(0) = (\sigma_k^1)'(1) = 0, \quad (50a)$$

while, for all integers $n \geq 2$ and for all $k \in \mathbb{N}$,

$$(\sigma_k^n)'' + k^2 \pi^2 \sigma_k^n = \sum_{i=1}^{n-1} \nu_k^i \sigma_k^{n-i}, \quad (50b)$$

$$(\sigma_k^n)'(0) = \sigma_k^{n-1}(1), \quad (\sigma_k^n)'(1) = 0. \quad (50c)$$

In particular,

$$\|\sigma_k^n\|_{L^\infty} = O(1) \quad \text{and} \quad \nu_k^n = O(1) \quad \text{if } n \geq 1, \quad (51a)$$

$$\|\sigma_k^n\|_{L^\infty} = O(1/k) \quad \text{if } n \geq 2, \quad (51b)$$

as $k \rightarrow +\infty$. Finally,

$$\sigma_k^n(x) = P_k^n(x) \cos(k\pi x) + Q_k^n(x) \sin(k\pi x). \quad (52)$$

for some polynomials $P_k^n, Q_k^n \in \mathbb{R}[X]$.

Proof. First, (50a) follows from the definitions (48a) and (49a) of σ_k^1 . The proof now consists, for $n \geq 2$, of applying iteratively Lemma 8 by setting $g_0(x) = \sum_{i=1}^{n-2} \nu_k^i \sigma_k^{n-i}$ and $g_1(x) = \sigma_k^1$ while taking $c = 0$. It has only to be checked that $\int_0^1 \cos(k\pi s) g_1(s) ds \neq 0$. This is satisfied because for $k = 0$, due to (48a), $\int_0^1 \sigma_0^1(s) ds = 1$ and, for $k \geq 1$, due to (49a), $\int_0^1 \cos(k\pi s) \sigma_k^1(s) ds = \sqrt{2}/2$. Finally, (51) and (52) are obtained by an induction argument based on (48) and (49). This completes the proof. \square

Remark 4. The functions σ_k^n can be iteratively computed using (48) and (49), adequately tuning the quantities ν_k^n so as to satisfy the first boundary condition of (50c). For a given number N of heat equations in the PDE cascade (1), we will see next that only the computation of σ_k^n for $1 \leq n \leq N$ is required. The first four computations give

$$\begin{aligned}\sigma_0^1(x) &= 1, \\ \sigma_0^2(x) &= -\frac{x^2}{2} + x, \\ \sigma_0^3(x) &= \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^2}{12} + \frac{x}{2}, \\ \sigma_0^4(x) &= -\frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{72} - \frac{x^3}{9} - \frac{17x^2}{720} + \frac{7x}{24},\end{aligned}$$

with $\nu_0^1 = -1$, $\nu_0^2 = -1/6$, $\nu_0^3 = -17/360$, $\nu_0^4 = -31/1890$, and, for $k \geq 1$,

$$\begin{aligned}\sigma_k^1(x) &= \sqrt{2} \cos(k\pi x), \\ \sigma_k^2(x) &= \frac{(-1)^k \sqrt{2}}{k\pi} (1-x) \sin(k\pi x), \\ \sigma_k^3(x) &= \frac{\sqrt{2}}{k^3 \pi^3} (x-1) \sin(k\pi x) + \frac{\sqrt{2}}{2k^2 \pi^2} x(2-x) \cos(k\pi x), \\ \sigma_k^4(x) &= (-1)^k \sqrt{2} \left(\frac{3-x-3x^2+x^3}{6k^3 \pi^3} + \frac{2-2x}{k^5 \pi^5} \right) \sin(k\pi x) + (-1)^k \sqrt{2} \frac{x^2-2x}{k^4 \pi^4} \cos(k\pi x)\end{aligned}$$

with $\nu_k^1 = 2(-1)^{k+1}$, $\nu_k^2 = \frac{1}{k^2 \pi^2}$, $\nu_k^3 = (-1)^{k+1} \frac{k^2 \pi^2 + 6}{3k^4 \pi^4}$, $\nu_k^4 = \frac{k^2 \pi^2 + 5}{k^6 \pi^6}$.

Lemma 10. Under Assumption 2, the eigenvalues of \mathcal{A} are given by

$$\lambda_k = a - k^2 \pi^2, \quad k \geq 0. \quad (53)$$

They are of geometric multiplicity 1 and of algebraic multiplicity N with associated generalized eigenvectors

$$\phi_{i,k} = (\phi_{i,k}^j)_{1 \leq j \leq N} = \underbrace{(\sigma_k^i, \sigma_k^{i-1}, \dots, \sigma_k^1)}_{i \text{ elements}}, \underbrace{(0, \dots, 0)}_{N-i \text{ times}} \in \mathcal{D}(A) \quad (54)$$

for integers $1 \leq i \leq N$ and $k \in \mathbb{N}$, that satisfy

$$\mathcal{A}\phi_{i,k} = \lambda_k \phi_{i,k} + \sum_{j=1}^{i-1} \nu_k^{i-j} \phi_{j,k}. \quad (55)$$

Hence, in $(\phi_{i,k})_{1 \leq i \leq N}$, the matrix of (the restriction of) \mathcal{A} is given by

$$M_k = \begin{bmatrix} \lambda_k & \nu_k^1 & \nu_k^2 & \nu_k^3 & \dots & \nu_k^{N-1} \\ 0 & \lambda_k & \nu_k^1 & \nu_k^2 & \dots & \nu_k^{N-2} \\ 0 & 0 & \lambda_k & \nu_k^1 & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_k & \ddots & \nu_k^2 \\ \vdots & \vdots & \ddots & \ddots & \lambda_k & \nu_k^1 \\ 0 & 0 & \dots & 0 & 0 & \lambda_k \end{bmatrix} \quad (56)$$

which has λ_k as unique eigenvalue, of geometric multiplicity 1 and of algebraic multiplicity N .

Remark 5. Lemma 9 is derived by iteratively applying Lemma 8 with $c = 0$ at each step. This choice is arbitrary: a different value of c could be chosen at each step of the iteration. This would change the eigenelements $\phi_{i,k}$ defined in Lemma 10. This reflects the fact that there is one degree of freedom² in the definition of each of the generalized eigenvectors $\phi_{i,k}$ for $2 \leq i \leq N$. This degree of freedom corresponds to the fact that a generalized eigenvector translated by an eigenvector associated with the same eigenvalue remains a generalized eigenvector. Since constants $c \neq 0$ would make the expressions of $\phi_{i,k}$ more complex, there is no technical benefit of choosing $c \neq 0$. Note however that this choice fixes the corresponding constants in the dual construction (see Remark 7).

Remark 6. Since M_k has one single eigenvalue, of geometric multiplicity 1, M_k is similar to a Jordan block.

Lemma 11. $\Phi = \{\phi_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ is a Riesz basis of \mathcal{H}^0 . Moreover, \mathcal{A} generates a C_0 -semigroup.

Proof. Define $\hat{\Phi} = \{\hat{\phi}_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ with $\hat{\phi}_{i,k} = (\hat{\phi}_{i,k}^j)_{1 \leq j \leq N}$ where $\hat{\phi}_{i,k}^j(x) = \delta_{i,j} \varphi_k(x)$ and φ_k is defined as in (5) for $j = i$. Here $\delta_{i,i} = 1$ while $\delta_{i,j} = 0$ whenever $i \neq j$. It is easy to see that $\hat{\Phi}$ is a Hilbert basis of \mathcal{H}^0 . Hence, noting that $\sigma_k^1 = \varphi_k$ and invoking (51b), we infer that

$$\sum_{i=1}^N \sum_{k \geq 0} \|\phi_{i,k} - \hat{\phi}_{i,k}\|_{\mathcal{H}^0}^2 = \sum_{i=1}^N \sum_{j=1}^{i-1} \sum_{k \geq 0} \|\phi_{i,k}^j\|_{L^2}^2 = \sum_{i=1}^N \sum_{j=2}^i \sum_{k \geq 0} \|\sigma_k^j\|_{L^2}^2 < \infty.$$

Since Φ is ω -linearly independent, Bari's theorem implies that Φ is a Riesz basis of \mathcal{H}^0 . Finally, a similar approach to [12, Lem. 2.1] shows that \mathcal{A} generates a C_0 -semigroup because $\|e^{M_k t}\| \leq g(t)$ for all $k \in \mathbb{N}$ and for some real-valued continuous function g (see the beginning of the proof of Theorem 4). \square

The previous lemma provides a suitable framework for studying the solutions in L^2 norm. The tool for studying the solutions in H^1 norm is provided by the following lemma.

Lemma 12. $\tilde{\Phi} = \{\tilde{\phi}_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$, where $\tilde{\phi}_{i,k} = \frac{1}{\sqrt{1+k^2\pi^2}} \phi_{i,k}$, is a Riesz basis of \mathcal{H}^1 (defined by (14)).

Proof. Define $\check{\Phi} = \{\check{\phi}_{i,k} \mid 1 \leq i \leq N, k \in \mathbb{N}\}$ with $\check{\phi}_{i,k} = (\check{\phi}_{i,k}^j)_{1 \leq j \leq N}$ where $\check{\phi}_{i,k}^j(x) = \frac{1}{\sqrt{1+k^2\pi^2}} \delta_{i,j} \varphi_k(x)$. We know that $\check{\Phi}$ is a Riesz basis of \mathcal{H}^1 . We apply Bari's theorem by comparing $\tilde{\Phi}$ to the Riesz basis $\check{\Phi}$. Since $\tilde{\Phi}$ is ω -linearly independent, it remains to show that

$$\sum_{1 \leq i \leq N} \sum_{k \geq 0} \|\tilde{\phi}_{i,k} - \check{\phi}_{i,k}\|_{\mathcal{H}^1}^2 = \sum_{i=1}^N \sum_{j=1}^{i-1} \sum_{k \geq 0} \frac{1}{1+k^2\pi^2} \|\phi_{i,k}^j\|_{H^1}^2 = \sum_{i=1}^N \sum_{j=2}^i \sum_{k \geq 0} \frac{1}{1+k^2\pi^2} \|\sigma_k^j\|_{H^1}^2 < \infty. \quad (57)$$

Based on (51b), we already know that, for any fixed $n \geq 2$, $\|\sigma_k^n\|_{L^2} = O(1/k)$ as $k \rightarrow +\infty$. It remains to study the terms $\|(\sigma_k^n)'\|_{L^2}$ for $n \geq 2$ as $k \rightarrow +\infty$. We infer from (49b) that

$$(\sigma_k^n)'(x) = \sigma_k^{n-1}(1) \cos(k\pi x) + \sum_{i=1}^{n-1} \nu_k^i \int_0^x \cos(k\pi(x-s)) \sigma_k^{n-i}(s) ds.$$

Based on (51a), we thus infer that $\|(\sigma_k^n)'\|_{L^\infty} = O(1)$ as $k \rightarrow +\infty$ for any given $n \geq 2$. Hence (57) holds true, which concludes the proof. \square

²The number of degrees of freedom coincides with the geometric multiplicity of the associated eigenvalues, which is 1 in our case.

In order to compute the dual Riesz basis of Φ , we introduce the adjoint operator $\mathcal{A}^*((g^j)_{1 \leq j \leq N}) = ((g^j)'' + ag^j)_{1 \leq j \leq N}$ defined on the domain (6).

Lemma 13. *Let $k \in \mathbb{N}$, $\gamma \in \mathbb{R}$, and $g_0, g_1 \in L^2(0, 1)$ be such that $\int_0^1 \cos(k\pi s)g_1(s) ds \neq 0$. Then, the functions $f : [0, 1] \rightarrow \mathbb{R}$ and real numbers $\nu \in \mathbb{R}$ that satisfy $f'' + k^2\pi^2 f = g_0 + \nu g_1$, $f'(0) = 0$, $f'(1) = \gamma$, are given by*

$$\nu = \frac{\gamma - \int_0^1 g_0(s) ds}{\int_0^1 g_1(s) ds}, \quad f(x) = c + \int_0^x (x-s)(g_0(s) + \nu g_1(s)) ds,$$

if $k = 0$, and by

$$\nu = \frac{(-1)^k \gamma - \int_0^1 \cos(k\pi s)g_0(s) ds}{\int_0^1 \cos(k\pi s)g_1(s) ds},$$

$$f(x) = c\sqrt{2} \cos(k\pi x) + \frac{1}{k\pi} \int_0^x \sin(k\pi(x-s))(g_0(s) + \nu g_1(s)) ds,$$

if $k \geq 1$, where $c \in \mathbb{R}$ is an arbitrary constant.

Lemma 14. *We define a sequence of functions as follows. For $k = 0$, we first set*

$$\tau_0^1(x) = 1. \tag{58a}$$

Then, we iteratively define for integers $n \geq 2$

$$\tau_0^n(x) = c_0^n + \underbrace{\sum_{i=1}^{n-1} \tilde{\nu}_0^i \int_0^x (x-s)\tau_0^{n-i}(s) ds}_{=\tau_{0,0}^n(x)}. \tag{58b}$$

where

$$\tilde{\nu}_0^{n-1} = -\tau_0^{n-1}(0) - \sum_{i=1}^{n-2} \tilde{\nu}_0^i \int_0^1 \tau_0^{n-i}(s) ds, \tag{58c}$$

and

$$c_0^n = -\sum_{i=1}^{n-1} \int_0^1 \sigma_0^{n-i+1}(s)\tau_0^i(s) ds - \int_0^1 \sigma_0^1(s)\tau_{0,0}^n(s) ds. \tag{58d}$$

For integers $k \geq 1$, we first set

$$\tau_k^1(x) = \sqrt{2} \cos(k\pi x). \tag{59a}$$

Then, we iteratively define for integers $n \geq 2$

$$\tau_k^n(x) = c_k^n \sqrt{2} \cos(k\pi x) + \underbrace{\frac{1}{k\pi} \sum_{i=1}^{n-1} \tilde{\nu}_k^i \int_0^x \sin(k\pi(x-s))\tau_k^{n-i}(s) ds}_{=\tau_{k,0}^n(x)} \tag{59b}$$

where

$$\tilde{\nu}_k^{n-1} = -\sqrt{2} \left((-1)^k \tau_k^{n-1}(0) + \sum_{i=1}^{n-2} \tilde{\nu}_k^i \int_0^1 \cos(k\pi s)\tau_k^{n-i}(s) ds \right) \tag{59c}$$

and

$$c_k^n = - \sum_{i=1}^{n-1} \int_0^1 \sigma_k^{n-i+1}(s) \tau_k^i(s) ds - \int_0^1 \sigma_k^1(s) \tau_{k,0}^n(s) ds. \quad (59d)$$

Then we have, for all $k \in \mathbb{N}$,

$$(\tau_k^1)'' + k^2 \pi^2 \tau_k^1 = 0, \quad (\tau_k^1)'(0) = (\tau_k^1)'(1) = 0, \quad (60a)$$

and

$$(\tau_k^n)'' + k^2 \pi^2 \tau_k^n = \sum_{i=1}^{n-1} \tilde{\nu}_k^i \tau_k^{n-i}, \quad (\tau_k^n)'(0) = 0, \quad (\tau_k^n)'(1) = -\tau_k^{n-1}(0), \quad (60b)$$

for all $n \geq 2$. Furthermore we have the biorthogonality condition:

$$\sum_{i=1}^n \int_0^1 \sigma_k^{n-i+1}(s) \tau_k^i(s) ds = 0 \quad \forall n \geq 2. \quad (61)$$

Finally, τ_k^n can be expressed as

$$\tau_k^n(x) = \tilde{P}_k^n(x) \cos(k\pi x) + \tilde{Q}_k^n(x) \sin(k\pi x). \quad (62)$$

for some polynomials $\tilde{P}_k^n, \tilde{Q}_k^n \in \mathbb{R}[X]$.

Proof. First, (60a) follows from the definitions (58a) and (59a) of τ_k^1 . The proof now consists, for $n \geq 2$, of applying iteratively Lemma 13 by setting $g_0(x) = \sum_{i=1}^{n-2} \tilde{\nu}_i \tau_k^{n-i}$ and $g_1(x) = \tau_k^1$. It only has to be checked that $\int_0^1 \cos(k\pi s) g_1(s) ds \neq 0$. For $k = 0$, due to (58a), $\int_0^1 \sigma_0^1(s) ds = 1$ and, for $k \geq 1$, due to (59a), $\int_0^1 \cos(k\pi s) \sigma_k^1(s) ds = \sqrt{2}/2$. The biorthogonality condition (61) holds true by fixing the arbitrary constant c appearing in Lemma 13 as (58d) and (59d). \square

Remark 7. The construction of the Riesz basis Φ relied on the iterative application of Lemma 8, giving Lemma 9. In that construction, we fixed the arbitrary constant c from Lemma 8 as $c = 0$. Note that for the dual construction of Lemma 14, the arbitrary constant c appearing in Lemma 13 still corresponds to the solutions to the homogeneous ODE, i.e., for $g_0 = g_1 = 0$ and $\gamma = 0$. However, in the iterative application of Lemma 13, giving Lemma 14, c is not defined as 0 but iteratively defined by (58d) and (59d). Such a definition of the c constant is made to ensure that the biorthogonality condition (61) holds. This latter condition will be key in the construction of the dual Riesz basis Ψ of Φ to ensure that Φ and Ψ are biorthogonal.

Remark 8. The functions τ_k^n can be iteratively computed using (58) and (59). For a given number N of heat equations in the PDE cascade (1), only the computation of τ_k^n for $1 \leq n \leq N$ is required. The first four computations give

$$\begin{aligned} \tau_0^1(x) &= 1, \\ \tau_0^2(x) &= -\frac{x^2}{2} - \frac{1}{6}, \\ \tau_0^3(x) &= \frac{x^4}{24} - \frac{1}{15}, \\ \tau_0^4(x) &= -\frac{x^6}{720} + \frac{x^4}{144} + \frac{17x^2}{720} - \frac{457}{15120} \end{aligned}$$

along with $\nu_0^n = \tilde{\nu}_0^n$ for $1 \leq n \leq 4$, and, for $k \geq 1$,

$$\begin{aligned}\tau_k^1(x) &= \sqrt{2} \cos(k\pi x), \\ \tau_k^2(x) &= \frac{(-1)^{k+1} \sqrt{2} x}{k\pi} \sin(k\pi x) + \frac{4 - (-1)^k \sqrt{2}}{2k^2 \pi^2} \cos(k\pi x), \\ \tau_k^3(x) &= \frac{(3\sqrt{2} + 4(-1)^{k+1})x}{2k^3 \pi^3} \sin(k\pi x) + \sqrt{2} \left(\frac{1 - 3x^2}{6k^2 \pi^2} + 3 \frac{1 + (-1)^{k+1} \sqrt{2}}{k^4 \pi^4} \right) \cos(k\pi x), \\ \tau_k^4(x) &= \left((-1)^k \sqrt{2} \frac{x(-2 + x^2)}{6k^3 \pi^3} + \frac{16 + 11\sqrt{2}(-1)^{k+1}}{2k^5 \pi^5} x \right) \sin(k\pi x) \\ &\quad + \left(\frac{8 + 6\sqrt{2}(-1)^{k+1} + (-12 + 15\sqrt{2}(-1)^k)x^2}{12k^4 \pi^4} + \sqrt{2} \frac{135\sqrt{2} + 162(-1)^{k+1}}{12k^6 \pi^6} \right) \cos(k\pi x),\end{aligned}$$

with $\nu_k^n = \tilde{\nu}_k^n$ for $1 \leq n \leq 4$.

Lemma 15. Under Assumption 2, the dual Riesz basis $\Psi = \{\psi_{i,k}, | 1 \leq i \leq N, k \in \mathbb{N}\}$ of Φ is described by

$$\psi_{i,k} = (\psi_{i,k}^j)_{1 \leq j \leq N} = \underbrace{(0, \dots, 0)}_{i-1 \text{ times}}, \underbrace{(\tau_k^1, \tau_k^2, \dots, \tau_k^{N-i+1})}_{N-i+1 \text{ elements}} \in \mathcal{D}(A^*) \quad (63)$$

for integers $1 \leq i \leq N, k \in \mathbb{N}$. Furthermore we have $\nu_k^i = \tilde{\nu}_k^i$ and

$$\mathcal{A}^* \psi_{i,k} = \lambda_k \psi_{i,k} + \sum_{j=1}^{N-i} \nu_k^j \psi_{i+j,k}. \quad (64)$$

Hence, in $(\psi_{i,k})_{1 \leq i \leq N}$, the matrix of \mathcal{A} is M_k^\top as defined by (56).

Proof. By definition of \mathcal{A}^* and the ODEs (60) satisfied by τ_k^n , we infer that

$$\mathcal{A}^* \psi_{i,k} = \lambda_k \psi_{i,k} + \sum_{j=1}^{N-i} \tilde{\nu}_k^j \psi_{i+j,k}. \quad (65)$$

We now show that Φ and Ψ are biorthogonal, i.e., $\langle \phi_{i,k}, \psi_{i',k'} \rangle = \delta_{(i,k), (i',k')}$. Using the definitions (54) and (63) of $\phi_{i,k}$ and $\psi_{i,k}$, respectively, the fact that $\sigma_0^1(x) = \tau_0^1(x) = 1$ while $\sigma_k^1(x) = \tau_k^1(x) = \sqrt{2} \cos(k\pi x)$ for $k \in \mathbb{N}^*$, and the biorthogonality property (61), we infer that $\langle \phi_{i,k}, \psi_{i',k} \rangle = \delta_{i,i'}$. For $k \neq k'$, the computation of the two sides of the identity $\langle \mathcal{A} \phi_{i,k}, \psi_{i',k'} \rangle = \langle \phi_{i,k}, \mathcal{A}^* \psi_{i',k'} \rangle$, for $1 \leq i, i' \leq N$, by using (55) and (65) shows that $\langle \phi_{i,k}, \psi_{i',k'} \rangle = 0$. Hence Φ and Ψ are biorthogonal. It remains to show that $\nu_k^i = \tilde{\nu}_k^i$. This is inferred from the computation of the two sides of the identity $\langle \mathcal{A} \phi_{i,k}, \psi_{i',k} \rangle = \langle \phi_{i,k}, \mathcal{A}^* \psi_{i',k} \rangle$. \square

3.2 Spectral reduction of the system

Consider again the change of variable (18) that gives the homogenous representation (19). Defining the states $\mathcal{X} = (y^j)_{1 \leq j \leq N}$ (original coordinates) and $\tilde{\mathcal{X}} = ((y^j)_{1 \leq j \leq N-1}, \tilde{y}^N)$ (homogeneous coordinates), we infer the change of variable formula (20) along with the abstract system dynamics (21). Define the coefficients of projection (22) and the vectors

$$\mathbf{x}_k = \begin{bmatrix} x_{1,k} \\ \vdots \\ x_{N,k} \end{bmatrix}, \quad \tilde{\mathbf{x}}_k = \begin{bmatrix} \tilde{x}_{1,k} \\ \vdots \\ \tilde{x}_{N,k} \end{bmatrix}, \quad \alpha_k = \begin{bmatrix} \alpha_{1,k} \\ \vdots \\ \alpha_{N,k} \end{bmatrix}, \quad \beta_k = \begin{bmatrix} \beta_{1,k} \\ \vdots \\ \beta_{N,k} \end{bmatrix}.$$

We infer from (20) that $\tilde{\mathbf{x}}_k = \mathbf{x}_k + \beta_k u$ while (21) gives

$$\dot{\tilde{\mathbf{x}}}_k = M_k \tilde{\mathbf{x}}_k + \alpha_k u + \beta_k v. \quad (66)$$

Combining the two latter identities, we infer that

$$\dot{\mathbf{x}}_k = M_k \mathbf{x}_k + N_k u \quad (67)$$

where $N_k = \alpha_k + M_k \beta_k$. Hence, the dynamics of the mode λ_k is described by (67). The study of its controllability properties relies on the following lemma.

Lemma 16. *We have $N_k = -\mu_k [c_k^N \ c_k^{N-1} \ \dots \ c_k^1 \ 1]^\top$ for all $k \in \mathbb{N}$. Here $\mu_0 = 1$ and $\mu_k = \sqrt{2}$ for all $k \in \mathbb{N}^*$.*

Proof. By the definition (22) of $\alpha_{i,k}$ and using integrations by parts, we infer that

$$\alpha_{i,k} = \int_0^1 (\varphi'' + a\varphi) \psi_{i,k}^N dx = [\varphi' \psi_{i,k}^N - \varphi (\psi_{i,k}^N)']_0^1 + \int_0^1 \varphi ((\psi_{i,k}^N)'' + a\psi_{i,k}^N) dx = -\psi_{i,k}^N(0) - \langle \beta, \mathcal{A}^* \psi_{i,k} \rangle$$

where we have used that $\varphi(1) = \varphi'(1) = 0$, $\varphi'(0) = 1$, and $(\psi_{i,k}^N)'(0) = 0$. In view of (64) we have $\langle \beta, \mathcal{A}^* \psi_{i,k} \rangle = (M_k)_{L_i} \beta_k$ where $(M_k)_{L_i}$ denotes the i -th line of the matrix M_k , which gives $\alpha_{i,k} + (M_k)_{L_i} \beta_k = -\psi_{i,k}^N(0)$. Hence, for all $1 \leq i \leq N$, we have

$$N_k = -[\psi_{1,k}^N(0) \ \psi_{2,k}^N(0) \ \dots \ \psi_{N,k}^N(0)]^\top = -[\tau_k^N(0) \ \tau_k^{N-1}(0) \ \dots \ \tau_k^1(0)]^\top$$

where we have used (63). The conclusion follows from (58)-(59). \square

Noting that $\nu_k^1 \neq 0$ and $\mu_k \neq 0$, the application of the Hautus test shows that (67) is controllable for all $k \in \mathbb{N}$.

3.3 Output feedback control strategy

Consider the output (26) measured on the last component y^1 of the PDE cascade (1).

Let $\delta > 0$ be the targeted exponential decay rate for the closed-loop system. Using (53), we choose a large enough integer $n_0 \geq 0$ such that $\lambda_{n_0} < -\delta$, implying that $\lambda_k \leq \lambda_{n_0} < -\delta$ for all integers $k \geq n_0$. We now introduce a finite-dimensional dynamics that captures the n_0 first modes of the plant. Defining the state vector $X_1 = \text{col}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{n_0-1})$, we infer from (66) that

$$\dot{X}_1 = A_1 X_1 + B_{1,u} u + B_{1,v} v \quad (68)$$

where

$$A_1 = \text{diag}(M_0, M_1, \dots, M_{n_0-1}), \quad B_{1,u} = \text{col}(\alpha_0, \alpha_1, \dots, \alpha_{n_0-1}), \quad B_{1,v} = \text{col}(\beta_0, \beta_1, \dots, \beta_{n_0-1}).$$

Recalling that $v = \dot{u}$, we define the augmented state vector $X_{1,a} = \text{col}(X_1, u)$, obtaining the dynamics

$$\dot{X}_{1,a} = A_{1,a} X_{1,a} + B_{1,a} v \quad (69)$$

where the matrices $A_{1,a}$ and $B_{1,a}$ are defined by (29).

Under Assumption 2, $k \neq l \Rightarrow \text{sp}(M_k) \cap \text{sp}(M_l) = \emptyset$. Therefore, the application of the Hautus test shows that the pair $(A_{1,a}, B_{1,a})$ is controllable if and only if the last entry of $N_k = \alpha_k + M_k \beta_k \neq 0$ is not zero for all $0 \leq k \leq n_0 - 1$. The result of Lemma 16 implies the following statement.

Lemma 17. *Under Assumption 2, the pair $(A_{1,a}, B_{1,a})$ is controllable.*

Remark 9. At this step, it is possible to design a state-feedback control strategy as in Remark 1.

Proceeding as in the previous section, the measurement (26) is expanded as

$$z(t) = \sum_{i=1}^N \sum_{k \geq 0} c_{i,k} \tilde{x}_{i,k}(t) = C_1 X_1(t) + \sum_{k \geq n_0} c_k \tilde{\mathbf{x}}_k(t) \quad (70)$$

where $c_{i,k}$ is defined by (31) and

$$c_k = [c_{1,k} \quad c_{2,k} \quad \dots \quad c_{N,k}], \quad C_1 = [c_0 \quad c_1 \quad \dots \quad c_{n_0-1}].$$

Again, under Assumption 2, $k \neq l \Rightarrow \text{sp}(M_k) \cap \text{sp}(M_l) = \emptyset$. Hence, the application of the Hautus test shows that the pair (A_1, C_1) is observable if and only if $c_{1,k} \neq 0$ for all $0 \leq k \leq n_0 - 1$. By the definition (31) of $c_{i,k}$ and the expression of $\phi_{1,k}$ given by (54), we infer the following result.

Lemma 18. Under Assumption 1, the pair (A_1, C_1) is observable if and only if

$$\int_0^1 c(x) \cos(k\pi x) dx \neq 0, \quad 0 \leq k \leq n_0 - 1. \quad (71)$$

We are now in position to define our output feedback control strategy. Let $n_1 \geq n_0$ be an integer to be chosen large enough later on. We defined the following controller dynamics:

$$\dot{u} = v \quad (72a)$$

$$\dot{\hat{X}}_1 = A_1 \hat{X}_1 + B_{1,u} u + B_{1,v} v - L \left(C_1 \hat{X}_1 + \sum_{k=n_0}^{n_1} c_k \hat{\mathbf{x}}_k - z \right) \quad (72b)$$

$$\dot{\hat{\mathbf{x}}}_k = M_k \hat{\mathbf{x}}_k + \alpha_k u + \beta_k v, \quad n_0 \leq k \leq n_1 \quad (72c)$$

$$v = -K_x \hat{X}_1 - k_u u \quad (72d)$$

where $k_u \in \mathbb{R}$ and $K_x \in \mathbb{R}^{1 \times n_0 N}$ are the feedback gains and $L \in \mathbb{R}^{n_0 N}$ is the observer gain.

Theorem 4. Let $a_1 = a_2 = \dots = a_N \in \mathbb{R}$ (Assumption 2) and let $c \in L^2(0, 1)$. Let $\delta > 0$ be arbitrary. Let $n_0 \in \mathbb{N}$ be large enough so that $\lambda_{n_0} < -\delta$. Assuming that the observability condition (71) holds, let $K = [K_x \quad k_u] \in \mathbb{R}^{1 \times (n_0 N + 1)}$ and $L \in \mathbb{R}^{n_0 N}$ be matrices such that $A_{1,a} - B_{1,a} K$ and $A_1 - LC_1$ are Hurwitz with spectral abscissa less than $-\delta$.

Then, for any $n_1 \geq n_0$ large enough, there exists $C > 0$ such that, for all initial conditions $y_0^j \in L^2(0, 1)$, $1 \leq j \leq N$, $u(0) = u_0 \in \mathbb{R}$, $\hat{X}_1(0) \in \mathbb{R}^{n_0 N}$, and $\hat{x}_{i,k}(0) \in \mathbb{R}$, the solutions of the closed-loop system consisting of the PDE cascade (1), the output (26) and the controller (72), satisfy (34) with $Y = \mathcal{H}^0$ defined by (2), for all $t \geq 0$.

If the initial condition is such that $y_0^j \in H^2(0, 1)$ with $(y^j)'(0) = y^{j+1}(1)$ and $(y^j)'(1) = 0$ for all $1 \leq j \leq N - 1$ while $(y^N)'(0) = u(0)$ and $(y^N)'(1) = 0$, the conclusion holds with $Y = \mathcal{H}^1$ defined by (14).

Proof. We define $\hat{X}_1 = \text{col}(\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_{n_0-1})$, $\mathbf{e}_k = \tilde{\mathbf{x}}_k - \hat{\mathbf{x}}_k$ and

$$\begin{aligned} E_1 &= \text{col}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n_0-1}), \quad E_2 = \text{col}((n_0 + 1)^\kappa \mathbf{e}_{n_0}, \dots, (n_1 + 1)^\kappa \mathbf{e}_{n_1}), \\ \hat{X}_{1,a} &= \text{col}(\hat{X}_1^1, u), \quad \hat{X}_2 = \text{col}(\hat{\mathbf{x}}_{n_0}, \hat{\mathbf{x}}_{n_0+1}, \dots, \hat{\mathbf{x}}_{n_1}), \quad X = \text{col}(\hat{X}_{1,a}, E_1, \hat{X}_2, E_2). \end{aligned}$$

Here we set $\kappa = 0$. A different value will be set later on in the proof of Theorem 5. Combining (66), (68), (69), (70), and (72), we obtain

$$\dot{X} = FX + \mathcal{L}\zeta \quad (73)$$

where the measurement residue is defined by

$$\zeta = \sum_{k \geq n_1+1} c_k \tilde{x}_k \quad (74)$$

and

$$F = \begin{bmatrix} A_{1,a} - B_{1,a}K & L_a C_1 & 0 & L_a C_2 \\ 0 & A_1 - LC_1 & 0 & -LC_2 \\ [0 \ B_{2,u}] - B_{2,v}K & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}, \quad \mathcal{L} = \text{col}(L_a, -L, 0, 0),$$

where

$$\begin{aligned} K &= [K_x \ k_u], \quad L_a = \text{col}(L, 0), \quad A_2 = \text{diag}(M_{n_0}, M_{n_0+1}, \dots, M_{n_1}), \\ B_{2,u} &= \text{col}(\alpha_{n_0}, \alpha_{n_0+1}, \dots, \alpha_{n_1}), \quad B_{2,v} = \text{col}(\beta_{n_0}, \beta_{n_0+1}, \dots, \beta_{n_1}), \\ C_2 &= \begin{bmatrix} \frac{c_{n_0}}{(n_0+1)^\kappa} & \frac{c_{n_0+2}}{(n_0+1)^\kappa} & \dots & \frac{c_{n_1}}{(n_1+1)^\kappa} \end{bmatrix}. \end{aligned}$$

The input u and its time derivative v are expressed as (37).

Since, for $k \geq n_0$, the matrix M_k defined by (56) has a spectral abscissa less than $-\delta$, it follows that $\tilde{M}_k = M_k + \delta I$ is Hurwitz. We can thus define $P_k \succ 0$ as the solution of the Lyapunov equation

$$M_k^\top P_k + P_k M_k + 2\delta P_k = \tilde{M}_k^\top P_k + P_k \tilde{M}_k = -|\lambda_k|I \quad (75)$$

for all $k \geq n_0$. Defining $\Delta_k = P_k - \frac{1}{2}I$, let us prove that $\|\Delta_k\| = O(1/k^2)$ as $k \rightarrow +\infty$. Using (51a) and (56), the matrix $R_k = M_k - \lambda_k I$ satisfies $\|R_k\| = O(1)$. Substituting $\tilde{M}_k = M_k + \delta I = (-|\lambda_k| + \delta)I + R_k$ and $P_k = \frac{1}{2}I + \Delta_k$ into (75) while using (53), we infer that $\|\Delta_k\| = O(1/k^2)$. Hence there exist $\gamma_m, \gamma_M > 0$ such that

$$\|P_k\| \leq \gamma_M \quad \text{and} \quad \gamma_m I \preceq P_k \preceq \gamma_M I \quad \forall k \geq n_0 + 1. \quad (76)$$

Noting that $R_k \in \mathbb{R}^{N \times N}$ is nilpotent (because upper triangular with zeros on the diagonal), we obtain $e^{(M_k + \delta I)t} = e^{(\lambda_k + \delta)t} e^{R_k t} = e^{(\lambda_k + \delta)t} \sum_{i=0}^{N-1} \frac{R_k^i t^i}{i!}$. Since $\|R_k\| = O(1)$, we infer the existence of a polynomial function $P_\infty \in \mathbb{R}_{N-1}[X]$, not depending on k , such that $\|e^{(M_k + \delta I)t}\| \leq P_\infty(t) e^{(\lambda_k + \delta)t} \leq P_\infty(t) e^{(\lambda_{n_0} + \delta)t}$ for all $t \geq 0$ and $k \geq n_0$. Since $\lambda_{n_0} < -\delta$, there exist $c, \epsilon > 0$, not depending on n_1 , such that

$$\|e^{(A_2 + \delta I)t}\| \leq c e^{-\epsilon t} \quad \forall t \geq 0 \quad \forall n_1 \geq n_0. \quad (77)$$

We now define the Lyapunov functional candidates to establish the stability estimate (34). In \mathcal{H}^0 norm, we define

$$V = X^\top P X + \sum_{k \geq n_1+1} \tilde{x}_k^\top P_k \tilde{x}_k \quad (78)$$

for some matrix $P \succ 0$. The equivalence between (78) and the (squared) \mathcal{H}^0 norm follows from (76) and Lemma 11. In \mathcal{H}^1 norm, we define for classical solutions

$$V = X^\top P X + \sum_{k \geq n_1+1} (1 + k^2) \tilde{x}_k^\top P_k \tilde{x}_k \quad (79)$$

for some matrix $P \succ 0$. The equivalence between (39) and the (squared) \mathcal{H}^1 norm follows from (76) and Lemma 12 since the series expansion (40) holds.

In what follows we focus on the \mathcal{H}^1 norm. We compute the derivative of V defined by (79) along the solutions of (66) and (73). Using the Lyapunov equations (75), the estimates (76), Young's inequality and (37), we obtain

$$\begin{aligned}\dot{V} + 2\delta V &= \tilde{X}^\top \begin{bmatrix} F^\top P + PF + 2\delta P & P\mathcal{L} \\ \mathcal{L}^\top P & 0 \end{bmatrix} \tilde{X} + 2 \sum_{k \geq n_1+1} (1+k^2) \tilde{x}_k^\top (M_k^\top P_k + P_k M_k + 2\delta P_k) \tilde{x}_k \\ &\quad + 2 \sum_{k \geq n_1+1} (1+k^2) \tilde{x}_k^\top P_k (\alpha_k u + \beta_k v) \\ &\leq \tilde{X}^\top \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} \\ \mathcal{L}^\top P & 0 \end{bmatrix} \tilde{X} + 2 \sum_{k \geq n_1+1} (1+k^2) \left(\lambda_k + \frac{\gamma_M^2(1+k^2)}{\epsilon} \right) \|\tilde{x}_k\|^2\end{aligned}$$

for all $\epsilon > 0$, where $\tilde{X} = \text{col}(X, \zeta)$ and

$$\Theta_{1,1} = F^\top P + PF + 2\delta P + \epsilon \left(S_{\alpha, n_1} \tilde{E}^\top \tilde{E} + S_{\beta, n_1} \tilde{K}^\top \tilde{K} \right)$$

where $S_{\alpha, n_1} = \sum_{k \geq n_1+1} \|\alpha_k\|^2 < \infty$ and $S_{\beta, n_1} = \sum_{k \geq n_1+1} \|\beta_k\|^2 < \infty$. Using (74) and the Cauchy-Schwarz inequality, we infer that

$$\zeta^2 \leq \underbrace{\sum_{k \geq n_1+1} \|c_k\|^2}_{=S_{\zeta, n_1} < \infty} \sum_{k \geq n_1+1} \|\tilde{x}_k\|^2.$$

Combining the two latter estimates, we obtain that

$$\dot{V} + 2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \sum_{k \geq n_1+1} (1+k^2) \Gamma_k \|\tilde{x}_k\|^2 \quad (80)$$

where

$$\Theta_1 = \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} \\ \mathcal{L}^\top P & -\eta \end{bmatrix}, \quad (81a)$$

$$\Gamma_k = 2 \left(\lambda_k + \frac{\gamma_M^2(1+k^2)}{\epsilon} \right) + \frac{\eta S_{\zeta, n_1}}{1+k^2} = 2 \left(-k^2 \left(\pi^2 - \frac{\gamma_M^2}{\epsilon} \right) + a + \frac{\gamma_M^2}{\epsilon} \right) + \frac{\eta S_{\zeta, n_1}}{1+k^2} \quad (81b)$$

for all $\epsilon, \eta > 0$. In particular, for all $\epsilon > \gamma_M^2/\pi^2$, we have $\Gamma_k \leq \Gamma_{n_1+1}$ for all $k \geq n_1 + 1$. We thus infer that

$$\dot{V} + 2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \Gamma_{n_1+1} \sum_{k \geq n_1+1} (1+k^2) \|\tilde{x}_k\|^2$$

for all $\epsilon > \gamma_M^2/\pi^2$ and all $\eta > 0$. Hence $\dot{V} + 2\delta V \leq 0$, giving the claimed stability estimate (34), provided that there exists an integer $n_1 \geq n_0 + 1$, real numbers $\epsilon > \gamma_M^2/\pi^2$ and $\eta > 0$ and a matrix $P \succ 0$ such that $\Theta_1 \preceq 0$ and $\Gamma_{n_1+1} \leq 0$.

Let us prove the feasibility of these constraints. Recall that F is Hurwitz with spectral abscissa less than $-\delta$. Note also that $\|C_2\| = O(1)$, $\|B_{2,u}\| = O(1)$ and $\|B_{2,v}\| = O(1)$ as $n_1 \rightarrow +\infty$. Finally, recall that (77) holds for some constants $c, \epsilon > 0$ not depending on n_1 . Hence, the application of the lemma in Appendix of [13] shows that the solution $P \succ 0$ to $F^\top P + PF + 2\delta P = -I$ satisfies $\|P\| = O(1)$ as $n_1 \rightarrow +\infty$. We now arbitrarily fix $\epsilon > \gamma_M^2/\pi^2$ and define

$$\eta = \begin{cases} \frac{1}{\sqrt{S_{\zeta, n_1}}} & \text{if } S_{\zeta, n_1} \neq 0 \\ n_1 & \text{otherwise} \end{cases}.$$

We infer that $\Gamma_{n_1+1} \rightarrow -\infty$ as $n_1 \rightarrow +\infty$. Moreover, the Schur complement shows that $\Theta_1 \preceq 0$ if n_1 is chosen sufficiently large. This completes the proof. \square

3.4 Extension to pointwise Dirichlet measurement

Consider the case of the pointwise measurement of y^1 given by (43). Then the series expansion (30) still holds true but with $c_{i,k} = \phi_{i,k}^1(\xi_p)$. We thus infer, similarly to Lemma 18 and in view of the expression of $\phi_{i,k}^1$ given by (54), that the pair (A_1, C_1) is observable if and only if

$$\cos(k\pi\xi_p) \neq 0, \quad 0 \leq k \leq N_{0,1} - 1. \quad (82)$$

Theorem 5. *Considering the pointwise measurement (43) for some $\xi_p \in [0, 1]$, replacing the observability condition (71) by (82), the statement of Theorem 4 holds true with $Y = \mathcal{H}^0$ defined by (2).*

The proof follows the arguments developed in the proof of Theorems 2 and 4 by setting $\kappa = 1$.

Remark 10. *In the case of N identical heat equations (Assumption 2), the pointwise measurement of y_x^1 given by (45) is never an appropriate output for the feedback stabilization of the PDE cascade (1). Indeed, in view of (53), the open-loop system is unstable if and only if $\lambda_0 = a > 0$. But this mode is never observable for the pointwise measurement (45) because $c_{1,0} = (\phi_{1,0}^1)'(\xi_p) = 0$. Hence, an output feedback strategy based on the pointwise measurement of y_x^1 given by (45) is not sufficient to stabilize the PDE cascade (1).*

4 Conclusion

We have studied the boundary output feedback stabilization problem for a cascade on N heat equations, with a spectral approach, using the key property that the (generalized) eigenlements of the underlying operator form a Riesz basis. Our presentation focused on the two extreme case of either two-by-two distinct eigenvalues or eigenvalues of multiplicity N , but the intermediate cases can be handled similarly. Future work may address other PDE cascades.

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