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# The influence of advection on the propagation phenomena of reaction-diffusion equations with KPP-bistable nonlinearity

Xing Liang\*, Lei Zhang<sup>†</sup> and Mingmin Zhang<sup>‡</sup>

## Abstract

This paper is devoted to propagation phenomena for reaction-diffusion-advection equations in one-dimensional heterogeneous environments, where heterogeneity is reflected by the nonlinearity term - being KPP type on  $(-\infty, -L]$  and being bistable type on  $[L, +\infty)$  for some  $L > 0$ . A comprehensive analysis is presented on the influence of advection and heterogeneous reactions, based on various values of the advection rate  $c$ . Denote by  $c_m$  the minimal wave speed of KPP equation and by  $c_b$  the unique wave speed of bistable equation, respectively. When  $c > -c_m$ , it is shown that propagation can always occur with leftward spreading speed  $c_m + c$  and rightward spreading speed  $\min(\max(c_b - c, 0), c_m - c)$ . Moreover, a logarithmic delay of the level sets in the left direction is discovered. When  $c \leq -c_m$ , propagation phenomena are determined by the initial data and by the sign of  $c_b$ . In particular, when  $c_b > 0$ , the leftward propagation speed is  $c_b + c$  if the initial population is “large enough”; whereas extinction occurs if the initial value is located in the bistable region and is “relatively small”. In addition, the attractiveness of the bistable traveling wave is obtained when the leftward spreading speed is  $c_b + c$  and/or when the rightward spreading speed is  $c_b - c$ .

*Mathematics Subject Classification (2020):* 35B40; 35K57; 35B35; 92D25.

*Key words:* Reaction-diffusion-advection equations; Propagation phenomena; Shifting environment; KPP-bistable.

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## 1 Introduction and main results

We address in this paper the propagation phenomena for reaction-diffusion-advection equations in a heterogeneous framework. The heterogeneous character arises in the equation through continuously varying (in space) reaction term between KPP (for Kolmogorov, Petrovsky and Piskunov) type and bistable type. The type of equations we consider here is:

$$u_t - u_{xx} - cu_x = f(x, u) \quad t > 0, x \in \mathbb{R}, \quad (1.1)$$

in which  $u(t, x)$  denotes the species density at time  $t$  and location  $x$ , the constant  $c \in \mathbb{R}$  is advection rate,  $f(x, u)$  describes population dynamics. From the biological point of view, advection can arise either from the behavior of individuals or from physical transport processes, such as winds, currents in rivers, for which the population may have a tendency to move along or against the gradient of  $u$ . Our paper is intended to understand the influence of the advection on propagation phenomena in the face of a heterogeneous environment of KPP-transition-bistable type.

Throughout this work, we assume that the function  $f(y, s) : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is of class  $C^2(\mathbb{R} \times \mathbb{R}_+)$ , and  $s \mapsto f(y, s)$  is locally Lipschitz continuous uniformly for  $y \in \mathbb{R}$ . Moreover,  $f$  satisfies

$$\begin{cases} \forall y \in \mathbb{R}, f(y, 0) = f(y, 1) = 0, \partial_s f(y, 1) < 0 \text{ and } f(y, s) \leq 0 \text{ for } s \geq 1, \\ \forall s \in \mathbb{R}_+, \exists L > 0, f(y, s) = f_m(s) \text{ for } y \in (-\infty, -L], f(y, s) = f_b(s) \text{ for } y \in [L, +\infty), \\ \forall s \in \mathbb{R}_+, f(y, s) \text{ is decreasing in } y \in [-L, L]. \end{cases} \quad (1.2)$$

Here,  $f_m$  denotes the KPP nonlinearity:

$$f_m(0) = f_m(1) = 0, 0 < f_m(s) \leq f'_m(0)s \text{ in } (0, 1), f'_m(1) < 0, f_m < 0 \text{ in } (1, +\infty), \quad (1.3)$$

while  $f_b$  represents bistable reaction:

$$\begin{aligned} f_b(0) = f_b(\theta) = f_b(1) = 0 \text{ for some } \theta \in (0, 1), \\ f'_b(0) < 0, f'_b(\theta) > 0, f'_b(1) < 0, f_b > 0 \text{ in } (\theta, 1), f_b < 0 \text{ in } (0, \theta) \cup (1, +\infty). \end{aligned} \quad (1.4)$$

We then refer to  $(-\infty, -L]$  as the *KPP region*, while  $[L, +\infty)$  would be regarded as the *bistable region*.

The spatial-temporal dynamics for (1.1) with nontrivial nonnegative continuous and compactly supported initial data shall be investigated in the following sense:

- *extinction*:  $u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$  uniformly in  $x \in \mathbb{R}$ ;
- *rightward (resp. leftward) blocking*:  $u(t, x) \rightarrow 0$  as  $x \rightarrow +\infty$  (resp.  $x \rightarrow -\infty$ ) uniformly in  $t \geq 0$ ;  
Here, we can easily see that extinction is a special case of (leftward and rightward) blocking.

- *propagation*: there exist  $\varepsilon > 0$  and a family of interval  $[a(t), b(t)] \subset S_\varepsilon$  such that  $\lim_{t \rightarrow +\infty} (b(t) - a(t)) = +\infty$ , where  $S_\varepsilon(t) := \{x \in \mathbb{R} : u(t, x) \geq \varepsilon\}$  for any  $t \geq 0$ . Furthermore, *leftward* (resp. *rightward*) *propagation* occurs, if  $\lim_{t \rightarrow +\infty} a(t) = -\infty$  (resp.  $\lim_{t \rightarrow +\infty} b(t) = +\infty$ ); *complete propagation* occurs, if  $\lim_{t \rightarrow +\infty} a(t) = -\infty$  and  $\lim_{t \rightarrow +\infty} b(t) = +\infty$ .

When propagation occurs, we can further consider the possible asymptotic spreading speeds. Specifically, if there exist two constants  $c_r$  and  $c_l$  such that for  $\varepsilon > 0$  small enough,

$$\lim_{t \rightarrow +\infty} \inf_{-(c_l - \varepsilon)t \leq x \leq (c_r - \varepsilon)t} u(t, x) > 0, \quad \lim_{t \rightarrow +\infty} \sup_{x \leq -(c_l + \varepsilon)t} u(t, x) = 0 = \lim_{t \rightarrow +\infty} \sup_{x \geq (c_r + \varepsilon)t} u(t, x),$$

then  $c_r$  and  $c_l$  are called the rightward and leftward spreading speeds, respectively.

Based on the definition of propagation, one can see that  $\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} u(t, x) > 0$ . Combining this and  $\lim_{t \rightarrow +\infty} \sup_{x \leq -(c_l + \varepsilon)t} u(t, x) = 0 = \lim_{t \rightarrow +\infty} \sup_{x \geq (c_r + \varepsilon)t} u(t, x)$ , it yields that  $c_l + c_r \geq 0$ . In the special case where  $c_l + c_r = 0$ , the interval  $[-(c_l - \varepsilon)t, (c_r - \varepsilon)t]$  is empty for any  $\varepsilon > 0$  and  $t > 0$ , then  $\lim_{t \rightarrow +\infty} \inf_{-(c_l - \varepsilon)t \leq x \leq (c_r - \varepsilon)t} u(t, x) = +\infty > 0$  is satisfied automatically. Moreover, we remark that complete propagation occurs when  $c_l > 0$  and  $c_r > 0$ , whereas propagation with rightward blocking takes place when  $c_l > 0$  and  $c_r < 0$ . However, either of two situations may occur when  $c_l \geq 0$  and  $c_r = 0$ .

Before stating our main result, let us review the fundamental results on the classical homogeneous reaction-diffusion equation

$$u_t = u_{xx} + f(u), \quad t > 0, \quad x \in \mathbb{R}, \quad (1.5)$$

where  $f$  is a  $C^1(\mathbb{R})$  function satisfying  $f(0) = f(1) = 0$ . This equation has been extensively studied in the mathematical, physical and biological literature since the pioneering works of Fisher [23] and Kolmogorov, Petrovskii and Piskunov [34].

With the KPP reaction  $f = f_m$  in (1.5), it is shown in [34] that (1.5) admits traveling front solutions  $u(t, x) = \varphi_\nu(x \cdot e - \nu t)$  where  $e = \pm 1$  denotes the direction of propagation and  $\nu$  is the wave speed, with  $\varphi_\nu : \mathbb{R} \rightarrow (0, 1)$  and  $\varphi_\nu(-\infty) = 1$ ,  $\varphi_\nu(+\infty) = 0$ , if and only if  $\nu \geq c_m := 2\sqrt{f'_m(0)}$ . For each  $\nu \geq c_m$ , the wave profile  $\varphi_\nu$  satisfies

$$\varphi_\nu'' + \nu \varphi_\nu' + f(\varphi_\nu) = 0 \text{ in } \mathbb{R}, \quad \varphi_\nu' < 0 \text{ in } \mathbb{R}, \quad \varphi_\nu(-\infty) = 1, \quad \varphi_\nu(+\infty) = 0, \quad (1.6)$$

and has translation invariance. Moreover,  $\varphi_\nu$  has the following asymptotics:

$$\varphi_\nu(s) \underset{s \rightarrow +\infty}{\sim} \begin{cases} Ae^{-\lambda_\nu s} & \text{if } \nu > c_m, \\ A^* s e^{-\lambda_\nu s} & \text{if } \nu = c_m, \end{cases} \quad (1.7)$$

where  $A, A^*$  are positive constants and the decay rate  $\lambda_\nu > 0$  is obtained from the linearized equation  $u_t = u_{xx} + f'_m(0)u$  and is given by  $\lambda_\nu = (\nu - \sqrt{\nu^2 - 4f'_m(0)})/2$ . It was proved in [30, 36, 46] that the front with minimal speed  $c_m$  attracts, in some sense, the solutions of the Cauchy problem (1.5) associated with nonnegative bounded nontrivial compactly supported initial data  $u_0$  in  $\mathbb{R}$ . Furthermore, Aronson and Weinberger [4] proved the *spreading property*, stating that the solution  $u$  to the Cauchy problem (1.5) with a nontrivial nonnegative compactly supported initial datum  $u_0$  in  $\mathbb{R}$  admits an asymptotic spreading speed  $c_m = 2\sqrt{f'_m(0)}$ , in the following sense:

$$\begin{cases} \sup_{|x| \geq \omega t} u(t, x) \rightarrow 0, & \text{if } \omega > c_m \\ \inf_{|x| \leq \omega t} u(t, x) \rightarrow 1 & \text{if } 0 \leq \omega < c_m, \end{cases} \quad \text{as } t \rightarrow +\infty. \quad (1.8)$$

The minimal traveling wave speed  $c_m$  for (1.5) can therefore also be considered as the asymptotic spreading speed for the Cauchy problem associated with (1.5).

In contrast, when the reaction is of bistable type, i.e.  $f = f_b$ , equation (1.5) has a unique (up to translation) traveling front solution  $u(t, x) = \phi(x \cdot e - c_b t)$ , where  $\phi : \mathbb{R} \rightarrow (0, 1)$  and satisfies (1.6),  $e = \pm 1$  is the direction of propagation, and  $c_b \in \mathbb{R}$  is the wave speed (has the sign of  $\int_0^1 f_b(s) ds$ ), depending only on  $f_b$  [4, 22]. It is known [22] that

$$\begin{cases} a_0 e^{-\alpha s} \leq \phi(s) \leq a_1 e^{-\alpha s}, & s \geq 0, \\ b_0 e^{\beta s} \leq 1 - \phi(s) \leq b_1 e^{\beta s}, & s \leq 0, \end{cases} \quad (1.9)$$

where  $a_0, a_1, b_0$  and  $b_1$  are some positive constants,  $\alpha$  and  $\beta$  are given by  $\alpha = (c_b + \sqrt{c_b^2 - 4f_b'(0)})/2 > 0$  and  $\beta = (-c_b + \sqrt{c_b^2 - 4f_b'(1)})/2 > 0$ .

Throughout this paper, we denote by  $\varphi_\nu(x - \nu t)$  the KPP traveling front satisfying (1.6) with  $f = f_m$  and with wave speed  $\nu \geq c_m$ , whereas by  $\phi(x - c_b t)$  the unique bistable traveling front satisfying (1.6) with  $f = f_b$ , with  $\varphi_\nu$  replaced by  $\phi$ , with wave speed  $\nu = c_b$  and with the normalization  $\phi(0) = \theta$ . Let us stress that  $c_m > c_b$ , which is shown in Lemma 2.1. Finally, we always denote by  $u$  the solution to (1.1) with a nontrivial nonnegative continuous and compactly supported initial datum  $u_0$ .

**Main results.** First of all, let us provide some intuition regarding the propagation dynamics of (1.1). Compared with the case of  $c = 0$ , the appearance of the advection now forces the solution  $u$  towards the left direction with an extra speed  $c \in \mathbb{R}$ . Therefore, the large-time dynamics of (1.1) in principle is dominated by the KPP region when  $c \gg 1$ , by the bistable region when  $c \ll -1$ , and by the KPP plus the bistable regions when  $|c|$  is intermediate. One may already have a global picture in mind for each situation, based on the known results in [4, 22, 29]. Although, let us propose the following fundamental questions with respect to the large-time dynamics of (1.1):

- when  $c \gg 1$ , the environment is almost the KPP region for large times, however the bistable region still remains on the right. Then, we wonder whether the bistable region can be negligible? In other words, what is the influence of the bistable region?
- when  $c \ll -1$ , the environment is asymptotically governed by the bistable region, but there is still the KPP region on the left. Similar as above, will the KPP region make any difference?
- when  $|c|$  is intermediate, the environment is composed of KPP-transition-bistable regions. Suppose the bistable region is unfavorable, i.e.  $\int_0^1 f_b(s) ds < 0$ , will the solution be blocked in the right direction?

As a matter of fact, it is not easy to give immediate answers without a serious thinking, in that, very different from homogeneous monostable or bistable setting, the effect of advection here significantly complicates the long-time dynamics of (1.1). Our goal is to give a comprehensive analysis for (1.1) for large times according to different ranges of the advection  $c$ , from which we expect to understand as much as possible the above-mentioned questions.

## Propagation with rightward blocking when $c \in [c_m, +\infty)$

We begin by showing the propagation with rightward blocking result when  $c \geq c_m$ .

**Theorem 1.1** (Propagation with rightward blocking). *Assume that  $c \geq c_m$ . Then,  $u$  propagates to the left with speed  $c_m + c$  and to the right with speed  $c_m - c$ , in the sense that*

$$\begin{cases} \forall \varepsilon > 0, & \lim_{t \rightarrow +\infty} \left( \sup_{x \in (-\infty, -(c_m + c + \varepsilon)t] \cup [(c_m - c + \varepsilon)t, +\infty)} u(t, x) \right) = 0, \\ \forall \varepsilon \in (0, c_m), & \lim_{t \rightarrow +\infty} \left( \sup_{-(c_m + c - \varepsilon)t \leq x \leq (c_m - c - \varepsilon)t} |u(t, x) - 1| \right) = 0. \end{cases} \quad (1.10)$$

## Complete propagation vs. rightward blocking when $c \in (-c_m, c_m)$

When  $c \in (-c_m, c_m)$ , we shall prove that the solution can always spread to the left, whereas rightward propagation or rightward blocking are both possible, depending on the sign of  $c_b - c$ . Let us point out that at this stage there is no further requirements on  $u_0$ .

### • Leftward propagation when $c \in (-c_m, c_m)$

**Theorem 1.2** (Leftward propagation). *Assume that  $c \in (-c_m, c_m)$ . Then,  $u$  propagates to the left with speed  $c_m + c > 0$  in the sense that*

$$\begin{cases} \forall \varepsilon > 0, \quad \lim_{t \rightarrow +\infty} \left( \sup_{x \leq -(c_m + c + \varepsilon)t} u(t, x) \right) = 0, \\ \forall \varepsilon \in (0, c_m + c), \quad \forall \delta > 0, \quad \exists x_1 < -L, \quad \limsup_{t \rightarrow +\infty} \left( \sup_{-(c_m + c - \varepsilon)t \leq x \leq x_1} |u(t, x) - 1| \right) < \delta. \end{cases} \quad (1.11)$$

In particular,  $\sup_{-c_2 t \leq x \leq -c_1 t} |u(t, x) - 1| \rightarrow 0$  as  $t \rightarrow +\infty$  for every  $0 < c_1 \leq c_2 < c_m + c$ .

### • Rightward propagation vs. rightward blocking with speed zero when $c \in (-c_m, c_m)$

We will distinguish our analysis into two cases: either  $c > c_b$  or  $c \leq c_b$ , where the former means  $\max(c_b, -c_m) < c < c_m$ , and for the latter it suffices to consider the situation when  $-c_m < c_b$  and  $c \in (-c_m, c_b]$ , in that the case of  $c \leq c_b \leq -c_m$  is outside the range  $(-c_m, c_m)$ .

**Theorem 1.3** (Rightward blocking with speed zero). *Assume that  $c \in (-c_m, c_m)$  and  $c > c_b$ . Then,  $u$  is blocked in the right direction, that is,*

$$u(t, x) \rightarrow 0 \quad \text{as } x \rightarrow +\infty, \quad \text{uniformly in } t \geq 0. \quad (1.12)$$

Furthermore,  $u$  satisfies

$$u(t, \cdot) - U \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \quad \text{locally uniformly in } x \in \mathbb{R}, \quad (1.13)$$

where  $U$  is the unique positive bounded stationary solution of (1.1) such that  $U(-\infty) = 1$  and  $U(+\infty) = 0$ , given in Proposition 4.1.

**Theorem 1.4** (Rightward propagation). *Assume that  $-c_m < c_b$  and  $c \in (-c_m, c_b]$ . Then,  $u$  propagates completely, namely,*

$$u(t, x) \rightarrow 1 \quad \text{as } t \rightarrow +\infty, \quad \text{locally uniformly in } x \in \mathbb{R}. \quad (1.14)$$

Furthermore,

- (i) if  $c \in (-c_m, c_b)$ , then  $u$  propagates to the right with speed  $c_b - c > 0$ . Moreover, there exist  $X > L$  and  $\xi \in \mathbb{R}$  such that

$$\lim_{t \rightarrow +\infty} \left( \sup_{x \geq X} |u(t, x) - \phi(x - (c_b - c)t + \xi)| \right) = 0,$$

where  $\phi$  is the unique bistable traveling wave profile;

- (ii) if  $c = c_b$ , then  $u$  propagates to the right with speed zero, in the sense that (1.14) holds and  $\sup_{x \geq \nu t} u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$  for every  $\nu > 0$ .

**Remark.** Combining the above three theorems, we can conclude that complete propagation occurs when  $c \in (-c_m, c_b]$  provided that  $-c_m < c_b$  and propagation with rightward blocking takes place when  $c \in (-c_m, c_m)$  and  $c > c_b$ . Moreover, Theorems 1.3–1.4 demonstrate that the rightward propagation phenomena is totally determined by the sign of  $c_b - c$ , provided that  $c \in (-c_m, c_m)$ .

- Theorem 1.3: We notice that  $c_m > c > c_b$ , which means on the one hand the advection pushes the solution to leave the bistable region, but on the other hand the solution grows in the KPP region and invades towards the right direction. Thus a saturation state - blocking in the right direction - is formed, in the sense that the solution eventually converges to a positive steady state  $U$ . This implies that the sign of  $c_b$  (equivalently whether the bistable region is favorable or not) is not important, and it is the effect of advection plus the power of the KPP region that leads to the only possibility - blocking.
- Theorem 1.4: In the particular case  $c = 0 \leq c_b$ , this result coincides, among other things, with that of [29, Theorem 2.13], which means that propagation is due to the favorableness of the bistable region. However, we find that here it is not the case in general. Again, the sign of  $c_b$  does not matter. Propagation may occur even when the bistable region is unfavorable, no matter how small the initial datum  $u_0$  is. This reflects the power of both the advection and the KPP region.

### Conditional complete propagation vs. extinction when $c \in (-\infty, -c_m]$

Eventually, let us deal with the case that  $c \leq -c_m$ . Here, the bistable region dominates the environment, therefore whether the bistable region is favorable or not, i.e. sign of  $c_b$ , really matters now. It turns out that propagation phenomena are determined by the sign of  $c_b$  and/or by the size and the location of  $u_0$ .

#### • Conditional complete propagation

In the case where  $c \leq -c_m$  and  $c_b > 0$ , the bistable region can be either favorable or unfavorable depending on the initial data. Our following result says that for initially “large enough” localized population set in bistable region, the species can persist and spread. Furthermore, we show that  $u$  eventually approaches two diverging bistable traveling waves with speeds  $c_b + c$  and  $c_b - c$ , respectively.

**Theorem 1.5** (Conditional complete propagation). *Assume that  $c \leq -c_m$  and  $c_b > 0$ . Then for any  $\eta > 0$ , there is  $L^* > 0$  such that the following holds: for any nonnegative continuous and compactly supported initial datum<sup>1</sup> satisfying  $u_0 \geq \theta + \eta$  on an interval of size  $L^*$ , the solution  $u$  of (1.1) with initial datum  $u_0$  propagates to the right with speed  $c_b - c$  and to the left with speed  $c_b + c$ . Moreover, there are  $X > L$  and  $z_i \in \mathbb{R}$  ( $i = 1, 2$ ) such that*

$$\lim_{t \rightarrow +\infty} \left( \sup_{x \geq X - ct} |u(t, x) - \phi(x - (c_b - c)t + z_1)| \right) = 0. \quad (1.15)$$

and

$$\lim_{t \rightarrow +\infty} \left( \sup_{x \leq X - ct} |u(t, x) - \phi(-x - (c_b + c)t + z_2)| \right) = 0, \quad (1.16)$$

where  $\phi$  is the bistable traveling wave profile.

Let us comment that the rightward convergence result (1.15) is indeed in the same spirit as [22, 29], where the bistable region can be analyzed somehow in a standard way. In contrast, the study of the leftward propagation phenomena needs much effort in that the KPP region still exists on the left side of the environment, as already pointed out at the very beginning. The challenge of our analysis is therefore a simultaneous consideration of both the KPP region and the bistable region. Our conclusion (1.16) implies that, although the KPP region cannot be ignored in the proof, essentially it does not affect the leftward front position at all.

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<sup>1</sup>Here we do not restrict the location of the initial datum  $u_0$ . In fact, this theorem is still true, when such  $u_0$  is fully set in the bistable region.

• **Conditional extinction**

In the end, we provide some sufficient conditions for extinction under the assumption that  $c \leq -c_m$ .

**Theorem 1.6** (Conditional extinction). *Assume that  $c \leq -c_m$ . Then,  $u$  will extinct, namely,  $u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$  uniformly in  $x \in \mathbb{R}$ , provided one of the following conditions is satisfied:*

(i)  $c_b < 0$ ;

(ii)  $\text{spt}(u_0)$  is included in the bistable region and  $\|u_0\|_{L^\infty(\mathbb{R})} < \theta$ .

Now that Theorem 1.6 is parallel to the standard result for the classical bistable equation, one may wonder what is the trouble of the advection and of the KPP region here? Roughly speaking, in the course of building upper barriers, it is necessary to distinguish  $c > c_b$ ,  $c = c_b$  and  $c < c_b$ , under the assumption (i); while when assuming (ii), the KPP region cannot be overlooked so that part of the supersolution has to work in the KPP region, which reflects the concern of the advection and of the KPP region. Moreover, a discussion on Theorem 1.6 (ii) is that we believe it can be relaxed but somehow it is not easy to find an optimal one. A heuristic explanation is the following: suppose that  $\text{spt}(u_0)$  is large enough and fully located in the KPP region, i.e., far to the left, and  $\|u_0\|_{L^\infty(\mathbb{R})} < \theta$ , the solution may have a chance to persist with  $u(T, \cdot) \geq \theta + \eta$  on an interval of size  $L^*$  for some large  $T > 0$ , which will lead to propagation provided that  $c_b > 0$ , due to Theorem 1.5.

**Sharp estimate of the level sets in the left direction when  $c \in (-c_m, +\infty)$**

Given  $\varrho \in (0, 1)$ , define the level set  $E_\varrho^-(t) = \inf\{x \in \mathbb{R} | u(t, x) = \varrho\}$  for any  $t > 0$ .

**Theorem 1.7.** *Assume that  $c > -c_m$ . Then,  $u$  propagates to the left with speed  $c_m + c$ , thanks to Theorems 1.1–1.2, and for every  $\varrho \in (0, 1)$ ,*

$$E_\varrho^-(t) = -(c_m + c)t + \frac{3}{2\lambda^*} \ln t + O_{t \rightarrow +\infty}(1)$$

with  $\lambda^* = c_m/2$ .

This theorem demonstrates that whenever the KPP region plays a role as (part of) the habitat, the leftward propagation will be very similar to the homogeneous KPP case in the sense that the same logarithmic correction appears regarding the asymptotic position of the level sets in the left direction, which means that the effect of the bistable region is rather weak. Then, one can ask whether such logarithmic delay of the level sets remains true in terms of the rightward propagation with speed  $c_m - c$  when  $c \geq c_m$ ? As a matter of fact, assume that  $c \geq c_m$ , by defining  $E_\varrho^+(t) = \sup\{x \in \mathbb{R} | u(t, x) = \varrho\}$  for any  $t > 0$  and for any  $\varrho \in (0, 1)$ , one readily observes that  $E_\varrho^+(t) \leq (c_m - c)t - 3/(2\lambda^*) \ln t + C$  for some  $C \in \mathbb{R}$ , since equation (8.1) with  $f_m(v)$  instead of  $f(y - ct, v)$  produces such an upper bound. This means that there exists *at least* a logarithmic delay for the level sets in the rightward propagation. However, how to catch a lower bound of  $E_\varrho^+(t)$  is far from clear, which also strongly rests on the effect of the bistable region. This is beyond the scope of this work and is left as an open question.

We are now in a position to comment on the aforementioned questions in order:

- when  $c \geq c_m$ , the environment is almost the KPP region for large times, the solution behaves the KPP property globally, see Theorem 1.1. Moreover, the famous Bramson's result in  $O(1)$  precision remains true when considering the leftward front position (actually this is true when  $c > -c_m$ ), see Theorem 1.7, which means that the bistable region, if makes any difference for the leftward position of the level sets when  $c > -c_m$ , is at most of  $O(1)$  order. Nevertheless, we are still unclear about the influence of the bistable region on the precision position in the rightward direction when  $c \geq c_m$ .

- when  $c \leq -c_m$ , the bistable region dominates the environment, and the propagation phenomena are similar to the standard results for classical bistable equation, see Theorems 1.5-1.6. Unlike the situation mentioned above, we are fortunate to be able to simultaneously address both the KPP and bistable regions when analyzing convergence in the left direction. At this time, the KPP region, although exists on the left and causes challenging difficulties in the course of our analysis, does not affect global dynamics of the solution.
- when  $c \in (-c_m, c_m)$ , the environment is composed of KPP-transition-bistable regions. The rightward propagation phenomena - either blocking or propagation - are actually determined by the sign of  $c_b - c$ , see Theorems 1.3-1.4. Even though the bistable region is assumed unfavorable, the KPP region plus the effect of the advection can drive the solution to propagate towards the right direction. For more careful discussion in this situation, we refer to the remark right after Theorem 1.4.

In addition, it is also noteworthy that our results, in the specific case  $c = 0$ , align with those in [29] which indeed provides foundational insight for our work.

Compared with existing literature for analyzing the effect of heterogeneous advection on nonlinear spreading and propagation phenomena for reaction-diffusion equations with a single dynamical mechanism (either KPP or bistable or combustion type) of the form

$$u_t - \nabla \cdot (A(x)\nabla u) + q(x) \cdot \nabla u = f(x, u), \quad (1.17)$$

with diffusion matrix  $A(x)$  and drift  $q(x)$  [5, 6, 10–12, 19, 26–28, 33, 41–44], here we consider the simple constant advection mode for the reaction-diffusion-advection equation (1.1) in a heterogeneous environment of KPP-transition-bistable type, and investigate all the possibilities of propagation phenomena by exhausting different values of the advection. To the best of our knowledge, our work is the first to address the possible effects of the advection on the propagation phenomena and the speeds of propagation in such a heterogeneous dynamical setting. More generally, our problem can be easily generalized to a high dimensional situation:

$$u_t - \Delta u + \alpha u_x = f(x, u), \quad \text{in } \Omega,$$

where  $\Omega \subset \mathbb{R}^N$  is a straight cylinder and  $q(x) = \alpha e_1$  represents a shear flow, which can be studied in a similar way but one needs to pay attention to the effect of the geometry of the domains.

**Organization of the paper.** The main body of the paper is devoted to the proofs of theorems in this section.

## 2 Preliminary results

**Lemma 2.1.** *There holds  $c_m > c_b$ .*

*Proof.* Let  $w$  and  $z$  be respectively the solution to the Cauchy problem (1.5) in  $\mathbb{R}_+ \times \mathbb{R}$  with  $f$  replaced by  $f_m$  and  $f_b$ , associated with the same initial data. First of all, one readily infers from the comparison principle that  $w(t, x) > z(t, x)$  for  $t > 0$  and  $x \in \mathbb{R}$ , which implies  $c_m \geq c_b$ .

Assume towards contradiction that  $c_m = c_b$ . Let us focus on the region  $x > 0$ . It follows from [22, Theorem 3.2], there exists  $\xi \in \mathbb{R}$  such that  $\sup_{x>0} |z(t, x) - \phi(x - c_b t + \xi)| \rightarrow 0$  as  $t \rightarrow +\infty$ . This implies that the level sets of  $z$  is asymptotically  $c_b t + O_{t \rightarrow +\infty}(1)$ . On the other hand, it is known from [30] that the level sets of  $w$  behave asymptotically as  $c_m t - 3/(2\lambda^*) \ln t + O_{t \rightarrow +\infty}(1)$ , which move apparently slower than the front propagation of  $z$ . This is a contradiction. We then conclude that  $c_m > c_b$ .  $\square$

**Lemma 2.2.** *The solution  $u$  to (1.1) satisfies*

$$\sup_{x \in \mathbb{R}} u(t, x) \leq 1 \quad \text{as } t \rightarrow +\infty.$$

*Namely, for any  $\varepsilon > 0$ , there exists  $T > 0$  such that  $\sup_{x \in \mathbb{R}} u(t, x) \leq 1 + \varepsilon$  for all  $t \geq T$ .*

**Remark.** Based upon Lemma 2.2, we observe that the solution of (1.1) will eventually be bounded from above by 1 for large times albeit with a small nuance, no matter how large the  $L^\infty$  norm of  $u_0$  is. Therefore, we assume with no loss of generality that the initial condition  $u_0$  is nontrivial such that, for any  $\varepsilon > 0$ , there holds  $0 \leq u_0(x) < 1 + \varepsilon$  for  $x \in \mathbb{R}$  throughout this paper.

*Proof.* Let  $\xi$  be the solution to the ODE  $\xi'(t) = f_m(\xi(t))$  for  $t \geq 0$  with initial condition  $\xi(0) = \max(1, \|u_0\|_{L^\infty(\mathbb{R})})$ , it follows that  $\xi(t) \searrow 1$  as  $t \rightarrow +\infty$ . By the comparison principle, one has  $0 < u(t, x) \leq \xi(t)$  for all  $(t, x) \in (0, +\infty) \times \mathbb{R}$ , which implies that  $u(t, x) \leq 1$  as  $t \rightarrow +\infty$  uniformly for  $x \in \mathbb{R}$ . This completes the proof.  $\square$

The following lemma provides the Gaussian upper bounds for the solution  $u$  to (1.1).

**Lemma 2.3.** *Let  $L_1 > 0, L_2 > 0$ , and let  $u$  be the solution to the Cauchy problem  $u_t = u_{xx} + cu_x + f(u)$ , with a nonnegative continuous and compactly supported initial datum  $u_0$  satisfying  $\text{spt}(u_0) \subset [-L_1, L_2]$ , where the  $C^1(\mathbb{R}_+)$  function  $f$  is assumed to satisfy  $f(s) \leq Ks$  for all  $s \geq 0$  with some constant  $K > 0$ . Then there holds, for all  $t > 0$ ,*

$$u(t, x) \leq Me^{Kt} e^{-\frac{(x+ct+L_1)^2}{4t}} \quad \text{for all } x \leq -ct - L_1, \quad \text{and} \quad u(t, x) \leq Me^{Kt} e^{-\frac{(x+ct-L_2)^2}{4t}} \quad \text{for all } x \geq L_2 - ct,$$

with  $M := \max(1, \|u_0\|_{L^\infty(\mathbb{R})})$ .

*Proof.* Set  $z = x + ct$  and  $v(t, z) = u(t, z - ct)$ , then  $v(t, z)$  satisfies

$$v_t = v_{zz} + f(v) \quad t > 0, \quad z \in \mathbb{R},$$

with initial condition  $u_0$  satisfying  $\text{spt}(u_0) \subset [-L_1, L_2]$ . By the comparison principle, for any  $z \in \mathbb{R}$  and  $t > 0$ ,

$$v(t, z) \leq \frac{e^{Kt}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4t}} u_0(y) dy = \frac{e^{Kt}}{\sqrt{4\pi t}} \int_{-L_1}^{L_2} e^{-\frac{(z-y)^2}{4t}} u_0(y) dy.$$

This gives that for  $t > 0$ ,

$$v(t, z) \leq Me^{Kt} e^{-\frac{(z+L_1)^2}{4t}} \quad \text{for all } z \leq -L_1, \quad \text{and} \quad v(t, z) \leq Me^{Kt} e^{-\frac{(z-L_2)^2}{4t}} \quad \text{for all } z \geq L_2,$$

Turning back to the function  $u$ , the conclusion immediately follows.  $\square$

**Lemma 2.4.** *Assume that  $c \in (-c_m, c_m)$ . Then  $u(t, x)$  is semi-persistent, that is, for every  $\bar{x} \in \mathbb{R}$ ,*

$$\inf_{x \leq \bar{x}} \left( \liminf_{t \rightarrow +\infty} u(t, x) \right) > 0.$$

*Moreover, (1.1) admits positive stationary solutions, and any positive stationary solution  $p$  of (1.1) satisfies  $p(-\infty) = 1$ .*

*Proof.* For any  $0 < \varepsilon < \frac{1}{4}(c_m^2 - c^2)$ , one can choose  $R > 0$  large enough such that

$$\frac{\pi}{2R} < \sqrt{f'_m(0) - \varepsilon - \frac{c^2}{4}}. \quad (2.1)$$

Then, define  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  as

$$\Psi(x) = \begin{cases} e^{-\frac{\varepsilon}{2}x} \cos\left(\frac{\pi}{2R}x\right) & \text{in } [-R, R], \\ 0 & \text{elsewhere.} \end{cases} \quad (2.2)$$

Then there exists  $\eta_0 > 0$  such that  $-(\eta\Psi)'' - c(\eta\Psi)' \leq f_m(\eta\Psi)$  in  $(-R, R)$  for all  $0 < \eta \leq \eta_0$ . Fix now any  $x_0 \leq -L - R$  and pick  $\eta \in (0, \eta_0]$  such that  $\eta\Psi(\cdot - x_0) < u(1, \cdot)$  in  $\mathbb{R}$ .

Let  $z$  be the solution to (1.1) with initial datum  $z(0, \cdot) = \eta\Psi(\cdot - x_0)$  in  $\mathbb{R}$ . The strong maximum principle applied in  $(0, +\infty) \times [-L - 2R, -L]$  yields that  $z(t, x) > z(0, x) = \eta\Psi(x - x_0)$  for  $t > 0$  and  $x \in [-L - 2R, -L]$ . Therefore,  $z(t + h, \cdot) > z(t, \cdot)$  in  $\mathbb{R}$  for every  $h > 0$  and  $t \geq 0$ . Namely,  $z$  is increasing with respect to  $t$ . Moreover, the comparison principle implies that  $0 < z(t, x) < u(t + 1, x)$  for all  $t > 0$  and  $x \in \mathbb{R}$ . It then follows from parabolic estimates that  $z(t, \cdot)$  converges as  $t \rightarrow +\infty$ , locally uniformly in  $\mathbb{R}$ , to a positive bounded stationary solution  $p$  of (1.1). Clearly,  $p(x) \geq z(t, x) > 0$  for  $t > 0$  and  $x \in \mathbb{R}$ . Furthermore, together with Lemma 2.2, it follows that

$$0 < p(x) \leq \liminf_{t \rightarrow +\infty} u(t, x) \leq \limsup_{t \rightarrow +\infty} u(t, x) \leq 1, \quad \text{locally uniformly for } x \in \mathbb{R}. \quad (2.3)$$

Since  $p$  is continuous and positive in  $\mathbb{R}$ , one gets from (2.3) that, for any given  $\bar{x} > x_0$ ,

$$\liminf_{t \rightarrow +\infty} u(t, x) \geq \min_{[x_0, \bar{x}]} p > 0 \quad \text{for all } x \in [x_0, \bar{x}]. \quad (2.4)$$

In view of (2.4), to prove the semi-persistence result, it suffices to show

$$\inf_{x \leq x_0} \liminf_{t \rightarrow +\infty} u(t, x) > 0. \quad (2.5)$$

Notice also that  $p(x) > z(0, x)$  for  $x \in \mathbb{R}$ . By continuity of  $p$  and  $z(0, \cdot)$ , there is  $\hat{\kappa} > 1$  such that  $p > \eta\Psi(\cdot - \kappa x_0)$  in  $[\kappa x_0 - R, \kappa x_0 + R]$  for all  $\kappa \in [1, \hat{\kappa}]$ . Define

$$\kappa^* := \sup \{ \kappa \geq 1 : p > \eta\Psi(\cdot - \tilde{\kappa}x_0) \text{ in } [\tilde{\kappa}x_0 - R, \tilde{\kappa}x_0 + R] \text{ for all } \tilde{\kappa} \in [1, \kappa] \}.$$

It follows that  $\kappa^* \geq \hat{\kappa} > 1$ . We only need to prove  $\kappa^* = +\infty$ . Assuming by contradiction that  $\kappa^* < +\infty$ , we see from the definition of  $\kappa^*$  that  $p \geq \eta\Psi(\cdot - \kappa^*x_0)$  in  $[\kappa^*x_0 - R, \kappa^*x_0 + R]$  and there is  $x^* \in [\kappa^*x_0 - R, \kappa^*x_0 + R]$  such that  $p(x^*) = \eta\Psi(x^* - \kappa^*x_0)$ . Since  $p > 0$  in  $\mathbb{R}$  and  $\Psi(\cdot - \kappa^*x_0) = 0$  at  $\kappa^*x_0 \pm R$ , one has  $x^* \in (\kappa^*x_0 - R, \kappa^*x_0 + R)$ . Then the strong elliptic maximum principle implies that  $p \equiv \eta\Psi(\cdot - \kappa^*x_0)$  in  $(\kappa^*x_0 - R, \kappa^*x_0 + R)$  and then in  $[\kappa^*x_0 - R, \kappa^*x_0 + R]$  by continuity, which is impossible. Thus,  $\kappa^* = +\infty$  and  $p > \eta\Psi(\cdot - \kappa x_0)$  in  $[\kappa x_0 - R, \kappa x_0 + R]$  for all  $\kappa \geq 1$ . This implies particularly that  $p(x) > \eta\Psi(0) = \eta$  for  $x \leq x_0$ . It follows from (2.3) that  $\liminf_{t \rightarrow +\infty} u(t, x) > \eta$  for all  $x \leq x_0$ , and hence, (2.5) holds. We then obtain the semi-persistence result, as well as the existence of positive stationary solutions to (1.1).

Assume that  $p$  is a positive stationary solution of (1.1) in  $\mathbb{R}$ , let us show that  $p(-\infty) = 1$ . Fix  $x_1 < -L - R$ . Since the function  $\Psi$  given in (2.2) is compactly supported in  $\mathbb{R}$ , we can choose  $\eta \in (0, \eta_0]$  such that  $\eta\Psi(\cdot - x_1) < p$  in  $\mathbb{R}$ . By repeating the arguments to obtain the semi-persistence result, we have  $\eta\Psi(\cdot - \tilde{x}) < p$  in  $\mathbb{R}$  for any  $\tilde{x} < -L - R$ , which further implies that  $\liminf_{x \rightarrow -\infty} p(x) > 0$ . We now claim that

$$\liminf_{x \rightarrow -\infty} p(x) > 0 \quad \implies \quad p(-\infty) = 1.$$

Indeed, let us consider an arbitrary sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}$  diverging to  $-\infty$  as  $n \rightarrow +\infty$  and define  $p_n := p(\cdot + x_n)$  in  $\mathbb{R}$  for each  $n \in \mathbb{N}$ . By standard elliptic estimates, the sequence  $(p_n)_{n \in \mathbb{N}}$  converges as  $n \rightarrow +\infty$ , up to extraction of some subsequence, in  $C_{loc}^2(\mathbb{R})$  to a bounded function  $p_\infty$  which solves  $p_\infty'' + cp_\infty' + f_m(p_\infty) = 0$  in  $\mathbb{R}$ . In view of  $\liminf_{x \rightarrow -\infty} p(x) > 0$ , we have  $\inf_{\mathbb{R}} p_\infty > 0$ . Notice that

$f_m > 0$  in  $(0, 1)$  and  $f_m < 0$  in  $(1, +\infty)$ , it then follows that the equilibrium 1 attracts all solutions of  $\xi'(t) = f_m(\xi(t))$  with any positive initial data  $\xi(0)$ . Therefore, a comparison argument implies  $p_\infty \equiv 1$  in  $\mathbb{R}$ . That is,  $p_n \rightarrow 1$  as  $n \rightarrow +\infty$  in  $C_{loc}^2(\mathbb{R})$ . Since the sequence  $(x_n)_{n \in \mathbb{N}}$  was arbitrarily chosen, we conclude that  $p(x) \rightarrow 1$  and  $p'(x) \rightarrow 0$  as  $x \rightarrow -\infty$ . Our claim is achieved. This completes the proof of Lemma 2.4.  $\square$

**Lemma 2.5.** *Assume that  $c \leq -c_m$ . Then, for any  $x_0 \in \mathbb{R}$ ,*

$$\lim_{t \rightarrow +\infty} u(t, x) = 0, \quad \text{uniformly in } x \leq -(c_m + c)t + x_0. \quad (2.6)$$

*In particular,  $\lim_{t \rightarrow +\infty} u(t, x) = 0$  uniformly for  $x \leq x_0$ .*

*Proof.* Let  $w$  be the solution of  $w_t = w_{xx} + cw_x + f'_m(0)w$  for  $t > 0$  and  $x \in \mathbb{R}$ , associated with initial condition  $w(0, \cdot) = u_0$  in  $\mathbb{R}$ . Then,

$$\begin{aligned} w(t, x) &= \frac{e^{f'_m(0)t}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{[x+(c_m+c)t-c_mt-y]^2}{4t}} u_0(y) dy \\ &= \frac{e^{\frac{c_m^2 t}{4}}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{[x+(c_m+c)t-y]^2 - 2[x+(c_m+c)t-y](c_mt) + (c_mt)^2}{4t}} u_0(y) dy \\ &\leq \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{2[x+(c_m+c)t-y](c_mt)}{4t}} u_0(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} e^{\frac{[x+(c_m+c)t]c_m}{2}} \int_{\mathbb{R}} e^{-\frac{c_my}{2}} u_0(y) dy \end{aligned}$$

from which we deduce that  $\sup_{x \leq -(c_m+c)t+x_0} w(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$ , for any  $x_0 \in \mathbb{R}$ . This, together with the comparison principle, implies (2.6).  $\square$

### 3 Propagation in the KPP region when $c \geq c_m$ and when $c \in (-c_m, c_m)$ : Proofs of Theorems 1.1–1.2

We now give the proofs of Theorem 1.1 – describing the complete propagation when  $c \geq c_m$  and of Theorem 1.2 – concerning the spreading property in the left direction when  $c \in (-c_m, c_m)$ .

*Proofs of Theorems 1.1–1.2.* Assume that  $c > -c_m$ . The conclusion can be easily reached by comparison arguments.

Let  $g(x, s) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$  be a  $C^1$  function. Moreover, we also require that  $g$  satisfies  $g(x, 0) = g(x, 1) = 0$  for each  $x \in \mathbb{R}$ , and  $g(x, s) \leq f(x, s)$  for  $(x, s) \in \mathbb{R} \times \mathbb{R}_+$ ,  $g(x, s) = f_m(s)$  for  $x \in (-\infty, -L]$ ,  $g(x, s) = -rs$  for  $y \in [L, +\infty)$ ,  $g(x, s)$  is nonincreasing in  $x \in \mathbb{R}$  for each  $s \in \mathbb{R}_+$ . Let  $z$  be the solution to problem  $z_t = z_{xx} + cz_x + g(x, z)$  starting with initial value  $z_0 = u_0$  in  $\mathbb{R}$ . Notice that  $z$  behaves as a lower barrier and it is known from, for instance, Theorem 1.3 (ii) and (iii) of [24] that  $z$  satisfies (1.10)–(1.11) when  $c \geq c_m$  and when  $c \in (-c_m, c_m)$ , respectively.

On the other hand, define by  $w$  the solution of  $w_t = w_{xx} + cw_x + f_m(w)$  in  $\mathbb{R}_+ \times \mathbb{R}$  with initial datum  $w_0 = u_0$  in  $\mathbb{R}$ , it follows from [3] that  $w$  satisfies property (1.10). By the comparison principle, we have  $z(t, x) \leq u(t, x) \leq w(t, x)$  for  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ . As a consequence, taking into account the properties satisfied by  $z$  and  $w$ , the conclusion in Theorems 1.1–1.2 immediately follows.  $\square$

### 4 Blocking in bistable region when $c \in (-c_m, c_m)$ and $c > c_b$ : Proof of Theorem 1.3

As preliminaries, we first show the existence and uniqueness of positive bounded stationary solutions for the above two cases respectively, which will play crucial roles in the course of our investigation.

**Proposition 4.1.** *Assume that  $c \in (-c_m, c_m)$  and  $c > c_b$ . Then problem (1.1) admits a unique positive bounded stationary solution  $U$  such that  $U(-\infty) = 1$  and  $U(+\infty) = 0$ . Such  $U$  satisfies  $0 < U < 1$  and  $U' < 0$  in  $\mathbb{R}$ .*

Based on Proposition 4.1, we point out that if  $u_0$  satisfies  $0 \leq u_0 \leq U$  in  $\mathbb{R}$ , then the comparison principle immediately implies that the associated solution  $u$  of the Cauchy problem (1.1) will satisfy  $0 \leq u(t, x) \leq U(x)$  for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , hence it is blocked in the bistable region.

*Proof.* We divide our proof into four steps.

*Step 1. Existence.* For  $\varepsilon \in (0, (1 - \theta)/2)$ , let  $f_{b,\varepsilon}$  be a  $C^1(\mathbb{R})$  function such that

$$\begin{cases} f_{b,\varepsilon}(0) = f_{b,\varepsilon}(\theta) = f_{b,\varepsilon}(1 + \varepsilon) = 0, & f'_{b,\varepsilon}(0) < 0, & f'_{b,\varepsilon}(1 + \varepsilon) < 0, \\ f_{b,\varepsilon} > 0 \text{ in } (-\infty, 0) \cup (\theta, 1 + \varepsilon), & f_{b,\varepsilon} < 0 \text{ in } (0, \theta) \cup (1 + \varepsilon, +\infty). \end{cases} \quad (4.1)$$

Moreover, we assume that  $f_{b,\varepsilon} \geq f_b$  in  $\mathbb{R}$ ,  $f_{b,\varepsilon} = f_b$  in  $(0, 1 - \varepsilon)$ , and  $f_{b,\varepsilon}$  is decreasing in  $[1 - \varepsilon, 1 + \varepsilon]$ . For each  $\varepsilon \in (0, (1 - \theta)/2)$ , let  $\bar{\phi}_\varepsilon$  be the unique traveling front profile of  $u_t = u_{xx} + f_{b,\varepsilon}(u)$  such that

$$\bar{\phi}_\varepsilon'' + c_{b,\varepsilon}\bar{\phi}_\varepsilon' + f_{b,\varepsilon}(\bar{\phi}_\varepsilon) = 0 \text{ in } \mathbb{R}, \quad \bar{\phi}_\varepsilon' < 0 \text{ in } \mathbb{R}, \quad \bar{\phi}_\varepsilon(L) = 1, \quad \bar{\phi}_\varepsilon(-\infty) = 1 + \varepsilon, \quad \bar{\phi}_\varepsilon(+\infty) = 0, \quad (4.2)$$

with speed  $c_{b,\varepsilon}$ . It is standard to see that  $\bar{\phi}_\varepsilon \rightarrow \bar{\phi}$  in  $C_{loc}^2(\mathbb{R})$  and  $c_{b,\varepsilon} \rightarrow c_b$  as  $\varepsilon \rightarrow 0$ . Since  $c_b < c$ , we can fix  $\varepsilon \in (0, (1 - \theta)/2)$  small enough such that  $c_{b,\varepsilon}$  is close enough to  $c_b$  and  $c_{b,\varepsilon} < c$ .

Let  $\bar{u}$  be the solution of (1.1) in  $\mathbb{R}_+ \times \mathbb{R}$  with initial condition

$$\bar{u}_0(x) = \min(\bar{\phi}_\varepsilon(x), 1) \quad \text{for } x \in \mathbb{R}. \quad (4.3)$$

We observe that  $\bar{u}_0 \equiv 1$  in  $(-\infty, L]$  and  $\bar{u}_0 = \bar{\phi}_\varepsilon$  in  $(L, +\infty)$ . As long as  $\bar{u}_0 = \bar{\phi}_\varepsilon < 1$ , since  $\bar{\phi}_\varepsilon'' + c\bar{\phi}_\varepsilon' + f_b(\bar{\phi}_\varepsilon) = \bar{\phi}_\varepsilon'' + c_{b,\varepsilon}\bar{\phi}_\varepsilon' + f_{b,\varepsilon}(\bar{\phi}_\varepsilon) + (c - c_{b,\varepsilon})\bar{\phi}_\varepsilon' + f_b(\bar{\phi}_\varepsilon) - f_{b,\varepsilon}(\bar{\phi}_\varepsilon) < 0$  in  $\mathbb{R}$ , it follows that  $\bar{u}_0$  is a stationary supersolution of (1.1) in  $\mathbb{R}$ .

Let  $\underline{u}$  be the solution of (1.1) with initial condition  $\underline{u}_0 = \eta\Psi(\cdot - x_0)$  given in (2.2) for  $x_0 \leq -L - R$  with  $R > 0$  given in (2.1). We can choose  $\eta > 0$  sufficiently small such that  $\underline{u}_0 < \bar{u}_0$  in  $\mathbb{R}$ . The comparison principle implies that

$$0 < \underline{u}(t, x) < \bar{u}(t, x) < \bar{u}_0(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \quad (4.4)$$

Moreover,  $\underline{u}$  is increasing with respect to  $t$  and  $\bar{u}$  is decreasing with respect to  $t$  in  $\mathbb{R}_+ \times \mathbb{R}$ . From standard parabolic estimates,  $\underline{u}(t, \cdot)$  and  $\bar{u}(t, \cdot)$  converge as  $t \rightarrow +\infty$ , locally uniformly in  $\mathbb{R}$ , to classical positive stationary solutions  $p$  and  $q$  of (1.1), respectively. The comparison principle implies that

$$0 < \underline{u}(t, x) < p(x) \leq q(x) < \bar{u}_0, \quad \text{for } t > 0, \quad x \in \mathbb{R}. \quad (4.5)$$

Moreover, since  $\max(-c_m, c_b) < c < c_m$ , we infer from Lemma 2.4 that  $p(-\infty) = q(-\infty) = 1$ . On the other hand, since  $\lim_{x \rightarrow +\infty} \bar{u}_0(x) = 0$ , we deduce from (4.5) that  $p(+\infty) = q(+\infty) = 0$ . Therefore, we have shown the existence of positive bounded stationary solution  $U$  of (1.1) such that  $U(-\infty) = 1$  and  $U(+\infty) = 0$ .

*Step 2.*  $0 < U < 1$  in  $\mathbb{R}$ . Let now  $U$  be a positive stationary solution of (8.1) such that  $U(-\infty) = 1$  and  $U(+\infty) = 0$ . Indeed, (4.5) implies that  $U > 0$  in  $\mathbb{R}$ . It suffices to show that  $U < 1$  in  $\mathbb{R}$ . To do so, we first suppose that there is  $\hat{x} \in \mathbb{R}$  such that  $U(\hat{x}) = \max_{\mathbb{R}} U > 1$ . Hence,  $U'(\hat{x}) = 0$  and  $U''(\hat{x}) \leq 0$ . We then observe that  $0 = U''(\hat{x}) + cU'(\hat{x}) + f(\hat{x}, U(\hat{x})) < 0$ , which is impossible. Therefore,  $U \leq 1$  in  $\mathbb{R}$ . If there is  $\bar{x} \in \mathbb{R}$  such that  $U(\bar{x}) = 1$ , then it is necessarily a local maximum of  $U$ , whence  $U'(\bar{x}) = 0$ . The Cauchy-Lipschitz theorem implies that  $U \equiv 1$  in  $[\bar{x}, +\infty)$ . This contradicts  $U(+\infty) = 0$ . Consequently,  $0 < U < 1$  in  $\mathbb{R}$ .

*Step 3. Uniqueness.* Suppose that  $U$  and  $W$  are two distinct positive bounded stationary solutions of (1.1) connecting 1 and 0. We notice that  $U$  and  $W$  have the same decay rate at  $\pm\infty$ , that is,

$$1 - U(x) = O(e^{\eta x}) = 1 - W(x) \text{ as } x \rightarrow -\infty, \quad U(x) = O(e^{-\zeta x}) = W(x) \text{ as } x \rightarrow +\infty \quad (4.6)$$

with  $\eta = (-c + \sqrt{c^2 - 4f'_m(1)})/2 > 0$  and  $\zeta = (c + \sqrt{c^2 - 4f'_b(0)})/2 > 0$ . Therefore, one can choose  $\kappa_0 > 0$  sufficiently large such that  $W > U(\cdot + \kappa_0)$  in  $\mathbb{R}$ , thanks to (4.6). Define

$$\kappa^* = \inf\{\kappa \in \mathbb{R} : W > U(\cdot + \kappa) \text{ in } \mathbb{R}\}.$$

We observe that  $-\infty < \kappa^* \leq \kappa_0$ . Set  $w := W - U(\cdot + \kappa^*)$ . By continuity, it follows that the function  $w$  is nonnegative in  $\mathbb{R}$  and vanishes at some point  $x_0 \in \mathbb{R}$ . Moreover,  $w$  satisfies

$$-w''(x) - cw'(x) = f(x, W(x)) - f(x + \kappa^*, U(x + \kappa^*)) \geq f(x, W(x)) - f(x, U(x + \kappa^*)) = \gamma(x)w(x) \quad x \in \mathbb{R},$$

for some bounded function  $\gamma$  in  $\mathbb{R}$ . Since  $w(x_0) = 0$ , the strong maximum principle implies that  $w \equiv 0$  in  $\mathbb{R}$ . That is,  $W(x) = U(x + \kappa^*)$  for  $x \in \mathbb{R}$ . Suppose that  $\kappa^* > 0$ , then we derive that both  $U$  and  $U(\cdot + \kappa^*)$  are stationary solutions of (1.1), which is impossible since

$$0 = U''(x + \kappa^*) + cU'(x + \kappa^*) + f(x, U(x + \kappa^*)) \not\geq U''(x + \kappa^*) + cU'(x + \kappa^*) + f(x + \kappa^*, U(x + \kappa^*)) = 0 \text{ for } x \in \mathbb{R},$$

contradiction, thanks to the hypothesis (1.2) on  $f$ . Similarly, we also exclude the case that  $\kappa^* < 0$ . Therefore,  $\kappa^* = 0$ , which simply means that  $W \equiv U$  in  $\mathbb{R}$ . Consequently, such stationary solution of (1.1) connecting 1 and 0 is unique.

*Step 4.  $U' < 0$  in  $\mathbb{R}$ .* Assume by contradiction that the unique positive stationary solution of (8.1) such that  $U(-\infty) = 1$  and  $U(+\infty) = 0$  is not strictly decreasing in  $\mathbb{R}$ , there then exists  $a > 0$  such that  $U \geq U(\cdot + a)$  in  $\mathbb{R}$  and  $U(x^*) = U(x^* + a)$  for some  $x^* \in \mathbb{R}$ . In particular, the functions  $U$  and  $U(\cdot + a)$  are strictly separated and ordered for all  $|x|$  large enough. Define  $z := U - U(\cdot + a)$  in  $\mathbb{R}$ , it follows that  $z \geq \neq 0$  in  $\mathbb{R}$  and  $z(x^*) = 0$ . Moreover,  $z$  satisfies

$$-z''(x) - cz'(x) = f(x, U(x)) - f(x + a, U(x + a)) \geq f(x, U(x)) - f(x, U(x + a)) = \mu(x)z(x), \quad x \in \mathbb{R},$$

for some bounded function  $\mu$  in  $\mathbb{R}$ . Since  $z(x^*) = 0$ , the strong maximum principle implies that  $z \equiv 0$  in  $\mathbb{R}$ . With the same reasoning as in Step 3, we arrive at a contradiction. Consequently,  $U$  is strictly decreasing in  $\mathbb{R}$ . The proof of Proposition 4.1 is complete.  $\square$

We now carry out the proof of Theorem 1.3. The proof, among other things, is based on a comparison with a traveling front with zero speed as a barrier.

*Proof of Theorem 1.3.* The strategy of the proof consists in constructing a supersolution which blocks the solution  $u(t, x)$  for all times  $t \geq 0$  as  $x \rightarrow +\infty$ .

*Step 1. Blocking.* Set  $M = \max(\|u_0\|_{L^\infty(\mathbb{R})}, 1) + 1$ . For any  $\varepsilon \in (0, (1 - \theta)/2)$ , let  $f_{b,\varepsilon} \in C^1(\mathbb{R})$  be defined as in the beginning of the proof of Proposition 4.1, with  $M$  this time instead of  $1 + \varepsilon$ . There is then a decreasing front profile  $\bar{\phi}_\varepsilon$  solving (4.2). It is seen that  $\bar{\phi}_\varepsilon \rightarrow \bar{\phi}$  in  $C^2_{loc}(\mathbb{R})$  and  $c_{b,\varepsilon} \rightarrow c_b$  as  $\varepsilon \rightarrow 0$ . We then fix  $\varepsilon \in (0, (1 - \theta)/2)$  small enough such that  $c_{b,\varepsilon}$  is close enough to  $c_b$  and  $c_{b,\varepsilon} < c$ . One can choose  $A > 0$  sufficiently large such that  $\max(u_0(x), 1) \leq \bar{\phi}_\varepsilon(-A)$  for all  $x \leq L$  and  $u_0(x) \leq \bar{\phi}_\varepsilon(x - A - L)$  for all  $x \geq L$ .

Define  $\bar{u}$  by<sup>2</sup>

$$\bar{u}(x) = \begin{cases} \bar{\phi}_\varepsilon(-A) & \text{if } x < L, \\ \bar{\phi}_\varepsilon(x - A - L) & \text{if } x \geq L. \end{cases} \quad (4.7)$$

<sup>2</sup>We remark here that the construction of  $\bar{u}$  relies only on the assumption that  $c > c_b$ .

For  $x \leq L$ , due to  $\bar{u}(x) = \bar{\phi}_\varepsilon(-A) \geq 1$ , one has that  $f(x, \bar{\phi}_\varepsilon(-A)) \leq 0$ . Moreover, since  $\bar{u}(x) = \bar{\phi}_\varepsilon(x - A - L)$  for  $x \geq L$ , it is observed that

$$\bar{u}_t - \bar{u}_{xx} - c\bar{u}_x - f(x, \bar{u}) = -\bar{\phi}_\varepsilon'' - c\bar{\phi}_\varepsilon' - f_b(\bar{\phi}_\varepsilon) \geq -\bar{\phi}_\varepsilon'' - c_{b,\varepsilon}\bar{\phi}_\varepsilon' - \bar{f}_{b,\varepsilon}(\bar{\phi}_\varepsilon) = 0, \quad x \geq L, \quad (4.8)$$

by noticing that  $\bar{\phi}_\varepsilon' < 0$ ,  $c > c_{b,\varepsilon}$  and  $f_b \leq \bar{f}_{b,\varepsilon}$  in  $\mathbb{R}$ . Therefore,  $\bar{u}$  is a stationary supersolution of (1.1) in  $\mathbb{R}$ , with  $u_0 \leq \bar{u}$  in  $\mathbb{R}$ . The comparison principle implies that  $u(t, x) \leq \bar{u}(x)$  for  $t \geq 0$  and  $x \in \mathbb{R}$ . Consequently,  $u$  will be blocked. That is,

$$u(t, x) \rightarrow 0 \quad \text{as } x \rightarrow +\infty, \quad \text{uniformly in } t \geq 0.$$

*Step 2. Convergence to the stationary solution  $U$ .* We denote by  $u_1$  the solution to the Cauchy problem (1.1) with initial datum  $u_1(0, \cdot) = \bar{u}$  in  $\mathbb{R}$  with  $\bar{u}$  given in (4.7) such that  $u_1(0, \cdot) > u_0$  in  $\mathbb{R}$ , and by  $u_2$  the solution to the Cauchy problem (1.1) with initial condition  $u_2(0, \cdot) = \eta\Psi(\cdot - x_0) < u(1, \cdot)$  in  $\mathbb{R}$  for  $\eta > 0$  small enough and for some  $x_0 \leq -L - R$ , where  $R > 0$  and  $\Psi$  are given as in (2.1)–(2.2). The comparison principle implies that

$$0 < u_2(t, x) < u(t + 1, x) < u_1(t + 1, x) < u_1(0, x) = \bar{u}(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \quad (4.9)$$

Moreover,  $u_2$  is increasing with respect to  $t$  and  $u_1$  is decreasing with respect to  $t$  in  $(0, +\infty) \times \mathbb{R}$ . From standard parabolic estimates,  $u_2(t, \cdot)$  and  $u_1(t, \cdot)$  converge as  $t \rightarrow +\infty$ , locally uniformly in  $\mathbb{R}$ , to classical stationary solutions  $p$  and  $q$  of (1.1), respectively. Moreover, there holds

$$0 < p(x) \leq \liminf_{t \rightarrow +\infty} u(t, x) \leq \limsup_{t \rightarrow +\infty} u(t, x) \leq q(x) \leq \bar{u}(x), \quad (4.10)$$

locally uniformly for  $x \in \mathbb{R}$ .

In view of  $\max(-c_m, c_b) < c < c_m$ , by Lemma 2.4, we eventually arrive at  $p(-\infty) = q(-\infty) = 1$ . On the other hand, since  $\underline{u}$  is nonnegative and  $\bar{\phi}_\varepsilon(+\infty) = 0$ , one also derives that  $p(+\infty) = q(+\infty) = 0$ . Furthermore, it follows from Proposition 4.1 that  $p = q = U$  in  $\mathbb{R}$ . Therefore, the local uniform convergence of  $u(t, \cdot)$  to  $U$  as  $t \rightarrow +\infty$  follows immediately from (4.10). The proof of Theorem 1.3 is therefore complete.  $\square$

## 5 Propagation in bistable region when $-c_m < c_b$ and $c \in (-c_m, c_b]$ : Proof of Theorem 1.4

In this section, we study the rightward spreading speed and the attractiveness of the bistable traveling wave when  $-c_m < c_b$ , motivated by [22, 29].

**Proposition 5.1.** *Assume that  $-c_m < c_b$  and  $c \in (-c_m, c_b]$ , then  $V \equiv 1$  is the unique positive stationary solution of (1.1).*

*Proof.* Suppose that  $V$  is a positive bounded stationary solution of (1.1). Since  $c_b < c_m$ , we have  $c \in (-c_m, c_m)$ . It then follows from Lemma 2.4 that  $V(-\infty) = 1$ . We prove now that  $V \leq 1$  in  $\mathbb{R}$ . In fact, let  $\xi(t)$  be the solution of the ODE  $\xi'(t) = f_m(\xi(t))$  for  $t > 0$  associated with  $\xi(0) = \|V\|_{L^\infty(\mathbb{R})} + 1$ . We notice that  $\xi(t) \searrow 1$  as  $t \rightarrow +\infty$ . By comparison principle, there holds  $V(x) < \xi(t)$  for  $t \geq 0$  and  $x \in \mathbb{R}$ , whence  $V(x) \leq 1$  for  $x \in \mathbb{R}$ . To complete the proof, it remains to prove that  $V \geq 1$  in  $\mathbb{R}$ . In the sequel, we deal with the case that  $c < c_b$  and  $c = c_b$ , respectively.

**Step 1.** *The case that  $c < c_b$ .* To show that  $V \geq 1$  in  $\mathbb{R}$ , we build a subsolution as a lower barrier which moves to the right with speed  $\delta > 0$ . To be specific, for  $\varepsilon \in (0, \min(\theta, 1 - \theta)/2)$ , let  $f_{b,\varepsilon}$  be a  $C^1(\mathbb{R})$  function such that

$$\begin{cases} f_{b,\varepsilon}(-\varepsilon) = f_{b,\varepsilon}(\theta) = f_{b,\varepsilon}(1 - \varepsilon) = 0, & f'_{b,\varepsilon}(-\varepsilon) < 0, & f'_{b,\varepsilon}(1 - \varepsilon) < 0, \\ f_{b,\varepsilon} = f_b \text{ in } (\varepsilon, 1 - 2\varepsilon), & f_{b,\varepsilon} > 0 \text{ in } (-\infty, -\varepsilon) \cup (\theta, 1 - \varepsilon), & f_{b,\varepsilon} < 0 \text{ in } (-\varepsilon, \theta) \cup (1 - \varepsilon, +\infty). \end{cases}$$

We further assume that  $f_{b,\varepsilon} \leq f_b$  in  $\mathbb{R}$  and that  $f_{b,\varepsilon}$  is decreasing in  $[-\varepsilon, \varepsilon]$  and in  $[1 - 2\varepsilon, 1]$ . For each  $\varepsilon \in (0, \min(\theta, 1 - \theta)/2)$ , let  $\tilde{\phi}_\varepsilon$  be the unique traveling front profile of  $u_t = u_{xx} + f_{b,\varepsilon}(u)$  such that

$$\tilde{\phi}_\varepsilon'' + c_{b,\varepsilon}\tilde{\phi}_\varepsilon' + f_{b,\varepsilon}(\tilde{\phi}_\varepsilon) = 0 \text{ in } \mathbb{R}, \quad \tilde{\phi}_\varepsilon' < 0 \text{ in } \mathbb{R}, \quad \tilde{\phi}_\varepsilon(0) = \theta, \quad \tilde{\phi}_\varepsilon(-\infty) = 1 - \varepsilon, \quad \tilde{\phi}_\varepsilon(+\infty) = -\varepsilon, \quad (5.1)$$

with speed  $c_{b,\varepsilon}$ . It is standard to see that  $\tilde{\phi}_\varepsilon \rightarrow \phi$  in  $C_{loc}^2(\mathbb{R})$  and  $c_{b,\varepsilon} \rightarrow c_b$  as  $\varepsilon \rightarrow 0$ . We then fix  $\varepsilon \in (0, \min(\theta, 1 - \theta)/2)$  small enough such that  $c_{b,\varepsilon} > c$ .

Since  $V(-\infty) = 1$  and  $V > 0$  in  $\mathbb{R}$ , one can choose  $x_0 \in \mathbb{R}$  such that  $\tilde{\phi}_\varepsilon(\cdot - x_0) < V$  in  $\mathbb{R}$ . Fix now  $0 < \delta < c_{b,\varepsilon} - c$ , and define

$$\underline{u}(t, x) = \max(\tilde{\phi}_\varepsilon(x - \delta t - x_0), 0), \quad t \geq 0, \quad x \in \mathbb{R}. \quad (5.2)$$

Let us now check that  $\underline{u}$  is a subsolution to (1.1) in  $\mathbb{R}_+ \times \mathbb{R}$ . To do so, it suffices to check the situation of  $\underline{u}(t, x) = \tilde{\phi}_\varepsilon(x - \delta t - x_0) > 0$ , where we have

$$\underline{u}_t - \underline{u}_{xx} - c\underline{u}_x - f(x, \underline{u}) \leq -\tilde{\phi}_\varepsilon'' - (c + \delta)\tilde{\phi}_\varepsilon' - f_b(\tilde{\phi}_\varepsilon) < -\tilde{\phi}_\varepsilon'' - c_{b,\varepsilon}\tilde{\phi}_\varepsilon' - f_{b,\varepsilon}(\tilde{\phi}_\varepsilon) = 0,$$

due to the fact that  $f(x, s)$  is nonincreasing with respect to  $x$  for each  $s > 0$ , and that  $c + \delta < c_{b,\varepsilon}$  and  $f_{b,\varepsilon} \leq f_b$ . Therefore,  $\underline{u}$  is a subsolution to (1.1) for  $t \geq 0$  and  $x \in \mathbb{R}$ . By comparison principle, it follows that

$$\underline{u}(t, x) < V(x), \quad t \geq 0, \quad x \in \mathbb{R}. \quad (5.3)$$

However, we observe that such a subsolution  $\underline{u}$  moves to the right with speed  $\delta > 0$ , which implies that  $\sup_{x < \delta t} \underline{u} \rightarrow 1 - \varepsilon$  as  $t \rightarrow +\infty$ . Since  $\varepsilon > 0$  is arbitrarily chosen and small enough, together with (5.3), we exclude the possibility that  $V < 1$  somewhere in  $\mathbb{R}$ . Thus,  $V \geq 1$  in  $\mathbb{R}$ . Consequently, we have achieved  $V \equiv 1$  in the case of  $c \in (-c_m, c_b)$ .

**Step 2.** *The case that  $c = c_b$ .* Assume by contradiction that  $V(x_0) < 1$  for some point  $x_0 \in \mathbb{R}$ . Let  $\phi$  be the bistable traveling wave profile such that

$$\phi'' + c_b\phi' + f_b(\phi) = 0 \text{ in } \mathbb{R}, \quad \phi(0) = \theta, \quad \phi(-\infty) = 1, \quad \phi(+\infty) = 0.$$

By observing that  $\phi(x)$  satisfies

$$-\phi'' - c_b\phi' - f(x, \phi) \leq -\phi'' - c_b\phi' - f_b(\phi) = 0 \text{ in } \mathbb{R},$$

it follows that  $\phi$  is a stationary subsolution of (1.1) in  $\mathbb{R}$ . Moreover, we observe that  $V(x) \geq O(e^{-\alpha x}) = \phi(x)$  as  $x \rightarrow +\infty$ , with  $\alpha$  given in (1.9), whereas as  $x \rightarrow -\infty$  one has  $1 - \phi(x) = O(e^{\beta x})$  and  $1 - V(x) = O(e^{\gamma x})$  with  $\beta$  given in (1.9) and  $\gamma = (-c_b + \sqrt{c_b^2 - 4f'_m(1)})/2 > \beta$ , due to  $f'_m(1) \leq f'_b(1) < 0$ . It follows that  $1 - V(x)$  decays faster than  $1 - \phi(x)$  as  $x \rightarrow -\infty$ . There then exists  $\eta_0 \in \mathbb{R}$  such that  $V > \phi(\cdot + \eta_0)$  in  $\mathbb{R}$ . Now define

$$\eta^* = \inf\{\eta \in \mathbb{R} : V > \phi(\cdot + \eta) \text{ in } \mathbb{R}\}.$$

One has that  $-\infty < \eta^* \leq \eta_0$  by noticing that  $V(x_0) < \phi(-\infty) = 1$  in  $\mathbb{R}$ , as assumed. By continuity, the function  $w := V - \phi(\cdot + \eta^*)$  is nonnegative, nontrivial and vanishes somewhere. Namely, there is  $x_0 \in \mathbb{R}$  such that  $w(x_0) = 0$  and  $w$  satisfies

$$-w''(x) - c_b w'(x) = f(x, V) - f_b(\phi(x + \eta^*)) \geq f_b(V) - f_b(\phi(x + \eta^*)) = r(x)w(x), \quad x \in \mathbb{R},$$

for some bounded function  $r$  in  $\mathbb{R}$ . The strong maximum principle implies that  $w \equiv 0$  in  $\mathbb{R}$ , i.e.  $V \equiv \phi(\cdot + \eta^*)$  in  $\mathbb{R}$ . This is impossible, contradicting the asymptotic behaviors of  $V$  and  $\phi$  as  $x \rightarrow -\infty$ . We have proven that  $V \geq 1$  in  $\mathbb{R}$ , whence  $V \equiv 1$  also holds true in the case of  $-c_m < c = c_b$ . The proof of Proposition 5.1 is complete.  $\square$

*Proof of Theorem 1.4.* Proposition 5.1 gives that  $V \equiv 1$  is the unique positive bounded stationary solution of (1.1). The comparison principle then implies that  $0 < u(t, x) \leq M := \max(1, \|u_0\|_{L^\infty(\mathbb{R})})$  for all  $t > 0$  and  $x \in \mathbb{R}$ .

Let  $z$  be the solution to the Cauchy problem (1.1) with initial condition  $z(0, \cdot) = \eta\Psi(\cdot - x_0) < u(1, \cdot)$  in  $\mathbb{R}$  for  $\eta > 0$  small enough and for any arbitrary  $x_0 \leq -L - R$ , where  $R > 0$  and  $\Psi$  are given as in (2.1)–(2.2), thanks to the condition that  $c \in (-c_m, c_b] \subset (-c_m, c_m)$ ; while let  $w$  denote the solution to (1.1) with initial condition  $w(0, \cdot) = M$  in  $\mathbb{R}$ . The comparison principle implies that  $0 < z(t, x) < u(t+1, x) < w(t+1, x) \leq M$  for all  $t > 0$  and  $x \in \mathbb{R}$ . Moreover,  $z$  is increasing with respect to  $t$  and  $w$  is nonincreasing with respect to  $t$  in  $\mathbb{R}_+ \times \mathbb{R}$ . From parabolic estimates,  $z(t, \cdot)$  and  $w(t, \cdot)$  converge as  $t \rightarrow +\infty$ , locally uniformly in  $\mathbb{R}$ , to positive classical stationary solutions  $p$  and  $q$  of (1.1), respectively. Moreover, there holds

$$0 < p \leq \liminf_{t \rightarrow +\infty} u(t, \cdot) \leq \limsup_{t \rightarrow +\infty} u(t, \cdot) \leq q \leq M, \quad (5.4)$$

locally uniformly in  $\mathbb{R}$ . Based on Proposition 5.1, it is immediate to see that  $p \equiv 1 = q$  in  $\mathbb{R}$ . Consequently, it follows that  $u(t, x) \rightarrow 1$  as  $t \rightarrow +\infty$ , locally uniformly for  $x \in \mathbb{R}$ . This proves (1.14).

Next, we shall prove Theorem 1.4 (i), which relies on three preliminary lemmas.

**Lemma 5.2.** *Assume that  $c < c_b$ . The following holds true for some  $\mu > 0$  and  $\delta > 0$ :*

(i) *there exist  $X_1 > L$ ,  $T_1 > 0$ ,  $z_1 \in \mathbb{R}$  such that*

$$u(t, x) \leq \phi(x - (c_b - c)(t - T_1) + z_1) + \delta e^{-\delta(t - T_1)} + \delta e^{-\mu(x - X_1)} \quad \text{for all } t \geq T_1 \text{ and } x \geq X_1, \quad (5.5)$$

(ii) *assume further that  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ , there exist  $X_2 > L$ ,  $T_2 > 0$ ,  $z_2 \in \mathbb{R}$  such that*

$$u(t, x) \geq \phi(x - (c_b - c)(t - T_2) + z_2) - \delta e^{-\delta(t - T_2)} - \delta e^{-\mu(x - X_2)} \quad \text{for all } t \geq T_2 \text{ and } x \geq X_2, \quad (5.6)$$

where  $\phi$  is the unique bistable traveling front profile solving (1.6) with  $f_b$  instead of  $f$ , with speed  $\nu = c_b$  and with  $\phi(0) = \theta$ .

*Proof.* We first introduce some parameters. Choose  $\mu > 0$  such that

$$0 < \mu < \min \left( \frac{\max_{[-2\delta, 1+2\delta]} |f'_b|}{c_m}, \frac{1}{2} \left( c + \sqrt{c^2 + 2 \min(|f'_b(0)|, |f'_b(1)|)} \right) \right). \quad (5.7)$$

Then we take  $\delta > 0$  such that

$$\begin{cases} 0 < \delta < \min \left( \mu(c_b - c), \frac{1}{2}, \frac{|f'_b(0)|}{2}, \frac{|f'_b(1)|}{2} \right), \\ f'_b \leq \frac{f'_b(0)}{2} \text{ in } [-2\delta, 3\delta], \quad f'_b \leq \frac{f'_b(1)}{2} \text{ in } [1 - 3\delta, 1 + 2\delta]. \end{cases} \quad (5.8)$$

Let  $C > 0$  be such that

$$\phi \geq 1 - \delta/2 \text{ in } (-\infty, -C] \quad \text{and} \quad \phi \leq \delta \text{ in } [C, +\infty). \quad (5.9)$$

Since  $\phi'$  is negative and continuous in  $\mathbb{R}$ , there is  $\kappa > 0$  such that

$$\phi' \leq -\kappa < 0 \text{ in } [-C, C]. \quad (5.10)$$

Finally, pick  $\omega > 0$  so large that

$$\kappa\omega \geq 2\delta + \max_{[-2\delta, 1+2\delta]} |f'_b|, \quad (5.11)$$

and  $B > \omega$  such that

$$\left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu B} < \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu(B-\omega)} \leq \delta. \quad (5.12)$$

*Step 1: proof of (5.5).* First of all, it follows from Lemma 2.2 that there are  $X_1 > L$  and  $T_1 > 0$  such that

$$u(t, x) \leq 1 + \frac{\delta}{2} \quad \text{for all } t \geq T_1 \text{ and } x \geq X_1. \quad (5.13)$$

Moreover, since  $u(t, x)$  has a Gaussian upper bound at each fixed  $t > 0$  for all  $|x|$  large enough by Lemma 2.3, whereas  $\phi(s)$  decays exponentially to 0 as  $s \rightarrow +\infty$  by (1.9), there is  $A \geq B$  such that

$$u(T_1, x) \leq \phi(x - X_1 - A - C) + \delta \quad \text{for all } x \geq X_1. \quad (5.14)$$

For  $t \geq T_1$  and  $x \geq X_1$ , let us define

$$\bar{u}(t, x) = \phi(\bar{\xi}(t, x)) + \delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)},$$

where

$$\bar{\xi}(t, x) = x - X_1 - (c_b - c)(t - T_1) + \omega e^{-\delta(t-T_1)} - \omega - A - C.$$

Let us check that  $\bar{u}(t, x)$  is a supersolution to  $u_t = u_{xx} + cu_x + f_b(u)$  for  $t \geq T_1$  and  $x \geq X_1$ . At time  $T_1$ , one has  $\bar{u}(T_1, x) \geq \phi(x - X_1 - A - C) + \delta \geq u(T_1, x)$  for all  $x \geq X_1$ , by (5.14). Moreover, since  $\bar{\xi}(t, X_1) \leq -A - C < -C$  for  $t \geq T_1$ , one gets that  $\bar{u}(t, X_1) \geq 1 - \delta/2 + \delta e^{-\delta(t-T_1)} + \delta \geq 1 + \delta/2 \geq u(t, X_1)$  by (5.9) and (5.13). Thus, it remains to check that  $\mathcal{N}\bar{u}(t, x) := \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for all  $t \geq T_1$  and  $x \geq X_1$ . After a straightforward computation, one derives

$$\mathcal{N}\bar{u}(t, x) = f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) - \phi'(\bar{\xi}(t, x))\omega\delta e^{-\delta(t-T_1)} - \delta^2 e^{-\delta(t-T_1)} - (\mu^2 - c\mu)\delta e^{-\mu(x-X_1)}.$$

We distinguish three cases:

- if  $\bar{\xi}(t, x) \leq -C$ , one has  $1 - \delta/2 \leq \phi(\bar{\xi}(t, x)) < 1$  by (5.9), hence  $1 + 2\delta > \bar{u}(t, x) \geq 1 - \delta/2$ ; it follows from (5.8) that  $f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) \geq -(f'_b(1)/2)(\delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)})$  and it then can be deduced from (5.7)–(5.8) as well as the negativity of  $\phi'$  and  $f'_b(1)$  that

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) &\geq -\frac{f'_b(1)}{2} \left( \delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)} \right) - \delta^2 e^{-\delta(t-T_1)} - (\mu^2 - c\mu)\delta e^{-\mu(x-X_1)} \\ &= \left( -\frac{f'_b(1)}{2} - \delta \right) \delta e^{-\delta(t-T_1)} + \left( -\frac{f'_b(1)}{2} - \mu^2 + c\mu \right) \delta e^{-\mu(x-X_1)} > 0; \end{aligned}$$

- if  $\bar{\xi}(t, x) \geq C$ , one derives  $0 < \phi(\bar{\xi}(t, x)) \leq \delta$  by (5.9), and then  $0 < \bar{u}(t, x) \leq 3\delta$ ; it follows from (5.8) that  $f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) \geq -(f'_b(0)/2)(\delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)})$ ; by virtue of (5.7)–(5.8) and the negativity of  $\phi'$  and  $f'_b(0)$ , there holds

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) &\geq -\frac{f'_b(0)}{2} \left( \delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)} \right) - \delta^2 e^{-\delta(t-T_1)} - (\mu^2 - c\mu)\delta e^{-\mu(x-X_1)} \\ &= \left( -\frac{f'_b(0)}{2} - \delta \right) \delta e^{-\delta(t-T_1)} + \left( -\frac{f'_b(0)}{2} - \mu^2 + c\mu \right) \delta e^{-\mu(x-X_1)} > 0; \end{aligned}$$

- if  $-C \leq \bar{\xi}(t, x) \leq C$ , it turns out that  $x - X_1 \geq (c_b - c)(t - T_1) - \omega e^{-\delta(t-T_1)} + \omega + A \geq (c_b - c)(t - T_1) + B$ , whence  $e^{-\mu(x-X_1)} \leq e^{-\mu((c_b - c)(t - T_1) + B)}$ . By (5.8) and (5.10)–(5.12), one infers that

$$\mathcal{N}\bar{u}(t, x) \geq -\max_{[-2\delta, 1+2\delta]} |f'_b| \left( \delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)} \right) + \kappa\omega\delta e^{-\delta(t-T_1)} - \delta^2 e^{-\delta(t-T_1)} - (\mu^2 - c\mu)\delta e^{-\mu(x-X_1)}$$

$$\begin{aligned}
&\geq \left( \kappa\omega - \delta - \max_{[-2\delta, 1+2\delta]} |f'_b| \right) \delta e^{-\delta(t-T_1)} - \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \delta e^{-\mu((c_b-c)(t-T_1)+B)} \\
&\geq \left( \kappa\omega - 2\delta - \max_{[-2\delta, 1+2\delta]} |f'_b| \right) \delta e^{-\delta(t-T_1)} \geq 0.
\end{aligned}$$

As a consequence, we have proven that  $\mathcal{N}\bar{u}(t, x) := \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for all  $t \geq T_1$  and  $x \geq X_1$ . The maximum principle implies that

$$u(t, x) \leq \bar{u}(t, x) = \phi(x - X_1 - (c_b - c)(t - T_1) + \omega e^{-\delta(t-T_1)} - \omega - A - C) + \delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)}$$

for all  $t \geq T_1$  and  $x \geq X_1$ , whence (5.5) is achieved by taking  $T_1 = X_1$  and  $z_1 = -X_1 - \omega - A - C$ , since  $\phi$  is decreasing.

*Step 2: proof of (5.6).* Assume that  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ . Since  $u(t, \cdot) \rightarrow 1$  as  $t \rightarrow +\infty$  locally uniformly in  $\mathbb{R}$  by (1.14), one can choose  $T_2 > 0$  large enough and  $X_2 > L$  such that

$$u(t, x) \geq 1 - \delta \quad \text{for all } t \geq T_2 \text{ and for all } x \in [X_2, X_2 + B + 2C]. \quad (5.15)$$

For  $t \geq T_2$  and  $x \geq X_2$ , we set

$$\underline{u}(t, x) = \phi(\underline{\xi}(t, x)) - \delta e^{-\delta(t-T_2)} - \delta e^{-\mu(x-X_2)},$$

in which

$$\underline{\xi}(t, x) = x - X_2 - (c_b - c)(t - T_2) - \omega e^{-\delta(t-T_2)} + \omega - B - C.$$

We shall check that  $\underline{u}(t, x)$  is a subsolution to  $u_t = u_{xx} + cu_x + f_b(u)$  for all  $t \geq T_2$  and  $x \geq X_2$ . At time  $t = T_2$ , one has  $\underline{u}(T_2, x) \leq 1 - \delta - \delta e^{-\mu(x-X_2)} \leq 1 - \delta \leq u(T_2, x)$  for  $X_2 \leq x \leq X_2 + B + 2C$  due to (5.15). For  $x \geq X_2 + B + 2C$ , one infers from  $\underline{\xi}(T_2, x) \geq X_2 + B + 2C - X_2 - B - C = C$  and (5.9) that  $\phi(\underline{\xi}(T_2, x)) \leq \delta$ , hence  $\underline{u}(T_2, x) \leq \delta - \delta - \delta e^{-\mu(x-X_2)} < 0 < u(T_2, x)$ . In conclusion,  $\underline{u}(T_2, x) \leq u(T_2, x)$  for all  $x \geq X_2$ . At  $x = X_2$ , we have  $\underline{u}(t, X_2) \leq 1 - \delta e^{-\delta(t-T_2)} - \delta < u(t, X_2)$  for all  $t \geq T_2$ , owing to (5.15). It thus suffices to check that  $\mathcal{N}\underline{u}(t, x) := \underline{u}_t(t, x) - \underline{u}_{xx}(t, x) - c\underline{u}_x(t, x) - f_b(\underline{u}(t, x)) \leq 0$  for all  $t \geq T_2$  and  $x \geq X_2$ . By a straightforward computation, one has

$$\mathcal{N}\underline{u}(t, x) = f_b(\phi(\underline{\xi}(t, x))) - f_b(\underline{u}(t, x)) + \phi'(\underline{\xi}(t, x))\omega\delta e^{-\delta(t-T_2)} + \delta^2 e^{-\delta(t-T_2)} + (\mu^2 - c\mu)\delta e^{-\mu(x-X_2)}$$

By analogy to Step 1, we consider three cases:

- if  $\underline{\xi}(t, x) \leq -C$ , then  $1 - \delta/2 \leq \phi(\underline{\xi}(t, x)) < 1$  by (5.9) and thus  $1 > \underline{u}(t, x) \geq 1 - 3\delta$ ; thanks to (5.8), one has  $f_b(\phi(\underline{\xi}(t, x))) - f_b(\underline{u}(t, x)) \leq (f'_b(1)/2)(\delta e^{-\delta(t-T_2)} + \delta e^{-\mu(x-X_2)})$ ; therefore, by using (5.7)–(5.8) as well as the negativity of  $\phi'$  and  $f'_b(1)$ , it comes that

$$\begin{aligned}
\mathcal{N}\underline{u}(t, x) &< \frac{f'_b(1)}{2} \left( \delta e^{-\delta(t-T_2)} + \delta e^{-\mu(x-X_2)} \right) + \delta^2 e^{-\delta(t-T_2)} + (\mu^2 - c\mu)\delta e^{-\mu(x-X_2)} \\
&= \left( \frac{f'_b(1)}{2} + \delta \right) \delta e^{-\delta(t-T_2)} + \left( \frac{f'_b(1)}{2} + \mu^2 - c\mu \right) \delta e^{-\mu(x-X_2)} < 0;
\end{aligned}$$

- if  $\underline{\xi}(t, x) \geq C$ , then  $0 < \phi(\underline{\xi}(t, x)) \leq \delta$  by (5.9) and thus  $-2\delta < \underline{u}(t, x) \leq \delta$ ; it follows from (5.8) that  $f_b(\phi(\underline{\xi}(t, x))) - f_b(\underline{u}(t, x)) \leq (f'_b(0)/2)(\delta e^{-\delta(t-T_2)} + \delta e^{-\mu(x-X_2)})$ ; therefore, owing to (5.7)–(5.8) as well as the negativity of  $\phi'$  and  $f'_b(0)$ , one infers that

$$\begin{aligned}
\mathcal{N}\underline{u}(t, x) &< \frac{f'_b(0)}{2} \left( \delta e^{-\delta(t-T_2)} + \delta e^{-\mu(x-X_2)} \right) + \delta^2 e^{-\delta(t-T_2)} + (\mu^2 - c\mu)\delta e^{-\mu(x-X_2)} \\
&= \left( \frac{f'_b(0)}{2} + \delta \right) \delta e^{-\delta(t-T_2)} + \left( \frac{f'_b(0)}{2} + \mu^2 - c\mu \right) \delta e^{-\mu(x-X_2)} < 0;
\end{aligned}$$

- if  $-C \leq \xi(t, x) \leq C$ , one has  $x - X_2 \geq (c_b - c)(t - T_2) + \omega e^{-\delta(t - T_2)} - \omega + B \geq (c_b - c)(t - T_2) - \omega + B$ , whence  $e^{-\mu(x - X_2)} \leq e^{-\mu((c_b - c)(t - T_2) + B - \omega)}$ ; by (5.8) and (5.10)–(5.12), one deduces that

$$\begin{aligned} \mathcal{N}\underline{u}(t, x) &\leq \max_{[-2\delta, 1+2\delta]} |f'_b| \left( \delta e^{-\delta(t - T_2)} + \delta e^{-\mu(x - X_2)} \right) - \kappa\omega\delta e^{-\delta(t - T_2)} + \delta^2 e^{-\delta(t - T_2)} + (\mu^2 - c\mu)\delta e^{-\mu(x - X_2)} \\ &\leq \left( \max_{[-2\delta, 1+2\delta]} |f'_b| - \kappa\omega + \delta \right) \delta e^{-\delta(t - T_2)} + \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \delta e^{-\mu((c_b - c)(t - T_2) + B - \omega)} \\ &\leq \left( \max_{[-2\delta, 1+2\delta]} |f'_b| - \kappa\omega + 2\delta \right) \delta e^{-\delta(t - T_2)} \leq 0. \end{aligned}$$

Consequently, one has  $\mathcal{N}\underline{u}(t, x) := \underline{u}_t(t, x) - \underline{u}_{xx}(t, x) - c\underline{u}_x(t, x) - f_b(\underline{u}(t, x)) \leq 0$  for all  $t \geq T_2$  and  $x \geq X_2$ . The maximum principle implies that

$$u(t, x) \geq \underline{u}(t, x) = \phi(x - X_2 - (c_b - c)(t - T_2) - \omega e^{-\delta(t - T_2)} + \omega - B - C) - \delta e^{-\delta(t - T_2)} - \delta e^{-\mu(x - X_2)}$$

for all  $t \geq T_2$  and  $x \geq X_2$ . Therefore, (5.6) is proved by taking  $z_2 = -X_2 + \omega - B - C$ , since  $\phi$  is decreasing. The proof of Lemma 5.2 is thereby complete.  $\square$

More general than Lemma 5.2, we have:

**Lemma 5.3.** *Assume that  $c < c_b$ . Then for any  $\varepsilon > 0$ ,*

(i) *there exist  $X_{1,\varepsilon} > L$ ,  $T_{1,\varepsilon} > 0$  and  $z_{1,\varepsilon} \in \mathbb{R}$  such that*

$$u(t, x) \leq \phi(x - (c_b - c)(t - T_{1,\varepsilon}) + z_{1,\varepsilon}) + \varepsilon e^{-\delta(t - T_{1,\varepsilon})} + \varepsilon e^{-\mu(x - X_{1,\varepsilon})}, \quad t \geq T_{1,\varepsilon}, \quad x \geq X_{1,\varepsilon}, \quad (5.16)$$

(ii) *assume further that  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ , there exist  $X_{2,\varepsilon} > L$ ,  $T_{2,\varepsilon} > 0$  and  $z_{2,\varepsilon} \in \mathbb{R}$  such that*

$$u(t, x) \geq \phi(x - (c_b - c)(t - T_{2,\varepsilon}) + z_{2,\varepsilon}) - \varepsilon e^{-\delta(t - T_{2,\varepsilon})} - \varepsilon e^{-\mu(x - X_{2,\varepsilon})}, \quad t \geq T_{2,\varepsilon}, \quad x \geq X_{2,\varepsilon}, \quad (5.17)$$

with  $\mu > 0$ ,  $\delta > 0$  as given in Lemma 5.2.

**Remark.** A straightforward consequence of Lemmas 5.2–5.3 is that (5.5)–(5.6) and (5.16)–(5.17) hold true under the assumption of Theorem 1.4 (i), i.e. when  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ .

*Proof.* Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  be defined as in (5.7)–(5.11) (notice that these parameters are independent of  $\varepsilon$ ). It is immediate to see from Lemma 5.2 that, when  $\varepsilon \geq \delta$ , the conclusion of Lemma 5.3 holds true with  $X_{i,\varepsilon} = X_i$ ,  $T_{i,\varepsilon} = T_i$  and  $z_{i,\varepsilon} = z_i$ , for  $i = 1, 2$ . It remains to discuss the case

$$0 < \varepsilon < \delta.$$

For convenience, let us introduce some further parameters. Pick  $C_\varepsilon > C > 0$  such that

$$\phi \geq 1 - \frac{\varepsilon}{2} \text{ in } (-\infty, -C_\varepsilon] \text{ and } \phi \leq \varepsilon \text{ in } [C_\varepsilon, +\infty).$$

Define

$$\omega_\varepsilon := \frac{\varepsilon\omega}{\delta} > 0. \quad (5.18)$$

Finally, let  $B_\varepsilon > \omega_\varepsilon$  be large enough such that

$$\left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu B_\varepsilon} < \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu(B_\varepsilon - \omega_\varepsilon)} \leq \delta.$$

*Step 1: proof of (5.16).* By repeating the arguments used in the proof of (5.13)–(5.14) in Step 1 of Lemma 5.2 and by replacing  $\delta$  with  $\varepsilon$ , there are  $T_{1,\varepsilon}$  large enough and  $X_{1,\varepsilon} > L$  such that  $u(t, X_{1,\varepsilon}) \leq 1 + \varepsilon/2$  for all  $t \geq T_{1,\varepsilon}$  and  $u(T_{1,\varepsilon}, x) \leq \phi(x - X_{1,\varepsilon} - A_\varepsilon - C_\varepsilon) + \varepsilon$  for all  $x \geq X_{1,\varepsilon}$ , for some  $A_\varepsilon \geq B_\varepsilon$ . Define

$$\bar{u}_\varepsilon(t, x) = \phi(\bar{\xi}_\varepsilon(t, x)) + \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \quad \text{for } t \geq T_{1,\varepsilon} \text{ and } x \geq X_{1,\varepsilon},$$

where

$$\bar{\xi}_\varepsilon(t, x) = x - X_{1,\varepsilon} - (c_b - c)(t - T_{1,\varepsilon}) + \omega_\varepsilon e^{-\delta(t-T_{1,\varepsilon})} - \omega_\varepsilon - A_\varepsilon - C_\varepsilon.$$

Following the same lines as in Step 1 of Lemma 5.2, one has  $\bar{u}_\varepsilon(X_{1,\varepsilon}, x) \geq u(X_{1,\varepsilon}, x)$  for all  $x \geq X_{1,\varepsilon}$ ,  $\bar{u}_\varepsilon(t, X_{1,\varepsilon}) \geq u(t, X_{1,\varepsilon})$  for all  $t \geq T_{1,\varepsilon}$ , and it can be deduced that  $\bar{u}_\varepsilon(t, x)$  is a supersolution to  $u_t = u_{xx} + cu_x + f_b(u)$  for all  $t \geq T_{1,\varepsilon}$  and  $x \geq X_{1,\varepsilon}$ , by dividing the calculations into three cases:

- if  $\bar{\xi}_\varepsilon(t, x) \leq -C$ , then  $1 - \delta/2 \leq \phi(\bar{\xi}_\varepsilon(t, x)) < 1$  by (5.9), hence  $1 + 2\delta \geq 1 + 2\varepsilon \geq \bar{u}_\varepsilon(t, x) \geq 1 - \delta/2$ ; therefore, by using (5.7)–(5.8) and the negativity of  $\phi'$  and  $f'_b(1)$ , it follows that

$$\begin{aligned} \mathcal{N}\bar{u}_\varepsilon(t, x) &\geq -\frac{f'_b(1)}{2} \left( \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \right) - \delta \varepsilon e^{-\delta(t-T_{1,\varepsilon})} - (\mu^2 - c\mu) \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \\ &= \left( -\frac{f'_b(1)}{2} - \delta \right) \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \left( -\frac{f'_b(1)}{2} - \mu^2 + c\mu \right) \varepsilon e^{-\mu(x-X_{1,\varepsilon})} > 0; \end{aligned}$$

- if  $\bar{\xi}_\varepsilon(t, x) \geq C$ , then  $0 < \phi(\bar{\xi}_\varepsilon(t, x)) \leq \delta$  by (5.9) and thus  $0 < \bar{u}_\varepsilon(t, x) \leq 3\delta$ ; therefore, owing to (5.7)–(5.8) as well as the negativity of  $\phi'$  and  $f'_b(0)$ , it follows that

$$\begin{aligned} \mathcal{N}\bar{u}_\varepsilon(t, x) &\geq -\frac{f'_b(0)}{2} \left( \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \right) - \delta \varepsilon e^{-\delta(t-T_{1,\varepsilon})} - (\mu^2 - c\mu) \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \\ &= \left( -\frac{f'_b(0)}{2} - \delta \right) \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \left( -\frac{f'_b(0)}{2} - \mu^2 + c\mu \right) \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \geq 0; \end{aligned}$$

- if  $-C \leq \bar{\xi}_\varepsilon(t, x) \leq C$ , one has  $x - X_{1,\varepsilon} \geq (c_b - c)(t - T_{1,\varepsilon}) - \omega_\varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \omega_\varepsilon + A_\varepsilon + C_\varepsilon - C \geq (c_b - c)(t - T_{1,\varepsilon}) + B_\varepsilon$ , hence  $e^{-\mu(x-X_{1,\varepsilon})} \leq e^{-\mu((c_b-c)(t-T_{1,\varepsilon})+B_\varepsilon)}$ ; since  $\omega_\varepsilon = \varepsilon\omega/\delta$ , one infers from (5.8), (5.10)–(5.11) and (5.21), that

$$\begin{aligned} \mathcal{N}\bar{u}_\varepsilon(t, x) &\geq -\max_{[-2\delta, 1+2\delta]} |f'_b| \left( \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \right) + \kappa\omega_\varepsilon \delta e^{-\delta(t-T_{1,\varepsilon})} \\ &\quad - \varepsilon \delta e^{-\delta(t-T_{1,\varepsilon})} - (\mu^2 - c\mu) \varepsilon e^{-\mu(x-X_{1,\varepsilon})} \\ &\geq \left( -\max_{[-2\delta, 1+2\delta]} |f'_b| + \kappa\omega - \delta \right) \varepsilon e^{-\delta(t-T_{1,\varepsilon})} - \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \varepsilon e^{-\mu((c_b-c)(t-T_{1,\varepsilon})+B_\varepsilon)} \\ &\geq \left( \max_{[-2\delta, 1+2\delta]} |f'_b| - \kappa\omega + 2\delta \right) \varepsilon e^{-\delta(t-T_{1,\varepsilon})} \geq 0. \end{aligned}$$

Therefore, the maximum principle implies that

$$u(t, x) \leq \phi(x - X_{1,\varepsilon} - (c_b - c)(t - T_{1,\varepsilon}) + \omega_\varepsilon e^{-\delta(t-T_{1,\varepsilon})} - \omega_\varepsilon - A_\varepsilon - C_\varepsilon) + \varepsilon e^{-\delta(t-T_{1,\varepsilon})} + \varepsilon e^{-\mu(x-X_{1,\varepsilon})}$$

for all  $t \geq T_{1,\varepsilon}$  and  $x \geq X_{1,\varepsilon}$ . Consequently, (5.16) follows by choosing  $z_{1,\varepsilon} = -X_{1,\varepsilon} - \omega_\varepsilon - A_\varepsilon - C_\varepsilon$ .

*Step 2: proof of (5.17).* Using the same argument as for the proof of (5.15) with  $\delta$  replaced by  $\varepsilon$ , one infers that there exist  $X_{2,\varepsilon} > L$  and  $T_{2,\varepsilon} > 0$  sufficiently large such that

$$u(t, x) \geq 1 - \varepsilon \quad \text{for all } t \geq T_{2,\varepsilon} \text{ and } x \in [X_{2,\varepsilon}, X_{2,\varepsilon} + B_\varepsilon + 2C_\varepsilon].$$

Then we set

$$\underline{u}_\varepsilon(t, x) = \phi(\underline{\xi}_\varepsilon(t, x)) - \varepsilon e^{-\delta(t-T_{2,\varepsilon})} - \varepsilon e^{-\mu(x-X_{2,\varepsilon})} \quad \text{for } t \geq T_{2,\varepsilon} \text{ and } x \geq X_{2,\varepsilon},$$

in which

$$\underline{\xi}_\varepsilon(t, x) = x - X_{2,\varepsilon} - (c_b - c)(t - T_{2,\varepsilon}) - \omega_\varepsilon e^{-\delta(t-T_{2,\varepsilon})} + \omega_\varepsilon - B_\varepsilon - C_\varepsilon.$$

As in the proof of (5.6), one can show that  $\underline{u}_\varepsilon(T_{2,\varepsilon}, x) \leq u(T_{2,\varepsilon}, x)$  for all  $x \geq X_{2,\varepsilon}$ , that  $\underline{u}_\varepsilon(t, X_{2,\varepsilon}) \leq u(t, X_{2,\varepsilon})$  for all  $t \geq T_{2,\varepsilon}$ , and that  $\underline{u}_\varepsilon(t, x)$  is a subsolution of  $u_t = u_{xx} + cu_x + f_b(u)$  for all  $t \geq T_{2,\varepsilon}$  and  $x \geq X_{2,\varepsilon}$ . By the maximum principle, one derives that

$$u(t, x) \geq \phi(x - X_{2,\varepsilon} - (c_b - c)(t - T_{2,\varepsilon}) - \omega_\varepsilon e^{-\delta(t-T_{2,\varepsilon})} + \omega_\varepsilon - B_\varepsilon - C_\varepsilon) - \varepsilon e^{-\delta(t-T_{2,\varepsilon})} - \varepsilon e^{-\mu(x-X_{2,\varepsilon})}$$

for all  $t \geq T_{2,\varepsilon}$  and  $x \geq X_{2,\varepsilon}$ . Then (5.17) follows by taking  $z_{2,\varepsilon} = -X_{2,\varepsilon} + \omega_\varepsilon - B_\varepsilon - C_\varepsilon$ , since  $\phi' < 0$ . The proof of Lemma 5.3 is thereby complete.  $\square$

Based on Lemmas 5.2 and 5.3, we now provide the stability result of the bistable traveling front  $\phi(x - (c_b - c)t)$  in the bistable region, under the assumption of Theorem 1.4 (i), i.e. when  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ .

**Lemma 5.4.** *Assume that  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ . Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  be as in (5.7)–(5.11) in the proof of Lemma 5.2. If there exist  $\varepsilon \in (0, \delta]$ ,  $t_0 > 0$ ,  $x_0 > L$  and  $\xi \in \mathbb{R}$  such that*

$$\sup_{x \geq x_0} |u(t_0, x) - \phi(x - (c_b - c)t_0 + \xi)| \leq \varepsilon, \quad (5.19)$$

$$1 - \varepsilon \leq u(t, x_0) \leq 1 + \frac{\varepsilon}{2} \quad \text{for all } t \geq t_0, \quad (5.20)$$

$$\phi(x_0 - (c_b - c)t_0 + \xi) \geq 1 - \frac{\varepsilon}{2},$$

and

$$\left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu((c_b - c)t_0 - x_0 - \omega_\varepsilon - \xi - C)} \leq \delta \quad (5.21)$$

with  $\omega_\varepsilon = \varepsilon\omega/\delta$ , then there exists  $\widetilde{M} > 0$  such that the following holds true:

$$\sup_{x \geq x_0} |u(t, x) - \phi(x - (c_b - c)t + \xi)| \leq \widetilde{M}\varepsilon \quad \text{for all } t \geq t_0.$$

*Proof.* We first claim that

$$\bar{u}(t, x) = \phi(x - (c_b - c)t + \omega_\varepsilon e^{-\delta(t-t_0)} - \omega_\varepsilon + \xi) + \varepsilon e^{-\delta(t-t_0)} + \varepsilon e^{-\mu(x-x_0)}$$

and

$$\underline{u}(t, x) = \phi(x - (c_b - c)t - \omega_\varepsilon e^{-\delta(t-t_0)} + \omega_\varepsilon + \xi) - \varepsilon e^{-\delta(t-t_0)} - \varepsilon e^{-\mu(x-x_0)}$$

are, respectively, a super- and a subsolution of  $u_t = u_{xx} + cu_x + f_b(u)$  for  $t \geq t_0$  and  $x \geq x_0$ . We just check that  $\underline{u}(t, x)$  is a subsolution in detail (the supersolution can be handled in a similar way).

At time  $t = t_0$ , one has  $\underline{u}(t_0, x) = \phi(x - (c_b - c)t_0 + \xi) - \varepsilon - \varepsilon e^{-\mu(x-x_0)} \leq u(t_0, x)$  for all  $x \geq x_0$  thanks to (5.19). Moreover,  $\underline{u}(t, x_0) = \phi(x_0 - (c_b - c)t - \omega_\varepsilon e^{-\delta(t-t_0)} + \omega_\varepsilon + \xi) - \varepsilon e^{-\delta(t-t_0)} - \varepsilon \leq 1 - \varepsilon \leq u(t, x_0)$  for all  $t \geq t_0$ , owing to (5.20). It then remains to show that  $\mathcal{N}\underline{u}(t, x) := \underline{u}_t(t, x) - \underline{u}_{xx}(t, x) - c\underline{u}_x(t, x) - f_b(\underline{u}(t, x)) \leq 0$  for all  $t \geq t_0$  and  $x \geq x_0$ . For convenience, we set

$$\underline{\xi}(t, x) := x - (c_b - c)t - \omega_\varepsilon e^{-\delta(t-t_0)} + \omega_\varepsilon + \xi.$$

By a straightforward computation, one has

$$\mathcal{N}\underline{u}(t, x) = f_b(\phi(\underline{\xi}(t, x))) - f_b(\underline{u}(t, x)) + \phi'(\underline{\xi}(t, x))\omega_\varepsilon \delta e^{-\delta(t-t_0)} + \varepsilon \delta e^{-\delta(t-t_0)} + (\mu^2 - c\mu)\varepsilon e^{-\mu(x-x_0)}.$$

We divide our analysis into three cases:

- if  $\underline{\xi}(t, x) \leq -C$ , then  $1 - \delta/2 \leq \phi(\underline{\xi}(t, x)) < 1$  by (5.9), hence  $1 > \underline{u}(t, x) \geq 1 - \delta/2 - 2\varepsilon \geq 1 - 3\delta$ ; therefore, by using (5.7)–(5.8) and the negativity of  $\phi'$  and  $f'_b(1)$ , it follows that

$$\begin{aligned} \mathcal{N}\underline{u}(t, x) &\leq \frac{f'_b(1)}{2} \left( \varepsilon e^{-\delta(t-t_0)} + \varepsilon e^{-\mu(x-x_0)} \right) + \varepsilon \delta e^{-\delta(t-t_0)} + (\mu^2 - c\mu) \varepsilon e^{-\mu(x-x_0)} \\ &= \left( \frac{f'_b(1)}{2} + \delta \right) \varepsilon e^{-\delta(t-t_0)} + \left( \frac{f'_b(1)}{2} + \mu^2 - c\mu \right) \varepsilon e^{-\mu(x-x_0)} \leq 0; \end{aligned}$$

- if  $\underline{\xi}(t, x) \geq C$ , then  $0 < \phi(\underline{\xi}(t, x)) \leq \delta$  by (5.9) and thus  $-2\delta \leq -2\varepsilon \leq \underline{u}(t, x) \leq \delta$ ; therefore, owing to (5.7)–(5.8) as well as the negativity of  $\phi'$  and  $f'_b(0)$ , it follows that

$$\begin{aligned} \mathcal{N}\underline{u}(t, x) &\leq \frac{f'_b(0)}{2} \left( \varepsilon e^{-\delta(t-t_0)} + \varepsilon e^{-\mu(x-x_0)} \right) + \varepsilon \delta e^{-\delta(t-t_0)} + (\mu^2 - c\mu) \varepsilon e^{-\mu(x-x_0)} \\ &= \left( \frac{f'_b(0)}{2} + \delta \right) \varepsilon e^{-\delta(t-t_0)} + \left( \frac{f'_b(0)}{2} + \mu^2 - c\mu \right) \varepsilon e^{-\mu(x-x_0)} \leq 0; \end{aligned}$$

- if  $-C \leq \underline{\xi}(t, x) \leq C$ , one has  $x - x_0 \geq (c_b - c)(t - t_0) + (c_b - c)t_0 - x_0 + \omega_\varepsilon e^{-\delta(t-t_0)} - \omega_\varepsilon - \xi - C \geq (c_b - c)(t - t_0) + (c_b - c)t_0 - x_0 - \omega_\varepsilon - \xi - C$ , hence  $e^{-\mu(x-x_0)} \leq e^{-\mu((c_b - c)(t-t_0) + (c_b - c)t_0 - x_0 - \omega_\varepsilon - \xi - C)}$ ; since  $\omega_\varepsilon = \varepsilon\omega/\delta$ , one infers from (5.8), (5.10)–(5.11) and (5.21), that

$$\begin{aligned} \mathcal{N}\underline{u}(t, x) &\leq \max_{[-2\delta, 1+2\delta]} |f'_b| \left( \varepsilon e^{-\delta(t-t_0)} + \varepsilon e^{-\mu(x-x_0)} \right) - \kappa\omega_\varepsilon \delta e^{-\delta(t-t_0)} + \varepsilon \delta e^{-\delta(t-t_0)} + (\mu^2 - c\mu) \varepsilon e^{-\mu(x-x_0)} \\ &\leq \left( \max_{[-2\delta, 1+2\delta]} |f'_b| - \kappa\omega + \delta \right) \varepsilon e^{-\delta(t-t_0)} + \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \varepsilon e^{-\mu((c_b - c)(t-t_0) + (c_b - c)t_0 - x_0 - \omega_\varepsilon - \xi - C)} \\ &\leq \left( \max_{[-2\delta, 1+2\delta]} |f'_b| - \kappa\omega + 2\delta \right) \varepsilon e^{-\delta(t-t_0)} \leq 0. \end{aligned}$$

Eventually, one concludes that  $\mathcal{N}\underline{u}(t, x) := \underline{u}_t(t, x) - c\underline{u}_x(t, x) - \underline{u}_{xx}(t, x) - f_b(\underline{u}(t, x)) \leq 0$  for all  $t \geq t_0$  and  $x \geq x_0$ . The maximum principle implies that

$$u(t, x) \geq \underline{u}(t, x) = \phi(x - (c_b - c)t - \omega_\varepsilon e^{-\delta(t-t_0)} + \omega_\varepsilon + \xi) - \varepsilon e^{-\delta(t-t_0)} - \varepsilon e^{-\mu(x-x_0)}$$

for all  $t \geq t_0$  and  $x \geq x_0$ . For these  $t$  and  $x$ , since  $\phi' < 0$ , one derives that

$$u(t, x) \geq \phi(x - (c_b - c)t + \omega_\varepsilon + \xi) - 2\varepsilon \geq \phi(x - (c_b - c)t + \xi) - \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} - 2\varepsilon.$$

Similarly, using especially that

$$\left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu((c_b - c)t_0 - x_0 - \xi - C)} \leq \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu((c_b - c)t_0 - x_0 - \omega_\varepsilon - \xi - C)} \leq \delta$$

by (5.21), one can also derive that  $u(t, x) \leq \bar{u}(t, x) = \phi(x - (c_b - c)t + \omega_\varepsilon e^{-\delta(t-t_0)} - \omega_\varepsilon + \xi) + \varepsilon e^{-\delta(t-t_0)} + \varepsilon e^{-\mu(x-x_0)}$  for all  $t \geq t_0$  and  $x \geq x_0$ , hence

$$u(t, x) \leq \phi(x - (c_b - c)t - \omega_\varepsilon + \xi) + 2\varepsilon \leq \phi(x - (c_b - c)t + \xi) + \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} + 2\varepsilon.$$

In conclusion, one has

$$\sup_{x \geq x_0} |u(t, x) - \phi(x - (c_b - c)t + \xi)| \leq \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} + 2\varepsilon = \widetilde{M}\varepsilon \text{ for all } t \geq t_0,$$

where  $\widetilde{M} := \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} / \varepsilon + 2 = \omega \|\phi'\|_{L^\infty(\mathbb{R})} / \delta + 2$  is independent of  $\varepsilon$ ,  $t_0$ ,  $x_0$  and  $\xi$ . The proof of Lemma 5.4 is thereby complete.  $\square$

Now we are in a position to complete the proof of Theorem 1.4.

*Proof of Theorem 1.4 (continued).* We first complete the proof of property (i). Remember that  $-c_m < c_b$  and  $c \in (-c_m, c_b)$ . Let  $X_1 > L$ ,  $X_2 > L$ ,  $T_1 > 0$ ,  $T_2 > 0$ ,  $z_1 \in \mathbb{R}$ ,  $z_2 \in \mathbb{R}$ ,  $\mu > 0$  and  $\delta > 0$  be as in Lemma 5.2, and let also  $C > 0$  be as in (5.9) in the proof of Lemma 5.2. For  $t \geq \max(T_1, T_2)$  and  $x \geq \max(X_1, X_2) > L$ , Lemma 5.2 implies that

$$\begin{aligned} \phi(x - (c_b - c)(t - T_2) + z_2) - \delta e^{-\delta(t-T_2)} - \delta e^{-\mu(x-X_2)} \\ \leq u(t, x) \leq \phi(x - (c_b - c)(t - T_1) + z_1) + \delta e^{-\delta(t-T_1)} + \delta e^{-\mu(x-X_1)}. \end{aligned} \quad (5.22)$$

Consider any given sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $t_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . By standard parabolic estimates, the functions

$$(t, z) \mapsto u_n(t, z) := u(t + t_n, z + (c_b - c)t_n)$$

converge as  $n \rightarrow +\infty$  up to extraction of a subsequence, locally uniformly in  $(t, z) \in \mathbb{R} \times \mathbb{R}$ , to a classical solution  $u_\infty$  of  $(u_\infty)_t = (u_\infty)_{zz} + c(u_\infty)_z + f_b(u_\infty)$  in  $\mathbb{R} \times \mathbb{R}$ . From (5.22) applied at  $(t + t_n, z + (c_b - c)t_n)$ , the passage to the limit as  $n \rightarrow +\infty$  gives

$$\phi(z - (c_b - c)(t - T_2) + z_2) \leq u_\infty(t, z) \leq \phi(z - (c_b - c)(t - T_1) + z_1) \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}.$$

Then, [9, Theorem 3.1] can be adapted to yield that there exists  $\xi \in \mathbb{R}$  such that  $u_\infty(t, z) = \phi(z - (c_b - c)t + \xi)$  for all  $(t, z) \in \mathbb{R} \times \mathbb{R}$ , whence

$$u_n(t, z) \rightarrow \phi(z - (c_b - c)t + \xi) \quad \text{as } n \rightarrow +\infty, \quad \text{locally uniformly in } (t, z) \in \mathbb{R} \times \mathbb{R}. \quad (5.23)$$

Consider now any  $\varepsilon \in (0, \delta/3]$ . Let  $A_\varepsilon > 0$  be such that

$$\phi \geq 1 - \frac{\varepsilon}{2} \quad \text{in } (-\infty, -A_\varepsilon] \quad \text{and} \quad \phi \leq \frac{\varepsilon}{2} \quad \text{in } [A_\varepsilon, +\infty). \quad (5.24)$$

Set  $E_1 := \max(A_\varepsilon - (c_b - c)T_1 - z_1, A_\varepsilon - \xi)$  and  $E_2 := \min(-A_\varepsilon - (c_b - c)T_2 - z_2, -A_\varepsilon - \xi) < E_1$ . Then, it can be deduced from (5.23) that

$$\sup_{E_2 \leq z \leq E_1} |u_n(0, z) - \phi(z + \xi)| \leq \varepsilon \quad \text{for all } n \text{ large enough.} \quad (5.25)$$

Since  $t_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ , (5.22) and (5.24) imply that, for all  $n$  large enough,

$$\begin{cases} 0 < u_n(0, z) \leq \varepsilon & \text{for all } z \geq E_1, \\ 1 - \varepsilon \leq u_n(0, z) \leq 1 + \varepsilon & \text{for all } E_2 - \frac{c_b - c}{2}t_n \leq z \leq E_2. \end{cases} \quad (5.26)$$

Furthermore, since  $E_1 \geq A_\varepsilon - \xi$  and  $E_2 \leq -A_\varepsilon - \xi$ , one has

$$\begin{cases} 0 < \phi(z + \xi) \leq \frac{\varepsilon}{2} < \varepsilon & \text{for all } z \geq E_1, \\ 1 - \varepsilon < 1 - \frac{\varepsilon}{2} \leq \phi(z + \xi) < 1 & \text{for all } z \leq E_2. \end{cases} \quad (5.27)$$

Then (5.26)–(5.27) imply that, for all  $n$  large enough,

$$|u_n(0, z) - \phi(z + \xi)| \leq 2\varepsilon \quad \text{for all } z \in \left[ E_2 - \frac{c_b - c}{2}t_n, E_2 \right] \cup [E_1, +\infty).$$

Together with (5.25) and the definition of  $u_n(t, z)$ , one has, for all  $n$  large enough,

$$|u(t_n, x) - \phi(x - (c_b - c)t_n + \xi)| \leq 2\varepsilon \quad \text{for all } x \geq E_2 + \frac{c_b - c}{2}t_n. \quad (5.28)$$

On the other hand, one infers from Lemma 5.3 that, for all  $n$  large enough,

$$\begin{aligned} 1 - 3\varepsilon &\leq \phi(x - (c_b - c)(t_n - T_{2,\varepsilon}) + z_{2,\varepsilon}) - \varepsilon e^{-\delta(t_n - T_{2,\varepsilon})} - \varepsilon e^{-\mu(x - X_{2,\varepsilon})} \leq u(t_n, x) \\ &\leq \phi(x - (c_b - c)(t_n - T_{1,\varepsilon}) + z_{1,\varepsilon}) + \varepsilon e^{-\delta(t_n - T_{1,\varepsilon})} + \varepsilon e^{-\mu(x - X_{1,\varepsilon})} \leq 1 + 2\varepsilon, \end{aligned} \quad (5.29)$$

for all  $\max(X_1, X_2, X_{1,\varepsilon}, X_{2,\varepsilon}) \leq x \leq E_2 + (c_b - c)t_n/2$ , where  $X_{1,\varepsilon} > L$ ,  $X_{2,\varepsilon} > L$ ,  $T_{1,\varepsilon} > 0$ ,  $T_{2,\varepsilon} > 0$ ,  $z_{1,\varepsilon} \in \mathbb{R}$  and  $z_{2,\varepsilon} \in \mathbb{R}$  were given in Lemma 5.3. Notice also that, for all  $n$  large enough,

$$1 - \varepsilon \leq \phi(x - (c_b - c)t_n + \xi) < 1 \quad \text{for all } \max(X_1, X_2, X_{1,\varepsilon}, X_{2,\varepsilon}) \leq x \leq E_2 + \frac{c_b - c}{2}t_n. \quad (5.30)$$

One deduces from (5.29)–(5.30) that, for all  $n$  large enough,

$$|u(t_n, x) - \phi(x - (c_b - c)t_n + \xi)| \leq 3\varepsilon \quad \text{for all } \max(X_1, X_2, X_{1,\varepsilon}, X_{2,\varepsilon}) \leq x \leq E_2 + \frac{c_b - c}{2}t_n.$$

Together with (5.28), one derives that, for all  $n$  large enough,

$$|u(t_n, x) - \phi(x - (c_b - c)t_n + \xi)| \leq 3\varepsilon \quad \text{for all } x \geq \max(X_1, X_2, X_{1,\varepsilon}, X_{2,\varepsilon}).$$

Choose  $x_\varepsilon = \max(X_1, X_2, X_{1,\varepsilon}, X_{2,\varepsilon})$ , then thanks to (1.14) and  $c_b - c > 0$ , it follows that for all  $n$  large enough,

$$1 - 3\varepsilon \leq u(t, x_\varepsilon) \leq 1 + \frac{3\varepsilon}{2} \quad \text{for all } t \geq t_n,$$

and

$$\phi(x_\varepsilon - (c_b - c)t_n + \xi) \geq 1 - \frac{3\varepsilon}{2}, \quad \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu((c_b - c)t_n - x_\varepsilon - 3\varepsilon\omega/\delta - \xi - C)} \leq \delta.$$

It then follows from Lemma 5.4 (applied with  $t_0 = t_n$ ,  $x_0 = x_\varepsilon$  and  $3\varepsilon$  instead of  $\varepsilon$ ) that, for all  $n$  large enough,

$$|u(t, x) - \phi(x - (c_b - c)t + \xi)| \leq 3\widetilde{M}\varepsilon \quad \text{for all } t \geq t_n \text{ and } x \geq x_\varepsilon,$$

with  $\widetilde{M}$  given in Lemma 5.4. Since  $\varepsilon \in (0, \delta/3]$  was arbitrary, one finally infers that

$$\lim_{t \rightarrow +\infty} \left( \sup_{x \geq x_\varepsilon} |u(t, x) - \phi(x - (c_b - c)t + \xi)| \right) = 0.$$

Therefore, property (i) is achieved.

It now remains to prove property (ii). Assume now that  $-c_m < c = c_b$ . Our goal is to show that  $\sup_{x \geq \nu t} u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$  for every  $\nu > 0$ . So let us fix  $\nu > 0$  in the sequel. For  $\varepsilon \in (0, (1 - \theta)/2)$ , let  $f_{b,\varepsilon}$  be a  $C^1(\mathbb{R})$  function such that

$$\begin{cases} f_{b,\varepsilon}(0) = f_{b,\varepsilon}(\theta) = f_{b,\varepsilon}(1 + \varepsilon) = 0, & f'_{b,\varepsilon}(0) < 0, & f'_{b,\varepsilon}(1 + \varepsilon) < 0, \\ f_{b,\varepsilon} = f_b \text{ in } (-\infty, 1 - \varepsilon), & f_{b,\varepsilon} > 0 \text{ in } (\theta, 1 + \varepsilon), & f_{b,\varepsilon} < 0 \text{ in } (1 + \varepsilon, +\infty). \end{cases}$$

We can also choose  $f_{b,\varepsilon}$  so that  $f_{b,\varepsilon} \geq f_b$  in  $\mathbb{R}$ , so that  $f_{b,\varepsilon}$  is decreasing in  $[1 - \varepsilon, 1 + \varepsilon]$ . For each  $\varepsilon \in (0, (1 - \theta)/2)$ , let  $\phi_\varepsilon$  be the unique traveling front profile of  $u_t = u_{xx} + f_{b,\varepsilon}(u)$  such that

$$\phi_\varepsilon'' + c_{b,\varepsilon}\phi_\varepsilon' + f_{b,\varepsilon}(\phi_\varepsilon) = 0 \text{ in } \mathbb{R}, \quad \phi_\varepsilon' < 0 \text{ in } \mathbb{R}, \quad \phi_\varepsilon(0) = \theta, \quad \phi_\varepsilon(-\infty) = 1 + \varepsilon, \quad \phi_\varepsilon(+\infty) = 0,$$

with speed  $c_{b,\varepsilon} > c_b$ . It is standard to see that  $\phi_\varepsilon \rightarrow \phi$  in  $C_{loc}^2(\mathbb{R})$  and  $c_{b,\varepsilon} \rightarrow c_b$  as  $\varepsilon \rightarrow 0$ . We then fix  $\varepsilon \in (0, (1 - \theta)/2)$  small enough such that  $0 < c_{b,\varepsilon} - c_b < \nu$ .

By Lemma 2.2, there are large enough  $T > 0$  and  $X > L$  such that  $u(t, x) \leq 1 + \varepsilon/2$  for all  $t \geq T$  and  $x \geq X$ . Since  $u(t, x)$  has a Gaussian upper bound as  $x \rightarrow +\infty$  at each fixed  $t > 0$  by Lemma 2.3,

whereas  $\phi_\varepsilon(s)$  has an exponential decay (similar to (1.9)) as  $s \rightarrow +\infty$ , it follows that there is  $A > 0$  such that  $u(T, x) \leq \phi_\varepsilon(x - (c_{b,\varepsilon} - c_b)T - A)$  for all  $x \geq X$ , and  $u(t, X) \leq \phi_\varepsilon(X - (c_{b,\varepsilon} - c_b)t - A)$  for all  $t \geq T$  (here we also use the fact  $c_{b,\varepsilon} > c_b$  and  $\phi_\varepsilon(-\infty) = 1 + \varepsilon$ ). By setting  $\underline{u}(t, x) := \phi_\varepsilon(x - (c_{b,\varepsilon} - c_b)t - A)$  for  $t \geq T$  and  $x \geq X$ , a direct computation yields that

$$\begin{aligned} \bar{u}_t - \bar{u}_{xx} - c_b \bar{u}_x - f_b(\bar{u}) &= -(c_{b,\varepsilon} - c_b)\phi'_\varepsilon(\xi(t, x)) - \phi''_\varepsilon(\xi(t, x)) - c_b\phi'_\varepsilon(\xi(t, x)) - f_b(\phi_\varepsilon(\xi(t, x))) \\ &= f_{b,\varepsilon}(\phi_\varepsilon(\xi(t, x))) - f_b(\phi_\varepsilon(\xi(t, x))) \geq 0, \quad t \geq T, x \geq X, \end{aligned}$$

with  $\xi(t, x) := x - (c_{b,\varepsilon} - c_b)t - A$ , since  $f_{b,\varepsilon} \geq f_b$  in  $\mathbb{R}$ . The comparison principle implies that  $0 < u(t, x) \leq \phi_\varepsilon(x - (c_{b,\varepsilon} - c_b)t - A)$  for all  $t \geq T$  and  $x \geq X$ , hence  $\sup_{x \geq \nu t} u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$ , since  $c_{b,\varepsilon} - c < \nu$  and  $\phi_\varepsilon(+\infty) = 0$ . This completes the proof of Theorem 1.4.  $\square$

## 6 Conditional complete propagation vs. extinction when $c \in (-\infty, -c_m]$

### 6.1 Conditional complete propagation: Proof of Theorem 1.5

In this subsection, we investigate the leftward propagation properties when  $c \in (-\infty, -c_m]$  for “large enough” initial data, inspired by [22, 29]. This is notably different from the preceding section. Here, we must consider both the KPP region and the bistable region simultaneously, leading to a more intricate analysis compared to the sections above.

**Lemma 6.1.** *Assume that  $c \leq -c_m$ . Then there exist  $T_3 > 0$ ,  $X_3 > L$ ,  $z_3 \in \mathbb{R}$ ,  $\mu > 0$  and  $\delta > 0$  such that*

$$u(t, x) \leq \phi(-x - (c_b + c)(t - T_3) + z_3) + \delta e^{-\delta(t - T_3)} + \delta e^{-\mu(x - X_3)} \quad \text{for } t \geq T_3 \text{ and } x \geq X_3. \quad (6.1)$$

*Proof.* We recall that  $c_b < c_m$  which is already proved in the introduction. It implies that  $-c_m < -c_b$ , whence  $c < -c_b$ , namely  $c + c_b < 0$ .

We first introduce some parameters. Choose  $\mu > 0$  such that

$$0 < \mu < \frac{1}{2} \left( c + \sqrt{c^2 + 2 \min(|f'_b(0)|, |f'_b(1)|)} \right). \quad (6.2)$$

Choose  $\delta > 0$  such that

$$\begin{cases} 0 < \delta < \min \left( -\mu(c_b + c), \frac{1}{5}, \frac{|f'_b(0)|}{2}, \frac{|f'_b(1)|}{2} \right), \\ f'_b \leq \frac{f'_b(0)}{2} \text{ in } [0, 3\delta], \quad f'_b \leq \frac{f'_b(1)}{2} \text{ in } [1 - \delta, 1 + 2\delta]. \end{cases} \quad (6.3)$$

Let  $C > 0$  be such that

$$\phi \geq 1 - \delta/2 \text{ in } (-\infty, -C] \text{ and } \phi \leq \delta \text{ in } [C, +\infty). \quad (6.4)$$

Since  $\phi'$  is negative and continuous in  $\mathbb{R}$ , there is  $\kappa > 0$  such that

$$\phi' \leq -\kappa < 0 \text{ in } [-C, C]. \quad (6.5)$$

Finally, pick  $\omega > 0$  so large that

$$\kappa\omega \geq 2\delta + \max_{[0, 1+2\delta]} |f'_b|. \quad (6.6)$$

and  $A > \omega + C$  such that (recall that  $c < 0$ )

$$\left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu(A - \omega - C)} \leq \delta. \quad (6.7)$$

By Lemma 2.2, we derive that there exists  $T_3 > 0$  large enough such that

$$u(T_3, x) < 1 + \delta/2 \quad \text{uniformly for } x \in \mathbb{R}. \quad (6.8)$$

Fix  $X_3 > L$  and  $\bar{X}_3 = X_3 + A + C$ . We infer from Lemma 2.5 that, up to increasing  $T_3$ ,

$$u(t, x) \leq \delta \quad \text{for } t \geq T_3, x \in [X_3, \bar{X}_3]. \quad (6.9)$$

When  $x \geq \bar{X}_3$ , we observe that  $-x + X_3 + A \leq -C$  and thus

$$\phi(-x + X_3 + A) \geq 1 - \delta/2 \quad \text{for } x \geq \bar{X}_3. \quad (6.10)$$

For  $t \geq T_3$  and  $x \geq X_3$ , let us now define

$$\bar{u}(t, x) = \phi(\bar{\xi}(t, x)) + \delta e^{-\delta(t-T_3)} + \delta e^{-\mu(x-X_3)},$$

where

$$\bar{\xi}(t, x) = -x + X_3 - (c_b + c)(t - T_3) + \omega e^{-\delta(t-T_3)} - \omega + A.$$

Let us check that  $\bar{u}(t, x)$  is a supersolution to (1.1) for  $t \geq T_3$  and  $x \geq X_3$ . At time  $t = T_3$ , one has  $\bar{u}(T_3, x) \geq \phi(-x + X_3 + A) + \delta \geq 1 - \delta/2 + \delta = 1 + \delta/2 > u(T_3, x)$  for  $x \geq \bar{X}_3$ , by (6.8) and (6.10). Thanks to (6.9), we have  $\bar{u}(T_3, x) \geq \delta \geq u(T_3, x)$  for  $x \in [X_3, \bar{X}_3]$ . This implies that  $\bar{u}(T_3, x) \geq u(T_3, x)$  for  $x \geq X_3$ . Moreover, for  $t \geq T_3$ , it is obvious that  $\bar{u}(t, X_3) \geq \delta \geq u(t, X_3)$ , due to (6.9). Therefore, it remains to check that  $\mathcal{N}\bar{u}(t, x) := \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for all  $t \geq T_3$  and  $x \geq X_3$ . To this end, one derives from a straightforward computation that for  $t \geq T_3$  and  $x \geq X_3$ ,

$$\mathcal{N}\bar{u}(t, x) = f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) - \phi'(\bar{\xi}(t, x))\omega\delta e^{-\delta(t-T_3)} - \delta^2 e^{-\delta(t-T_3)} - (\mu^2 - c\mu)\delta e^{-\mu(x-X_3)}.$$

We distinguish three cases:

- if  $\bar{\xi}(t, x) \leq -C$ , one has  $1 - \delta/2 \leq \phi(\bar{\xi}(t, x)) < 1$  by (6.4), hence  $1 - \delta/2 \leq \bar{u}(t, x) \leq 1 + 2\delta$ ; it follows from (6.3) that  $f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) \geq -(f'_b(1)/2)(\delta e^{-\delta(t-T_3)} + \delta e^{-\mu(x-X_3)})$  and it then can be deduced from the negativity of  $\phi'$  and  $f'_b(1)$  as well as (6.2)–(6.3) that

$$\mathcal{N}\bar{u}(t, x) \geq \left(-\frac{f'_b(1)}{2} - \delta\right)\delta e^{-\delta(t-T_3)} + \left(-\frac{f'_b(1)}{2} - \mu^2 + c\mu\right)\delta e^{-\mu(x-X_3)} > 0;$$

- if  $\bar{\xi}(t, x) \geq C$ , one derives  $0 < \phi(\bar{\xi}(t, x)) \leq \delta$  by (6.4), and then  $0 < \bar{u}(t, x) \leq 3\delta$ ; it follows from (6.3) that  $f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) \geq -(f'_b(0)/2)(\delta e^{-\delta(t-T_3)} + \delta e^{-\mu(x-X_3)})$ ; by virtue of the negativity of  $\phi'$  and  $f'_b(0)$  as well as (6.2)–(6.3), there holds

$$\mathcal{N}\bar{u}(t, x) \geq \left(-\frac{f'_b(0)}{2} - \delta\right)\delta e^{-\delta(t-T_3)} + \left(-\frac{f'_b(0)}{2} - \mu^2 + c\mu\right)\delta e^{-\mu(x-X_3)} > 0;$$

- if  $-C \leq \bar{\xi}(t, x) \leq C$ , then  $0 < \bar{u}(t, x) \leq 1 + 2\delta$ . By noticing that  $x - X_3 \geq -(c_b + c)(t - T_3) - \omega + A - C$ , we have  $e^{-\mu(x-X_3)} \leq e^{-\mu(-(c_b+c)(t-T_3)-\omega+A-C)}$ . One infers from (6.3) and (6.5) – (6.7) that

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) &\geq -\max_{[0, 1+2\delta]} |f'_b| \left(\delta e^{-\delta(t-T_3)} + \delta e^{-\mu(x-X_3)}\right) + \kappa\omega\delta e^{-\delta(t-T_3)} - \delta^2 e^{-\delta(t-T_3)} - (\mu^2 - c\mu)\delta e^{-\mu(x-X_3)} \\ &= \left(\kappa\omega - \max_{[0, 1+2\delta]} |f'_b| - \delta\right)\delta e^{-\delta(t-T_3)} - \left(\max_{[0, 1+2\delta]} |f'_b| + \mu^2 - c\mu\right)\delta e^{-\mu(x-X_3)} \\ &\geq \left(\kappa\omega - \max_{[0, 1+2\delta]} |f'_b| - \delta\right)\delta e^{-\delta(t-T_3)} - \left(\max_{[0, 1+2\delta]} |f'_b| + \mu^2 - c\mu\right)\delta e^{-\mu(-(c_b+c)(t-T_3)-\omega+A-C)} \\ &\geq \left(\kappa\omega - \max_{[0, 1+2\delta]} |f'_b| - 2\delta\right)\delta e^{-\delta(t-T_3)} \geq 0. \end{aligned}$$

As a consequence, we have proven that  $\mathcal{N}\bar{u}(t, x) = \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for all  $t \geq T_3$  and  $x \geq X_3$ . The comparison principle yields that

$$u(t, x) \leq \bar{u}(t, x) = \phi(-x + X_3 - (c_b + c)(t - T_3) + \omega e^{-\delta(t-T_3)} - \omega + A) + \delta e^{-\delta(t-T_3)} + \delta e^{-\mu(x-X_3)}$$

for all  $t \geq T_3$  and  $x \geq X_3$ . By picking  $z = A - \omega + X_3$ , the conclusion follows.  $\square$

*Proof of Theorem 1.5.* Assume now that  $c \leq -c_m$  and  $c_b > 0$ . Remember that  $u$  is the solution of (1.1) with any nonnegative continuous and compactly supported initial datum  $u_0$  satisfying  $u_0 \geq \theta + \eta$  for any  $\eta > 0$  on an interval of size  $L^*$ .

*Step 1. Propagation.* Let us look at the Cauchy problem (8.1) with initial datum  $v_0 = u_0$  in  $\mathbb{R}$ .

First of all, according to the hypothesis (1.2) of  $f$ , the solution  $v$  of (8.1) will be bounded from below by the function  $w$  which solves  $w_t = w_{yy} + f_b(w)$  (the mass of  $f_b$  is positive, since  $c_b > 0$ ) for  $t > 0$  and  $x \in \mathbb{R}$  with initial condition  $w_0 = v_0$  in  $\mathbb{R}$ , thanks to the comparison principle. By virtue of Theorem 3.2 in Fife and McLeod [22], there exist some constants  $\xi_1 \in \mathbb{R}$ ,  $\xi_2 \in \mathbb{R}$ ,  $M > 0$  and  $\eta > 0$  such that

$$\sup_{y < 0} |w(t, y) - \phi(-y - c_b t + \xi_1)| + \sup_{y > 0} |w(t, y) - \phi(y - c_b t + \xi_2)| < M e^{-\eta t} \quad \text{for } t > 0. \quad (6.11)$$

This implies that the function  $w$  develops into a pair of diverging fronts, moving in opposite directions. Since  $w$  plays the role of a lower barrier for the function  $v$ , one concludes that  $v$  indeed propagates to the right and to the left, with propagation speeds no smaller than  $c_b > 0$ .

Thanks to the upper barrier of  $u$  constructed in Lemma 5.2 (i) (by noticing  $c < c_b$ ), one uses the transformation  $v(t, y) = u(t, y - ct)$  for  $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$  and immediately deduces that there exist  $X_1 > L$ ,  $T_1 > 0$ ,  $\tau_1 \in \mathbb{R}$ ,  $\bar{\mu} > 0$  and  $\bar{\delta} > 0$  such that

$$v(t, y) \leq \phi(y - c_b t + \tau_1) + \bar{\delta} e^{-\bar{\delta}(t-T_1)} + \bar{\delta} e^{-\bar{\mu}(y-ct-X_1)} \quad \text{for all } t \geq T_1 \text{ and } y \geq X_1 + ct, \quad (6.12)$$

The remaining part of the proof will rely on the following several lemmas.

**Lemma 6.2.** *Assume that  $c \leq -c_m$  and  $c_b > 0$ . Let  $v$  be the solution of (8.1) with any nonnegative continuous and compactly supported initial datum  $v_0 = u_0$  in  $\mathbb{R}$ , with  $u_0$  as given in Theorem 1.5. Then there exist  $X_2 > L$ ,  $X_4 > L$ ,  $T_2 > 0$ ,  $T_4 > 0$ ,  $\tau_2 \in \mathbb{R}$ ,  $z_4 \in \mathbb{R}$ ,  $\mu > 0$  and  $\delta > 0$  such that*

$$v(t, y) \geq \phi(y - c_b t + \tau_2) - \delta e^{-\delta(t-T_2)} - \delta e^{-\mu(y-X_2)} \quad \text{for all } t \geq T_2 \text{ and } y \geq X_2, \quad (6.13)$$

$$v(t, y) \geq \phi(-y - c_b t + z_4) - \delta e^{-\delta(t-T_4)} - \delta e^{-\mu(X_4-y)} \quad \text{for all } t \geq T_4 \text{ and } y \leq X_4. \quad (6.14)$$

**Remark.** Let  $u$  be as given in Theorem 1.5. It is obvious that (6.14) is equivalent to

$$u(t, x) \geq \phi(-x - (c_b + c)t + z_4) - \delta e^{-\delta(t-T_4)} - \delta e^{-\mu(X_4-x-ct)} \quad \text{for all } t \geq T_4 \text{ and } x \leq X_4 - ct, \quad (6.15)$$

which will be more convenient in the investigation of leftward propagation of Theorem 1.5 later on.

*Proof.* We first introduce some parameters. Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega$  satisfy (5.7)–(5.11) and (6.2)–(6.6). We further assume that

$$0 < \mu < \sqrt{\min\left(\frac{|f'_b(0)|}{2}, \frac{|f'_b(1)|}{2}\right)}, \quad (6.16)$$

$$\begin{cases} 0 < \delta < \min\left(\mu c_b, \frac{1}{5}, \frac{|f'_b(0)|}{2}, \frac{|f'_b(1)|}{2}\right), \\ f'_b \leq \frac{f'_b(0)}{2} \text{ in } [-2\delta, \delta], \quad f'_b \leq \frac{f'_b(1)}{2} \text{ in } [1 - 3\delta, 1], \end{cases} \quad (6.17)$$

and

$$\kappa\omega \geq 2\delta + \max_{[-2\delta, 1]} |f'_b|. \quad (6.18)$$

We also choose some constant  $B > \omega$  such that

$$\left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) e^{-\mu(B-\omega)} \leq \delta. \quad (6.19)$$

*Step 1. Proof of (6.13).* First, (6.11) implies that  $v(t, y) \geq w(t, y) \geq \phi(y - c_b t + \xi_2) - Me^{-\eta t}$  for  $t > 0$  and  $y \geq 0$ . In particular, there exist  $X_2 > L$  and  $T_2 \geq \max(-\ln(\delta/(2M))/\eta, (X_2 + B + 3C + \xi_2)/c_b) > 0$  such that for  $X_2 \leq y \leq X_2 + B + 2C$  and  $t \geq T_2$ , there holds

$$\begin{aligned} v(t, y) &\geq w(t, y) \geq \phi(y - c_b t + \xi_2) - Me^{-\eta t} \\ &\geq \phi(X_2 + B + 2C - c_b T_2 + \xi_2) - Me^{-\eta T_2} \geq \phi(-C) - \delta/2 \geq 1 - \delta. \end{aligned} \quad (6.20)$$

For  $t \geq T_2$  and  $y \geq X_2$ , we define

$$\underline{v}(t, y) = \phi(\xi(t, y)) - \delta e^{-\delta(t-T_2)} - \delta e^{-\mu(y-X_2)},$$

with  $\xi(t, y) = y - X_2 - c_b(t - T_2) - \omega e^{-\delta(t-T_2)} + \omega - B - C$ . We are going to verify that  $\underline{v}$  is a subsolution for (8.1) for  $t \geq T_2$  and  $y \geq X_2$ . At  $t = T_2$ , we notice that  $\underline{v}(T_2, y) \leq 1 - \delta \leq v(T_2, y)$  for  $X_2 \leq y \leq X_2 + B + 2C$  by virtue of (6.20), while in the region where  $y \geq X_2 + B + 2C$ , since  $\xi(T_2, y) \geq B + 2C - B - C = C$ , it follows that  $\underline{v}(T_2, y) \leq \delta - \delta - \delta e^{-\mu(y-X_2)} < 0 < v(T_2, y)$ . Therefore, we have  $\underline{v}(T_2, y) \leq v(T_2, y)$  for  $y \geq X_2$ . Moreover, we also observe from (6.20) that  $\underline{v}(t, X_2) \leq 1 - \delta e^{-\delta(t-T_2)} - \delta \leq v(t, X_2)$  for  $t \geq T_2$ . It remains to check that  $\underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f(y - ct, \underline{v}(t, y)) \leq 0$  for  $t \geq T_2$  and  $y \geq X_2$ . By noticing that  $\underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f(y - ct, \underline{v}(t, y)) \leq \underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f_b(\underline{v}(t, y)) =: \mathcal{L}\underline{v}(t, y)$  for  $t \geq 0$  and  $y \in \mathbb{R}$ , it suffices to prove that  $\mathcal{L}\underline{v}(t, y) \leq 0$  for  $t \geq T_2$  and  $y \geq X_2$ . After a straightforward computation, one derives

$$\mathcal{L}\underline{v}(t, y) = f_b(\phi(\xi(t, y))) - f_b(\underline{v}(t, y)) + \phi'(\xi(t, y))\omega\delta e^{-\delta(t-T_2)} + \delta^2 e^{-\delta(t-T_2)} + \mu^2 \delta e^{-\mu(y-X_2)}.$$

By doing the analysis according to three cases, namely,  $\xi(t, y) \leq -C$ ,  $\xi(t, y) \geq C$  and  $-C \leq \xi(t, y) \leq C$  based on the choices of the parameters above, eventually we obtain that  $\mathcal{L}\underline{v}(t, y) = \underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f_b(\underline{v}(t, y)) \leq 0$  for  $t \geq T_2$  and  $y \geq X_2$ . The comparison principle then implies that for  $t \geq T_2$  and  $y \geq X_2$

$$v(t, y) \geq \phi(y - X_2 - c_b(t - T_2) - \omega e^{-\delta(t-T_2)} + \omega - B - C) - \delta e^{-\delta(t-T_2)} - \delta e^{-\mu(y-X_2)}.$$

By taking  $\tau_2 = -X_2 + c_b T_2 + \omega - B - C$ , we then reach (6.13), as stated.

*Step 2. Proof of (6.14).* Fix  $X_4 > L + B + C$ . We infer from (6.11) that  $v(t, y) \geq w(t, y) \geq \phi(-y - c_b t + \xi_1) - Me^{-\eta t}$  for  $t > 0$  and  $y \leq 0$ , and that  $v(t, y) \geq w(t, y) \geq \phi(y - c_b t + \xi_2) - Me^{-\eta t}$  for  $t > 0$  and  $y \geq 0$ . Then, there exists  $T_4 > \max((2C + \xi_1)/c_b, -(1/\eta) \ln(\delta/(2M)), (C + X_4 + \xi_2)/c_b) > 0$  such that

$$\begin{aligned} v(t, y) &\geq w(t, y) \geq \phi(-y - c_b t + \xi_1) - Me^{-\eta t} \\ &\geq \phi(C - c_b T_4 + \xi_1) - Me^{-\eta T_4} \geq \phi(-C) - \delta/2 \geq 1 - \delta, \quad t \geq T_4, \quad -C \leq y \leq 0, \end{aligned}$$

and

$$\begin{aligned} v(t, y) &\geq w(t, y) \geq \phi(y - c_b t + \xi_2) - Me^{-\eta t} \\ &\geq \phi(X_4 - c_b T_4 + \xi_2) - Me^{-\eta T_4} \geq \phi(-C) - \delta/2 \geq 1 - \delta, \quad t \geq T_4, \quad 0 \leq y \leq X_4. \end{aligned}$$

Combining the above two inequalities, it follows that

$$v(t, y) \geq 1 - \delta \quad \text{for } t \geq T_4 \text{ and } -C \leq y \leq X_4. \quad (6.21)$$

For  $t \geq T_4$  and  $y \leq X_4$ , set now

$$\check{v}(t, y) = \phi(\zeta(t, y)) - \delta e^{-\delta(t-T_4)} - \delta e^{-\mu(X_4-y)},$$

with  $\zeta(t, y) = -y + X_4 - c_b(t - T_4) - \omega e^{-\delta(t-T_4)} + \omega - B - C$ . We are going to show that  $\check{v}$  is a subsolution for (8.1) for  $t \geq T_4$  and  $y \leq X_4$ . Indeed, at time  $t = T_4$ , we notice that  $\check{v}(T_4, y) \leq 1 - \delta - \delta e^{-\mu(X_4-y)} \leq v(T_4, y)$  for  $-C \leq y \leq X_4$  by virtue of (6.21), and moreover, the observation that  $\zeta(T_4, y) \geq C$  for  $y \leq -C$  implies that  $\check{v}(T_4, y) \leq \delta - \delta - \delta e^{-\mu(X_4-y)} \leq 0 < v(T_4, y)$  for  $y \leq -C$ . Thus,  $\check{v}(T_4, y) \leq v(T_4, y)$  for  $y \leq X_4$ . On the other hand, (6.21) also implies that  $v(t, X_4) \geq 1 - \delta \geq \check{v}(t, X_4)$  for  $t \geq T_4$ . The remaining parts can be proved similarly to those in Step 1 with  $z_4 = X_4 + c_b T_4 + \omega - B - C$ .  $\square$

At this stage, let us combine (6.12) and (6.13), by repeating the lines as in the proof of Theorem 1.4 (together with parallel results to the technical Lemmas 5.3–5.4 and with [9, Theorem 3.1]), eventually we reach the following stability result: there exist  $X > L$  and some constant  $z_1 \in \mathbb{R}$  such that

$$\sup_{y \geq X} |v(t, y) - \phi(y - c_b t + z_1)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

Equivalently,

$$\sup_{x \geq X - ct} |u(t, x) - \phi(x - (c_b - c)t + z_1)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (6.22)$$

This proves the stability result (1.15) in Theorem 1.5, which also demonstrates that the rightward propagation speed is  $c_b - c$ .

From now on, we will focus on the proof of the “leftward” propagation with speed  $c_b + c < 0$ . To start with, we give a stronger estimate than (6.1) in Lemma 6.1.

**Lemma 6.3.** *Assume that  $c \leq -c_m$  and  $c_b > 0$ . Then for any  $\varepsilon > 0$ , there exist  $X_{3,\varepsilon} > L$ ,  $T_{3,\varepsilon} > 0$  and  $z_{3,\varepsilon} \in \mathbb{R}$  such that*

$$u(t, x) \leq \phi(-x - (c_b + c)(t - T_{3,\varepsilon}) + z_{3,\varepsilon}) + \varepsilon e^{-\delta(t-T_{3,\varepsilon})} + \varepsilon e^{-\mu(x-X_{3,\varepsilon})}, \quad t \geq T_{3,\varepsilon}, \quad x \geq X_{3,\varepsilon}, \quad (6.23)$$

with  $\mu > 0$  and  $\delta > 0$  as given in Lemma 6.2.

*Proof.* Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  be defined as in the proof of Lemma 6.2 (all independent of  $\varepsilon$ ). It follows from Lemma 6.1 that the conclusion of Lemma 6.3 immediately holds true when  $\varepsilon \geq \delta$ , with  $z_{3,\varepsilon} = X$  and  $T_{3,\varepsilon} = T$ . It is left to deal with the case where

$$0 < \varepsilon < \delta.$$

Let us introduce for convenience further parameters. Fix  $C_\varepsilon > C > 0$  such that

$$\phi \geq 1 - \frac{\varepsilon}{2} \quad \text{in } (-\infty, -C_\varepsilon], \quad \phi \leq \frac{\varepsilon}{2} \quad \text{in } [C_\varepsilon, +\infty). \quad (6.24)$$

Denote

$$\omega_\varepsilon := \frac{\varepsilon \omega}{\delta}. \quad (6.25)$$

Finally, we choose  $A_\varepsilon > \omega_\varepsilon + C_\varepsilon$  such that (notice that  $\mu^2 - c\mu + \max_{[0,1+2\delta]} |f'_b| > 0$ )

$$\left( \mu^2 - c\mu + \max_{[0,1+2\delta]} |f'_b| \right) e^{-\mu(A_\varepsilon - \omega_\varepsilon - C_\varepsilon)} < \delta. \quad (6.26)$$

We simply repeat the arguments used in the proofs of (6.8) and (6.9) by replacing  $\delta$  with  $\varepsilon$ . Lemma 2.2 gives the existence of  $T_{3,\varepsilon} > 0$  large enough such that

$$u(T_{3,\varepsilon}, x) < 1 + \frac{\varepsilon}{2} \quad \text{uniformly for } x \in \mathbb{R}. \quad (6.27)$$

Fix  $X_{3,\varepsilon} > L$  and  $\bar{X}_{3,\varepsilon} := X_{3,\varepsilon} + A_\varepsilon + C_\varepsilon$ . We deduce from Lemma 2.5 that, up to increasing  $T_{3,\varepsilon}$ ,

$$u(t, x) < \varepsilon \quad \text{for } t \geq T_{3,\varepsilon} \text{ and } x \in [X_{3,\varepsilon}, \bar{X}_{3,\varepsilon}]. \quad (6.28)$$

Define now for  $t \geq T_{3,\varepsilon}$  and  $x \geq X_{3,\varepsilon}$ ,

$$\bar{u}(t, x) = \phi(\bar{\xi}(t, x)) + \varepsilon e^{-\delta(t-T_{3,\varepsilon})} + \varepsilon e^{-\mu(x-X_{3,\varepsilon})},$$

with

$$\bar{\xi}(t, x) := -x + X_{3,\varepsilon} - (c_b + c)(t - T_{3,\varepsilon}) + \omega_\varepsilon e^{-\delta(t-T_{3,\varepsilon})} - \omega_\varepsilon + A_\varepsilon.$$

We now check that  $\bar{u}$  is a supersolution to (1.1) for  $t \geq T_{3,\varepsilon}$  and  $x \geq X_{3,\varepsilon}$ . In fact, at time  $T_{3,\varepsilon}$ , we infer from (6.28) that

$$\bar{u}(T_{3,\varepsilon}, x) \geq \varepsilon > u(T_{3,\varepsilon}, x) \quad \text{for } X_{3,\varepsilon} \leq x \leq \bar{X}_{3,\varepsilon}.$$

On the other hand, in the case where  $x \geq \bar{X}_{3,\varepsilon}$ , that is,  $-x + X_{3,\varepsilon} + A_\varepsilon \leq -C_\varepsilon$ , (6.24) and (6.27) imply

$$\bar{u}(T_{3,\varepsilon}, x) \geq \phi(-x + X_{3,\varepsilon} + A_\varepsilon) + \varepsilon \geq \phi(-C_\varepsilon) + \varepsilon \geq 1 + \frac{\varepsilon}{2} > u(T_{3,\varepsilon}, x).$$

Moreover, at  $x = X_{3,\varepsilon}$ , it is easy to see that  $\bar{u}(t, X_{3,\varepsilon}) \geq \varepsilon > u(t, X_{3,\varepsilon})$  for all  $t \geq T_{3,\varepsilon}$  due to (6.28). It is then left to check that  $\mathcal{N}\bar{u}(t, x) := \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for  $t \geq T_{3,\varepsilon}$  and  $x \geq X_{3,\varepsilon}$ .

After a straightforward computation, we arrive at

$$\mathcal{N}\bar{u}(t, x) = f_b(\phi(\bar{\xi}(t, x))) - f_b(\bar{u}(t, x)) - \delta\omega_\varepsilon e^{-\delta(t-T_{3,\varepsilon})} \phi'(\bar{\xi}(t, x)) - \delta\varepsilon e^{-\delta(t-T_{3,\varepsilon})} + (c\mu - \mu^2)\varepsilon e^{-\mu(x-X_{3,\varepsilon})}.$$

We divide into three cases.

- if  $\bar{\xi}(t, x) \leq -C$ , one has  $1 - \varepsilon/2 \leq \phi(\bar{\xi}(t, x)) < 1$ , therefore  $1 - \delta < 1 - \varepsilon/2 \leq \bar{u}(t, x) < 1 + 2\varepsilon < 1 + 2\delta$ . Thus, it follows from (6.2)–(6.3) that

$$\mathcal{N}\bar{u}(t, x) \geq \left( -\frac{f'_b(1)}{2} - \delta \right) \varepsilon e^{-\delta(t-T_{3,\varepsilon})} + \left( -\frac{f'_b(1)}{2} - \mu^2 + c\mu \right) \varepsilon e^{-\mu(x-X_{3,\varepsilon})} > 0;$$

- if  $\bar{\xi}(t, x) \geq C$ , one derives  $0 < \phi(\bar{\xi}(t, x)) \leq \varepsilon/2 < \delta$ , and then  $0 < \bar{u}(t, x) \leq 3\varepsilon < 3\delta$ . Together with (6.2)–(6.3), there then holds

$$\mathcal{N}\bar{u}(t, x) \geq \left( -\frac{f'_b(0)}{2} - \delta \right) \varepsilon e^{-\delta(t-T_{3,\varepsilon})} + \left( -\frac{f'_b(0)}{2} - \mu^2 + c\mu \right) \varepsilon e^{-\mu(x-X_{3,\varepsilon})} > 0;$$

- if  $-C \leq \bar{\xi}(t, x) \leq C$ , then  $0 < \bar{u}(t, x) \leq 1 + 2\varepsilon < 1 + 2\delta$ . By noticing that  $x - X_{3,\varepsilon} \geq -(c_b + c)(t - T_{3,\varepsilon}) - \omega_\varepsilon + A_\varepsilon - C \geq -(c_b + c)(t - T_{3,\varepsilon}) - \omega_\varepsilon + A_\varepsilon - C_\varepsilon$ , we have  $e^{-\mu(x-X_{3,\varepsilon})} \leq e^{-\mu(-(c_b+c)(t-T_{3,\varepsilon})-\omega_\varepsilon+A_\varepsilon-C_\varepsilon)}$ . One then infers from (6.3), (6.5), (6.6) and (6.26) that

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) \geq & - \max_{[0, 1+2\delta]} |f'_b| \left( \varepsilon e^{-\delta(t-T_{3,\varepsilon})} + \varepsilon e^{-\mu(x-X_{3,\varepsilon})} \right) + \kappa\omega_\varepsilon \delta e^{-\delta(t-T_{3,\varepsilon})} \\ & - \delta\varepsilon e^{-\delta(t-T_{3,\varepsilon})} - (\mu^2 - c\mu)\varepsilon e^{-\mu(x-X_{3,\varepsilon})} \end{aligned}$$

$$\begin{aligned}
&= \left( \kappa\omega - \max_{[0,1+2\delta]} |f'_b| - \delta \right) \varepsilon e^{-\delta(t-T_{3,\varepsilon})} - \left( \max_{[0,1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \varepsilon e^{-\mu(x-X_{3,\varepsilon})} \\
&\geq \left( \kappa\omega - \max_{[0,1+2\delta]} |f'_b| - \delta \right) \varepsilon e^{-\delta(t-T_{3,\varepsilon})} - \left( \max_{[0,1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \varepsilon e^{-\mu(-(c_b+c)(t-T_{3,\varepsilon})-\omega_\varepsilon+A_\varepsilon-C_\varepsilon)} \\
&\geq \left( \kappa\omega - \max_{[0,1+2\delta]} |f'_b| - 2\delta \right) \varepsilon e^{-\delta(t-T_{3,\varepsilon})} \geq 0.
\end{aligned}$$

Consequently, we have shown that  $\mathcal{N}\bar{u}(t, x) = \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for  $t \geq T_{3,\varepsilon}$  and  $x \geq X_{3,\varepsilon}$ . The maximum principle implies that

$$u(t, x) \leq \bar{u}(t, x) = \phi(-x + X_{3,\varepsilon} - (c_b + c)(t - T_{3,\varepsilon}) + \omega_\varepsilon e^{-\delta(t-T_{3,\varepsilon})} - \omega_\varepsilon + A_\varepsilon) + \varepsilon e^{-\delta(t-T_{3,\varepsilon})} + \varepsilon e^{-\mu(x-X_{3,\varepsilon})}$$

for  $t \geq T_{3,\varepsilon}$  and  $x \geq X_{3,\varepsilon}$ . Thus, (6.23) is achieved by taking  $z_{3,\varepsilon} = X_{3,\varepsilon} - \omega_\varepsilon + A_\varepsilon$ . This completes the proof of Lemma 6.3.  $\square$

In the same spirit of Lemma 6.3, we give below a strong version of (6.14) in Lemma 6.2, which concerns the lower estimate of the leftward propagation front.

**Lemma 6.4.** *Assume that  $c \leq -c_m$  and  $c_b > 0$ . Let  $v$  be as given in Lemma 6.2. Then for any  $\varepsilon > 0$ , there exist  $X_{4,\varepsilon} > L$ ,  $T_{4,\varepsilon} > 0$  and  $z_{4,\varepsilon} \in \mathbb{R}$  such that*

$$v(t, y) \geq \phi(-y - c_b(t - T_{4,\varepsilon}) + z_{4,\varepsilon}) - \varepsilon e^{-\delta(t-T_{4,\varepsilon})} - \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} \quad \text{for } t \geq T_{4,\varepsilon} \text{ and } y \leq X_{4,\varepsilon}, \quad (6.29)$$

with  $\mu > 0$  and  $\delta > 0$  as given in Lemma 6.2.

Let  $u$  be as given in Theorem 1.5. It turns out that (6.29) is parallel to the following estimate of  $u$  for  $t \geq T_{4,\varepsilon}$  and  $x \leq X_{4,\varepsilon} - ct$ ,

$$u(t, x) \geq \phi(-x - (c + c_b)(t - T_{4,\varepsilon}) - cT_{4,\varepsilon} + z_{4,\varepsilon}) - \varepsilon e^{-\delta(t-T_{4,\varepsilon})} - \varepsilon e^{-\mu(X_{4,\varepsilon}-x-ct)}, \quad (6.30)$$

with  $\mu > 0$  and  $\delta > 0$  as given in Lemma 6.2.

*Proof.* The proof is essentially very similar to that of Lemma 6.3. We sketch the details for the sake of completeness.

Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  be defined as in the proof of Lemma 6.2 (independent of  $\varepsilon$ ). We first notice that the case of  $\varepsilon \geq \delta$  is done in Lemma 6.2. Therefore, it is sufficient to consider the case of  $0 < \varepsilon < \delta$ . As before, we introduce parameters  $C_\varepsilon > 0$  and  $\omega_\varepsilon > 0$  as in (6.24)–(6.25) for convenience.

Fix  $X_{4,\varepsilon} > C_\varepsilon + \omega_\varepsilon + L$  such that

$$\left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) e^{-\mu(X_{4,\varepsilon}-C_\varepsilon-\omega_\varepsilon)} < \delta. \quad (6.31)$$

We infer from (6.11) that  $v(t, y) \geq w(t, y) \geq \phi(-y - c_b t + \xi_1) - Me^{-\eta t}$  for  $t > 0$  and  $y \leq 0$ , and that  $v(t, y) \geq w(t, y) \geq \phi(y - c_b t + \xi_2) - Me^{-\eta t}$  for  $t > 0$  and  $y \geq 0$ . We can then choose  $T_{4,\varepsilon} > \max((2C_\varepsilon + \xi_1)/c_b, (C_\varepsilon + X_{4,\varepsilon} + \xi_2)/c_b, -(1/\eta) \ln(\varepsilon/(2M))) > 0$  such that

$$\begin{aligned}
v(t, y) &\geq w(t, y) \geq \phi(-y - c_b t + \xi_1) - Me^{-\eta t} \\
&\geq \phi(C_\varepsilon - c_b T_{4,\varepsilon} + \xi_1) - Me^{-\eta T_{4,\varepsilon}} \geq \phi(-C_\varepsilon) - \frac{\varepsilon}{2} \geq 1 - \varepsilon, \quad t \geq T_{4,\varepsilon}, \quad -C_\varepsilon \leq y \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
v(t, y) &\geq w(t, y) \geq \phi(y - c_b t + \xi_2) - Me^{-\eta t} \\
&\geq \phi(X_{4,\varepsilon} - c_b T_{4,\varepsilon} + \xi_2) - Me^{-\eta T_{4,\varepsilon}} \geq \phi(-C_\varepsilon) - \frac{\varepsilon}{2} \geq 1 - \varepsilon, \quad t \geq T_{4,\varepsilon}, \quad 0 \leq y \leq X_{4,\varepsilon}.
\end{aligned}$$

Combining the above two inequalities, we obtain that

$$v(t, y) \geq 1 - \varepsilon \quad \text{for } t \geq T_{4,\varepsilon} \text{ and } -C_\varepsilon \leq y \leq X_{4,\varepsilon}. \quad (6.32)$$

For  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ , let us define

$$\underline{v}(t, y) = \phi(\zeta(t, y)) - \varepsilon e^{-\delta(t-T_{4,\varepsilon})} - \varepsilon e^{-\mu(X_{4,\varepsilon}-y)},$$

with  $\zeta(t, y) = -y - c_b(t - T_{4,\varepsilon}) - \omega_\varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \omega_\varepsilon$ . We shall prove that  $\underline{v}$  is a subsolution for (8.1) for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ . Indeed, at time  $t = T_{4,\varepsilon}$ , we notice that  $\underline{v}(T_{4,\varepsilon}, y) \leq 1 - \varepsilon - \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} \leq v(T_{4,\varepsilon}, y)$  for  $-C_\varepsilon \leq y \leq X_{4,\varepsilon}$  by virtue of (6.32). For  $y \leq -C_\varepsilon$ , the observation that  $\zeta(T_{4,\varepsilon}, y) \geq C_\varepsilon$  implies that  $\underline{v}(T_{4,\varepsilon}, y) \leq 0 < v(T_{4,\varepsilon}, y)$  for  $y \leq -C_\varepsilon$ . Consequently, we derive that  $\underline{v}(T_{4,\varepsilon}, y) \leq v(T_{4,\varepsilon}, y)$  for  $y \leq X_{4,\varepsilon}$ . At  $x = X_{4,\varepsilon}$ , (6.32) implies that  $v(t, X_{4,\varepsilon}) \geq 1 - \varepsilon \geq \underline{v}(t, X_{4,\varepsilon})$  for all  $t \geq T_{4,\varepsilon}$ . Therefore, it remains to verify that  $\underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f(y - ct, \underline{v}(t, y)) \leq 0$  for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ . Since  $\underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f(y - ct, \underline{v}(t, y)) \leq \underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f_b(\underline{v}(t, y)) =: \mathcal{L}\underline{v}(t, y)$  for  $t \geq 0$  and  $y \in \mathbb{R}$ , it is therefore sufficient to check that  $\mathcal{L}\underline{v}(t, y) \leq 0$  for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ . By a direct computation, we have that for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ ,

$$\mathcal{L}\underline{v}(t, y) = f_b(\phi(\zeta(t, y))) - f_b(\underline{v}(t, y)) + \phi'(\zeta(t, y))\omega_\varepsilon \delta e^{-\delta(t-T_{4,\varepsilon})} + \delta \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \mu^2 \varepsilon e^{-\mu(X_{4,\varepsilon}-y)}.$$

Again, we distinguish between three cases:

- if  $\zeta(t, y) \leq -C$ ,  $1 > \phi(\zeta(t, y)) \geq 1 - \varepsilon/2$  and  $1 > \underline{v} \geq 1 - 3\varepsilon$ . One infers from (6.16)–(6.17) that

$$\mathcal{L}\underline{v} \leq \left( \frac{f'_b(1)}{2} + \delta \right) \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \left( \frac{f'_b(1)}{2} + \mu^2 \right) \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} < 0;$$

- if  $\zeta(t, y) \geq C$ , we have  $\varepsilon > \phi(\zeta(t, y)) > 0$  and  $\varepsilon > \underline{v} \geq -2\varepsilon$ . Thanks to (6.16)–(6.17), one has

$$\mathcal{L}\underline{v} \leq \left( \frac{f'_b(0)}{2} + \delta \right) \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \left( \frac{f'_b(0)}{2} + \mu^2 \right) \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} < 0;$$

- if  $-C \leq \zeta(t, y) \leq C$ , it follows that  $-y + X_{4,\varepsilon} \geq c_b(t - T_{4,\varepsilon}) + X_{4,\varepsilon} - \omega_\varepsilon - C \geq c_b(t - T_{4,\varepsilon}) + X_{4,\varepsilon} - \omega_\varepsilon - C_\varepsilon$ . One deduces from (6.17)–(6.18) and (6.31) that

$$\begin{aligned} \mathcal{L}\underline{v} &\leq \max_{[-2\delta, 1]} |f'_b| \left( \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} \right) - \kappa \omega_\varepsilon \delta e^{-\delta(t-T_{4,\varepsilon})} + \delta \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \mu^2 \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} \\ &= \left( \max_{[-2\delta, 1]} |f'_b| + \delta - \kappa \omega \right) \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) \varepsilon e^{-\mu(X_{4,\varepsilon}-y)} \\ &\leq \left( \max_{[-2\delta, 1]} |f'_b| + \delta - \kappa \omega \right) \varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) \varepsilon e^{-\mu(c_b(t-T_{4,\varepsilon}) + X_{4,\varepsilon} - \omega_\varepsilon - C_\varepsilon)} \\ &\leq \left( \max_{[-2\delta, 1]} |f'_b| + 2\delta - \kappa \omega \right) \varepsilon e^{-\delta(t-T_{4,\varepsilon})} \leq 0. \end{aligned}$$

We conclude that  $\mathcal{L}\underline{v}(t, y) \leq 0$  for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ , which implies that  $\underline{v}$  is a subsolution of (8.1) for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ . The comparison principle then gives that for  $t \geq T_{4,\varepsilon}$  and  $y \leq X_{4,\varepsilon}$ ,

$$v(t, y) \geq \underline{v}(t, y) = \phi(-y - c_b(t - T_{4,\varepsilon}) - \omega_\varepsilon e^{-\delta(t-T_{4,\varepsilon})} + \omega_\varepsilon) - \varepsilon e^{-\delta(t-T_{4,\varepsilon})} - \varepsilon e^{-\mu(X_{4,\varepsilon}-y)}.$$

Therefore, (6.29) is achieved by taking  $z_{4,\varepsilon} = \omega_\varepsilon$  and by noticing that  $\phi' < 0$  in  $\mathbb{R}$ . This completes the proof.  $\square$

**Lemma 6.5.** Assume that  $c \leq -c_m$  and  $c_b > 0$ . Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  satisfy (6.2)–(6.6) and (6.16)–(6.18). If there exist  $\varepsilon \in (0, \delta]$ ,  $t_1 > 0$ ,  $x_1 > L$  and  $\xi_1 \in \mathbb{R}$  such that

$$\sup_{x \leq x_1 - ct_1} \left( u(t_1, x) - \phi(-x - (c_b + c)t_1 + \xi_1) \right) \leq \varepsilon, \quad (6.33)$$

$$\sup_{x \leq x_1} u(t, x) \leq \varepsilon \quad \text{and} \quad u(t, x_1 - ct) \leq 1 + \varepsilon/2 \quad \text{for } t \geq t_1, \quad (6.34)$$

$$\phi(-x_1 - c_b t_1 + \xi_1) \geq 1 - \varepsilon/2, \quad (6.35)$$

and

$$\left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu(-(c_b+c)t_1 - x_1 - \omega_\varepsilon + \xi_1 - C)} + \left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 \right) e^{-\mu(c_b t_1 + x_1 - \xi_1 - C)} \leq \delta \quad (6.36)$$

with  $\omega_\varepsilon = \varepsilon\omega/\delta$ , then there exists  $M_1 > 0$  such that the following holds true:

$$\sup_{x \leq x_1 - ct} \left( u(t, x) - \phi(-x - (c_b + c)t + \xi_1) \right) \leq M_1 \varepsilon \quad \text{for all } t \geq t_1.$$

*Proof.* The proof is based on a comparison argument. First of all, we observe that  $\sup_{x \leq x_1} (u(t, x) - \phi(-x - (c_b + c)t + \xi_1)) \leq \varepsilon$  for all  $t \geq t_1$ , thanks to (6.34). Let us now consider  $t \geq t_1$  and  $x_1 \leq x \leq x_1 - ct$ . Define

$$\bar{u}(t, x) = \phi(-x - (c_b + c)t + \omega_\varepsilon e^{-\delta(t-t_1)} - \omega_\varepsilon + \xi_1) + \varepsilon e^{-\delta(t-t_1)} + \varepsilon e^{-\mu(x-x_1)} + \varepsilon e^{-\mu(x_1-ct-x)}$$

for  $t \geq t_1$  and  $x_1 \leq x \leq x_1 - ct$ . Let us prove that  $\bar{u}$  is a supersolution of (1.1) for  $t \geq t_1$  and  $x_1 \leq x \leq x_1 - ct$ .

In fact, at time  $t = t_1$ , one infers from (6.33) that  $\bar{u}(t_1, x) \geq \phi(-x - (c_b + c)t_1 + \xi_1) + \varepsilon \geq u(t_1, x)$  for all  $x_1 \leq x \leq x_1 - ct_1$ . On the other hand, at  $x = x_1$ , it follows from (6.34)–(6.35) that  $\bar{u}(t, x_1) \geq \varepsilon \geq u(t, x_1)$  and  $\bar{u}(t, x_1 - ct) \geq \phi(-x_1 - c_b t_1 + \xi_1) + \varepsilon \geq 1 + \varepsilon/2 \geq u(t, x_1 - ct)$  for all  $t \geq t_1$ . It then remains to show that  $\mathcal{N}\bar{u}(t, x) := \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for all  $t \geq t_1$  and  $x_1 \leq x \leq x_1 - ct$ . For convenience, we set

$$\xi(t, x) := -x - (c_b + c)t + \omega_\varepsilon e^{-\delta(t-t_1)} - \omega_\varepsilon + \xi_1.$$

A straightforward computation implies that

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) &= f_b(\phi(\xi(t, x))) - f_b(\bar{u}(t, x)) - \phi'(\xi(t, x))\omega_\varepsilon \delta e^{-\delta(t-t_1)} - \delta \varepsilon e^{-\delta(t-t_1)} \\ &\quad - (\mu^2 - c\mu)\varepsilon e^{-\mu(x-x_1)} - \mu^2 \varepsilon e^{-\mu(x_1-ct-x)}. \end{aligned}$$

We distinguish three cases:

- if  $\xi(t, x) \leq -C$ , then  $1 - \delta/2 \leq \phi(\xi(t, x)) < 1$  by (6.4), hence  $1 - \delta/2 \leq \bar{u}(t, x) \leq 1 + 2\delta$ ; therefore, by using (6.2)–(6.3) and (6.16), along with the negativity of  $\phi'$  and  $f'_b(1)$ , it follows that

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) &\geq \left( -\frac{f'_b(1)}{2} - \delta \right) \varepsilon e^{-\delta(t-t_1)} + \left( -\frac{f'_b(1)}{2} - \mu^2 + c\mu \right) \varepsilon e^{-\mu(x-x_1)} \\ &\quad + \left( -\frac{f'_b(1)}{2} - \mu^2 \right) \varepsilon e^{-\mu(x_1-ct-x)} > 0; \end{aligned}$$

- if  $\xi(t, x) \geq C$ , then  $0 < \phi(\xi(t, x)) \leq \delta$  by (6.4) and thus  $0 \leq \bar{u}(t, x) \leq 3\delta$ ; therefore, owing to (6.2)–(6.3) and (6.16) as well as the negativity of  $\phi'$  and  $f'_b(0)$ , it follows that

$$\begin{aligned} \mathcal{N}\bar{u}(t, x) &\geq \left( -\frac{f'_b(0)}{2} - \delta \right) \varepsilon e^{-\delta(t-t_1)} + \left( -\frac{f'_b(0)}{2} - \mu^2 + c\mu \right) \varepsilon e^{-\mu(x-x_1)} \\ &\quad + \left( -\frac{f'_b(0)}{2} - \mu^2 \right) \varepsilon e^{-\mu(x_1-ct-x)} > 0; \end{aligned}$$

- if  $-C \leq \xi(t, x) \leq C$ , one has  $x - x_1 \geq -(c + c_b)(t - t_1) - (c + c_b)t_1 - x_1 + \omega_\varepsilon e^{-\delta(t-t_1)} - \omega_\varepsilon + \xi_1 - C \geq -(c_b + c)(t - t_1) - (c_b + c)t_1 - x_1 - \omega_\varepsilon + \xi_1 - C$ , hence  $e^{-\mu(x-x_1)} \leq e^{-\mu(-(c_b+c)(t-t_1)-(c_b+c)t_1-x_1-\omega_\varepsilon+\xi_1-C)}$ . On the other hand,  $x_1 - ct - x \geq x_1 - ct + (c_b + c)t - \omega_\varepsilon e^{-\delta(t-t_1)} + \omega_\varepsilon - \xi_1 - C \geq x_1 + c_b t - \xi_1 - C$ , which implies that  $e^{-\mu(x_1-ct-x)} \leq e^{-\mu(c_b(t-t_1)+c_b t_1+x_1-\xi_1-C)}$ . In view of  $\omega_\varepsilon = \varepsilon\omega/\delta$ , one infers from (6.3), (6.5), (6.6), (6.17) and (6.1), that

$$\begin{aligned}
\mathcal{N}\bar{u}(t, x) &\geq - \max_{[0, 1+2\delta]} |f'_b| \left( \varepsilon e^{-\delta(t-t_1)} + \varepsilon e^{-\mu(x-x_1)} + \varepsilon e^{-\mu(x_1-ct-x)} \right) + \kappa\omega\varepsilon e^{-\delta(t-t_1)} \\
&\quad - \delta\varepsilon e^{-\delta(t-t_1)} - (\mu^2 - c\mu)\varepsilon e^{-\mu(x-x_1)} - \mu^2\varepsilon e^{-\mu(x_1-ct-x)} \\
&= \left( \kappa\omega - \max_{[0, 1+2\delta]} |f'_b| - \delta \right) \varepsilon e^{-\delta(t-t_1)} - \left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \varepsilon e^{-\mu(x-x_1)} \\
&\quad - \left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 \right) \varepsilon e^{-\mu(x_1-ct-x)} \\
&\geq \left( \kappa\omega - \max_{[0, 1+2\delta]} |f'_b| - \delta \right) \varepsilon e^{-\delta(t-t_1)} - \left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 \right) \varepsilon e^{-\mu(c_b(t-t_1)+c_b t_1+x_1-\xi_1-C)} \\
&\quad - \left( \max_{[0, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) \varepsilon e^{-\mu(-(c_b+c)(t-t_1)-(c_b+c)t_1-x_1-\omega_\varepsilon+\xi_1-C)} \\
&\geq \left( \kappa\omega - \max_{[0, 1+2\delta]} |f'_b| - 2\delta \right) \varepsilon e^{-\delta(t-t_1)} \geq 0.
\end{aligned}$$

We then obtain that  $\mathcal{N}\bar{u}(t, x) = \bar{u}_t(t, x) - \bar{u}_{xx}(t, x) - c\bar{u}_x(t, x) - f_b(\bar{u}(t, x)) \geq 0$  for all  $t \geq t_1$  and  $x_1 \leq x \leq x_1 - ct$ . The comparison principle implies that

$$u(x, t) \leq \bar{u}(t, x) = \phi(-x - (c_b + c)t + \omega_\varepsilon e^{-\delta(t-t_1)} - \omega_\varepsilon + \xi_1) + \varepsilon e^{-\delta(t-t_1)} + \varepsilon e^{-\mu(x-x_1)} + \varepsilon e^{-\mu(x_1-ct-x)}$$

for  $t \geq t_1$  and  $x_1 \leq x \leq x_1 - ct$ . For these  $t$  and  $x$ , since  $\phi' < 0$ , one derives that

$$u(x, t) \leq \phi(-x - (c_b + c)t - \omega_\varepsilon + \xi_1) + 3\varepsilon \leq \phi(-x - (c_b + c)t + \xi_1) + \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} + 3\varepsilon.$$

In conclusion, one has

$$\sup_{x_1 \leq x \leq x_1 - ct} \left( u(t, x) - \phi(-x - (c_b + c)t_1 + \xi_1) \right) \leq M_1 \varepsilon \quad \text{for all } t \geq t_1,$$

where  $M_1 := \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} / \varepsilon + 3 = \omega \|\phi'\|_{L^\infty(\mathbb{R})} / \delta + 3$  is independent of  $\varepsilon$ ,  $t_1$ ,  $x_1$  and  $\xi_1$ . Lemma 6.5 is therefore achieved, with  $M_1$  chosen above.  $\square$

**Lemma 6.6.** *Assume that  $c \leq -c_m$  and  $c_b > 0$ . Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  satisfy (6.2)–(6.6) and (6.16)–(6.18). If there exist  $\varepsilon \in (0, \delta]$ ,  $t_2 > 0$ ,  $x_2 \in \mathbb{R}$  and  $\xi_2 \in \mathbb{R}$  such that*

$$-\varepsilon \leq \sup_{y \leq x_2} \left( v(t_2, y) - \phi(-y - c_b t_2 + \xi_2) \right), \quad (6.37)$$

$$1 - \varepsilon \leq v(t, x_2) \quad \text{for all } t \geq t_2, \quad (6.38)$$

and

$$\left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) e^{-\mu(c_b t_2 + x_2 - \omega_\varepsilon - \xi_2 - C)} \leq \delta \quad (6.39)$$

with  $\omega_\varepsilon = \varepsilon\omega/\delta$ , then there exists  $M_2 > 0$  such that the following holds true:

$$-M_2 \varepsilon \leq \sup_{y \leq x_2} \left( v(t, y) - \phi(-y - c_b t + \xi_2) \right) \quad \text{for all } t \geq t_2.$$

*Proof.* We first claim that

$$\underline{v}(t, y) = \phi(-y - c_b t - \omega_\varepsilon e^{-\delta(t-t_2)} + \omega_\varepsilon + \xi_2) - \varepsilon e^{-\delta(t-t_2)} - \varepsilon e^{-\mu(x_2-y)}$$

is a subsolution of (8.1) for  $t \geq t_2$  and  $y \leq x_2$ , for which it is sufficient to show that  $\bar{v}$  is a subsolution to  $v_t = v_{yy} + f_b(v)$  for  $t \geq t_2$  and  $y \leq x_2$ .

At time  $t = t_2$ , one has  $\underline{v}(t_2, y) \leq \phi(-y - c_b t_2 + \xi_2) - \varepsilon \leq v(t_2, y)$  for all  $y \leq x_2$ , thanks to (6.37). Moreover,  $\underline{v}(t, x_2) \leq 1 - \varepsilon \leq v(t, x_2)$  for all  $t \geq t_2$ , owing to (6.38). It remains to show that  $\mathcal{N}\underline{v}(t, y) := \underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f_b(\underline{v}(t, y)) \leq 0$  for all  $t \geq t_2$  and  $y \leq x_2$ . For convenience, we set

$$\eta(t, y) := -y - c_b t - \omega_\varepsilon e^{-\delta(t-t_2)} + \omega_\varepsilon + \xi_2.$$

By a straightforward computation, one has

$$\mathcal{N}\underline{v}(t, y) = f_b(\phi(\eta(t, y))) - f_b(\underline{v}(t, y)) + \phi'(\eta(t, y))\omega_\varepsilon \delta e^{-\delta(t-t_2)} + \varepsilon \delta e^{-\delta(t-t_2)} + \mu^2 \varepsilon e^{-\mu(x_2-y)}.$$

We divide our analysis into three cases:

- if  $\eta(t, y) \leq -C$ , then  $1 - \delta/2 \leq \phi(\eta(t, y)) < 1$  by (6.4), hence  $1 > \underline{v}(t, y) \geq 1 - \delta/2 - 2\varepsilon \geq 1 - 3\delta$ ; therefore it follows from (6.16)–(6.17), together with the negativity of  $\phi'$  and  $f'_b(1)$ , that

$$\begin{aligned} \mathcal{N}\underline{v}(t, y) &\leq \frac{f'_b(1)}{2} \left( \varepsilon e^{-\delta(t-t_2)} + \varepsilon e^{-\mu(x_2-y)} \right) + \varepsilon \delta e^{-\delta(t-t_2)} + \mu^2 \varepsilon e^{-\mu(x_2-y)} \\ &= \left( \frac{f'_b(1)}{2} + \delta \right) \varepsilon e^{-\delta(t-t_2)} + \left( \frac{f'_b(1)}{2} + \mu^2 \right) \varepsilon e^{-\mu(x_2-y)} \leq 0; \end{aligned}$$

- if  $\eta(t, y) \geq C$ , then  $0 < \phi(\eta(t, y)) \leq \delta$  by (6.4) and thus  $-2\delta \leq -2\varepsilon \leq \underline{v}(t, y) \leq \delta$ ; therefore, owing to (6.16)–(6.17) as well as the negativity of  $\phi'$  and  $f'_b(0)$ , it follows that

$$\begin{aligned} \mathcal{N}\underline{v}(t, y) &\leq \frac{f'_b(0)}{2} \left( \varepsilon e^{-\delta(t-t_2)} + \varepsilon e^{-\mu(x_2-y)} \right) + \varepsilon \delta e^{-\delta(t-t_2)} + \mu^2 \varepsilon e^{-\mu(x_2-y)} \\ &= \left( \frac{f'_b(0)}{2} + \delta \right) \varepsilon e^{-\delta(t-t_2)} + \left( \frac{f'_b(0)}{2} + \mu^2 \right) \varepsilon e^{-\mu(x_2-y)} \leq 0; \end{aligned}$$

- if  $-C \leq \eta(t, y) \leq C$ , one has  $x_2 - y \geq c_b(t - t_2) + c_b t_2 + x_2 + \omega_\varepsilon e^{-\delta(t-t_2)} - \omega_\varepsilon - \xi_2 - C \geq c_b(t - t_2) + c_b t_2 + x_2 - \omega_\varepsilon - \xi_2 - C$ , hence  $e^{-\mu(x_2-y)} \leq e^{-\mu(c_b(t-t_2)+c_b t_2+x_2-\omega_\varepsilon-\xi_2-C)}$ ; since  $\omega_\varepsilon = \varepsilon\omega/\delta$ , one infers from (6.5)–(6.6), (6.17)–(6.18) and (6.39) that

$$\begin{aligned} \mathcal{N}\underline{v}(t, y) &\leq \max_{[-2\delta, 1]} |f'_b| \left( \varepsilon e^{-\delta(t-t_2)} + \varepsilon e^{-\mu(x_2-y)} \right) - \kappa\omega_\varepsilon \delta e^{-\delta(t-t_2)} + \varepsilon \delta e^{-\delta(t-t_2)} + \mu^2 \varepsilon e^{-\mu(x_2-y)} \\ &\leq \left( \max_{[-2\delta, 1]} |f'_b| - \kappa\omega + \delta \right) \varepsilon e^{-\delta(t-t_2)} + \left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) \varepsilon e^{-\mu(c_b(t-t_2)+c_b t_2+x_2-\omega_\varepsilon-\xi_2-C)} \\ &\leq \left( \max_{[-2\delta, 1]} |f'_b| - \kappa\omega + 2\delta \right) \varepsilon e^{-\delta(t-t_2)} \leq 0. \end{aligned}$$

Eventually, one concludes that  $\mathcal{N}\underline{v}(t, y) = \underline{v}_t(t, y) - \underline{v}_{yy}(t, y) - f_b(\underline{v}(t, y)) \leq 0$  for all  $t \geq t_2$  and  $y \leq x_2$ . The maximum principle implies that

$$v(t, y) \geq \underline{v}(t, y) = \phi(-y - c_b t - \omega_\varepsilon e^{-\delta(t-t_2)} + \omega_\varepsilon + \xi_2) - \varepsilon e^{-\delta(t-t_2)} - \varepsilon e^{-\mu(x_2-y)}$$

for all  $t \geq t_2$  and  $y \leq x_2$ . For these  $t$  and  $y$ , since  $\phi' < 0$ , one derives that

$$v(t, y) \geq \phi(-y - c_b t + \omega_\varepsilon + \xi_2) - 2\varepsilon \geq \phi(-y - c_b t + \xi_2) - \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})} - 2\varepsilon.$$

In conclusion, one has

$$-M_2\varepsilon \leq \sup_{y \leq x_2} \left( v(t, y) - \phi(-y - c_b t_2 + \xi_2) \right) \quad \text{for all } t \geq t_2,$$

where  $M_2 := \omega_\varepsilon \|\phi'\|_{L^\infty(\mathbb{R})}/\varepsilon + 2 = \omega \|\phi'\|_{L^\infty(\mathbb{R})}/\delta + 2$  is independent of  $\varepsilon$ ,  $t_2$ ,  $x_2$  and  $\xi_2$ . The proof of Lemma 6.6 is thereby complete.  $\square$

Lemma 6.6 can be equivalently written as:

**Lemma 6.7.** *Assume that  $c \leq -c_m$  and  $c_b > 0$ . Let  $\mu > 0$ ,  $\delta > 0$ ,  $C > 0$ ,  $\kappa > 0$  and  $\omega > 0$  satisfy (6.2)–(6.6) and (6.16)–(6.18). If there exist  $\varepsilon \in (0, \delta]$ ,  $t_2 > 0$ ,  $x_2 \in \mathbb{R}$  and  $\xi_2 \in \mathbb{R}$  such that*

$$-\varepsilon \leq \sup_{x \leq x_2 - ct_2} \left( u(t_2, x) - \phi(-x - (c_b + c)t_2 + \xi_2) \right), \quad (6.40)$$

$$1 - \varepsilon \leq u(t, x_2 - ct) \quad \text{for all } t \geq t_2, \quad (6.41)$$

and

$$\left( \max_{[-2\delta, 1]} |f'_b| + \mu^2 \right) e^{-\mu(c_b t_2 + x_2 - \omega_\varepsilon - \xi_2 - C)} \leq \delta$$

with  $\omega_\varepsilon = \varepsilon\omega/\delta$ , then there exists  $M_2 > 0$  such that the following holds true:

$$-M_2\varepsilon \leq \sup_{x \leq x_2 - ct} \left( u(t, x) - \phi(-x - (c_b + c)t + \xi_2) \right) \quad \text{for all } t \geq t_2.$$

We are now in a position to finish the proof of Theorem 1.5, by Lemma 6.5 and Lemma 6.7.

*End of proof of Theorem 1.5.* Let  $X_3 > L$ ,  $T_3 > 0$ ,  $z_3 \in \mathbb{R}$  be given as in Lemma 6.1, and let  $X_4 > L$ ,  $T_4 > 0$ ,  $z_4 \in \mathbb{R}$ ,  $\mu > 0$  and  $\delta > 0$  be given as in the proof of Lemma 6.2. It follows from (6.1) and (6.15) that, for  $t \geq T := \max(T_3, T_4)$  such that  $X_3 < X_4 - cT$  and for  $X_3 \leq x \leq X_4 - ct$ ,

$$\begin{aligned} & \phi(-x - (c_b + c)t + z_4) - \delta e^{-\delta(t-T_4)} - \delta e^{-\mu(X_4 - x - ct)} \\ & \leq u(t, x) \leq \phi(-x - (c_b + c)(t - T_3) + z_3) + \delta e^{-\delta(t-T_3)} + \delta e^{-\mu(x-X_3)}. \end{aligned} \quad (6.42)$$

Consider any sequence  $(t_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}$  such that  $t_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . From parabolic estimates, the functions

$$(t, z) \in \mathbb{R}^2 \mapsto u_n(t, z) := u(t + t_n, z - (c_b + c)t_n)$$

converge as  $n \rightarrow +\infty$ , locally uniformly in  $\mathbb{R}^2$ , to a classical solution  $u_\infty$  of  $(u_\infty)_t = (u_\infty)_{zz} + c(u_\infty)_z + f_b(u_\infty)$  in  $\mathbb{R}^2$ . By applying (6.42) at  $(t + t_n, z - (c_b + c)t_n)$  and then passing to the limit as  $n \rightarrow +\infty$ , it follows that

$$\phi(-z - (c_b + c)t + z_4) \leq u_\infty(t, z) \leq \phi(-z - (c_b + c)(t - T_3) + z_3) \quad \text{in } \mathbb{R}^2. \quad (6.43)$$

Then, [9, Theorem 3.1] implies that there exists  $\eta \in \mathbb{R}$  such that

$$u_\infty(t, z) = \phi(-z - (c_b + c)t + \eta) \quad \text{for } (t, z) \in \mathbb{R}^2. \quad (6.44)$$

whence

$$u_n(t, z) \rightarrow \phi(-z - (c_b + c)t + \eta) \quad \text{as } n \rightarrow +\infty \quad \text{locally uniformly in } (t, z) \in \mathbb{R} \times \mathbb{R}. \quad (6.45)$$

Consider now any  $\varepsilon \in (0, \delta/3]$ . Let  $B_\varepsilon > 0$  be such that

$$\phi \geq 1 - \frac{\varepsilon}{2} \quad \text{in } (-\infty, -B_\varepsilon] \quad \text{and} \quad \phi \leq \frac{\varepsilon}{2} \quad \text{in } [B_\varepsilon, +\infty). \quad (6.46)$$

Set  $E_1 := \min(-B_\varepsilon + \eta, -B_\varepsilon + (c_b + c)T_3 + z_3)$  and  $E_2 := \max(B_\varepsilon + \eta, B_\varepsilon + z_4) > E_1$ . It then follows from (6.45) that

$$\sup_{E_1 \leq z \leq E_2} |u_n(0, z) - \phi(-z + \eta)| \leq \varepsilon \quad (6.47)$$

for all  $n$  large enough. In view of  $t_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ , (6.42), (6.46) and the definition of  $E_1$  and  $E_2$ , we then have

$$\begin{cases} 0 < u_n(0, z) \leq \varepsilon & \text{for all } z \leq E_1, \\ 1 - \varepsilon \leq u_n(0, z) \leq 1 + \varepsilon & \text{for all } E_2 \leq z \leq E_2 + \frac{c_b}{2}t_n, \end{cases} \quad (6.48)$$

for all  $n$  large enough. Furthermore, since  $E_2 \geq B_\varepsilon + \eta$  and  $E_1 \leq -B_\varepsilon + \eta$ ,

$$\begin{cases} 0 < \phi(-z + \eta) \leq \frac{\varepsilon}{2} < \varepsilon & \text{for all } z \leq E_1, \\ 1 - \varepsilon < 1 - \frac{\varepsilon}{2} \leq \phi(-z + \eta) < 1 & \text{for all } z \geq E_2. \end{cases} \quad (6.49)$$

It then follows from (6.47)-(6.49) that, for all  $n$  large enough

$$|u_n(0, z) - \phi(-z + \eta)| \leq 2\varepsilon \quad \text{for all } z \leq E_2 + \frac{c_b}{2}t_n.$$

Due to the definition of  $u_n(t, z)$ , one has

$$|u(t_n, x) - \phi(-x - (c_b + c)t_n + \eta)| \leq 2\varepsilon \quad \text{for all } x \leq E_2 - \frac{c_b}{2}t_n - ct_n. \quad (6.50)$$

On the other hand, one infers from (6.23) and (6.30) that for all  $n$  large enough,

$$\begin{aligned} 1 - 3\varepsilon &\leq \phi(-x - (c_b + c)(t_n - T_{4,\varepsilon}) - cT_{4,\varepsilon} + z_{4,\varepsilon}) - \varepsilon e^{-\delta(t_n - T_{4,\varepsilon})} - \varepsilon e^{-\mu(X_{4,\varepsilon} - x - ct_n)} \\ &\leq u(t_n, x) \leq \phi(-x - (c_b + c)(t_n - T_{3,\varepsilon}) + z_{3,\varepsilon}) + \varepsilon e^{-\delta(t_n - T_{3,\varepsilon})} + \varepsilon e^{-\mu(x - X_{3,\varepsilon})} \leq 1 + 2\varepsilon \end{aligned}$$

for  $E_2 - \frac{c_b}{2}t_n - ct_n \leq x \leq X_{4,\varepsilon} - ct_n$ , where  $X_{3,\varepsilon} > L$ ,  $X_{4,\varepsilon} > L$ ,  $T_{3,\varepsilon} > 0$ ,  $T_{4,\varepsilon} > 0$ ,  $z_{3,\varepsilon} \in \mathbb{R}$ ,  $z_{4,\varepsilon} \in \mathbb{R}$  were given in Lemmas 6.3–6.4. Notice also that for all  $n$  large enough

$$1 - \varepsilon \leq \phi(-x - (c_b + c)t_n + \eta) < 1 \quad \text{for all } E_2 - \frac{c_b}{2}t_n - ct_n \leq x \leq X_{4,\varepsilon} - ct_n.$$

Combining the above two inequalities yields that for all  $n$  large enough,

$$|u(t_n, x) - \phi(-x - (c_b + c)t_n + \eta)| \leq 3\varepsilon \quad \text{for all } x \in \left[ E_2 - \frac{c_b}{2}t_n - ct_n, X_{4,\varepsilon} - ct_n \right].$$

Together with (6.50), one derives that for all  $n$  large enough,

$$\sup_{x \leq X_{4,\varepsilon} - ct_n} |u(t_n, x) - \phi(-x - (c_b + c)t_n + \eta)| \leq 3\varepsilon.$$

We are now in a position to prove (1.16) by applying Lemmas 6.5 and 6.7. Let  $x_1 = x_2 = X_{4,\varepsilon}$  and  $\xi_1 = \xi_2 = \eta$ . We infer from Lemma 2.2 and (6.11) that for all  $n$  large enough,

$$1 - \varepsilon \leq u(t, X_{4,\varepsilon} - ct) \leq 1 + \frac{\varepsilon}{2} \quad \text{for } t \geq t_n.$$

Moreover, one has for all  $n$  large enough that  $\sup_{x \leq X_{4,\varepsilon}} u(t, x) \leq \varepsilon$  for all  $t \geq t_n$  by Lemma 2.5, and  $\phi(-X_{4,\varepsilon} - c_b t_n + \eta) \geq 1 - \frac{\varepsilon}{2}$ , thanks to  $c_b > 0$ , and that

$$\left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 - c\mu \right) e^{-\mu(-(c_b+c)t_n - X_{4,\varepsilon} - \omega_\varepsilon + \eta - C)} + \left( \max_{[-2\delta, 1+2\delta]} |f'_b| + \mu^2 \right) e^{-\mu(c_b t_n + X_{4,\varepsilon} - \omega_\varepsilon - \eta - C)} \leq \delta$$

It then follows from Lemmas 6.5 and 6.7 (applied with  $t_1 = t_2 = t_n$ ,  $x_1 = x_2 = X_{4,\varepsilon}$  and  $\xi_1 = \xi_2 = \eta$  and  $3\varepsilon$  instead of  $\varepsilon$ ) that for  $n$  large enough,

$$|u(t, x) - \phi(-x - (c_b + c)t + \xi_2)| \leq 3M\varepsilon \quad \text{for all } t \geq t_n \text{ and } x \leq X_{4,\varepsilon} - ct,$$

for  $M := \max(M_1, M_2)$ , where  $M_1$  and  $M_2$  were given in Lemmas 6.5 and 6.7 respectively. Since  $\varepsilon \in (0, \delta/3]$  was arbitrarily chosen, one eventually infers that

$$\sup_{x \leq X_{4,\varepsilon} - ct} |u(t, x) - \phi(-x - (c_b + c)t + \xi_2)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Thus, (1.16) is achieved, with  $z_2 = \xi_2$ . This completes the proof of Theorem 1.5.  $\square$

## 6.2 Extinction: Proof of Theorem 1.6

*Proof of Theorem 1.6.* Remember that  $c \leq -c_m$ .

*Step 1: proof under condition (i).* We first consider the case where  $c_b < 0$ . Our proof includes two parts:  $c_b < -c_m$  and  $c \in (c_b, -c_m]$ ;  $c_b < 0$  and  $c \leq \min(-c_m, c_b)$ .

*Step 1.1.* Assume that  $c_b < -c_m$  and  $c \in (c_b, -c_m]$ . Since  $c_b < c$ , we can construct  $\bar{u}$  as in (4.7) in the proof of Theorem 1.3 (which relies only on  $c_b < c$ ). We readily check that  $\bar{u}$  is a supersolution such that the solution  $u$  is blocked in the right direction, i.e.,  $u(t, x) \rightarrow 0$  as  $x \rightarrow +\infty$  uniformly in  $t \geq 0$ . Then, for any  $\varepsilon \in (0, \theta)$ , there exists  $x_0 > L$  sufficiently large such that

$$0 < \sup_{x \geq x_0} u(t, x) < \varepsilon \quad \text{for } t \geq 0.$$

On the other hand, since  $c \leq -c_m$ , we infer from Lemma 2.5 that there exists  $t_0 > 0$  large enough such that

$$0 < \sup_{x \leq x_0} u(t, x) < \varepsilon \quad \text{for } t \geq t_0.$$

Therefore, one concludes that  $u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$ , uniformly for  $x \in \mathbb{R}$ .

*Step 1.2.* Assume that  $c_b < 0$  and  $c \leq \min(-c_m, c_b)$ . We divide into two cases.

*Case 1.* Assume that  $c \leq -c_m$  and  $c < c_b$ . Choose  $\varepsilon_0 \in (0, -c_b)$ . Lemma 6.1 immediately implies that there exists  $X > L$  such that

$$\limsup_{t \rightarrow +\infty} \sup_{X \leq x < -(c_b + c + \varepsilon_0)t} u(t, x) = 0. \quad (6.51)$$

On the other hand, it follows from (5.5) in Lemma 5.2 (i) that

$$\limsup_{t \rightarrow +\infty} \sup_{x > (c_b - c + \varepsilon_0)t} u(t, x) = 0. \quad (6.52)$$

Combining (6.51)–(6.52), along with the fact that  $c_b - c + \varepsilon_0 < -c_b - c - \varepsilon_0$  as well as Lemma 2.5, one reaches  $\lim_{t \rightarrow +\infty} u(t, x) = 0$  uniformly in  $x \in \mathbb{R}$ .

*Case 2.* Assume that  $c = c_b \leq -c_m$ . For any  $\varepsilon \in (0, \theta/2)$ , let  $f_{b,\varepsilon}$  be a  $C^1(\mathbb{R})$  function such that  $f_{b,\varepsilon} \geq f_b$  in  $\mathbb{R}$ , and

$$\begin{cases} f_{b,\varepsilon}(0) = f_{b,\varepsilon}(\theta - \varepsilon) = f_{b,\varepsilon}(1) = 0, & f'_{b,\varepsilon}(0) < 0, \quad f'_{b,\varepsilon}(\theta) > 0, \quad f'_{b,\varepsilon}(1) < 0, \\ f_{b,\varepsilon} < 0 \text{ in } (0, \theta - \varepsilon), \quad f_{b,\varepsilon} > 0 \text{ in } (\theta - \varepsilon, 1), \quad \int_0^1 f_{b,\varepsilon}(s) ds < 0. \end{cases}$$

For each  $\varepsilon \in (0, \theta/2)$  small enough, let  $\phi_\varepsilon$  be the unique traveling front profile of  $u_t = u_{xx} + f_{b,\varepsilon}(u)$  such that

$$\phi_\varepsilon'' + c_{b,\varepsilon} \phi_\varepsilon' + f_{b,\varepsilon}(\phi_\varepsilon) = 0 \text{ in } \mathbb{R}, \quad \phi_\varepsilon(0) = \theta, \quad \phi_\varepsilon(-\infty) = 1, \quad \phi_\varepsilon(+\infty) = 0,$$

with speed  $c_{b,\varepsilon} > c_b$ . Then,  $\phi_\varepsilon \rightarrow \phi$  in  $C_{loc}^2(\mathbb{R})$  and  $c_{b,\varepsilon} \rightarrow c_b$  as  $\varepsilon \rightarrow 0$ . We then fix  $\varepsilon \in (0, \theta/2)$  small enough such that  $c_b < c_{b,\varepsilon} < 0$ .

Let now  $u_\varepsilon$  be the solution to the Cauchy problem (1.5) with  $f_\varepsilon(x, s)$  instead of  $f(x, s)$  for  $(x, s) \in \mathbb{R} \times \mathbb{R}_+$  starting from the initial condition  $u_0$ , where we assume that  $f_\varepsilon(x, s) = f(x, s)$  for  $(x, s) \in (-\infty, L] \times \mathbb{R}_+$ , while  $f(x, s) = f_{b,\varepsilon}(s)$  for  $x \in [L, +\infty)$ . By the comparison principle, we have that  $u_\varepsilon(t, x) > u(t, x)$  for  $t > 0$  and  $x \in \mathbb{R}$ . We now claim that  $u_\varepsilon$  extincts, therefore so does  $u$ .

In fact, we notice that  $c \leq -c_m$  and  $c < c_{b,\varepsilon}$ . On the one hand, since  $c \leq -c_m$ , we apply Lemma 6.1 and derive that there exist  $T > 0$ ,  $X > L$ ,  $z \in \mathbb{R}$  and  $\delta > 0$  such that

$$u_\varepsilon(t, x) \leq \phi_\varepsilon(-x + X - (c_{b,\varepsilon} + c)(t - T) + z) + \delta e^{-\delta(t-T)} \quad \text{for } t \geq T \text{ and } x \geq X.$$

This implies that  $\lim_{t \rightarrow +\infty} \sup_{X \leq x < -(c_{b,\varepsilon} + c)t} u_\varepsilon(t, x) = 0$ . Together with Lemma 2.5, we then obtain

$$\lim_{t \rightarrow +\infty} \sup_{x < -(c_{b,\varepsilon} + c)t} u_\varepsilon(t, x) = 0 \quad (6.53)$$

On the other hand, due to  $c < c_{b,\varepsilon}$ , we adapt Lemma 5.2 (i) and deduce that there exist  $X_1 > L$ ,  $T_1 > 0$ ,  $z_1 \in \mathbb{R}$ ,  $\mu_1 > 0$  and  $\delta_1 > 0$  such that

$$u_\varepsilon(t, x) \leq \phi_\varepsilon(x - (c_{b,\varepsilon} - c)(t - T_1) + z_1) + \delta_1 e^{-\delta_1(t-T_1)} + \delta_1 e^{-\mu_1(x-X_1)} \quad \text{for all } t \geq T_1 \text{ and } x \geq X_1,$$

which yields that

$$\lim_{t \rightarrow +\infty} \sup_{x > (c_{b,\varepsilon} - c)t} u_\varepsilon(t, x) = 0. \quad (6.54)$$

Combining (6.53)–(6.54), along with the fact that  $c_{b,\varepsilon} < 0$ , we immediately get  $\sup_{x \in \mathbb{R}} u_\varepsilon(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$ . That is,  $u_\varepsilon$  goes extinction.

*Step 2: proof under condition (ii).* Assume now that  $\text{spt}(u_0)$  is and  $\|u_0\|_{L^\infty(\mathbb{R})} < \theta$ . For any  $\varepsilon \in (0, \theta)$ , we choose  $f_{b,\varepsilon} \in C^1(\mathbb{R})$  such that  $f_{b,\varepsilon} \geq f_b$  and

$$\begin{cases} f_{b,\varepsilon}(\varepsilon) = f_{b,\varepsilon}(\theta) = f_{b,\varepsilon}(1) = 0, & f'_{b,\varepsilon}(\varepsilon) < 0, & f'_{b,\varepsilon}(\theta) > 0, & f'_{b,\varepsilon}(1) < 0, \\ f_{b,\varepsilon} > 0 \text{ in } (-\infty, \varepsilon) \cup (\theta, 1), & f_{b,\varepsilon} < 0 \text{ in } (\varepsilon, \theta) \cup (1, +\infty). \end{cases} \quad (6.55)$$

We now prove that there exist monostable decreasing traveling waves of

$$w_t = w_{xx} + f_{b,\varepsilon}(w), \quad (t, x) \in \mathbb{R}^2, \quad (6.56)$$

connecting  $\theta$  and  $\varepsilon$ , and moving to the left, one of which will serve as an upper barrier for the function  $u$ . To do so, set  $z(t, x) = \theta - \omega(t, x)$  for  $(t, x) \in \mathbb{R}^2$ , then  $z$  satisfies

$$z_t = z_{xx} + g(z), \quad (t, x) \in \mathbb{R}^2, \quad (6.57)$$

with  $g(z) = -f_{b,\varepsilon}(\theta - z)$  being positive in  $(0, \theta - \varepsilon)$ . Then it is known that (6.57) admits traveling front solutions  $\psi(x - \nu t)$  with  $\psi : \mathbb{R} \rightarrow (0, \theta - \varepsilon)$  such that  $\psi'' + \nu \psi' + g(\psi) = 0$  in  $\mathbb{R}$ ,  $0 < \psi < \theta - \varepsilon$ ,  $\psi(-\infty) = \theta$  and  $\psi(+\infty) = \varepsilon$  if and only if  $\nu \geq \nu_{\min}$  for some  $\nu_{\min} \geq 2\sqrt{f'_{b,\varepsilon}(\theta)}$ . Fix  $\nu_0 > \max(-c, \nu_{\min})$ . It is straightforward to see that  $\theta - \psi(-x - \nu_0 t)$  is a decreasing traveling wave for (6.56) connecting  $\theta$  and  $\varepsilon$ , and moving to the left with speed  $\nu_0 > 0$ .

On the other hand, one can easily verify that  $\varphi_{c_m+c}(-x - (c_m + c)t)$  is a supersolution to (1.1) for  $(t, x) \in \mathbb{R}^2$  which has an increasing profile and moves to the left with speed  $c_m + c \leq 0$ .

Since  $\text{spt}(u_0)$  is included in the bistable region and  $\|u_0\|_{L^\infty(\mathbb{R})} < \theta$ , one can fix  $B \in \mathbb{R}$  such that  $\theta - \psi(-x - B) > u_0(x)$  for  $x \geq L$ . Then, choose  $A \in \mathbb{R}$  such that  $\varphi_{c_m+c}(-L + A) = \theta - \psi(-L - B)$ . Then, one can check that

$$\bar{u}(t, x) := \min(\varphi_{c_m+c}(-x - (c_m + c)t + A), \theta - \psi(-x - (\nu_0 + c)t - B)), \quad t \geq 0, x \in \mathbb{R},$$

is a supersolution to (1.1) such that  $\bar{u}(0, x) > u_0(x)$  for  $x \in \mathbb{R}$ . The comparison principle gives that  $\bar{u}(t, x) > u(t, x)$  for  $t > 0$  and  $x \in \mathbb{R}$ . Noticing that  $\theta - \psi(-x - (\nu_0 + c)t - B)$  indeed moves to the left with speed  $\nu_0 + c > 0$ , it implies that  $\sup_{x \geq X} u(t, x) \leq \varepsilon$  as  $t \rightarrow +\infty$  for some  $X > L$ . Since  $\varepsilon > 0$  was arbitrarily chosen, together with  $\sup_{x \leq -X} u(t, x) \leq \varepsilon$  as  $t \rightarrow +\infty$  by Lemma 2.5, it follows that  $u(t, x) \rightarrow 0$  as  $t \rightarrow +\infty$ , uniformly for  $x \in \mathbb{R}$ . This completes the proof of Theorem 1.6.  $\square$

## 7 Sharp estimate of the level sets in the left direction when $c \in (-c_m, +\infty)$ : Proof of Theorem 1.7

The proof of Theorem 1.7 is highly motivated by [30]. In fact, the estimate of the level sets from exterior (upper bound) comes from the supersolution and is a straightforward application of [30]. Regarding the estimate from interior (lower bound), the idea in [30] can be adapted with a slight modification, so that it works for every  $c > -c_m$ . The details will be sketched below for the sake of completeness.

*Proof of Theorem 1.7.* Remember that the level set is defined as  $E_\varrho^-(t) = \inf\{x \in \mathbb{R} | u(t, x) = \varrho\}$  for any  $t > 0$  and for any given  $\varrho \in (0, 1)$ .

First of all, consider (8.1) with  $f_m(v)$  instead of  $f(y - ct, v)$ , for which let us define  $S_\varrho(t) = \inf\{y \in \mathbb{R} | v(t, y) = \varrho\}$  for  $t > 0$  and for any  $\varrho \in (0, 1)$ . Then it follows from [30] that  $S_\varrho(t) = -c_m t + 3/(2\lambda^*) \ln t + O_{t \rightarrow +\infty}(1)$ , with  $2\lambda^* = c_m$ . Correspondingly, the level sets of equation (1.1) with  $f_m(u)$  instead of  $f(x, u)$  in the left direction behave asymptotically like  $-(c_m + c)t + 3/(2\lambda^*) \ln t + O_{t \rightarrow +\infty}(1)$ . This implies that  $E_\varrho^-(t) \geq -(c_m + c)t + 3/(2\lambda^*) \ln t + C_1$  for some  $C_1 \in \mathbb{R}$ . In what follows, it remains to prove that  $E_\varrho^-(t) \leq -(c_m + c)t + 3/(2\lambda^*) \ln t + C_2$  for some  $C_2 \in \mathbb{R}$ .

*Step 1.* The linearized problem with Dirichlet boundary condition at  $-(c + c_m)t$ . It is easy to verify that the function  $w(t, \hat{x}) = u(t, -(c + c_m)t - \hat{x} - L)$  solves

$$w_t - w_{\hat{x}\hat{x}} - c_m w_{\hat{x}} - f(-(c + c_m)t - \hat{x} - L, w) = 0, \quad t > 0, \hat{x} \in \mathbb{R}.$$

In particular, we have  $w_t - w_{\hat{x}\hat{x}} - c_m w_{\hat{x}} - f_m(w) = 0$  for  $t > 0$  and  $\hat{x} > 0$ . Consider the following linear equation with Dirichlet boundary condition at  $\hat{x} = 0$ :

$$\begin{cases} z_t - z_{\hat{x}\hat{x}} - c_m z_{\hat{x}} - f'_m(0)z = 0, & t > 0, \hat{x} > 0, \\ z(t, 0) = 0, & t > 0. \end{cases} \quad (7.1)$$

In view of  $c_m = 2\sqrt{f'_m(0)} = 2\lambda^*$ , the function  $p(t, \hat{x}) = e^{\lambda^* \hat{x}} z(t, \hat{x})$  solves

$$\begin{cases} p_t = p_{\hat{x}\hat{x}}, & t > 0, \hat{x} > 0, \\ p(t, 0) = 0, & t > 0, \end{cases}$$

hence

$$z(t, \hat{x}) = \frac{e^{-\lambda^* \hat{x}}}{\sqrt{4\pi t}} \int_0^{+\infty} \left( e^{-\frac{(\hat{x}-v)^2}{4t}} - e^{-\frac{(\hat{x}+v)^2}{4t}} \right) p(0, v) dv \quad \text{for all } t > 0 \text{ and } \hat{x} \geq 0,$$

which implies that

$$z(t, \hat{x}) \sim C \hat{x} e^{-\lambda^* \hat{x} - \frac{\hat{x}^2}{4t}} t^{-\frac{3}{2}} \text{ as } t \rightarrow +\infty$$

in the interval  $\hat{x} \in [0, \sqrt{t}]$ , where  $C$  depends only on  $p(0, \cdot)$ .

*Step 2.* Lower bound at  $x = -(c + c_m)t - O(\sqrt{t})$ . In view of  $f_m \in C^2([0, 1])$ , there exists  $M > 0$  so that

$$f(s) - f'_m(0)s \geq -Ms^2 \text{ for } s \in [0, s_0) \quad \text{for some } s_0 > 0.$$

Notice that  $z(t, \hat{x}) \leq C(t+1)^{-\frac{3}{2}}$  for  $t > 0$ . Let  $a(t)$  solve  $a'(t) = -CM(t+1)^{-\frac{3}{2}}a(t)^2$ ,  $t > 0$ . Such  $a(t)$  can be chosen uniformly bounded from above and below:  $0 < a_0 < a(t) < a_1 < +\infty$ . Then, the function  $\underline{w}(t, \hat{x}) = a(t)z(t, \hat{x})$  satisfies

$$\begin{aligned} \underline{w}_t - \underline{w}_{\hat{x}\hat{x}} - c_m \underline{w}_{\hat{x}} - f_m(\underline{w}) &= a'(t)z + a(t)(z_t - z_{\hat{x}\hat{x}} - c_m z_{\hat{x}} - f'_m(0)z) + f'(0)az - f(az) \\ &\leq a'(t)z + M(az)^2 = (a'(t) + Ma(t)^2)z = 0. \end{aligned}$$

Therefore, for any  $\sigma > 0$ , there exists  $\hat{\delta} > 0$  such that

$$u(t, -(c+c_m)t - \hat{x} - L) = w(t, \hat{x}) \geq \underline{w}(t, \hat{x}) \geq \hat{\delta} \hat{x} e^{-\lambda^* \hat{x}} t^{-\frac{3}{2}} \quad \text{for all } t \geq 2 \text{ and } \hat{x} \in [0, \sigma\sqrt{t}]. \quad (7.2)$$

*Step 3.* The approximate traveling fronts are subsolutions of (1.1) for  $-(c+c_m)t - O(\sqrt{t}) - L \leq x \leq \mu_0 t - L$  with any fixed  $-(c+c_m) < \mu_0 < \min(0, c_m - c)$ . Fix  $\sigma > 0$  and let  $\xi(t) = \sigma\sqrt{t}$ . By the estimate (7.2), we will construct an explicit subsolution of (1.1) on the interval  $-(c+c_m)t - \xi(t) - L \leq x \leq \mu_0 t - L$  for  $t$  large enough. This subsolution will be an approximate traveling front, moving with leftward speed  $c_m + c > 0$ .

According to (7.2), there exist  $\tilde{\delta} > 0$  and  $T_1 \geq 0$  such that

$$u(t, -(c+c_m)t - \xi(t) - L) \geq \underline{w}(t, \xi(t)) \geq \tilde{\delta} \xi(t) e^{-\lambda^* \xi(t)} t^{-\frac{3}{2}} \quad (7.3)$$

for all  $t \geq T_1$ . It follows from Theorems 1.1–1.2 that

$$\inf_{\mu_1 t \leq x \leq \mu_2 t} u(t, x) \rightarrow 1 \quad \text{for any } -(c_m + c) < \mu_1 < \mu_2 < \min(0, c_m - c).$$

Given any  $\varrho \in (0, 1)$ , fix  $\bar{\varrho} \in (\varrho, 1)$ . Therefore, there is  $T_2 (\geq T_1)$  such that  $u(t, \mu_0 t - L) \geq \bar{\varrho}$  for all  $t \geq T_2$ .

Let  $f_1$  be a  $C^1$  function such that  $f_1 \leq f_m$  for  $u \in [0, \bar{\varrho}]$ ,  $f_1(0) = f_1(\bar{\varrho}) = 0$ ,  $f'_1(0) = f'_m(0)$  and  $f_1(s) > 0$  on  $(0, \bar{\varrho})$ . The function  $f_1$  then satisfies

$$f_1(s) \leq f_m(s) \leq f'_m(0)s = f'_1(0)s \quad \text{for all } s \in [0, \bar{\varrho}].$$

Then there exists a traveling front  $U_{c_m}(x - c_m t)$  of  $u_t = u_{xx} + f_1(u)$  such that  $0 < U_{c_m} < \bar{\varrho}$  in  $\mathbb{R}$ ,  $U_{c_m}(-\infty) = \bar{\varrho}$ ,  $U_{c_m}(+\infty) = 0$  with speed  $c_m = 2\sqrt{f'_m(0)}$ . The profile  $U_{c_m}$  is decreasing in  $\mathbb{R}$  and is such that

$$U_{c_m}(s) \sim \tilde{B} s e^{-\lambda^* s} \text{ as } s \rightarrow +\infty, \quad \text{for some } \tilde{B} > 0.$$

Let now  $\gamma > 0$  and fix  $x_1 \in \mathbb{R}$  large enough so that  $\tilde{B}(\gamma+1)e^{-\lambda^* x_1} \leq \tilde{\delta}$ . Since there exists  $T_3 \geq T_2$  such that

$$\frac{3}{2\lambda^*} \ln t + x_1 < \gamma \xi(t) \quad \text{for } t \geq T_3,$$

we have

$$U_{c_m} \left( \frac{3}{2\lambda^*} \ln t + \xi(t) + x_1 \right) \leq \tilde{\delta} \xi(t) e^{-\lambda^* \xi(t)} t^{-\frac{3}{2}}, \quad t \geq T_3. \quad (7.4)$$

On the other hand, in view of  $U_{c_m}(+\infty) = 0$  and  $\min_{x \in [-(c+c_m)T_3 - \xi(T_3) - L, \mu_0 T_3 - L]} u(T_3, x) > 0$ , there exists  $x_2 (\geq x_1)$  such that

$$U_{c_m} \left( -x - (c+c_m)T_3 + \frac{3}{2\lambda^*} \ln T_3 + x_2 - L \right) \leq u(T_3, x)$$

for all  $x \in [-(c+c_m)T_3 - \xi(T_3) - L, \mu_0 T_3 - L]$ . Define the subsolution  $\underline{u}$  as follows

$$\underline{u}(t, x) = U_{c_m} \left( -x - (c+c_m)t + \frac{3}{2\lambda^*} \ln t + x_2 - L \right)$$

for  $t \geq T_3$  and  $x \in [-(c + c_m)t - \xi(t) - L, \mu_0 t - L]$ .

It is easy to verify that  $\underline{u}(T_3, x) \leq u(T_3, x)$  for all  $x \in [-(c + c_m)T_3 - \xi(T_3) - L, \mu_0 T_3 - L]$ . Since  $x_2 \geq x_1$  and  $U_{c_m}$  is decreasing, along with (7.3)–(7.4), we have

$$\underline{u}(t, -(c + c_m)t - \xi(t) - L) = U_{c_m} \left( \xi(t) + \frac{3}{2\lambda^*} \ln t + x_2 \right) \leq \tilde{\delta} \xi(t) e^{-\lambda^* \xi(t)} t^{-\frac{3}{2}} \leq u(t, -(c + c_m)t - \xi(t) - L)$$

for all  $t \geq T_3 \geq T_1$ . Besides,  $\underline{u}(t, \mu_0 t - L) < \bar{\varrho} \leq u(t, \mu_0 t - L)$  for all  $t \geq T_3 \geq T_2$ .

Lastly, since  $f_1 \leq f_m$  in  $[0, \bar{\varrho}]$  and since  $U_{c_m}$  is decreasing and satisfies  $U_{c_m}'' + c_m U_{c_m}' + f_1(U_{c_m}) = 0$  for all  $t \geq T_3$  and  $x \in [-(c + c_m)t - \xi(t) - L, \mu_0 t - L]$ , we get

$$\underline{u}_t - \underline{u}_{xx} - c \underline{u}_x - f_m(\underline{u}) = \left( -(c + c_m) + \frac{3}{2\lambda^* t} \right) U_{c_m}'(\eta) - U_{c_m}''(\eta) + c U_{c_m}'(\eta) - f_1(U_{c_m}(\eta)) = \frac{3}{2\lambda^* t} U_{c_m}'(\eta) \leq 0$$

where  $\eta = -x - (c + c_m)t + 3/(2\lambda^*) \ln t + x_2 - L$ . Therefore, the function  $\underline{u}$  is a subsolution of (1.1) for all  $t \geq T_3$  and  $x \in [-(c + c_m)t - \xi(t) - L, \mu_0 t - L]$ . The comparison principle yields that

$$u(t, x) \geq \underline{u}(t, x) = U_{c_m} \left( -x - (c + c_m)t + \frac{3}{2\lambda^*} \ln t + x_2 - L \right) \quad (7.5)$$

for all  $t \geq T_3$  and  $x \in [-(c + c_m)t - \xi(t) - L, \mu_0 t - L]$ .

*Step 4.* Conclusion of the proof. The inequality (7.5) implies that for any given  $x' \in \mathbb{R}$

$$u \left( t, -(c + c_m)t + \frac{3}{2\lambda^*} \ln t + x' \right) \geq \underline{u} \left( t, -(c + c_m)t + \frac{3}{2\lambda^*} \ln t + x' \right) = U_{c_m}(-x' + x_2 - L) > 0$$

for  $t$  large enough. This proves that  $E_{\varrho}^-(t) \leq -(c_m + c)t + 3/(2\lambda^*) \ln t + C_2$  for some  $C_2 \in \mathbb{R}$ . This completes the proof of Theorem 1.7.  $\square$

In addition, assume that  $c > -c_m$ , it follows from the lines of [30, Theorem 1.2], one can further show that the solution  $u$  of (1.1) approaches the family of shifted traveling waves  $\varphi_{c_m}(-x - (c_m + c)t + 3/(2\lambda^*) \ln t + \xi(t))$  uniformly for  $x < [\min(0, c_m - c) - \varepsilon]t$  for  $\varepsilon > 0$  small enough, where  $\xi : (0, +\infty) \rightarrow \mathbb{R}$  is such that  $|\xi(t)| \leq C$  with some  $C > 0$ , and that the solution  $u$  converges along its level sets to the profile of the minimal traveling wave, for which we refer readers to [30] and will not give further details.

## 8 Discussion

In this section, we apply our main result to the shifting environment and making a comparison between our result and the existing ones.

By setting  $u(t, x) = v(t, x + ct)$ , equation (1.1) is recast as:

$$v_t = v_{yy} + f(y - ct, v), \quad t > 0, \quad y \in \mathbb{R}, \quad (8.1)$$

in which the parameter  $c$  is understood as the shifting speed, and the shifting growth  $f(s, v)$  is assumed decreasing in  $s \in [-L, L]$ , being of KPP type on the left semi-infinite region, while being of bistable type on the right semi-infinite region. With fixed diffusion, we aim to investigate the effect of shifting reaction. Of particular interest here is to give a full classification of the propagation phenomena for such a model subject to KPP-transition-bistable nonlinearities with any value of shifting speeds. The novelty is that here the strong Allee effect is taken into account with the KPP reaction as the complement so that the whole habitat can be either favorable plus favorable, or favorable plus unfavorable. In this sense, we can summarize the propagation dynamics for (8.1) as follows:

**Theorem 8.1.** *The solution  $v$  of (8.1) with a nonnegative continuous and compactly supported initial datum  $u_0 \not\equiv 0$  has the following propagation phenomena:*

- When  $c_b > 0$ , there holds

Range of $c$	$(-\infty, -c_m]$	$(-c_m, c_b]$	$(c_b, c_m)$	$[c_m, +\infty)$
Leftward speed	$c_b - \star$ ; Extinction-♣	$c_m$	$c_m$	$c_m$
	Theorem 1.5; Theorem 1.6	Theorem 1.2	Theorem 1.2	Theorem 1.1
Rightward speed	$c_b - \star$ ; Extinction-♣	$c_b$	$c$	$c_m$
	Theorem 1.5; Theorem 1.6	Theorem 1.4	Theorem 1.3	Theorem 1.1

Here, the notation  $\star$  represents that the localized initial condition is further assumed to be “large enough”, while  $\clubsuit$  denotes that  $\text{spt}(u_0)$  is included in the bistable region and  $\|u_0\|_{L^\infty(\mathbb{R})} < \theta$ .

- When  $-c_m < c_b < 0$ , there holds

Range of $c$	$(-\infty, -c_m]$	$(-c_m, c_b]$	$(c_b, c_m)$	$[c_m, +\infty)$
Leftward speed	Extinction	$c_m$	$c_m$	$c_m$
	Theorem 1.6	Theorem 1.2	Theorem 1.2	Theorem 1.1
Rightward speed	Extinction	$c_b$	$c$	$c_m$
	Theorem 1.6	Theorem 1.4	Theorem 1.3	Theorem 1.1

- When  $c_b \leq -c_m$ , there holds

Range of $c$	$(-\infty, c_b]$	$(c_b, -c_m]$	$(-c_m, c_m)$	$[c_m, +\infty)$
Leftward speed	Extinction	Extinction	$c_m$	$c_m$
	Theorem 1.6	Theorem 1.6	Theorem 1.2	Theorem 1.1
Rightward speed	Extinction	Extinction	$c$	$c_m$
	Theorem 1.6	Theorem 1.6	Theorem 1.3	Theorem 1.1

‘ Moreover,

- ▲ when  $v$  is propagating in the right direction with speed  $c$ , then  $v$  converges to the unique forced wave solution  $U$  where  $U(-\infty) = 1$  and  $U(+\infty) = 0$  (see Theorem 1.3);
- ▲ when  $v$  propagates to the left with speed  $c_b$  or propagates to the right with speed  $c_b$ ,  $u$  will eventually converge to the unique bistable traveling wave with a constant shift, see Theorems 1.5 and 1.4;
- ▲ when  $v$  propagates to the left with speed  $c_m$ , the level sets always admit a logarithmic time delay, see Theorem 1.7.

Given different values of the shifting speed  $c$ , the species might reside within the KPP and/or the bistable regions, resulting in significantly varying propagation dynamics after a long time. Theorem 8.1 shows that, on the one hand, very similar to the non-shifting case, the population localization and total population size can increase the possibility of persistence and propagation under a moving climate, and, on the other hand, mobility can both reduce and enhance the ability of population to track climate change in order for persistence and even propagation.

In 2009, Berestycki et al. [7] studied (8.1) by assuming that  $f(y, v)$  is of KPP type for  $0 < y < L$ , and  $f(y, v) < 0$  for  $y < 0$  and for  $y > L$  which embodies the assumption that the population grows logistically in a favorable region of length  $L$  but declines exponentially outside of that region. The authors [7] proved that the global dynamics (extinction, persistence as well as convergence to the unique positive solution of  $U'' + cU' + f(x, U) = 0$ ) is determined by the sign of the generalized

principal eigenvalue of the linearized problem around zero state. Berestycki and Rossi extended the large time dynamics result to high dimension [13] and to infinite cylindrical type domains [14], which was then investigated by Vo [47] for more general unfavorable media at infinity. In a particular situation  $f(y-ct, v) = v(r(y-ct)-v)$  in (8.1) with  $r : \mathbb{R} \rightarrow \mathbb{R}$  continuous, bounded and nonincreasing<sup>3</sup>, the KPP-decay case, i.e.  $r(-\infty) > 0 > r(+\infty)$ , was studied by Li et al. [38] and Fang et al. [21], showing that the species will extinct if  $c \leq -c_-$  and persist if  $c > -c_-$ , and when propagation occurs, the leftward speed is  $c_-$  and rightward speed is  $\min(c, c_-)$ . Fang et al. [24] further considered the spreading properties and forced waves in a time-periodic setting. Besides, the KPP-KPP case, i.e.  $r(-\infty) > r(+\infty) > 0$ , was investigated by Hu et al. [31], and by Lam and Yu [35] in which the spreading properties were studied, and moreover the frame of reaction-diffusion equations and integro-differential equations with a distributed time-delay were addressed in [35].

Yet, other types of growth functions (especially Allee effect) are also interesting in the investigation of population dynamics in moving habitats, which is known for instance in [2, 45]. Bouhours and Giletti [15] studied the propagation phenomena for (8.1) in cylindrical type domains with general monostable reaction  $f$  (including Allee effect). Bouhours and Nadin [16] considered (8.1) when the size of the favorable zone is bounded with general  $f$  (including monostable and bistable cases) in the favorable zone, and proved the existence of two speeds  $0 < \underline{c} \leq \bar{c} < +\infty$  such that the population persists for large enough initial data when  $0 < c < \underline{c}$  and goes extinction when  $c > \bar{c}$ . Recently, Li and Otto [39] investigated forced waves for (8.1) where the favorable region, characterized by strong Allee effect, is a bounded interval surrounded by the unfavorable ones. Beyond this, there have been extensive investigations on questions concerning the existence and further properties of forced waves for (8.1) of the form  $v(t, y) = V(y - ct)$  with prescribed speed  $c$ , see e.g. [8, 14, 32]. Spreading speeds in shifting environments have also been concerned, for instance, with nonlocal feature [1, 40, 50], with shifting diffusion [25], in free boundary problems [37], in systems [17, 20], in abstract setting [49], etc. We refer the readers to [18, 48] for comprehensive description of recent development and open challenging questions on reaction-diffusion problems in shifting environments.

We conclude with a brief discussion. The propagation dynamics resemble those in the KPP-decay case [21, 24, 38] when  $c_b \leq -c_m$ , where the bistable environment approximates the decay one. When  $c_b \in (-c_m, c_m)$ , the spreading properties are dominated by the KPP region for a positive and sufficiently large advection, i.e.  $c \geq c_m$ ; the propagation dynamics are determined by the sign of  $c_b$  and by the initial value for a negative and sufficiently large advection, i.e.  $c \leq -c_m$ ; the spreading phenomena are characterized by the sign of  $c_b - c$  when  $c$  is not large enough in either direction, i.e.  $c \in (-c_m, c_m)$ .

## Ethical Statement.

- Originality: The work presented in this manuscript is original and has not been published elsewhere in any form or language.
- Simultaneous Submission: This manuscript has not been submitted to more than one journal for simultaneous consideration.
- Availability of data and material: Not applicable.
- Conflict of Interest: There are no conflicts of interest related to this manuscript.

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<sup>3</sup>For convenience, here we restate the conclusions of some references by assuming that  $r$  is nonincreasing instead of nondecreasing.

## References

- [1] M. Alfaro, H. Berestycki, G. Raoul, *The effect of climate shift on a species submitted to dispersion, evolution, growth and nonlocal competition*, SIAM J. Math. Anal., 49 (2017), 562-596.
- [2] W. C. Allee, *The social life of animals*, W. W. Norton and Co., New York (1938).
- [3] D. G. Aronson, H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, In: Partial Differential Equations and Related Topics, Lecture Notes in Math. **446** (1975), Springer, Berlin, 5-49.
- [4] D. G. Aronson, H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. **30** (1978), 33-76.
- [5] B. Audoly, H. Berestycki, Y. Pomeau, *Réaction-diffusion en écoulement stationnaire rapide*, C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre **328** (2000), 255-262.
- [6] H. Berestycki, *The influence of advection on the propagation of fronts in reaction-diffusion equations*, In: Nonlinear PDEs in Condensed Matter and Reactive Flows, H. Berestycki and Y. Pomeau (eds.), Kluwer, 1-45 (2002).
- [7] H. Berestycki, O. Diekmann, C. J. Nagelkerke, P. A. Zegeling, *Can a species keep pace with a shifting climate?* Bull. Math. Biol., 71 (2009), 399-429.
- [8] H. Berestycki, J. Fang, *Forced waves of the Fisher-KPP equation in a shifting environment*, J. Differential Equations, 264 (2018), 2157-2183.
- [9] H. Berestycki, F. Hamel, *Generalized travelling waves for reaction-diffusion equations*, In: *Perspectives in Nonlinear Partial Differential Equations, in honor of H. Brezis*, Amer. Math. Soc. Contemp. Math. **446** (2007), 101-123.
- [10] H. Berestycki, F. Hamel, N. Nadirashvili, *The speed of propagation for KPP type problems. I - Periodic framework*, J. European Math. Soc. **7** (2005), 173-213.
- [11] H. Berestycki, B. Larrouturou, P.-L. Lions, *Multi-dimensional traveling-wave solutions of a flame propagation model*, Arch. Ration. Mech. Anal. **111** (1990), 3349.
- [12] H. Berestycki, B. Larrouturou, J.-M. Roquejoffre, *Stability of traveling fronts in a model for flame propagation. I. Linear analysis*, Arch. Ration. Mech. Anal. **117** (1992), 971-117.
- [13] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed. I. The case of the whole space*, Discrete Contin. Dyn. Syst., **21** (2008), 41-67.
- [14] H. Berestycki, L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed. II. Cylindrical-type domains*, Discrete Contin. Dyn. Syst., **25** (2009), 19-61.
- [15] J. Bouhours, T. Giletti, *Spreading and vanishing for a monostable reaction-diffusion equation with forced speed*, J. Dynam. Differential Equations, **31** (2019), 247-286.
- [16] J. Bouhours, G. Nadin, *A variational approach to reaction-diffusion equations with forced speed in dimension 1*, Discrete Contin. Dyn. Syst., **35** (2015), 1843-1872.
- [17] X. Chen, J.-C. Tsai, Y. Wu, *Longtime behavior of solutions of an SIS epidemiological model*, SIAM J. Math. Anal., 49 (2017), 3925-3950.
- [18] C. Cosner, *Challenges in modeling biological invasions and population distributions in a changing climate*, Ecol. Complex., **20** (2014), 258-263.
- [19] P. Constantin, A. Kiselev, L. Ryzhik, *Quenching of flames by fluid advection*, Commun. Pure Appl. Math. **54** (2001), 13201-1342.
- [20] F. Dong, J. Shang, W. Fagan, B. Li, *Persistence and spread of solutions in a two-species Lotka-Volterra competition-diffusion model with a shifting habitat*, SIAM J. Appl. Math. **81** (2021), 16001622.
- [21] J. Fang, Y. Lou, J. Wu, *Can pathogen spread keep pace with its host invasion?* SIAM J. Appl. Math., **76** (2016), 1633-1657.

- [22] P. C. Fife, B. McLeod, *The approach of solutions of nonlinear diffusion equations to traveling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335-361.
- [23] R. A. Fisher, *The advance of advantageous genes*, Ann. Eugenics **7** (1937), 355-369.
- [24] J. Fang, R. Peng, X.-Q. Zhao, *Propagation dynamics of a reaction-diffusion equation in a time-periodic shifting environment*, J. Math. Pures Appl. **147** (2021), 1-28.
- [25] G. Faye, T. Giletti, M. Holzer, *Asymptotic spreading for Fisher-KPP reaction-diffusion equations with heterogeneous shifting diffusivity*, Discrete Contin. Dyn. Syst. Ser. S **15** (2022), 24672496.
- [26] M. Freidlin, *On wavefront propagation in periodic media*, in *Stochastic analysis and applications*, editor M. Pinsky, Advances in Probability and related topics **7**, M. Dekker, New-York, 147-166, 1984.
- [27] F. Hamel, *Reaction-diffusion problems in cylinders with no invariance by translation, Part I: Small perturbations*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **14** (1997), 457-498.
- [28] F. Hamel, *Reaction-diffusion problems in cylinders with no invariance by translation. II. Monotone perturbations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **14** (1997), 555-596.
- [29] F. Hamel, F. Lutscher, M. Zhang, *Propagation and blocking in a two-patch reaction-diffusion model*, J. Math. Pures Appl. **168** (2022), 213-267.
- [30] F. Hamel, J. Nolen, J.-M. Roquejoffre, L. Ryzhik, *A short proof of the logarithmic Bramson correction in Fisher-KPP equations*, Netw. Heterog. Media **8** (2013), 275-289.
- [31] C. Hu, J. Shang, B. Li, *Spreading Speeds for ReactionDiffusion Equations with a Shifting Habitat*, J. Dynam. Differential Equations, **32** (2020), 1941-1964.
- [32] H. Hu, X. Zou, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc., **145** (2017), 4763-4771.
- [33] A. Kiselev, L. Ryzhik, *Enhancement of the traveling front speeds in reaction-diffusion equations with advection*, Ann. Inst. H. Poincaré, Analyse Non Linéaire **18** (2001), 309358.
- [34] A. N. Kolmogorov, I. G. Petrovskii, N. S. Piskunov, *Étude de l'équation de la chaleur de matière et son application à un problème biologique*, Bull. Moskov. Gos. Univ. Mat. Mekh. **1** (1937), 1-25.
- [35] K.-Y. Lam, X. Yu, *Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments*, J. Math. Pures Appl. **167** (2022), 147.
- [36] K.-S. Lau, *On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov*, J. Diff. Equations **59** (1985), 44-70.
- [37] C. Lei, Y. Du, *Asymptotic profile of the solution to a free boundary problem arising in a shifting climate model*, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), 895-911.
- [38] B. Li, S. Bewick, M. R. Barnard, W. F. Fagan, *Persistence and spread of a species with a shifting habitat edge*, SIAM J. Appl. Math., **74** (2014), 1397-1417.
- [39] B. Li, G. Otto, *Forced traveling waves in a reaction-diffusion equation with strong Allee effect and shifting habitat*, Bull. Math. Biol., **85** (2023), Paper No. 121.
- [40] W.-T. Li, J.-B. Wang, X.-Q. Zhao, *Spatial dynamics of a nonlocal dispersal population model in a shifting environment*, J. Nonlinear Sci. **28** (2018), 1189-1219.
- [41] J.-F. Mallordy, J.-M. Roquejoffre, *A parabolic equation of the KPP type in higher dimensions*, SIAM J. Math. Anal. **26** (1995), 120.
- [42] G. Papanicolaou, X. Xin, *Reaction-diffusion fronts in periodically layered media*, J. Statist. Phys. **63** (1991), 915931.
- [43] J.-M. Roquejoffre, *Stability of traveling fronts in a model for flame propagation. II. Nonlinear stability*, Arch. Rational Mech. Anal. **117** (1992), 119153.
- [44] J.-M. Roquejoffre, *Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 499552.

- [45] L. Roques, A. Roques, H. Berestycki, A. Kretzschmar, *A population facing climate change: joint influences of Allee effects and environmental boundary geometry*, Popul. Ecol. 50 (2008), 215-225.
- [46] K. Uchiyama, *The behavior of solutions of some nonlinear diffusion equations for large time*, J. Math. Kyoto Univ. 18 (1978), 453-508.
- [47] H.-H. Vo, *Persistence versus extinction under a climate change in mixed environments*, J. Differential Equations, 259 (2015), 4947-4988.
- [48] J.-B. Wang, W.-T. Li, F.-D. Dong, S.-X. Qiao, *Recent developments on spatial propagation for diffusion equations in shifting environments*, Discrete Contin. Dyn. Syst. Ser. B, 27(2022), 5101-5127.
- [49] T. Yi, X.-Q. Zhao, *Propagation dynamics for monotone evolution systems without spatial translation invariance*, J. Funct. Anal. 279 (2020), 108722.
- [50] Y. Zhou, M. Kot, *Life on the move: Modeling the effects of climate-driven range shifts with integrodifference equations*, In: Dispersal, Individual Movement and Spatial Ecology, 2071 (2013), 263-292.