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When Periodicity Fails to Guarantee the Existence of Rotation: A Counterexample on \mathbb{T}^3

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Abstract

In this manuscript, we construct an explicit counterexample of a smooth C^∞ , periodic dynamical system on the torus \mathbb{T}^3 for which the rotation vector exists in a weak sense, but fails to exist in the strong sense of bounded deviation (also referred to as *frequencies* in parts of the physics and biology literature). The construction exploits Liouville-type arithmetic properties and demonstrates that smoothness and periodicity alone do not ensure bounded deviation, even within the class of integrable systems.

Keywords: Periodic system, rotation vector, Liouville number, almost-periodic function, trigonometric series, bounded deviation.

MSC subject classifications: 34D05, 37B65, 34C15, 37C10.

1 Introduction

For vector fields on the torus defined by trigonometric polynomials, the existence of the rotation vector in the strong sense is well established [6]. In this setting, trajectories admit a linear asymptotic drift with uniformly bounded deviation.

For general smooth periodic vector fields, however, the situation is more subtle. In the context of homogenization theory and the asymptotic analysis of periodic flows, the existence of an effective drift vector is often assumed [2, 3]. Classical results on almost periodic differential equations identify regimes where rotation-like behavior holds [5, 8], but they do not exclude the arithmetic effects.

There is existing work on the rotation vector in discrete dynamical systems. In this paper, we focus instead on the continuous case, namely continuous dynamical systems, or ordinary differential equations.

The dimension 3 in this paper is essential because the first two components generate a Liouville resonance in the third component: on \mathbb{T}^1 the flow is periodic, and on \mathbb{T}^2 the Denjoy theory [4] implies that a C^∞ flow without fixed points has bounded deviation, even for Liouville rotation numbers.

Rotation vectors also arise naturally as effective frequencies in coupled oscillator models such as the Winfree and Kuramoto systems, which play a central role in synchronization theory [1, 7, 9].

The purpose of this work is to show that periodicity and C^∞ regularity do not suffice to guarantee the existence of a rotation vector in the strong sense. We construct an explicit integrable example on \mathbb{T}^3 where the weak rotation vector exists, but the deviation from the linear drift is unbounded due to *Liouville-type resonances*.

2 Definitions and Framework

Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ denote the 3-dimensional torus. For a flow $\varphi^t : \mathbb{T}^3 \rightarrow \mathbb{T}^3$, consider a lift $X : \mathbb{R} \rightarrow \mathbb{R}^3$ satisfying $\pi(X(t)) = \varphi^t(\pi(X(0)))$ where $\pi : \mathbb{R}^3 \rightarrow \mathbb{T}^3$ is the canonical projection.

Definition 1 (Rotation vectors). 1. The rotation vector exists in the weak sense if

$$\rho := \lim_{t \rightarrow \infty} \frac{X(t) - X(0)}{t}$$

exists in \mathbb{R}^3 .

2. The rotation vector exists in the strong sense if ρ exists and moreover

$$\sup_{t \geq 0} \|X(t) - X(0) - \rho t\| < \infty.$$

3 The counterexample

3.1 Construction and regularity

Let $r \in \mathbb{R}$ be a Liouville number. By definition, for each $n \geq 1$ there exist integers $p_n \geq 2$ and $q_n \geq 1$ such that

$$0 < |rp_n - q_n| < p_n^{-n}.$$

Using the fact that $p_n \geq 2$, fix a sequence $(p_m, q_m)_{m \in \mathbb{N}^*}$ such that

$$0 < |rp_m - q_m| < p_m^{-m} < |rp_{m-1} - q_{m-1}|, \quad \forall m \geq 1. \quad (1)$$

Define the vector field $h : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ by

$$h(z) = \left(r, 1, \sum_{m=1}^{\infty} m p_m^{-m} \cos(2\pi(z_1 p_m - z_2 q_m)) \right).$$

Lemma 2 (Smoothness). *The vector field h belongs to $C^\infty(\mathbb{T}^3)$.*

Proof. For each $k \geq 0$, we estimate the k -th derivative of the series termwise:

$$\sum_{m=1}^{\infty} m p_m^{-m} (2\pi \|(p_m, q_m)\|)^k \leq (2\pi)^k \sum_{m=1}^{\infty} m p_m^{-m} \max\{p_m^k, q_m^k\}.$$

From the Liouville condition we have $|rp_m - q_m| < p_m^{-m}$, which implies $q_m \leq |r|p_m + p_m^{-m}$. Since $p_m \geq 2$, the term p_m^{-m} is bounded and decreases as $m \rightarrow \infty$. For a fixed k and a large m , q_m is dominated by the linear term $|r|p_m$. More precisely,

$$q_m^k < (|r|p_m + p_m^{-m})^k = (|r| + p_m^{-m-1})^k p_m^k \leq (|r| + 1)^k p_m^k, \quad m \geq 1.$$

Given $p_m \geq 2$, that gives

$$m p_m^{-m} \max\{p_m^k, q_m^k\} \leq m (|r| + 1)^k p_m^{-m+k} \leq 2^k (|r| + 1)^k m 2^{-m}, \quad \forall m \geq k.$$

The series $\sum m 2^{-m}$ converges, so by the Weierstrass M-test, the series converges absolutely in $C^k(\mathbb{T}^3)$ for all fixed k , implying $h \in C^\infty(\mathbb{T}^3)$. \square

Consider the initial value problem

$$\dot{x} = h(x), \quad x(0) = 0. \quad (2)$$

The system is integrable with explicit solution

$$x_1(t) = rt, \quad x_2(t) = t, \quad x_3(t) = \int_0^t \sum_{m=1}^{\infty} m p_m^{-m} \cos(2\pi(rp_m - q_m)s) ds. \quad (3)$$

Lemma 3 (Termwise integration). *For each $t \geq 0$,*

$$x_3(t) = \sum_{m=1}^{\infty} \frac{m}{2\pi p_m^m (rp_m - q_m)} \sin(2\pi(rp_m - q_m)t).$$

Proof. Since $p_m \geq 2$ then $\sum m p_m^{-m} < \infty$. By the Weierstrass M-test, for every fixed $t > 0$ the series converges uniformly on $[0, t]$. Hence integration and summation commute. We obtain

$$\begin{aligned} x_3(t) &= \int_0^t \sum_{m=1}^{\infty} m p_m^{-m} \cos(2\pi(rp_m - q_m)s) ds \\ &= \sum_{m=1}^{\infty} m p_m^{-m} \int_0^t \cos(2\pi(rp_m - q_m)s) ds. \end{aligned}$$

By equation (1) we have $|rp_m - q_m| > 0$. It follows that

$$\int_0^t \cos(2\pi(rp_m - q_m)s) ds = \frac{1}{2\pi} \frac{1}{rp_m - q_m} \sin(2\pi(rp_m - q_m)t).$$

□

3.2 Weak rotation vector

Proposition 4 (Weak rotation vector). *The rotation vector of (2) exists in the weak sense and equals $\rho = (r, 1, 0)$.*

Proof. Thanks to the equation (3): For the first two components, $x_1(t)/t = r$ and $x_2(t)/t = 1$ exactly. For the third component, we have

$$x_3(t) = \int_0^t \sum_{m \geq 1} m p_m^{-m} \cos(2\pi(rp_m - q_m)s) ds.$$

Let $N \in \mathbb{N}^*$, then

$$\begin{aligned} x_3(t) &= \sum_{m=1}^N m p_m^{-m} \int_0^t \cos(2\pi(rp_m - q_m)s) ds \\ &\quad + \int_0^t \sum_{m>N} m p_m^{-m} \cos(2\pi(rp_m - q_m)s) ds. \end{aligned}$$

This implies

$$|x_3(t)| \leq I_N(t) + J_N(t),$$

Where

$$\begin{aligned} I_N(t) &:= \sum_{m=1}^N m p_m^{-m} \left| \int_0^t \cos(2\pi(rp_m - q_m)s) ds \right| \\ J_N(t) &:= \left| \int_0^t \sum_{m>N} m p_m^{-m} \cos(2\pi(rp_m - q_m)s) ds \right|. \end{aligned}$$

We recall that $|rp_m - q_m| > 0$ as given in equation (1). Since

$$I_N(t) = \sum_{m=1}^N m p_m^{-m} \left| \frac{\sin(2\pi(rp_m - q_m)t)}{2\pi(rp_m - q_m)} \right| \leq \sum_{m=1}^N \frac{m p_m^{-m}}{2\pi |rp_m - q_m|},$$

which is a finite quantity and independent of t . So for every $N \in \mathbb{N}^*$, we obtain $\lim_{t \rightarrow +\infty} t^{-1} I_N(t) = 0$. Since $t^{-1} J_N(t) \leq \sum_{m>N} m p_m^{-m}$, we get

$$\lim_{t \rightarrow +\infty} t^{-1} |x_3(t)| \leq \sum_{m>N} m p_m^{-m}, \quad \forall N > 1.$$

Using the fact that $\lim_{N \rightarrow +\infty} \sum_{m>N} m p_m^{-m} = 0$, because $p_m \geq 2$, we deduce that $\lim_{t \rightarrow +\infty} t^{-1} |x_3(t)| = 0$. \square

3.3 Failure of strong rotation vector

Proposition 5 (Unbounded deviation). *The function $t \mapsto x_3(t)$ is unbounded. Consequently, the rotation vector does not exist in the strong sense.*

Proof. We prove the present Proposition by contradiction. Suppose that $\sup_{t \geq 0} |x_3(t)| < +\infty$. Denote $\lambda_m := rp_m - q_m$ for all $m \geq 1$. We have $|\lambda_m| > 0$ for all $m \geq 1$. Using integration by parts, for all fixed $n \geq 1$ we obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t x_3(s) \cos(2\pi \lambda_n s) ds &= -\frac{1}{2\pi \lambda_n} \lim_{t \rightarrow +\infty} t^{-1} x_3(t) \sin(2\pi \lambda_n t) \\ &+ \frac{1}{2\pi \lambda_n} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \dot{x}_3(s) \cos(2\pi \lambda_n s) ds \\ &= \frac{1}{2\pi \lambda_n} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \dot{x}_3(s) \cos(2\pi \lambda_n s) ds. \end{aligned} \quad (4)$$

We recall that $\lambda_m := rp_m - q_m$. From the differential equation (2) and the equation (3), we have

$$\dot{x}_3(s) = \sum_{m=1}^{\infty} m p_m^{-m} \cos(2\pi \lambda_m s), \quad \forall s \geq 0.$$

Substitute this quantity into Equation (4) to obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t x_3(s) \cos(2\pi \lambda_n s) ds \\ = \frac{1}{2\pi \lambda_n} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \cos(2\pi \lambda_n s) \sum_{m \geq 1} m p_m^{-m} \cos(2\pi \lambda_m s) ds. \end{aligned}$$

As in the proof of the Proposition 4, for every $N > n$, we get

$$\begin{aligned} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t x_3(s) \cos(2\pi \lambda_n s) ds \\ = \frac{1}{2\pi \lambda_n} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \cos(2\pi \lambda_n s) \sum_{m=1}^N m p_m^{-m} \cos(2\pi \lambda_m s) ds \\ + \frac{1}{2\pi \lambda_n} \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \cos(2\pi \lambda_n s) \sum_{m > N} m p_m^{-m} \cos(2\pi \lambda_m s) ds. \end{aligned}$$

The series $\sum_{m > N} m p_m^{-m}$ converges since $p_m \geq 2$. We obtain the following estimate:

$$\lim_{t \rightarrow +\infty} t^{-1} \left| \int_0^t (\cos(2\pi \lambda_n s) x_3(s) - \psi_{N,n}(s)) ds \right| \leq \frac{1}{2\pi |\lambda_n|} \sum_{m > N} m p_m^{-m}. \quad (5)$$

where, in order to simplify the notation, we denote

$$\psi_{N,n}(s) := \frac{1}{2\pi\lambda_n} \cos(2\pi\lambda_n s) \sum_{m=1}^N m p_m^{-m} \cos(2\pi\lambda_m s), \quad \forall s \geq 0.$$

Since $N > n$, this can be written as

$$\begin{aligned} \psi_{N,n}(s) &= \frac{n}{2\pi p_n^{-n} \lambda_n} \cos^2(2\pi\lambda_n s) \\ &\quad + \frac{1}{2\pi\lambda_n} \sum_{\substack{m \neq n \\ 1 \leq m \leq N}} m p_m^{-m} \cos(2\pi\lambda_m s) \cos(2\pi\lambda_n s), \end{aligned}$$

Using the trigonometric identity $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$, we obtain the following equality.

$$\begin{aligned} \psi_{N,n}(s) &= \frac{n}{4\pi p_n^{-n} \lambda_n} - \frac{n}{4\pi p_n^{-n} \lambda_n} \cos(4\pi\lambda_n s) \\ &\quad + \frac{1}{4\pi\lambda_n} \sum_{\substack{m \neq n \\ 1 \leq m \leq N}} m p_m^{-m} \cos(2\pi(\lambda_m - \lambda_n)s) \\ &\quad + \frac{1}{4\pi\lambda_n} \sum_{\substack{m \neq n \\ 1 \leq m \leq N}} m p_m^{-m} \cos(2\pi(\lambda_m + \lambda_n)s). \end{aligned}$$

By equation (1), we have

$$0 < |rp_m - q_m| < p_m^{-m} < |rp_{m-1} - q_{m-1}| \quad \text{for all } m \geq 1.$$

Since we set $\lambda_m := rp_m - q_m$, it follows that $|\lambda_m| < |\lambda_{m-1}|$ for all $m \geq 1$. This implies that $\lambda_m \neq \pm\lambda_n$ for all $m, n \in \{1, \dots, N\}$ with $m \neq n$. Integrating the previous equation yields

$$\begin{aligned} \int_0^t \psi_{N,n}(s) ds &= \frac{n}{4\pi p_n^{-n} \lambda_n} t - \frac{n}{16\pi^2 p_n^{-n} \lambda_n^2} \sin(4\pi\lambda_n t) \\ &\quad + \frac{1}{8\pi^2 \lambda_n} \sum_{\substack{m \neq n \\ 1 \leq m \leq N}} \frac{m}{p_m^m (\lambda_m - \lambda_n)} \sin(2\pi(\lambda_m - \lambda_n)t) \\ &\quad + \frac{1}{8\pi^2 \lambda_n} \sum_{\substack{m \neq n \\ 1 \leq m \leq N}} \frac{m}{p_m^m (\lambda_m + \lambda_n)} \sin(2\pi(\lambda_m + \lambda_n)t) \end{aligned}$$

By consequence

$$\forall N > n : \quad \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \psi_{N,n}(s) ds = \frac{n}{4\pi p_n^n \lambda_n}.$$

For all $N > n$, equation (5) implies

$$\left| \lim_{t \rightarrow +\infty} t^{-1} \int_0^t \cos(2\pi \lambda_n s) x_3(s) ds - \frac{n}{4\pi p_n^n \lambda_n} \right| \leq \frac{1}{2\pi |\lambda_n|} \sum_{m>N} m p_m^{-m}.$$

At $N \rightarrow +\infty$, we obtain

$$\lim_{t \rightarrow +\infty} t^{-1} \int_0^t \cos(2\pi \lambda_n s) x_3(s) ds = \frac{n}{4\pi p_n^n \lambda_n}.$$

From equation (1) we have $p_n^n |\lambda_n| < 1$. Indeed, this inequality highlights the role of Liouville number. Then

$$\lim_{t \rightarrow +\infty} t^{-1} \left| \int_0^t \cos(2\pi \lambda_n s) x_3(s) ds \right| > \frac{n}{4\pi}, \quad \forall n \geq 1.$$

We recall that we assumed that $\sup_{t \geq 0} |x_3(t)| < +\infty$. We obtained

$$\frac{n}{4\pi} < \lim_{t \rightarrow +\infty} t^{-1} \int_0^t |x_3(s)| ds < \sup_{s \geq 0} |x_3(s)| < +\infty, \quad \forall n \geq 1.$$

We obtained a contradiction. Then $\sup_{t \geq 0} |x_3(t)| = +\infty$. \square

4 Conclusion and questions

We have shown that a smooth periodic vector field on \mathbb{T}^3 may admit a weak rotation vector while failing to have a strong one, due to Liouville-type arithmetic resonances. This demonstrates that bounded deviation is not automatic even for C^∞ integrable systems.

Several natural questions arise:

1. **Arithmetic conditions:** Characterize the Diophantine properties of frequency vectors that guarantee bounded deviation for smooth (or analytic) periodic systems. Is a Diophantine condition necessary and sufficient?

2. **Regularity versus decay:** How does the required decay rate of Fourier coefficients depend on the arithmetic type of the frequencies to ensure bounded deviation?
3. **Non-integrable perturbations:** Does the phenomenon persist under small non-integrable perturbations? What happens in nearly-integrable Hamiltonian systems?
4. **Higher-dimensional generalisations:** Can similar counterexamples be constructed on \mathbb{T}^n for $n > 3$ with rotation sets instead of vectors?

These questions highlight the interplay between arithmetic, regularity, and asymptotic dynamics in periodic systems, an interplay that is often implicit in applied studies of oscillatory networks.

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