



HAL
open science

What Is a Proper Graph Laplacian? An Operator-Theoretic Framework for Graph Diffusion

Ernesto Estrada

► **To cite this version:**

Ernesto Estrada. What Is a Proper Graph Laplacian? An Operator-Theoretic Framework for Graph Diffusion. 2025. ⟨hal-05089665v2⟩

HAL Id: hal-05089665

<https://hal.science/hal-05089665v2>

Preprint submitted on 23 Oct 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

What Is a Proper Graph Laplacian? An Operator-Theoretic Framework for Graph Diffusion

Ernesto Estrada

October 22, 2025

Institute for Cross-Disciplinary Physics and Complex Systems (IFISC), CSIC-UIB, Palma de Mallorca, Spain. E-mail: estrada@ifisc.uib-csic.es

Abstract

We introduce an operator-theoretic definition of a *proper graph Laplacian* as any matrix associated with a given graph that can be expressed as the composition of a divergence and a gradient operator, with the gradient acting between graph-related spaces and annihilating constant functions. This provides a unified framework for determining whether a matrix represents a genuine diffusive operator on a graph. Within this framework, we prove that the standard Laplacian, the fractional Laplacian, the d -path Laplacians, and the degree-attracting and degree-repelling Laplacians are all *proper diffusive Laplacians*. In contrast, the in-degree and out-degree Laplacians correspond to advection operators, while the signed, signless, magnetic, and deformed Laplacians are *improper*, as they cannot be written as the composition of a divergence and a true gradient. The magnetic Laplacian is shown to arise as the *Schur complement* of an extended proper Laplacian defined on a higher-dimensional space, a property also inherited by the signless Laplacian. The Lerman–Ghosh Laplacian is identified as a *nonconservative diffusive operator* coupled to an external reservoir. Finally, we prove that the Moore–Penrose pseudoinverse of the Laplacian is itself a proper Laplacian. This classification establishes a rigorous operator-theoretic foundation for distinguishing proper, nonconservative, and improper graph Laplacians.

Keywords: Graph Laplacian; pseudoinverse; diffusion operator; gradient–divergence formulation; magnetic Laplacian; Schur complement; operator theory.

MSC codes: 05C50; 15A09; 47A05; 35R02.

1 Introduction

The standard graph Laplacian L [45, 61] is a cornerstone of algebraic graph theory [47, 83, 25, 46, 68]. It plays a crucial role in understanding the structural properties of graphs and in modeling a wide range of dynamical processes on graphs and networks [60, 12, 75]. A classical example is the diffusion equation on an undirected graph $G = (V, E)$ [30]:

$$\partial_t x(t) + Lx(t) = 0, \tag{1}$$

with initial condition $x(0) = x^0$, where $x(t) = [x_1(t), \dots, x_n(t)]^T$ is a column vector of concentrations. The Laplacian matrix is defined as $L = K - A$, where K is the diagonal matrix of vertex degrees and A is the adjacency matrix of the graph.

Let $\mathbb{C}(V)$ denote the set of all complex-valued functions on V , and let $\ell_2(V)$ be the Hilbert space of square-summable functions on V . Then L acts as a mapping from $\mathbb{C}(V)$ into itself:

$$Lf(v) = \sum_{(v,w) \in E} (f(w) - f(v)), \quad f \in \mathbb{C}(V). \quad (2)$$

Beyond this standard definition, many alternative forms of graph Laplacians have been proposed in the literature. For directed graphs, two natural generalizations exist: the in-degree Laplacian $L^{\text{in}} = K^{\text{in}} - A$ and the out-degree Laplacian $L^{\text{out}} = K^{\text{out}} - A$, where the diagonal matrices K^{in} and K^{out} contain the in- and out-degrees of the vertices, respectively. A non-exhaustive list of Laplacian variants includes the deformed Laplacian [71, 37], the d -path Laplacian L_d [31, 32, 33], the fractional Laplacian L^α [77, 63], the hub-biased Laplacians L^{rep} and L^{attr} [29, 42], the Hermitian (magnetic) Laplacian L_B [11, 85, 28], the signless Laplacian Q [21, 22, 23], the Laplacian for signed graphs L^s [56, 5], and the Lerman–Ghosh Laplacian L_χ [57, 44].

Further extensions have been introduced through the Euclidean distance matrix by Balaji and Bapat [8], as well as by considering matrix functions of the standard Laplacian [78, 70]. More recently, Saxena *et al.* [79] proposed a diffusion-like model inspired by electrical network analysis, where an impedance network with shunt inductors connects each vertex to the ground. In matrix-vector form, the dynamics of this model can be written as

$$L\dot{x}(t) = -x(t), \quad (3)$$

and, by applying the Moore–Penrose pseudoinverse L^+ to both sides, one obtains

$$\partial_t x(t) + L^+ x(t) = 0, \quad (4)$$

in close analogy with Eq. (1).

The question of what qualifies as a graph Laplacian has been addressed differently by various authors. Balaji and Bapat [8] proposed that a Laplacian matrix should be positive semidefinite, have zero row-sum, and possess rank $n - 1$, where n is the number of vertices. In contrast, Riascos *et al.* [78, 70] and, more recently, Devriendt [26] required that a Laplacian must also have nonpositive off-diagonal entries. Devriendt further added that the matrix should be irreducible, a condition automatically satisfied for connected graphs and thus not discussed further here.

The standard Laplacian L satisfies all these requirements, as do L_d and L^α . However, several other variants do not: L^{in} and L^{out} may not have rank $n - 1$; L_B is Hermitian but not always positive semidefinite, and it lacks both the zero row-sum and rank properties; the signless Laplacian Q neither has zero row-sum nor nonpositive off-diagonal entries; the Lerman–Ghosh Laplacian fails all three conditions; and the pseudoinverse L^+ can have positive off-diagonal elements. Although Saxena *et al.* [79] regarded L^+ as a valid Laplacian operator within their diffusion framework—and it does satisfy the conditions proposed by Balaji and Bapat—it does not fulfill the stricter requirements imposed by Riascos *et al.* and Devriendt. Indeed, it has been explicitly stated that L^+ “is not always a Laplacian” because some of its off-diagonal elements can be positive (see pp. 133–134 in [64]). The Laplacian pseudoinverse is an important matrix in the study of graphs [39, 65], with applications in the definition of the resistance distance matrix of the graph [53, 69, 76, 49, 86, 87, 43], the calculation of the hitting/commuting times in Markov chains [14, 74, 18], as

well as for analyzing the robustness of networked control systems [89, 88, 59]. Thus, the question about whether it is a proper Laplacian operator or not is very relevant across the disciplines. The objective of this work is therefore twofold: first, to provide a clear and pedagogical discussion of what should properly be regarded as a Laplacian operator for a graph, and second, to address the question of whether the pseudoinverse of the graph Laplacian, L^+ , can itself be considered a genuine Laplacian operator.

2 Preliminaries

We introduce the basic notations and operator spaces used throughout this paper. Let $\mathbb{C}(V)$ denote the space of all complex-valued functions on the vertex set V , and $\mathbb{C}(E)$ (or more generally $\mathbb{C}(\Sigma)$) the space of edge-valued functions (or functions defined on subgraph elements). We endow $\ell_2(V)$ and $\ell_2(E)$ with their canonical inner products $\langle f, g \rangle_V = \sum_{v \in V} \overline{f(v)}g(v)$ and $\langle u, w \rangle_E = \sum_{e \in E} \overline{u(e)}w(e)$.

The discrete gradient $\Upsilon : \mathbb{C}(V) \rightarrow \mathbb{C}(E)$ acts as a difference operator between connected vertices, producing edgewise increments. Its adjoint Υ^* defines a canonical divergence operator, $\text{div}_0 := \Upsilon^* : \mathbb{C}(E) \rightarrow \mathbb{C}(V)$, which measures the net outflow from each vertex. A *proper gradient* satisfies $\Upsilon(\mathbf{1}) = 0$, ensuring that constant functions have zero gradient.

In general, a *proper Laplacian* is the composition $\mathcal{L} = \text{div} \Upsilon$, where div need not coincide with Υ^* . The familiar combinatorial Laplacian $L = \Upsilon^* \Upsilon$ corresponds to the special case in which the divergence equals the adjoint of the gradient, $\text{div} = \Upsilon^*$, thus yielding a self-adjoint diffusive operator. Other Laplacians, such as directed or magnetic variants, arise when div and Υ^* differ, breaking self-adjointness while preserving the diffusive structure.

Let us give here some indications about the notation used in the paper. Throughout the paper we distinguish four levels of gradient notation:

1. the abstract operator Υ acting between function spaces,
2. its matrix realization ∇ representing oriented vertex–edge incidence,
3. specialized variants $\nabla_\tau, \nabla_\sigma, \nabla_B, \nabla_d, \nabla^x$ used for directed, signed, magnetic, distance-based, or extended gradients, and
4. the roman symbols grad and div , which denote generic gradient/divergence operators in prose.

This hierarchy avoids ambiguity between abstract operator, matrix form, and contextual variants.

We collect a few standard facts that we will use repeatedly. Let $\mu_1 \leq \dots \leq \mu_n$ be the set of eigenvalues of the graph Laplacian $L(G)$. We have then the following known results.

Lemma 1 (Monotonicity under edge addition). [19, 38] *If $G' = (V, E')$ is obtained from $G = (V, E)$ by adding edges ($E \subseteq E'$), then $L(G') - L(G)$ is positive semidefinite. Consequently, $\mu_k(G') \geq \mu_k(G)$ for all k , and in particular $\mu_n(G') \geq \mu_n(G)$.*

Proof. Each added edge $\{i, j\}$ contributes the rank-1 PSD Laplacian $L_{\{i,j\}} = (e_i - e_j)(e_i - e_j)^\top$. Summing such contributions yields $L(G') - L(G) \succeq 0$. The eigenvalue monotonicity follows from the Courant–Fischer characterization. \square

Lemma 2 (Upper bound by maximum degree). [19, 81] *For any graph G , $\mu_n(G) \leq 2\Delta(G)$, where $\Delta(G)$ is the maximum degree.*

Proof. By Gershgorin disks for the Laplacian, every eigenvalue lies in $[0, 2\Delta]$. Equivalently, $x^\top Lx = \sum_{\{i,j\} \in E} (x_i - x_j)^2 \leq 2\Delta \sum_i x_i^2$. \square

Lemma 3 (Cheeger upper bound). [17, 19] Let $\phi(G) := \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$ where $\text{vol}(S) = \sum_{i \in S} k_i$. Then the (combinatorial) Laplacian satisfies $\mu_2(G) \leq 2\phi(G)$.

Proof. This classical bound follows by taking the Fiedler Rayleigh quotient with a test function f that is approximately constant on S and on \bar{S} with opposite signs, and optimizing over S (see, e.g., standard Cheeger-type inequalities for the combinatorial Laplacian [4, 19, 81]). \square

3 On Balaji–Bapat and Riascos *et al.*/Devriendt properties of a Laplacian

Here we explain briefly what the properties proposed by Balaji and Bapat [8] and by Riascos *et al.* [78, 70] and Devriendt [26] mean from the perspective of an operator for diffusive dynamics on a graph. Let us discuss these properties individually.

3.1 Positive definiteness and rank equal to $n - 1$

Let \mathcal{L} be a candidate Laplacian operator for a graph. The requirement that \mathcal{L} be positive semidefinite ensures that the diffusion dynamics

$$\partial_t x(t) + \mathcal{L}x(t) = 0, \quad x(0) = x^0,$$

converges to a steady state. Indeed, if $0 = \mu_{1_1}(\mathcal{L}) = \dots = \mu_{1_h}(\mathcal{L})$, i.e., there are h zero eigenvalues, and $\mu_{j>h} > 0$, then

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} e^{-t\mathcal{L}} x^0 = \sum_{j=1}^h \psi_{1_j} (\psi_{1_j}^\top x^0) = \kappa,$$

where ψ_{1_j} are the eigenvectors associated with the zero eigenvalues of \mathcal{L} . The vector κ is time-independent, representing the steady state of the diffusion. In general, its components may differ, depending on the multiplicity of the zero eigenvalue.

If, in addition, $\text{rank}(\mathcal{L}) = n - 1$, then the zero eigenvalue is simple, implying that all components of κ are identical. In this case, the diffusion converges to a unique steady state.

3.2 Zero row-sum

The zero row-sum property of \mathcal{L} ensures that $\mathcal{L}\mathbf{1} = 0$. Since we can write $\mathbf{1} = \sum_{k=1}^c \sqrt{n_k} u_k \in \ker(\mathcal{L})$ for $c \geq 1$, with $u_k = \frac{1}{\sqrt{n_k}} \mathbf{1}_{C_k}$ ($k = 1, \dots, c$), we have

$$\lim_{t \rightarrow \infty} x(t) = \sum_{k=1}^c (\psi_{1,k}^\top x^0) \psi_{1,k} = \sum_{k=1}^c \left(\frac{1}{\sqrt{n_k}} \sum_{i \in C_k} x_i^0 \right) \frac{1}{\sqrt{n_k}} \mathbf{1}_{C_k} = \sum_{k=1}^c \left(\frac{1}{n_k} \sum_{i \in C_k} x_i^0 \right) \mathbf{1}_{C_k}.$$

This means that the steady state of the diffusion dynamics on the graph consists of one or several “consensus” states among groups of vertices, each characterized by the average of the initial

condition over those vertices. Clearly, if the rank of \mathcal{L} is $n - 1$, there is a unique steady state, corresponding to the global average of the initial condition over all vertices, which justifies the name of “consensus” given to this dynamics in engineering contexts (see for instance [73]).

3.3 Non-positive off-diagonal entries

Assuming that the diagonal entries of \mathcal{L} are positive, the condition that off-diagonal entries are nonpositive ensures that when the operator is applied to a function defined on the vertices of a graph, it produces at each vertex the sum of the differences of the function values along the edges incident to that vertex. Indeed, let $\mathcal{L} = [\ell_{ij}]_{i,j=1}^n$ be a symmetric matrix satisfying

$$\ell_{ii} > 0, \quad \ell_{ij} \leq 0 \text{ for } i \neq j, \quad \sum_j \ell_{ij} = 0.$$

Then, for any vector $x \in \mathbb{C}^n$,

$$(\mathcal{L}x)_i = \sum_{j \neq i} (-\ell_{ij})(x_i - x_j).$$

Thus, the action of \mathcal{L} on x at vertex i consists of adding up the weighted differences between the value of x at i and those at its neighboring vertices. The positive diagonal term $\ell_{ii} = \sum_{j \neq i} (-\ell_{ij})$ balances the outgoing and incoming contributions, ensuring that $\mathcal{L}\mathbf{1} = 0$.

3.4 An additional condition for conservation of mass

A diffusive dynamics is said to be conservative if $\sum_{i=1}^n x_i(t) = \sum_{i=1}^n x_i^0$ for any $t \geq 0$, where the sum is carried out over all vertices of the graph. Then, if \mathcal{L} is symmetric the condition that it has zero row-sum is sufficient to guaranty that the diffusive process is conservative. If the matrix is not symmetric, then the conservation of mass is obtained if the matrix is also zero column-sum. That is, let us consider the sum of all the values of $x(t)$ at a given $t > 0$

$$\begin{aligned} \mathbf{1}^T x(t) &= \mathbf{1}^T e^{-t\mathcal{L}} x^0 \\ &= \mathbf{1}^T \left(I + t(-\mathcal{L}) + \frac{t^2(-\mathcal{L})^2}{2!} + \dots + \frac{t^k(-\mathcal{L})^k}{k!} + \dots \right) x^0 \\ &= \mathbf{1}^T x^0 + t\mathbf{1}^T (-\mathcal{L}) x^0 + t^2\mathbf{1}^T (-\mathcal{L}) ((-\mathcal{L}) x^0) + \dots + t^k\mathbf{1}^T (-\mathcal{L}) ((-\mathcal{L})^{k-1} x^0) + \dots \\ &= \mathbf{1}^T x^0, \end{aligned} \tag{5}$$

due to the fact that the condition $\mathbf{1}^T (-\mathcal{L}) = 0$ has been imposed.

4 General definition of graph Laplacians

The notion of a Laplacian on a graph should not be restricted to specific matrix sign patterns or spectral properties, but should instead be grounded on its functional construction as the composition of a divergence and a gradient operator. Accordingly, we propose the following general definition.

Definition 4. (Graph Laplacian). A matrix $\mathcal{L} \in \mathbb{C}^{n \times n}$ associated to a graph $G = (V, E)$ is said to be a *Laplacian operator* on G if there exist two linear operators

$$\text{grad} : \mathbb{C}(V) \rightarrow \mathbb{C}(\Sigma), \quad \text{div} : \mathbb{C}(\Sigma) \rightarrow \mathbb{C}(V),$$

acting between the space of complex-valued functions on vertices and that on a family of subgraph structures Σ of G (for instance, edges, paths, cycles, or higher-order motifs), such that

$$\mathcal{L} = \text{div grad}.$$

Here, Σ is a family of graph elements that connect groups of vertices, such as edges, paths, cycles, or any other subgraph structures of G . The gradient operator grad maps vertex functions into functions defined on Σ , thus quantifying the local variations of the field between connected graph elements, while div aggregates those variations back onto the vertices.

Remark 5. The structural definition above differs fundamentally from the usual algebraic characterizations of a Laplacian, such as those of Balaji–Bapat [8] and Riascos *et al.*[78]/Devriendt [26]. Because $\mathcal{L} = \text{div grad}$ and $\text{grad}(\mathbf{1}) = 0$, every proper Laplacian automatically satisfies $\mathcal{L}\mathbf{1} = 0$, i.e., it has zero row-sum, and its nullspace always contains the constants, so that $\text{rank}(\mathcal{L}) \leq n - 1$ for a connected graph. No other of the Balaji–Bapat [8] and Riascos *et al.*[78]/Devriendt [26] matrix property is required. In particular, positive semidefiniteness, symmetry, and nonpositive off-diagonal entries are *not implied* by the operator form div grad . They arise only when one further demands $\text{div} = \text{grad}^*$ with respect to some inner product, in which case $\mathcal{L} = \text{grad}^* \text{grad}$ becomes self-adjoint and positive semidefinite in that metric. Without this additional assumption, \mathcal{L} may be nonsymmetric or even indefinite, as exemplified by the degree-biased Laplacians or other anisotropic diffusions. Finally, the requirement that grad acts on a family Σ of subgraph structures ensures that \mathcal{L} is not an arbitrary zero-row-sum matrix, but one generated by local or higher-order interactions prescribed by the graph G .

Definition 6. (Graph gradient). The operator grad is a *proper gradient* if it satisfies the condition

$$\text{grad}(\mathbf{1}) = 0,$$

where $\mathbf{1}$ is the constant function on V . This condition ensures that the gradient measures *differences* between values of the function across connected elements of the graph and vanishes for constant functions. If $\text{grad}(\mathbf{1}) \neq 0$, the operator does not represent variations of the field and cannot be associated with a Laplacian of diffusive nature.

These conditions are well known to be obeyed by the standard graph Laplacian as can be seen in the following result, shown for completeness of the current work.

Proposition 7. *The standard graph Laplacian $L = K - A$ for an undirected graph is a Laplacian operator in the above sense.*

Proof. Let ∇ denote the oriented vertex–edge incidence matrix of the graph, defined by

$$\nabla_{ev} = \begin{cases} +1, & v = \text{head}(e), \\ -1, & v = \text{tail}(e), \\ 0, & \text{otherwise.} \end{cases}$$

Define the discrete gradient as

$$\text{grad } x = \nabla x,$$

so that $(\text{grad } x)_e = x_{\text{head}(e)} - x_{\text{tail}(e)}$, and its adjoint (negative divergence) as

$$\text{div} = \nabla^\top.$$

Then,

$$\mathcal{L} = \text{div grad} = \nabla^\top \nabla.$$

By construction,

$$\text{grad}(\mathbf{1}) = \nabla \mathbf{1} = 0,$$

so the gradient of a constant function vanishes. Therefore, $L = \nabla^\top \nabla$ satisfies the defining property of a Laplacian operator. \square

5 Are other “Laplacians” proper graph Laplacians?

Based on the previous definition we now prove a series of results about several known “Laplacian” on graphs. Let us start by the out- and in-degree Laplacians. The out-degree Laplacian has been previously studied by Chapman [15, 16] who effectively defines it as an “advection” operator. The in-degree Laplacian is commonly used as a “diffusive” (consensus) operator on directed graphs (see Chapter 3.2 in [62]). Hereafter we clarify the conceptual meaning of these two operators.

Proposition 8. *On a directed graph $G = (V, E)$ with adjacency matrix A , the out- and in-degree Laplacians*

$$L^{\text{out}} := K^{\text{out}} - A, \quad L^{\text{in}} := K^{\text{in}} - A$$

are not proper Laplacians, i.e., they cannot be written as $\mathcal{L} = \text{div grad}$ with a diffusive gradient grad satisfying $\text{grad}(\mathbf{1}) = 0$. Instead, they admit transport-type factorizations:

$$L^{\text{out}} = U \nabla_\tau, \quad L^{\text{in}} = \text{div}_\rho V,$$

where U and V are vertex-edge and edge-vertex “tail” selectors (transport fields), and $\nabla_\tau, \text{div}_\rho$ are oriented discrete derivatives along edges. Hence L^{out} acts as an advection operator (outflow) and L^{in} as its inflow/accumulation conjugate.

Proof. (1) *Out- and in-degree Laplacians are not diffusive.* Any operator of the form $\mathcal{L} = \text{div grad}$ with $\text{grad}(\mathbf{1}) = 0$ and $\text{div} = \text{grad}^*$ is Hermitian and positive semidefinite. For a genuinely directed graph ($A \neq A^\top$),

$$(L^{\text{out}})^* = K^{\text{out}} - A^\top \neq K^{\text{out}} - A = L^{\text{out}}, \quad (L^{\text{in}})^* = K^{\text{in}} - A^\top \neq L^{\text{in}},$$

so neither is self-adjoint or diffusive in the standard inner product. Thus they are not proper Laplacians.

(2) *Advection factorization of L^{out} .* Choose an orientation of edges and let ∇ be the vertex-edge incidence (difference) matrix,

$$(\nabla x)_e = x_{\text{head}(e)} - x_{\text{tail}(e)}, \quad \nabla \mathbf{1} = 0.$$

Let U be the vertex–edge “tail” selector,

$$U_{ve} = \begin{cases} 1, & v = \text{tail}(e), \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $x \in \mathbb{C}^n$,

$$(U(-\nabla)x)_v = \sum_{e: \text{tail}(e)=v} (x_{\text{tail}(e)} - x_{\text{head}(e)}) = k^{\text{out}}(v)x_v - \sum_w A_{vw}x_w = (L^{\text{out}}x)_v.$$

Hence

$$\boxed{L^{\text{out}} = U \nabla_{\tau}, \quad \nabla_{\tau} := -\nabla,}$$

which represents a discrete advection (outflow) operator with velocity field U .

(3) *Inflow/accumulation factorization of L^{in} .* Define the discrete divergence $\text{div} := \nabla^{\top}$ and the edge–vertex “tail” selector

$$V_{ev} = \begin{cases} -1, & v = \text{tail}(e), \\ 0, & \text{otherwise.} \end{cases}$$

Then componentwise,

$$(\nabla^{\top} Vx)_v = \sum_{e: \text{head}(e)=v} (-x_{\text{tail}(e)}) + \sum_{e: \text{tail}(e)=v} x_{\text{tail}(e)} = - \sum_{u: u \rightarrow v} x_u + k^{\text{out}}(v)x_v.$$

Reversing the orientation (equivalently, using the incidence of the reversed digraph) yields

$$(\nabla_{\text{rev}}^{\top} V_{\text{rev}}x)_v = k^{\text{in}}(v)x_v - \sum_w A_{vw}x_w = (L^{\text{in}}x)_v,$$

so that

$$\boxed{L^{\text{in}} = \text{div}_{\rho} V,}$$

with div_{ρ} the divergence in the reversed orientation. This operator aggregates incoming flux and thus acts as an inflow or accumulation operator—the natural conjugate of advection. \square

Remark 9. The pair $(L^{\text{out}}, L^{\text{in}})$ realizes a discrete transport duality

$$\mathbf{u} \cdot \nabla f \quad (\text{advection/outflow}) \quad \text{vs.} \quad -\nabla \cdot (\mathbf{u}f) \quad (\text{inflow/continuity}).$$

Neither operator is a proper Laplacian, since they lack the diffusive–conservative form div grad with $\text{grad}(\mathbf{1}) = 0$; they instead describe directed transport processes.

Let us now consider the so-called deformed Laplacian in its undirected graph-version [71, 37] which was used for studying “consensus” (diffusive) dynamics on graphs.

Proposition 10. *Let $G = (V, E)$ be a simple undirected graph with degree matrix $K = \text{diag}(k_v)$, adjacency A , and identity I . For a real parameter r , define the deformed Laplacian*

$$L(r) = (K - I)r^2 + Ar + I.$$

Then, except in two trivial regular cases, $L(r)$ is neither diffusive (positive semidefnite) nor a proper Laplacian of the form $\mathcal{L} = \text{div grad}$ with a proper gradient satisfying $\text{grad}(\mathbf{1}) = 0$.

Proof. If $L(r)$ were a proper Laplacian, the condition $\text{grad}(\mathbf{1}) = 0$ would imply $L(r)\mathbf{1} = 0$. For each vertex v ,

$$(L(r)\mathbf{1})_v = (k_v - 1)r^2 + k_v r + 1 =: q_v(r).$$

For a fixed r , $q_v(r) = 0$ for all v only if all degrees are equal (G is k -regular), since q_v depends on k_v . In that case, the roots of $(k - 1)r^2 + kr + 1 = 0$ are $r = -1$ and $r = -1/(k - 1)$. Only for these two values do we have

$$L(-1) = K - A = L, \quad L\left(-\frac{1}{k-1}\right) = \frac{1}{k-1} L,$$

which are (multiples of) the standard Laplacian and hence proper in the strict diffusive–self–adjoint sense. For every other value of r , $L(r)\mathbf{1} \neq 0$, so $L(r)$ cannot be expressed as div grad with $\text{grad}(\mathbf{1}) = 0$.

Regarding diffusivity, for general r the off–diagonal entries of $L(r)$ on edges equal $+r$, which become *positive* when $r > 0$. Consequently, $x^\top L(r)x$ need not be nonnegative, and the matrix $L(r)$ fails to be positive semidefinite except at the conservative points $r = -1$ and $r = -1/(k - 1)$ for regular graphs. In general, $L(r)$ is indefinite and therefore non-diffusive. Hence, except for the two regular conservative cases, $L(r)$ is neither diffusive nor a proper Laplacian. \square

Remark 11. Whenever the diagonal quantities

$$s_v := (k_v - 1)r^2 + k_v r + 1 \quad (\geq 0)$$

are nonnegative (e.g. for $r \leq -1$ or for $r \in [-1/(\Delta - 1), 0]$ with $\Delta = \max_v k_v$), one may write

$$L(r) = (-r) \nabla^\top \nabla + \text{diag}(s_v) = \begin{bmatrix} \sqrt{-r} \nabla \\ \text{diag}(\sqrt{s_v}) \end{bmatrix}^\top \begin{bmatrix} \sqrt{-r} \nabla \\ \text{diag}(\sqrt{s_v}) \end{bmatrix}.$$

Here ∇ is any oriented incidence matrix. This shows that $L(r)$ is *diffusive* (positive semidefinite), but the added unary rows do not compare pairs of vertices, so the corresponding “gradient” does not annihilate constants. Therefore, even in this case, $L(r)$ is not a *proper* Laplacian under the definition $\mathcal{L} = \text{div grad}$ with $\text{grad}(\mathbf{1}) = 0$.

We now consider now one of the emergent lines of research for the study of directed and mixed graphs: the use of complex-valued Hermitian Laplacian matrices [20, 67, 41, 91, 50, 1, 48, 58, 13], which possess many valuable algebraic properties. Additionally, complex-valued Hermitian Laplacian matrices can be related to the magnetic Schrödinger operator [7, 54, 55, 51, 52, 34, 80, 11, 85, 36], such they are known as magnetic Laplacians in the literature.

Proposition 12. *Let $G = (V, E)$ be a directed or mixed graph with complex edge phases $\omega_{ij} \in \mathbb{C}$ satisfying $|\omega_{ij}| = 1$ and $\omega_{ji} = \overline{\omega_{ij}}$ whenever both orientations (i, j) and (j, i) are present. Define the magnetic (Hermitian) Laplacian*

$$L_B = K - A_B, \quad (A_B)_{ij} = \begin{cases} \omega_{ij}, & (i, j) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

where $K = \text{diag}(k_v)$ is the degree matrix with $k_v = \sum_j |(A_B)_{vj}|$. Then L_B is Hermitian and diffusive (it admits the factorization $L_B = \nabla_B^* \nabla_B$), but it is not a proper Laplacian in the sense $\mathcal{L} = \text{div grad}$ with a proper gradient satisfying $\text{grad}(\mathbf{1}) = 0$, unless all phases are trivial ($\omega_{ij} \equiv 1$). As a limiting real case, the signless Laplacian $Q = K + A$ (corresponding to $\omega_{ij} \equiv -1$) is also diffusive but not proper.

Proof. Fix an orientation of E and define the magnetic gradient $\nabla_B : \mathbb{C}(V) \rightarrow \mathbb{C}(E)$ by

$$(\nabla_B x)_{(i,j)} = x_i - \omega_{ij} x_j.$$

A direct computation yields the Hermitian diffusive factorization

$$L_B = \nabla_B^* \nabla_B.$$

However, for the constant vector $\mathbf{1}$ we have

$$(\nabla_B \mathbf{1})_{(i,j)} = 1 - \omega_{ij},$$

which is nonzero whenever $\omega_{ij} \neq 1$. Thus ∇_B fails to be a *proper gradient* since it does not annihilate constants. Consequently, L_B , while diffusive and self-adjoint, cannot be expressed as div grad with a proper gradient satisfying $\text{grad}(\mathbf{1}) = 0$ unless all phases are trivial.

To exclude the possibility of another proper gradient Υ generating L_B , suppose $L_B = \Upsilon^* \Upsilon = \nabla_B^* \nabla_B$. By the polar decomposition there exists a partial isometry U such that $\nabla_B = U \Upsilon$. Hence $\nabla_B \mathbf{1} = U \Upsilon \mathbf{1} = 0$, contradicting the fact that $\nabla_B \mathbf{1} \neq 0$. Therefore, no proper gradient can yield L_B unless $\omega_{ij} \equiv 1$, in which case ∇_B reduces to the ordinary incidence matrix and L_B becomes the standard Laplacian.

For the uniform phase $\omega_{ij} \equiv -1$, one has $A_B = -A$ and thus $L_B = K - (-A) = K + A =: Q$. The associated incidence operator is

$$(\nabla_+ x)_{(i,j)} = x_i + x_j,$$

for which $\nabla_+ \mathbf{1} \neq 0$. Repeating the same argument shows that no proper gradient Υ with $\Upsilon \mathbf{1} = 0$ can satisfy $Q = \Upsilon^* \Upsilon$. Hence Q is also diffusive but not proper. \square

Remark 13. The magnetic Laplacian L_B can be interpreted as the Schur complement of a proper Laplacian on an extended vertex–edge space. Given an oriented or mixed graph $G = (V, E)$ and magnetic phases ω_{ij} with $|\omega_{ij}| = 1$, introduce auxiliary edge variables $z_e \in \mathbb{C}$ and define the extended gradient

$$\begin{aligned} \Upsilon : \mathbb{C}(V) \times \mathbb{C}(E) &\rightarrow \mathbb{C}^{2|E|}, & \Upsilon(x, z) &\text{ has, for each } e = (i, j), \text{ two rows} \\ & & & \sqrt{2}(x_i - z_e), \quad \sqrt{2}(\omega_{ij} x_j - z_e). \end{aligned}$$

In block form,

$$\Upsilon = [M \quad -D], \quad M = \sqrt{2} \bigoplus_{e=(i,j)} \begin{bmatrix} e_i^\top \\ \omega_{ij} e_j^\top \end{bmatrix}, \quad D = \begin{bmatrix} I_m \\ I_m \end{bmatrix}.$$

The corresponding extended Laplacian

$$\mathcal{L}_{\text{ext}} = \Upsilon^* \Upsilon = \begin{bmatrix} M^* M & -M^* D \\ -D^* M & D^* D \end{bmatrix}$$

is Hermitian positive semidefinite and of the diffusive form $\text{grad}^* \text{grad}$ on the extended space $V \cup E$. Eliminating the edge variables z gives the Schur complement

$$L_B = \mathcal{L}_{\text{ext}} / (D^* D) = M^* M - M^* D (D^* D)^{-1} D^* M.$$

Thus the magnetic Laplacian arises as the projection of a proper Laplacian defined on a higher-dimensional space. The signless Laplacian $Q = K + A$ corresponds to the real case with a uniform magnetic phase $\omega_{ij} = -1$ (a phase shift of π) along all oriented edges.

Example 14. Consider the undirected 3-cycle with edges $\{1, 2\}, \{2, 3\}, \{3, 1\}$ and magnetic phases $\omega_{12}, \omega_{23}, \omega_{31} \in \mathbb{C}$ satisfying $|\omega_{ij}| = 1$ and $\omega_{ji} = \overline{\omega_{ij}}$. The magnetic Laplacian is

$$L_B = K - A_B = \begin{pmatrix} 2 & -\omega_{12} & -\overline{\omega_{31}} \\ -\overline{\omega_{12}} & 2 & -\omega_{23} \\ -\omega_{31} & -\overline{\omega_{32}} & 2 \end{pmatrix}, \quad K = 2I_3.$$

We introduce one auxiliary variable z_{ij} for each oriented edge $(i \rightarrow j) \in \{(1, 2), (2, 3), (3, 1)\}$. Define the extended magnetic gradient $\Upsilon : \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}^6$ by

$$\Upsilon = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \omega_{12} & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & \omega_{23} & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \omega_{31} & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Upsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ z_{12} \\ z_{23} \\ z_{31} \end{pmatrix} = \begin{pmatrix} x_1 - z_{12} \\ \omega_{12}x_2 - z_{12} \\ x_2 - z_{23} \\ \omega_{23}x_3 - z_{23} \\ x_3 - z_{31} \\ \omega_{31}x_1 - z_{31} \end{pmatrix}.$$

In matrix form, this can be written as

$$\Upsilon = [M \mid -D],$$

where

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega_{23} \\ 0 & 0 & 1 \\ \omega_{31} & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$D^*D = 2I_3, \quad \mathcal{L}_{\text{ext}} = \Upsilon^*\Upsilon = \begin{pmatrix} M^*M & -M^*D \\ -D^*M & D^*D \end{pmatrix},$$

which for the example in question is given by:

$$\mathcal{L}_{\text{ext}} = \Upsilon^*\Upsilon = \begin{pmatrix} 2 & -\omega_{12} & -\overline{\omega_{13}} & 1 & 0 & 0 \\ -\overline{\omega_{12}} & 2 & -\omega_{23} & 0 & 1 & 0 \\ -\omega_{31} & -\overline{\omega_{23}} & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Eliminating the edge variables z gives the Schur complement

$$\mathcal{L}_{\text{ext}} / (D^*D) = M^*M - \frac{1}{2} M^*DD^*M = \sum_{\{i,j\} \in E} (e_i e_i^* + e_j e_j^* - \overline{\omega_{ij}} e_i e_j^* - \omega_{ij} e_j e_i^*) = L_B.$$

Therefore, the Hermitian magnetic Laplacian L_B is precisely the Schur complement of the proper extended Laplacian $\mathcal{L}_{\text{ext}} = \Upsilon^*\Upsilon$. Notice that in this particular example, due to the specific choice of orientation and normalization, L_B is also the upper-left block of \mathcal{L}_{ext} .

Remark 15. Let $L_B = K - A_B$ be the magnetic Laplacian as defined before, such that $A_B(i, j) = \omega_{ij} \in \mathbb{C}$ if $(i, j) \in E$ and $\omega_{ji} = \overline{\omega_{ij}}$. Let $\phi = \text{Re}(\omega_{ij})$ and let $\varphi = \text{Im}(\omega_{ij})$. Let R be the matrix whose entries are defined as

$$R_{ij} = \begin{cases} 1 & \text{iff } i \rightarrow j \\ -1 & \text{iff } i \leftarrow j \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can write

$$L_B = \phi \left(\frac{K}{\phi} - A \right) + \varphi i R =: \phi L_\phi + \varphi i R.$$

Let us plug this operator on the diffusion equation:

$$\partial_t x(t) + \phi L_\phi x(t) - \varphi i R x(t) = 0.$$

Then, L_ϕ is a diffusive Laplacian with diffusion coefficient ϕ and iR can be considered as an imaginary reaction potential which account for wave-like reactions. The equation is similar to the one proposed by Varshney et al. [84] to model a diffusive process plus a chemical potential in neuronal systems. The difference is that the model of Varshney et al. [84] does not include the term φi multiplying the chemical potential which makes that their operator is not symmetric, its eigenvalues may be complex, it could be nondiagonalizable. Additionally, the semigroup obtained from their model is neither stochastic nor sub-Markovian.

We now turn our attention towards the Laplacian matrix of a graph with edge signs, known as signed graphs (see [90]). The study of the signed Laplacian has received some attention in the literature for the analysis of real-world systems in which friendship-enmity relations are present [56, 5], particularly interesting is the concept of dissensus found in some graphs by Altafini [5].

Proposition 16. | Let $G^\sigma = (V, E, \sigma)$ be a signed graph with edge signs $\sigma_{ij} \in \{\pm 1\}$ on $\{i, j\} \in E$, and let

$$L^\sigma = D - A^\sigma, \quad (A^\sigma)_{ij} = \begin{cases} \sigma_{ij}, & \{i, j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Then L^σ is Hermitian and diffusive (it admits the factorization $L^\sigma = \nabla_\sigma^* \nabla_\sigma$), but it is not a proper Laplacian in the sense $\mathcal{L} = \text{div grad}$ with a proper gradient satisfying $\text{grad}(\mathbf{1}) = 0$, unless the signing is trivial ($\sigma_{ij} \equiv +1$ on all edges).

Proof. Fix an orientation of E and define the signed (“twisted”) gradient $\nabla_\sigma : \mathbb{C}(V) \rightarrow \mathbb{C}(E)$ by

$$(\nabla_\sigma x)_{(i,j)} = x_i - \sigma_{ij} x_j.$$

A direct computation gives the standard factorization

$$L^\sigma = \nabla_\sigma^* \nabla_\sigma.$$

Evaluating this operator on the constant vector $\mathbf{1}$ yields

$$(\nabla_\sigma \mathbf{1})_{(i,j)} = 1 - \sigma_{ij}.$$

Hence $\nabla_\sigma \mathbf{1} = 0$ if and only if $\sigma_{ij} = +1$ on every edge. Whenever a negative edge is present, ∇_σ fails to be a *proper gradient*, since it does not annihilate constants.

Suppose there existed another proper gradient Υ with $\Upsilon \mathbf{1} = 0$ such that $L^\sigma = \Upsilon^* \Upsilon = \nabla_\sigma^* \nabla_\sigma$. Then, by the polar decomposition, there would exist a partial isometry U such that $\nabla_\sigma = U \Upsilon$. Applying both sides to $\mathbf{1}$ gives $\nabla_\sigma \mathbf{1} = U \Upsilon \mathbf{1} = 0$, a contradiction when any negative edge exists. Thus L^σ can only be proper when all $\sigma_{ij} = +1$, in which case it reduces to the standard Laplacian. \square

Remark 17. The signed Laplacian $L^\sigma = D - A^\sigma$ can be decomposed as the difference of two proper Laplacians,

$$L^\sigma = L^+ - L^-,$$

where L^+ and L^- are the ordinary Laplacians of the subgraphs consisting of positive and negative edges, respectively. Hence L^σ represents the superposition of two diffusive processes of opposite sign: one promoting alignment across positive links, and the other promoting anti-alignment across negative ones. It is therefore not a purely diffusive operator but the generator of a *competition between cooperative and antagonistic diffusion*.

The so-called Lerman-Ghosh Laplacian L_χ was introduced by Lerman and Ghosh in 2012 [57] as an attempt to explore the nonconservative diffusion processes occurring in online social media. It was then further studied in [44, 30, 66].

Proposition 18. *Let $G = (V, E)$ be a (simple) undirected graph with degree matrix $K = \text{diag}(k_1, \dots, k_n)$ and standard incidence (edge-vertex) matrix ∇ (so $L = \nabla \nabla^\top = K - A$). Fix a parameter $\chi \in \mathbb{R}$ and define the augmented (“hairy”) incidence*

$$\nabla^\chi := [\nabla \mid \mathcal{R}] \in \mathbb{R}^{n \times (m+n)}, \quad \mathcal{R} := \text{diag}(\sqrt{\max\{\chi - k_i, 0\}}), \quad \mathcal{R}^2 = \text{diag}(\max\{\chi - k_i, 0\}),$$

interpreting each diagonal entry of \mathcal{R} as a semiedge connecting vertex i to an external reservoir. Then the Lerman-Ghosh operator

$$L_\chi := \nabla^\chi (\nabla^\chi)^\top$$

is self-adjoint and positive semidefinite (diffusive) and satisfies

$$L_\chi = L + (\chi I - K).$$

However, L_χ is not a proper Laplacian in the sense $\mathcal{L} = \text{grad}^ \text{grad}$ with a proper gradient such that $\text{grad}(\mathbf{1}) = 0$.*

Proof. Block multiplication gives

$$L_\chi = [\nabla \mid \mathcal{R}] \begin{bmatrix} \nabla^\top \\ \mathcal{R}^\top \end{bmatrix} = \nabla \nabla^\top + \mathcal{R} \mathcal{R}^\top = L + \mathcal{R}^2,$$

where $\mathcal{R}^2 = \text{diag}(\max\{\chi - k_i, 0\})$. Since L_χ is a Gram operator, it is symmetric and positive semidefinite:

$$x^\top L_\chi x = \|\nabla^\chi x\|^2 \geq 0, \quad \forall x \in \mathbb{R}^n.$$

Nevertheless,

$$\nabla^\chi \mathbf{1} = (\nabla \mathbf{1}; \mathcal{R} \mathbf{1}) = (0; \mathcal{R} \mathbf{1}) \neq 0$$

whenever $\mathcal{R} \neq 0$, which implies that the augmented gradient ∇^χ does not annihilate constants. Therefore L_χ cannot be expressed as $\text{grad}^* \text{grad}$ with a proper gradient $\text{grad}(\mathbf{1}) = 0$. In this sense, the Lerman-Ghosh operator is *diffusive but nonconservative*, and not a proper Laplacian. \square

Remark 19. If the system is extended by including an additional vertex representing the reservoir and defining the hair terms as true differences $x_i - y$, the corresponding augmented gradient

$$\nabla_{\text{ext}}^\chi x = (\nabla x; \mathcal{R}(x - \mathbf{1}y))$$

satisfies $\nabla_{\text{ext}}^\chi \mathbf{1} = 0$. The associated extended operator

$$L_\chi^{\text{ext}} = \nabla_{\text{ext}}^\chi (\nabla_{\text{ext}}^\chi)^\top = \begin{pmatrix} L + \mathcal{R}^2 & -\mathcal{R}^2 \mathbf{1} \\ -\mathbf{1}^\top \mathcal{R}^2 & \mathbf{1}^\top \mathcal{R}^2 \mathbf{1} \end{pmatrix}$$

is then a proper conservative Laplacian on the enlarged graph $(V \cup \{y\}, E \cup \{(i, y)\})$. Therefore, the Lerman–Ghosh operator is: $L_\chi = L + \mathcal{R}^2$, which means it is precisely the principal submatrix of the extended conservative Laplacian obtained by removing the row and column corresponding to the reservoir node. Hence L_χ describes the open (nonconservative) subsystem of the proper diffusive operator L_χ^{ext} acting on the extended graph $(V \cup \{y\}, E \cup \{(i, y)\})$.

Remark 20. Setting $\chi = 0$ gives $L_0 = L - K = -A$, so the (negative) adjacency operator appears as the χ –Laplacian with zero reservoir strength. Allowing complex hair entries or phases on edges turns ∇^χ into a complex (magnetic–type) incidence operator, as studied in related works on complex gradients.

The study of roots of the standard Laplacian matrix, a.k.a. fractional graph Laplacian has been developed by Riascos and collaborators [77, 63] as a way to account for nonlocal spatial effects on diffusive processes on graphs.

Proposition 21. *Let $G = (V, E)$ be an undirected graph and let $L = \nabla^\top \nabla$ be its standard Laplacian (with any fixed orientation of E), acting on $\mathbb{C}(V)$. For any $\alpha > 0$, define the fractional Laplacian by spectral calculus*

$$L^\alpha = U \Lambda^\alpha U^*,$$

where $L = U \Lambda U^*$ is a unitary diagonalization with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_k \geq 0$ and $\lambda_k = 0$ for k indexing the nullspace (spanned by constants on each connected component). Then L^α is a proper graph Laplacian in the sense that there exists a (possibly nonlocal) gradient

$$\text{grad}_\alpha : \mathbb{C}(V) \rightarrow \mathbb{C}^n, \quad \text{grad}_\alpha(\mathbf{1}) = 0, \quad L^\alpha = \text{grad}_\alpha^* \text{grad}_\alpha.$$

Proof. Let $L = U \Lambda U^*$ with U unitary and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_k \geq 0$. Define

$$\text{grad}_\alpha := \Lambda^{\alpha/2} U^* : \mathbb{C}(V) \longrightarrow \mathbb{C}^n.$$

Then

$$\text{grad}_\alpha^* \text{grad}_\alpha = U \Lambda^{\alpha/2} \Lambda^{\alpha/2} U^* = U \Lambda^\alpha U^* = L^\alpha,$$

so L^α is of the form $\text{grad}^* \text{grad}$. Moreover, if $\mathbf{1}$ denotes the (componentwise) constant vector on V , then $\mathbf{1}$ lies in the nullspace of L (one copy per connected component), hence $U^* \mathbf{1}$ has zero entries on all coordinates where $\lambda_k > 0$. Since $\Lambda^{\alpha/2}$ has zeros on those null coordinates, we obtain

$$\text{grad}_\alpha(\mathbf{1}) = \Lambda^{\alpha/2} U^* \mathbf{1} = 0.$$

Thus grad_α is a *proper gradient* (it annihilates constants). Consequently $L^\alpha = \text{grad}_\alpha^* \text{grad}_\alpha$ is self-adjoint, positive semidefinite, and conservative on V : $L^\alpha \mathbf{1} = 0$ (indeed, L^α acts as 0 on the nullspace of L by functional calculus). \square

Remark 22. The operator grad_α maps vertex functions to coefficients along the spectral (Fourier) modes of L , scaled by $\lambda_k^{\alpha/2}$. Hence, for $\alpha \neq 1$, the induced Laplacian L^α is *nonlocal* in the vertex domain, but remains a proper (diffusive–conservative) Laplacian under the general definition where the gradient may act between vertices and an abstract family of graph elements (here, the spectral modes). For a connected graph, the nullspace is $\text{span}\{\mathbf{1}\}$; for a graph with c components, grad_α annihilates the c indicator-constants of the components.

A different approach to account for nonlocal spatial effects on the diffusion on graphs has been developed under the name of d -path Laplacians [31, 32, 33]. In a series of papers it was proved analytically that a Mellin transformation of the d -path Laplacians produce superdiffusive behavior on graphs [32, 33], which combined with fractional temporal derivatives have conduced to a generalization of the diffusion equation on graph [27] capable to describe sub-, normal and super-diffusive dynamics on graphs.

Proposition 23. *Let $G = (V, E)$ be a (simple, undirected) graph with geodesic (shortest-path) distance $\text{dist}(\cdot, \cdot)$ and diameter $\text{diam}(G)$. For $d \in \{1, \dots, \text{diam}(G)\}$ define the d -path Laplacian $\mathcal{L}_d : \mathbb{C}(V) \rightarrow \mathbb{C}(V)$ by*

$$(\mathcal{L}_d f)(v) = \sum_{\substack{w \in V \\ \text{dist}(v, w) = d}} (f(v) - f(w)).$$

Then:

- (i) Each \mathcal{L}_d is a *proper conservative–diffusive Laplacian*, i.e., there exist linear operators

$$\text{grad}_d : \mathbb{C}(V) \rightarrow \mathbb{C}(\vec{E}_d), \quad \text{div}_d : \mathbb{C}(\vec{E}_d) \rightarrow \mathbb{C}(V),$$

with $\text{grad}_d(\mathbf{1}) = 0$ and $\mathcal{L}_d = \text{div}_d \text{grad}_d = \text{grad}_d^* \text{grad}_d$. In particular, \mathcal{L}_d is self-adjoint, positive semidefinite, and conservative: $\mathcal{L}_d \mathbf{1} = 0$.

- (ii) For any nonnegative coefficients $(c_d)_{d=1}^{\text{diam}(G)}$, the operator

$$\mathcal{L} := \sum_{d=1}^{\text{diam}(G)} c_d \mathcal{L}_d$$

is also a proper conservative–diffusive Laplacian. This covers, in particular, exponentially decaying weights $c_d = e^{-\lambda d}$ and polynomially decaying weights $c_d = d^{-s}$ with $\lambda > 0$ or $s > 0$.

Proof. (i) Fix d and form the d -sphere adjacency on V : connect v and w iff $\text{dist}(v, w) = d$. Let $\vec{E}_d := \{(v \rightarrow w) : \text{dist}(v, w) = d\}$ denote the set of oriented d -pairs. Define the discrete d -gradient and d -divergence by

$$(\text{grad}_d x)_{(v \rightarrow w)} := x_v - x_w, \quad (\text{div}_d y)_v := \sum_{\substack{w \in V \\ \text{dist}(v, w) = d}} y_{(v \rightarrow w)}.$$

Then for any $x \in \mathbb{C}(V)$,

$$(\text{div}_d \text{grad}_d x)_v = \sum_{\substack{w \\ \text{dist}(v, w) = d}} (x_v - x_w) = (\mathcal{L}_d x)_v.$$

Moreover, $\text{grad}_d(\mathbf{1}) = 0$ since $1 - 1 = 0$ on every oriented pair. Thus $\mathcal{L}_d = \text{div}_d \text{grad}_d$ with a proper gradient. With the standard inner products on $\mathbb{C}(V)$ and $\mathbb{C}(\vec{E}_d)$, we also have $\text{div}_d = \text{grad}_d^*$, hence

$$\mathcal{L}_d = \text{grad}_d^* \text{grad}_d, \quad \langle x, \mathcal{L}_d x \rangle = \|\text{grad}_d x\|^2 \geq 0,$$

which proves self-adjointness and positive semidefiniteness. Finally, $\mathcal{L}_d \mathbf{1} = \text{div}_d \text{grad}_d \mathbf{1} = 0$, so the operator is conservative.

(ii) Consider the block gradient

$$\text{grad} : \mathbb{C}(V) \longrightarrow \bigoplus_{d=1}^{\text{diam}(G)} \mathbb{C}(\vec{E}_d), \quad \text{grad} x := \bigoplus_{d=1}^{\text{diam}(G)} \sqrt{c_d} \text{grad}_d x,$$

with the convention $\sqrt{c_d} = 0$ if $c_d = 0$. Then $\text{grad}(\mathbf{1}) = \bigoplus_d \sqrt{c_d} \text{grad}_d(\mathbf{1}) = 0$ and

$$\text{grad}^* \text{grad} = \sum_{d=1}^{\text{diam}(G)} c_d \text{grad}_d^* \text{grad}_d = \sum_{d=1}^{\text{diam}(G)} c_d \mathcal{L}_d = \mathcal{L}.$$

Thus \mathcal{L} is of the form $\text{grad}^* \text{grad}$ with a proper gradient, hence self-adjoint, positive semidefinite, and conservative: $\mathcal{L} \mathbf{1} = \text{grad}^* \text{grad} \mathbf{1} = 0$. For the specific choices $c_d = e^{-\lambda d}$ ($\lambda > 0$) or $c_d = d^{-s}$ ($s > 0$), all $c_d \geq 0$, so the conclusion applies. \square

Remark 24. Each \mathcal{L}_d is the standard (combinatorial) Laplacian of the d -path graph whose edges connect pairs at geodesic distance d ; hence d -path Laplacians are nonlocal (for $d > 1$) yet remain proper Laplacians in the sense of $\text{div} \text{grad}$ with $\text{grad}(\mathbf{1}) = 0$. Weighted superpositions with $c_d \geq 0$ simply rescale the corresponding edge bundles and preserve the diffusive-conservative structure.

The hub-biased Laplacians L^{rep} and L^{attr} have been developed to account for the realistic scenarios in which a diffusive particle at a given vertex of a graph is biased away or toward nearest neighbor vertices of higher degree [29, 42]. They have found applications not only in diffusive processes but in nonlinear synchronization models (see references cited before).

Proposition 25. *Let $G = (V, E)$ be a simple undirected graph with vertex degrees $k_v > 0$, and let*

$$(\mathcal{L}_R f)(v) = \sum_{w \in \mathcal{N}_v} \frac{k_v}{k_w} (f(v) - f(w)), \quad (\mathcal{L}_A f)(v) = \sum_{w \in \mathcal{N}_v} \frac{k_w}{k_v} (f(v) - f(w)),$$

be respectively the hubs-repelling and hubs-attracting Laplacians acting on vertex functions $f \in \mathbb{C}(V)$. Then there exist linear operators

$$\text{grad}_\alpha : \mathbb{C}(V) \rightarrow \mathbb{C}(\vec{E}), \quad \text{div}_\alpha : \mathbb{C}(\vec{E}) \rightarrow \mathbb{C}(V),$$

where \vec{E} denotes the set of oriented edges and

$$c_\alpha(v, w) = \left(\frac{k_v}{k_w} \right)^\alpha, \quad \alpha \in \{+1, -1\},$$

such that

$$\mathcal{L}_\alpha = \text{div}_\alpha \text{grad}_\alpha, \quad \text{grad}_\alpha(\mathbf{1}) = 0.$$

Hence both \mathcal{L}_R and \mathcal{L}_A are proper diffusive Laplacians in the sense of Definition 4.

Proof. For each oriented edge $(v \rightarrow w) \in \vec{E}$, define the degree-weighted gradient by

$$(\text{grad}_\alpha x)_{(v \rightarrow w)} = \sqrt{c_\alpha(v, w)} (x_v - x_w) = \left(\frac{k_v}{k_w}\right)^{\alpha/2} (x_v - x_w), \quad x \in \mathbb{C}(V).$$

Then $\text{grad}_\alpha(\mathbf{1}) = 0$, since all vertex differences vanish for the constant function. Define the corresponding divergence as the aggregation of edge fluxes leaving each vertex,

$$(\text{div}_\alpha y)_v = \sum_{w \in \mathcal{N}_v} \sqrt{c_\alpha(v, w)} y_{(v \rightarrow w)}, \quad y \in \mathbb{C}(\vec{E}).$$

By direct substitution,

$$(\text{div}_\alpha \text{grad}_\alpha x)_v = \sum_{w \in \mathcal{N}_v} \sqrt{c_\alpha(v, w)} \left[\sqrt{c_\alpha(v, w)} (x_v - x_w) \right] = \sum_{w \in \mathcal{N}_v} c_\alpha(v, w) (x_v - x_w) = (\mathcal{L}_\alpha x)_v.$$

Thus $\mathcal{L}_\alpha = \text{div}_\alpha \text{grad}_\alpha$ with $\text{grad}_\alpha(\mathbf{1}) = 0$. Since every term acts as a weighted difference between neighboring vertices, \mathcal{L}_R and \mathcal{L}_A are diffusive operators, although generally asymmetric. \square

Remark 26. The asymmetry of \mathcal{L}_R and \mathcal{L}_A reflects degree-dependent anisotropy of the diffusion process rather than a failure of the Laplacian structure. Indeed, both operators are similar to the symmetric matrix $W - A$, where $\mathcal{L}_R = K(W - A)K^{-1}$ and $\mathcal{L}_A = K^{-1}(W - A)K$, so they are diagonalizable with real, nonnegative eigenvalues. Hence the degree-biased Laplacians remain proper within the operator-theoretic definition of this paper, even though they are not self-adjoint in the Euclidean metric. They represent biased diffusions in which probability fluxes are modulated by local vertex degrees.

6 The Moore–Penrose pseudoinverse of L as a proper Laplacian

The motivation of this section is twofold. On one hand we are interested in exploring the physically-sounded model proposed by Saxena *et al.* [79], which is a diffusion-like model inspired by electrical network based on the pseudoinverse L^+ of the graph Laplacian. On the other hand, is the statement in a spectral graph theory book that L^+ “is not always a Laplacian” because some of its off-diagonal elements can be positive (see pp. 133–134 in [64]). Therefore, we investigate analytically whether the pseudoinverse of the graph Laplacian is a proper diffusive Laplacian or not.

Let $G = (V, E)$ be a connected undirected graph with n vertices and m edges. Denote by $\nabla \in \mathbb{R}^{n \times m}$ its vertex–edge incidence matrix and by $L = \nabla \nabla^\top$ the combinatorial Laplacian of G . Let the singular value decomposition of ∇ be

$$\nabla = U \Sigma V^\top, \quad U \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{m \times m}, \Sigma = \text{diag}(\sigma_1, \dots, \sigma_{n-1}, 0),$$

where $\sigma_i > 0$ for $i \leq n - 1$ because G is connected.

Definition 27. Define the *pseudogradient* of G as

$$\tilde{\nabla} := U(\Sigma^+)^{\top} V^\top,$$

where $\Sigma^+ = \text{diag}(\sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}, 0)$ is the Moore–Penrose pseudoinverse of Σ .

Lemma 28. *The matrix $\tilde{\nabla}$ satisfies:*

- (a) $L\tilde{\nabla}\tilde{\nabla}^\top L = L$;
- (b) $\tilde{\nabla}\tilde{\nabla}^\top L\tilde{\nabla}\tilde{\nabla}^\top = \tilde{\nabla}\tilde{\nabla}^\top$;
- (c) $(L\tilde{\nabla}\tilde{\nabla}^\top)^\top = L\tilde{\nabla}\tilde{\nabla}^\top$ and $(\tilde{\nabla}\tilde{\nabla}^\top L)^\top = \tilde{\nabla}\tilde{\nabla}^\top L$.

Proof. Each statement follows directly by substitution of $\nabla = U\Sigma V^\top$, $\tilde{\nabla} = U(\Sigma^+)^\top V^\top$, and the orthogonality of U, V . The first equality reduces to $U\Sigma V^\top V\Sigma^\top U^\top U(\Sigma^+)^\top V^\top V\Sigma^+ U^\top U\Sigma V^\top V\Sigma^\top U^\top = U\Sigma\Sigma^\top U^\top = L$, and the remaining ones are analogous. \square

Theorem 29. *Let L^+ be the Moore–Penrose pseudoinverse of the Laplacian L . Then*

$$L^+ = \tilde{\nabla}\tilde{\nabla}^\top = (\nabla^+)^\top \nabla^+,$$

and L^+ satisfies the defining properties of a proper Laplacian operator:

$$L^+ = \text{grad}^* \text{grad} \quad \text{with} \quad \text{grad}(\mathbf{1}) = 0.$$

Proof. The Moore–Penrose pseudoinverse of any matrix product AA^\top satisfies $(AA^\top)^+ = (A^+)^\top A^+$. Applying this to $A = \nabla$ gives $L^+ = (\nabla^+)^\top \nabla^+ = (U(\Sigma^+)^\top V^\top)(V(\Sigma^+)U^\top) = U(\Sigma^+)^\top \Sigma^+ U^\top$. Setting $\text{grad} := \nabla^+$, we obtain $L^+ = \text{grad}^* \text{grad}$.

Since G is connected, $\text{Ran}(\nabla) = \mathbf{1}^\perp$ and therefore $\mathbf{1} \in \text{Ker}(\nabla^\top) = \text{Ran}(\nabla)^\perp$. The Moore–Penrose properties imply that $\nabla^+ \mathbf{1} = 0$, i.e. $\text{grad}(\mathbf{1}) = 0$. \square

Remark 30. (Spectral form and geometric meaning.) In the orthonormal basis of Laplacian eigenvectors $L = U\Lambda U^\top$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$, one has

$$L^+ = U\Lambda^+ U^\top, \quad \Lambda^+ = \text{diag}(0, \lambda_2^{-1}, \dots, \lambda_n^{-1}).$$

Thus L^+ plays the role of the inverse diffusion operator restricted to $\mathbf{1}^\perp$: it corresponds to the integral kernel of the Green’s function for the diffusion equation $\partial_t x + Lx = 0$.

Remark 31. (On nonlocality and diffusion.) Although L^+ is diffusive (being positive semidefinite and self-adjoint), its associated pseudogradient $\tilde{\nabla}$ is *nonlocal*: each row mixes all vertices rather than single edge differences. This agrees with the idea that L^+ represents the steady-state response or long-range coupling induced by the diffusion generated by L . Hence L^+ is a proper (but nonlocal) Laplacian under the general definition $\mathcal{L} = \text{div grad}$ with $\text{grad}(\mathbf{1}) = 0$.

Example 32 (Pseudogradient for the pseudoinverse Laplacian of a tree). Let $T = (V, E)$ be the tree with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (3, 4), (3, 5)\}$. The pseudogradient $\tilde{\nabla} = \tilde{\nabla}(T)$ such that $L^+(T) = \tilde{\nabla}\tilde{\nabla}^\top$ is given by

$$\tilde{\nabla}(T) = \frac{1}{5} \begin{bmatrix} 4 & -3 & 1 & -1 \\ -1 & -3 & 1 & -1 \\ -1 & 2 & 1 & -1 \\ -1 & 2 & -4 & -1 \\ -1 & 2 & 1 & 4 \end{bmatrix}.$$

This construction follows the method of Bapat [10], later extended by Alazemi *et al.* [3, 2] (see also [24]), who derived combinatorial and signed-graph generalizations of the Moore–Penrose pseudoinverse of graph Laplacians.

These results can be interpreted combinatorially as follows. For any edge $e = (i, j) \in E$, removing e splits T into two subtrees $T_+(i, j)$ and $T_-(i, j)$. Let n_+ and n_- be their respective numbers of vertices, and $n = |V|$. Then, following Bapat [10],

$$\tilde{\nabla}_{v_k, e(i, j)} = \begin{cases} n_-/n, & v_k \in T_+(i, j), \\ -n_+/n, & v_k \in T_-(i, j). \end{cases}$$

In words, each entry of the pseudogradient measures the size imbalance between the two subtrees created by deleting e , normalized by n . The sign encodes whether the vertex v_k belongs to the same component as the head or the tail of the oriented edge.

We can also give a path-count interpretation to the entries of the gradient, which is intimately related to that of Bapat [10] for the case of trees, but which we present here for the sake of interpretability. Let $\pi_{(i, j)}^k$ denote the number of paths (of any length) in T that start at vertex v_k and traverse the oriented edge (i, j) . Then,

$$\tilde{\nabla}_{v_k, e(i, j)} = \frac{1}{n} \begin{cases} \pi_{(i, j)}^k, & \text{if the path orientation agrees with } (i, j), \\ -\pi_{(i, j)}^k, & \text{if it is opposite.} \end{cases}$$

Thus, the pseudogradient $\tilde{\nabla}$ has a purely combinatorial interpretation: it encodes, for each vertex–edge pair, the normalized signed count of all paths passing through that edge.

In closing, the pseudogradient $\tilde{\nabla}(T)$ is *nonlocal*: every vertex interacts with every edge of T , with weights determined by the sizes of the complementary subtrees. Its entries are both positive and negative, reflecting the orientation and relative position of each vertex with respect to each cut defined by an edge. Therefore, although $L^+(T)$ represents a *signed complete graph*, it remains a *proper Laplacian* under our operator-based definition, since

$$L^+(T) = \tilde{\nabla} \tilde{\nabla}^\top \quad \text{and} \quad \tilde{\nabla}(\mathbf{1}) = 0.$$

Bapat’s formula [10] shows that the entries of the pseudoinverse of a tree Laplacian depend only on the sizes of the subtrees obtained by deleting an edge. This provides a concrete realization of the abstract operator framework introduced in this paper: the pseudogradient $\tilde{\nabla}$ is a nonlocal, signed gradient satisfying $\tilde{\nabla}(\mathbf{1}) = 0$ and yielding a diffusive Gram form $L^+ = \tilde{\nabla} \tilde{\nabla}^\top$. Hence, the pseudoinverse of the Laplacian of a tree is a *proper* Laplacian in the generalized sense. Alazemi *et al.* [3, 2] extended these constructions to threshold and signed graphs, demonstrating that such nonlocal pseudogradients arise naturally beyond trees.

6.1 Diffusive dynamics with L and L^+

We finally compare the standard diffusion governed by the graph Laplacian L with that induced by its Moore–Penrose pseudoinverse L^+ in undirected graphs. Both are proper Laplacians in the sense of this paper, but they operate in complementary geometric spaces: L diffuses over the adjacency (metric) structure of the graph, whereas L^+ diffuses over the effective–resistance (dual) geometry. Let us then consider $\mathcal{L} \in \{L, L^+\}$ and the diffusive process

$$\dot{x}(t) = -\mathcal{L}x(t), \quad x(0) = x^0,$$

which has solution $x(t) = e^{-t\mathcal{L}}x^0$. Because \mathcal{L} is positive semidefinite we have

$$\lim_{t \rightarrow \infty} x(t) = \frac{\mathbf{1}^\top x_0}{n} \mathbf{1},$$

for both Laplacians. However, because L has eigenvalues $0 = \mu_1 < \mu_2 \leq \dots \leq \mu_n$, the rate of convergence to the steady state is dictated by μ_2 (the algebraic connectivity): the larger μ_2 , the faster the convergence. On the other hand, $L^+ = U\Lambda^+U^\top$, where $\Lambda^+ = \text{diag}(0, 1/\mu_2, \dots, 1/\mu_n)$, and the rate of convergence of the diffusion is now controlled by $1/\mu_n$, the reciprocal of the largest eigenvalue of L . Hence, while L -diffusion emphasizes the tightest connected modes of the graph, L^+ -diffusion emphasizes its loosest (resistance-based) modes. Hence L^+ -diffusion is faster than L -diffusion iff

$$\frac{1}{\mu_n} > \mu_2. \quad (\star)$$

The quadratic form of L^+ is $x^\top L^+ x = \frac{1}{2} \sum_{i,j} \Omega_{ij} (x_i - x_j)^2$, with Ω_{ij} the effective resistance between i and j . Thus L^+ acts as the Laplacian of the complete weighted graph whose edge weights are the resistive distances of Υ . In stretched graphs (long paths, narrow bottlenecks), many Ω_{ij} are large, so L^+ strongly penalizes global discrepancies and accelerates homogenization relative to adjacency-based diffusion by L . That is, basically we need graphs with very small algebraic connectivity which also has relatively small Laplacian spectral radius. Graphs with small algebraic connectivity have been previously studied in the literature (see Chapter 6.8 in [82]). These graphs tend to be elongated graphs with large diameter. In the case of trees the minimum μ_2 for graph with fixed size and diameter is obtained by attaching an almost equal number of pendant vertices at each end of a path [35]. On the other hand, some of the graphs with small Laplacian spectral radius are the path and the cycle with n vertices (see Chapter 6.3 in [82]). Then we have the following example.

Example 33. For the path P_n , the Laplacian eigenvalues are $\mu_k = 2(1 - \cos(\frac{k\pi}{n+1}))$, so $\mu_2 \sim \pi^2/n^2$ and $\mu_n \rightarrow 4$ as $n \rightarrow \infty$, and thus $1/\mu_n \approx 0.25 > \mu_2$ for $n \gtrsim 6$. For the cycle C_n , $\mu_k = 2(1 - \cos(\frac{2\pi k}{n}))$, so $\mu_2 \sim 4\pi^2/n^2$ and $\mu_n = 4$, giving $1/\mu_n = 0.25 > \mu_2$ for $n \gtrsim 13$.

Inspired by the results of [35] we explore other types of graph which we prove to display $\mu_n \mu_2 < 1$.

6.1.1 Lollipop graphs $L(m, \ell)$

Let $L(m, \ell)$ be the graph obtained by attaching a path P_ℓ to a clique K_m by a single bridge edge. Denote by v_\star the clique vertex incident to the bridge. Then $k_{v_\star} = m$ and $k_v = m - 1$ for other clique vertices; the path has degrees 1 (ends) or 2 (internal). Hence $\Delta(L(m, \ell)) = m$.

Proposition 34 (Rates on lollipops). *For $L(m, \ell)$ one has*

$$m + 1 \leq \mu_n \leq 2m, \quad \mu_2 \leq \frac{2}{\text{vol}_{\min}},$$

where $\text{vol}_{\min} := \min\{\text{vol}(\text{clique side}), \text{vol}(\text{path side})\}$ for the cut that removes the single bridge edge. In particular, if the path side has larger volume (e.g., ℓ sufficiently large compared to m), then

$\text{vol}_{\min} = \text{vol}(\text{clique side}) = m + (m - 1)^2 = m^2 - m + 1$, and hence

$$\mu_2 \leq \frac{2}{m^2 - m + 1} \leq \frac{2}{m^2 - m}.$$

Consequently, for all $m \geq 2$,

$$\frac{1}{\mu_n} \geq \frac{1}{2m} > \frac{2}{m^2 - m + 1} \geq \mu_2$$

whenever m is sufficiently large; in particular (\star) holds for all large m uniformly in ℓ as soon as $\text{vol}(\text{path side}) \geq \text{vol}(\text{clique side})$.

Proof. Bounds on μ_n . The subgraph induced by v_\star and its m neighbors contains the star $K_{1,m}$. By Lemma 1, adding edges cannot decrease μ_n , hence $\mu_n(L(m, \ell)) \geq \mu_n(K_{1,m}) = m + 1$. The upper bound $\mu_n \leq 2\Delta = 2m$ follows from Lemma 2.

Bound on μ_2 . Consider the cut that removes the unique bridge edge; then $|E(S, \bar{S})| = 1$ and $\phi(\Upsilon) = \frac{1}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$. If the path side has larger volume than the clique side, then $\text{vol}_{\min} = \text{vol}(\text{clique side}) = k_{v_\star} + \sum_{\substack{v \in K_m \\ v \neq v_\star}} (m - 1) = m + (m - 1)^2 = m^2 - m + 1$. Hence $\phi \leq 1/(m^2 - m + 1)$ and Lemma 3 yields $\mu_2 \leq 2\phi \leq 2/(m^2 - m + 1)$. The final comparison with $1/\mu_n \geq 1/(2m)$ gives the claim for all sufficiently large m . \square

Remark 35. If the clique side were smaller in volume, one would obtain $\mu_2 \leq 2/\text{vol}(\text{path side})$, which can be even smaller for very long paths. Thus the conclusion (\star) is robust: any lollipop where the path side is not the bottleneck in volume satisfies faster L^+ -diffusion.

Example 36. We consider here a graph formed by one clique of 20 vertices connected by a single edge to a chain of 80 vertices. In this case $1/\mu_n \approx 0.0476$ and $\mu_2 \approx 0.0010$. In Fig. 1 we illustrate the progress of the diffusive dynamics on this graph using the standard Laplacian (panel a) and its pseudoinverse (panel b). The simulations are performed with exactly the same initial condition.

6.1.2 Two cliques joined by a chain (barbell-with-handle)

Let $B(m, \ell)$ be the graph obtained by connecting two disjoint cliques K_m and K_m by a simple path P_ℓ whose endpoints attach by a single edge to one vertex in each clique. The two junction vertices have degree m ; all other clique vertices have degree $m - 1$; path vertices have degree 1 or 2. Thus $\Delta(B(m, \ell)) = m$.

Proposition 37 (Rates on two cliques joined by a chain). *For $B(m, \ell)$ one has*

$$m + 1 \leq \mu_n \leq 2m, \quad \mu_2 \leq \frac{2}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}},$$

where the cut S is chosen across any single edge of the interconnecting chain. In particular, cutting the chain near its middle gives $|E(S, \bar{S})| = 1$ and

$$\min\{\text{vol}(S), \text{vol}(\bar{S})\} \geq m^2 - m + 1$$

(because one side contains an entire K_m), hence $\mu_2 \leq 2/(m^2 - m + 1)$. Therefore, for all sufficiently large m ,

$$\frac{1}{\mu_n} \geq \frac{1}{2m} > \frac{2}{m^2 - m + 1} \geq \mu_2,$$

so (\star) holds and L^+ -diffusion is faster than L -diffusion.

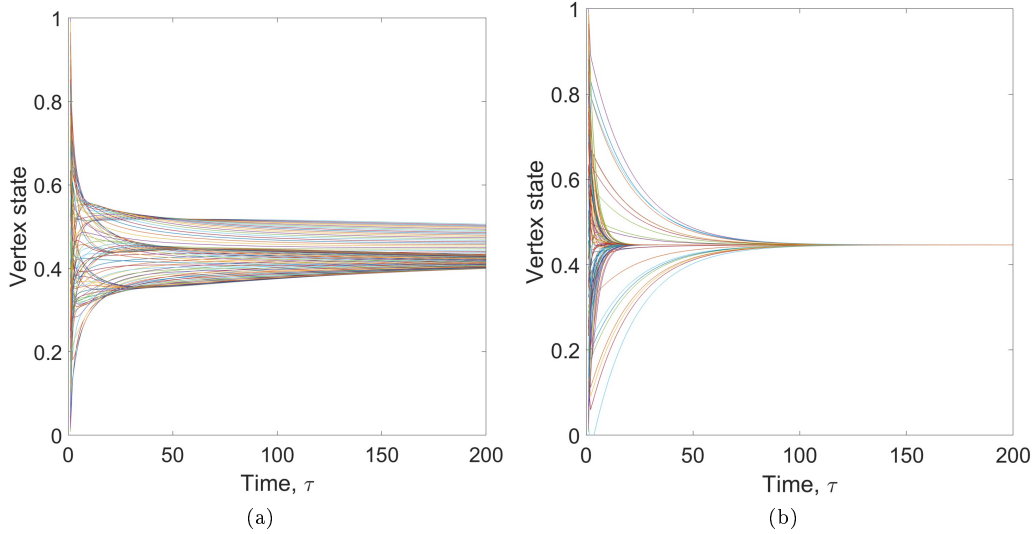


Figure 1: Illustration of the progress of the diffusive dynamics on a graph formed by one clique of 20 vertices connected by a single edge to a chain of 80 vertices. (a) Standard diffusion. (b) Diffusion based on the pseudoinverse of the Laplacian.

Proof. The bounds on μ_n are as in Proposition 34: the induced star around a junction vertex gives $\mu_n \geq m + 1$, and Lemma 2 gives $\mu_n \leq 2m$. For μ_2 , choose S so that $|E(S, \bar{S})| = 1$ across the chain. Then $\phi = 1/\min\{\text{vol}(S), \text{vol}(\bar{S})\}$. Either side contains at least one full K_m , whose volume is $m + (m - 1)^2 = m^2 - m + 1$, ignoring the chain contribution which only increases volume. Hence $\phi \leq 1/(m^2 - m + 1)$ and $\mu_2 \leq 2\phi$ by Lemma 3. \square

Remark 38. If the chain length equals the clique size ($\ell = m$), one still has $\mu_2 = O(m^{-2})$ while $\mu_n \in [m + 1, 2m]$, hence (\star) holds for all large m . As the chain grows, the minimal side volume in the middle cut only increases, so the upper bound on μ_2 remains $O(m^{-2})$ (or smaller), further favoring L^+ .

Example 39. We consider here a graph formed by two cliques of 33 vertices each connected by a chain of 33 vertices. The connection between the cliques and the chain are made by a single edge to one vertex in each clique. In this case $1/\mu_n \approx 0.0476$ and $\mu_2 \approx 0.0041$. In Fig. 2 we illustrate the progress of the diffusive dynamics on this graph using the standard Laplacian (panel a) and its pseudoinverse (panel b). The simulations were carried out with exactly the same initial condition.

In all these stretched families (lollipops, clique–chain–clique, paths, cycles), the bottleneck structure depresses μ_2 to $O(m^{-2})$ or $O(n^{-2})$, while μ_n remains $\Theta(\Delta)$; consequently (\star) holds and L^+ -diffusion equilibrates strictly faster than standard L -diffusion.

Remark 40. Both L and L^+ are proper Laplacians in the operator sense: they can be written as div grad with $\text{grad}(\mathbf{1}) = 0$, but they correspond to dual notions of locality. L acts on physical adjacency, while L^+ acts on resistance space. Together they form a complementary pair of diffusion generators on a graph and its dual metric.

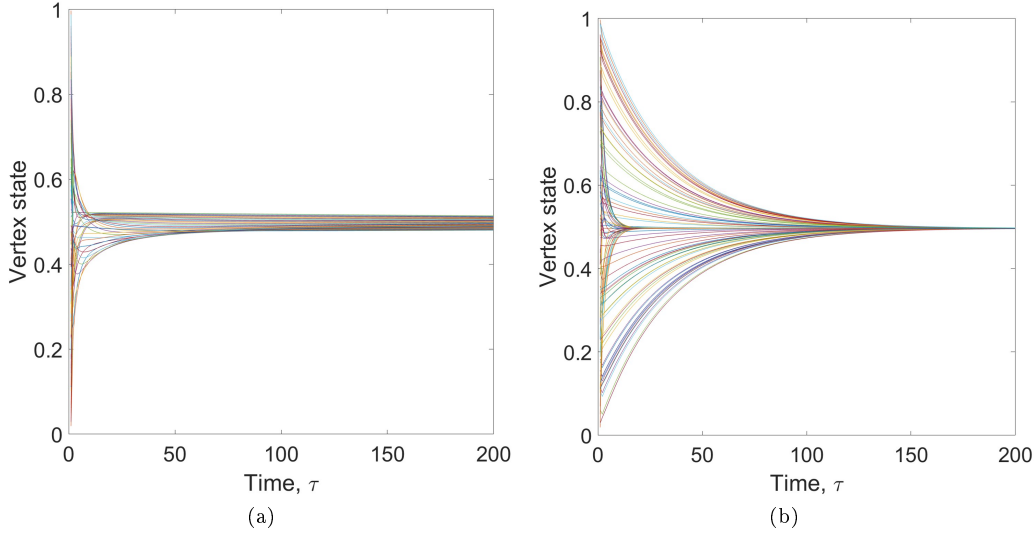


Figure 2: Illustration of the progress of the diffusive dynamics on a barbell graph formed by two cliques of 33 vertices each connected by a linear chain of 33 vertices. (a) Standard diffusion. (b) Diffusion based on the pseudoinverse of the Laplacian.

6.1.3 Example of a real-world network

It is possible to find real world networks in which $\mu_n \mu_2 < 1$? Here we consider the call network of the software MySQL consisting of 1480 vertices representing software modules and 4190 edges among them. We have computed that for this network

$$\mu_2(L) = 0.0039 \quad \text{and} \quad \frac{1}{\mu_n(L)} = 0.0045,$$

so the slowest exponential rate of L^+ -diffusion exceeds that of L -diffusion by $\frac{0.0045}{0.0039} - 1 \approx 15\%$. For any initial condition x_0 , writing $x^L(t) = e^{-tL}x_0$ and $x^{L^+}(t) = e^{-tL^+}x_0$, the deviations from the mean satisfy

$$\|x^L(t) - \bar{x} \mathbf{1}\| \leq e^{-\mu_2 t} \|x_0 - \bar{x} \mathbf{1}\|, \quad \|x^{L^+}(t) - \bar{x} \mathbf{1}\| \leq e^{-\frac{1}{\mu_n} t} \|x_0 - \bar{x} \mathbf{1}\|,$$

with $\bar{x} = \frac{1}{n} \mathbf{1}^\top x_0$. Thus the observed trajectories should display a visibly faster decay for L^+ , consistent with our theory that L^+ diffuses in effective-resistance geometry and can outpace adjacency-based diffusion on stretched or bottlenecked graphs.

We verify these predictions by simulating the temporal evolution of the diffusion in this network. As initial condition we select here that all the initial concentration was allocated in a single vertex. We perform the simulations using L and L^+ with exactly the same initial condition. As can be seen on Fig. 3 the rate of convergence—measured here as the standard deviation of all values of the vector $x(t)$ —is faster when the diffusion is controlled by L^+ than by L . The main reason of this difference in the studied network is that it consists of two main clusters which are very poorly connected among them. Therefore, the algebraic connectivity of the network is very small and $\mu_n(L)$ is not so big, such that $\mu_2(L)\mu_n(L) < 1$.

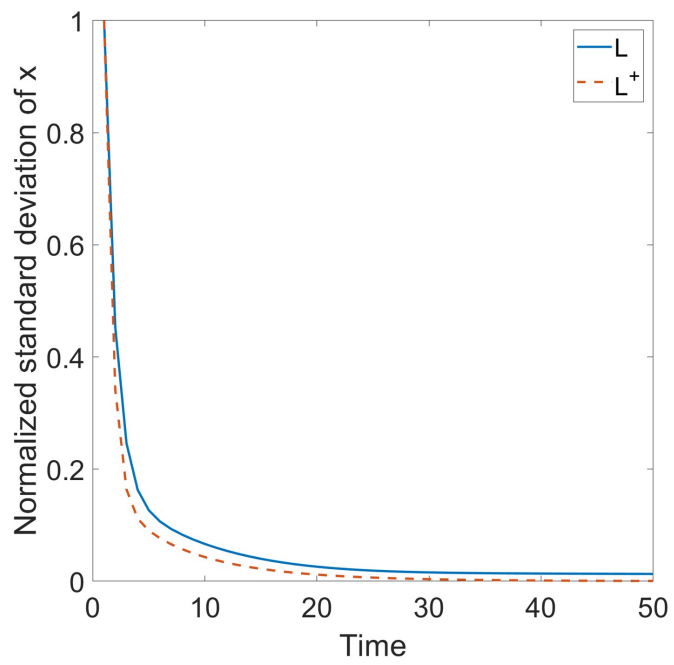


Figure 3: Illustration of the temporal evolution of the diffusive dynamics toward the steady state on the network of MySQL software.

Conclusions

We have introduced a unifying definition of a *proper graph Laplacian* as any operator that can be expressed in the form $\mathcal{L} = \text{div grad}$ with $\text{grad}(\mathbf{1}) = 0$, acting between vertex functions and functions defined on suitable graph structures. This operator-theoretic perspective recovers the classical Laplacian as a special case but extends naturally to nonlocal, weighted, and higher-order settings. It also allows a systematic distinction between *proper* and *improper* Laplacians on structural grounds, independent of ad hoc algebraic properties such as symmetry, sign patterns, or zero row-sum.

Within this framework we have shown that several matrices traditionally referred to as “Laplacians” are, in fact, *not* proper Laplacians. This includes the in- and out-degree Laplacians (which describe advection rather than diffusion), the signless and magnetic Laplacians (which arise as Schur complements of proper extended operators), and the deformed Laplacian. In contrast, the signed Laplacian, the Lerman–Ghosh Laplacian, and the fractional and degree-biased Laplacians were shown to be proper—though in some cases nonconservative–diffusive operators. Finally, we demonstrated that the Moore–Penrose pseudoinverse L^+ of the standard Laplacian is itself a proper (nonlocal) Laplacian, governing diffusion in the effective-resistance geometry of the graph. For certain classes of stretched or bottlenecked networks (paths, cycles, lollipops, and clique–chain–clique graphs), the L^+ -based diffusion equilibrates faster than the standard Laplacian flow, revealing a duality between physical and resistive diffusion. These results may be extended to other concepts of Laplacians not considered here like the “distance Laplacian” [6, 72, 9] as to the pseudoinverse of signed Laplacians just to mention a couple of examples [40].

Overall, this work establishes an operator-theoretic foundation for graph Laplacians that unifies classical and generalized constructions, clarifies long-standing ambiguities in terminology, and identifies a minimal structural criterion—existence of a genuine gradient annihilating constants for Laplacian-type operators. This approach opens the way to new forms of graph dynamics beyond the local edge-based paradigm, while preserving the geometric essence of diffusion on complex networks.

Declaration of competing interest

There is no competing interest.

Acknowledgments

The author thanks the Editor and referees for valuable comments and suggestions which improved significantly the presentation of the results. Financial support from Agencia Estatal de Investigación (AEI, MCI, Spain) MCIN/AEI/ 10.13039/501100011033 under grant PID2023-149473NB-I00 and by Agencia Estatal de Investigación (AEI, MCI, Spain) MCIN/AEI/ 10.13039/501100011033 and Fondo Europeo de Desarrollo Regional (FEDER, UE) under the María de Maeztu Program for units of Excellence in R&D, grant CEX2021-001164-M) are also acknowledged.

References

- [1] M. Abudayah, O. Alomari, and O. AbuGhneim. “Inverse of Hermitian Adjacency Matrix of a Mixed Graph”. In: *Applied Mathematics* 16.5 (2022), pp. 823–828. DOI: 10.4236/am.2022.1612059.
- [2] A. Alazemi, M. Anđelić, and S. Mallik. “Signed Graphs and Inverses of Their Incidence Matrices”. In: *Linear Algebra and its Applications* 694 (2024), pp. 78–100. DOI: 10.1016/j.laa.2023.11.011.
- [3] A. Alazemi et al. “Combinatorial Versus Algebraic Formulae for the Moore–Penrose Inverse of a Laplacian Matrix of a Threshold Graph”. In: *Journal of Computational and Applied Mathematics* 442 (2024), p. 115714. DOI: 10.1016/j.cam.2023.115714.
- [4] N. Alon and V. Milman. “ λ_1 , Isoperimetric Inequalities for Graphs, and Superconcentrators”. In: *Journal of Combinatorial Theory, Series B* 38.1 (1985), pp. 73–88. DOI: 10.1016/0095-8956(85)90092-9.
- [5] C. Altafini. “Consensus Problems on Networks with Antagonistic Interactions”. In: *IEEE Transactions on Automatic Control* 58.4 (2013), pp. 935–946. DOI: 10.1109/TAC.2012.2224251.
- [6] M. Aouchiche and P. Hansen. “Two Laplacians for the Distance Matrix of a Graph”. In: *Linear Algebra and its Applications* 439.1 (2013), pp. 21–33. DOI: 10.1016/j.laa.2012.12.024.
- [7] N. Athmouni et al. “The Magnetic Discrete Laplacian Inferred from the Gauß–Bonnet Operator and Application”. In: *Annals of Functional Analysis* 12.2 (2021), p. 33. DOI: 10.1007/s43034-021-00130-2.
- [8] R. Balaji and R. B. Bapat. “On Euclidean Distance Matrices”. In: *Linear Algebra and its Applications* 424.1 (2007), pp. 108–117. DOI: 10.1016/j.laa.2006.08.006.
- [9] R. Balaji and V. Gupta. “On distance Laplacian matrices of weighted trees”. In: *Operators & Matrices* 18.1 (2024), pp. 97–114. DOI: 10.7153/oam-2024-18-07.
- [10] R. B. Bapat. “Moore–Penrose Inverse of the Incidence Matrix of a Tree”. In: *Linear and Multilinear Algebra* 42.2 (1997), pp. 159–167. DOI: 10.1080/03081089708818532.
- [11] G. Berkolaiko. “Nodal Count of Graph Eigenfunctions via Magnetic Perturbation”. In: *Analysis & PDE* 6.5 (2013), pp. 1213–1233. DOI: 10.2140/apde.2013.6.1213.
- [12] E. Blåsten et al. “Inverse Problems for Discrete Heat Equations and Random Walks for a Class of Graphs”. In: *SIAM Journal on Discrete Mathematics* 37.2 (2023), pp. 831–863. DOI: 10.1137/21M1436936.
- [13] L. Böttcher and M. A. Porter. “Complex Networks with Complex Weights”. In: *Physical Review E* 109.2 (2024), p. 024314. DOI: 10.1103/PhysRevE.109.024314.
- [14] A. K. Chandra et al. “The Electrical Resistance of a Graph Captures Its Commute and Cover Times”. In: *Proceedings of the Twenty-First Annual ACM Symposium on Theory of Computing (STOC)*. ACM, 1989, pp. 574–586. DOI: 10.1145/73007.73062.
- [15] A. Chapman. “Advection on graphs”. In: *Semi-Autonomous Networks. Effective Control of Networked Systems through Protocols, Design, and Modeling*. Springer, 2015, pp. 3–16. DOI: 10.1007/978-3-319-15010-9_1.

- [16] A. Chapman, E. Schoof, and M. Mesbahi. “Advection on Networks with an Application to Decentralized Load Balancing”. In: *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2012, pp. 2680–2681. DOI: 10.1109/IR0S.2012.6385924.
- [17] J. Cheeger. “A Lower Bound for the Smallest Eigenvalue of the Laplacian”. In: *Problems in Analysis*. Ed. by R. C. Gunning. Princeton University Press, 1970, pp. 195–199.
- [18] H. Chen. “Random Walks and the Effective Resistance Sum Rules”. In: *Discrete Applied Mathematics* 158.15 (2010), pp. 1691–1700. DOI: 10.1016/j.dam.2010.06.013.
- [19] F. R. K. Chung. *Spectral Graph Theory*. CBMS Regional Conference Series in Mathematics 92. American Mathematical Society, 1997. DOI: 10.1090/cbms/092.
- [20] M. Cucuringu et al. “Hermitian Matrices for Clustering Directed Graphs: Insights and Applications”. In: *Proceedings of AISTATS*. PMLR, 2020, pp. 983–992.
- [21] D. Cvetković and S. K. Simić. “Towards a spectral theory of graphs based on the signless Laplacian, I”. In: *Publications de l’Institut Mathématique* 85 (99).105 (2009), pp. 19–33. DOI: 10.2298/PIM0999019C.
- [22] D. Cvetković and S. K. Simić. “Towards a Spectral Theory of Graphs Based on the Signless Laplacian, II”. In: *Linear Algebra and its Applications* 432.9 (2010), pp. 2257–2272. DOI: 10.1016/j.laa.2009.12.024.
- [23] Dragoš Cvetković and Slobodan K. Simić. “Towards a spectral theory of graphs based on the signless Laplacian, III”. In: *Applicable Analysis and Discrete Mathematics* 4.1 (2010), pp. 156–166. DOI: 10.2298/AADM1000001C.
- [24] Y. Dai and S. Chen. “Invariants for Incidence Matrix of a Tree”. In: *Journal of Algebraic Combinatorics* 60.4 (2024), pp. 1011–1029. DOI: 10.1007/s10801-024-01367-9.
- [25] K. C. Das. “The Laplacian spectrum of a graph”. In: *Computers & Mathematics with Applications* 48.5-6 (2004), pp. 715–724. DOI: 10.1016/j.compma.2004.03.111.
- [26] K. Devriendt. “Effective Resistance Is More Than Distance: Laplacians, Simplices and the Schur Complement”. In: *Linear Algebra and its Applications* 639 (2022), pp. 24–49. DOI: 10.1016/j.laa.2021.11.013.
- [27] F. Diaz-Diaz and E. Estrada. “Time and Space Generalized Diffusion Equation on Graph/Networks”. In: *Chaos, Solitons & Fractals* 156 (2022), p. 111791. DOI: 10.1016/j.chaos.2021.111791.
- [28] J. Dodziuk and V. Mathai. “Kato’s inequality and asymptotic spectral properties for discrete magnetic Laplacians”. In: *Contemporary Mathematics* 398 (2006), pp. 69–81. DOI: 10.1090/conm/398/07484.
- [29] E. Estrada. ““Hubs-Repelling” Laplacian and Related Diffusion on Graphs/Networks”. In: *Linear Algebra and its Applications* 596 (2020), pp. 256–280. DOI: 10.1016/j.laa.2020.03.010.
- [30] E. Estrada. “Conservative Versus Non-conservative Diffusion Toward a Target in a Networked Environment”. In: *Target Search Problems*. Springer, 2024, pp. 511–540.
- [31] E. Estrada. “Path Laplacian Matrices: Introduction and Application to the Analysis of Consensus in Networks”. In: *Linear Algebra and its Applications* 436.9 (2012), pp. 3373–3391. DOI: 10.1016/j.laa.2011.12.026.

- [32] E. Estrada et al. “Path Laplacian Operators and Superdiffusive Processes on Graphs. I. One-Dimensional Case”. In: *Linear Algebra and its Applications* 523 (2017), pp. 307–334. DOI: 10.1016/j.laa.2017.02.027.
- [33] E. Estrada et al. “Path Laplacian Operators and Superdiffusive Processes on Graphs. II. Two-Dimensional Lattice”. In: *Linear Algebra and its Applications* 555 (2018), pp. 373–397. DOI: 10.1016/j.laa.2018.05.006.
- [34] J. S. Fabila-Monroy, F. Lledó, and O. Post. “Matching Number, Hamiltonian Graphs and Discrete Magnetic Laplacians”. In: *Linear Algebra and its Applications* 648 (2022), pp. 86–100. DOI: 10.1016/j.laa.2022.07.025.
- [35] S. Fallat and S. Kirkland. “Extremizing Algebraic Connectivity Subject to Graph Theoretic Constraints”. In: *The Electronic Journal of Linear Algebra* 3 (1998), pp. 48–74. DOI: 10.13001/1081-3810.1008.
- [36] M. Fanuel, C. M. Alaiz, and J. A. K. Suykens. “Magnetic Eigenmaps for Community Detection in Directed Networks”. In: *Physical Review E* 95.2 (2017), p. 022302. DOI: 10.1103/PhysRevE.95.022302.
- [37] M. Fanuel and J. A. K. Suykens. “Deformed Laplacians and Spectral Ranking in Directed Networks”. In: *Applied and Computational Harmonic Analysis* 47.2 (2019), pp. 397–422. DOI: 10.1016/j.acha.2017.12.003.
- [38] M. Fiedler. “Algebraic connectivity of graphs”. In: *Czechoslovak Mathematical Journal* 23.2 (1973), pp. 298–305. DOI: 10.21136/CMJ.1973.101168.
- [39] A. Fontan and C. Altafini. “On the Properties of Laplacian Pseudoinverses”. In: *Proc. 60th IEEE Conference on Decision and Control (CDC)*. IEEE. 2021, pp. 5538–5543. DOI: 10.1109/CDC45484.2021.9683630.
- [40] A. Fontan and C. Altafini. “Pseudoinverses of Signed Laplacian Matrices”. In: *SIAM Journal on Matrix Analysis and Applications* 44.2 (2023), pp. 622–647. DOI: 10.1137/21M1441962.
- [41] S. Furutani et al. “Graph Signal Processing for Directed Graphs Based on the Hermitian Laplacian”. In: *Machine Learning and Knowledge Discovery in Databases: ECML PKDD 2019, Part I*. Springer. 2020, pp. 447–463. DOI: 10.1007/978-3-030-46150-8_29.
- [42] L. V. Gambuzza, M. Frasca, and E. Estrada. “Hubs-Attracting Laplacian and Related Synchronization on Networks”. In: *SIAM Journal on Applied Dynamical Systems* 19.2 (2020), pp. 1057–1079. DOI: 10.1137/19M1285727.
- [43] A. Ghosh, S. Boyd, and A. Saberi. “Minimizing Effective Resistance of a Graph”. In: *SIAM Review* 50.1 (2008), pp. 37–66. DOI: 10.1137/060651758.
- [44] R. Ghosh et al. “Non-conservative Diffusion and Its Application to Social Network Analysis”. In: *Journal of Complex Networks* 12.1 (2024), cnae006. DOI: 10.1093/comnet/cnae006.
- [45] C. Godsil and G. Royle. “The Laplacian of a Graph”. In: *Algebraic Graph Theory*. Springer, 2001, pp. 279–306. DOI: 10.1007/978-1-4613-0163-9_13.
- [46] R. Grone and R. Merris. “The Laplacian Spectrum of a Graph II”. In: *SIAM Journal on Discrete Mathematics* 7.2 (1994), pp. 221–229. DOI: 10.1137/S0895480192238932.
- [47] R. Grone, R. Merris, and V. S. Sunder. “The Laplacian spectrum of a graph”. In: *SIAM Journal on Matrix Analysis and Applications* 11.2 (1990), pp. 218–238. DOI: 10.1137/0611016.

- [48] K. Guo and B. Mohar. “Hermitian Adjacency Matrix of Digraphs and Mixed Graphs”. In: *Journal of Graph Theory* 85.1 (2017), pp. 217–248. DOI: 10.1002/jgt.22176.
- [49] I. Gutman and W. Xiao. “Generalized Inverse of the Laplacian Matrix and Some Applications”. In: *Bulletin de l’Académie Serbe des Sciences et des Arts. Classe des Sciences Mathématiques et Naturelles. Sciences Mathématiques* (2004), pp. 15–23.
- [50] K. Hayashi, S. G. Aksoy, and H. Park. “Skew-Symmetric Adjacency Matrices for Clustering Directed Graphs”. In: *Proc. 2022 IEEE International Conference on Big Data (Big Data)*. IEEE, 2022, pp. 555–564. DOI: 10.1109/BigData55660.2022.10020367.
- [51] Y. Higuchi, S. Kubota, and E. Segawa. “On Symmetric Spectra of Hermitian Adjacency Matrices for Non-Bipartite Mixed Graphs”. In: *Discrete Mathematics* 347.5 (2024), p. 113911. DOI: 10.1016/j.disc.2023.113911.
- [52] Y. Higuchi and T. Shirai. “A Remark on the Spectrum of Magnetic Laplacian on a Graph”. In: *Yokohama Mathematical Journal* 47.Special (1999), pp. 129–141.
- [53] D. J. Klein and M. Randić. “Resistance Distance”. In: *Journal of Mathematical Chemistry* 12.1 (1993), pp. 81–95. DOI: 10.1007/BF01164627.
- [54] E. Korotyaev and N. Saburova. “Magnetic Schrödinger Operators on Periodic Discrete Graphs”. In: *Journal of Functional Analysis* 272.4 (2017), pp. 1625–1660. DOI: 10.1016/j.jfa.2016.11.018.
- [55] E. Korotyaev and N. Saburova. “Trace Formulas for Magnetic Schrödinger Operators on Periodic Graphs and Their Applications”. In: *Linear Algebra and its Applications* 676 (2023), pp. 395–440. DOI: 10.1016/j.laa.2023.03.007.
- [56] J. Kunegis et al. “Spectral Analysis of Signed Graphs for Clustering, Prediction and Visualization”. In: *Proc. 2010 SIAM International Conference on Data Mining*. SIAM, 2010, pp. 559–570. DOI: 10.1137/1.9781611972801.48.
- [57] K. Lerman and R. Ghosh. “Network Structure, Topology, and Dynamics in Generalized Models of Synchronization”. In: *Physical Review E* 86.2 (2012), p. 026108. DOI: 10.1103/PhysRevE.86.026108.
- [58] S. Li and Y. Yu. “Hermitian Adjacency Matrix of the Second Kind for Mixed Graphs”. In: *Discrete Mathematics* 345.5 (2022), p. 112798. DOI: 10.1016/j.disc.2021.112798.
- [59] G. Lindmark and C. Altafini. “Investigating the Effect of Edge Modifications on Networked Control Systems: Stability Analysis”. In: *Automatica* 149 (2023), p. 110801. DOI: 10.1016/j.automatica.2022.110801.
- [60] N. Masuda, M. A. Porter, and R. Lambiotte. “Random walks and diffusion on networks”. In: *Physics Reports* 716 (2017), pp. 1–58. DOI: 10.1016/j.physrep.2017.07.007.
- [61] R. Merris. “Laplacian matrices of graphs: a survey”. In: *Linear Algebra and its Applications* 197 (1994), pp. 143–176. DOI: 10.1016/0024-3795(94)90486-3.
- [62] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.
- [63] T. Michelitsch et al. *Fractional Dynamics on Networks and Lattices*. John Wiley & Sons, 2019. DOI: 10.1002/9781119473961.
- [64] P. Van Mieghem. *Graph Spectra for Complex Networks*. Cambridge University Press, 2023. DOI: 10.1017/9781009107545.

- [65] P. Van Mieghem, K. Devriendt, and H. Cetinay. “Pseudoinverse of the Laplacian and Best Spreader Node in a Network”. In: *Physical Review E* 96.3 (2017), p. 032311. DOI: 10.1103/PhysRevE.96.032311.
- [66] M. Miranda, P. Moreno-Spiegelberg, and E. Estrada. “Heterogeneous Consensus Dynamics through Reaction–Diffusion Models on Graphs/Networks”. In: *Communications in Nonlinear Science and Numerical Simulation* 153 (2025), p. 109160. DOI: 10.1016/j.cnsns.2025.109160.
- [67] B. Mohar. “A New Kind of Hermitian Matrices for Digraphs”. In: *Linear Algebra and its Applications* 584 (2020), pp. 343–352. DOI: 10.1016/j.laa.2019.12.028.
- [68] B. Mohar. “The Laplacian spectrum of graphs”. In: *Graph Theory, Combinatorics, and Applications*. Ed. by Y. Alavi et al. Vol. 2. Wiley, 1991, pp. 871–898.
- [69] P. P. Mondal, R. B. Bapat, and F. Atik. “On the Inverse and Moore–Penrose Inverse of Resistance Matrix of Graphs with More General Matrix Weights”. In: *Journal of Applied Mathematics and Computing* 69.6 (2023), pp. 4805–4820. DOI: 10.1007/s12190-023-01769-0.
- [70] F. Morbidi. “Functions of the Laplacian Matrix with Application to Distributed Formation Control”. In: *IEEE Transactions on Control of Network Systems* 9.3 (2021), pp. 1459–1467. DOI: 10.1109/TCNS.2021.3117814.
- [71] F. Morbidi. “The Deformed Consensus Protocol”. In: *Automatica* 49.10 (2013), pp. 3049–3055. DOI: 10.1016/j.automatica.2013.06.022.
- [72] M. Nath and S. Paul. “On the Distance Laplacian Spectra of Graphs”. In: *Linear Algebra and its Applications* 460 (2014), pp. 97–110. DOI: 10.1016/j.laa.2014.06.020.
- [73] R. Olfati-Saber, J. A. Fax, and R. M. Murray. “Consensus and Cooperation in Networked Multi-Agent Systems”. In: *Proceedings of the IEEE* 95.1 (2007), pp. 215–233. DOI: 10.1109/JPROC.2006.887293.
- [74] J. L. Palacios. “Resistance Distance in Graphs and Random Walks”. In: *International Journal of Quantum Chemistry* 81.1 (2001), pp. 29–33. DOI: 10.1002/qua.1022.
- [75] L. Pan, H. Shao, and M. Mesbahi. “Laplacian dynamics on signed networks”. In: *Proceedings of the 55th IEEE Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 891–896. DOI: ???.
- [76] B. Ramamurthy, R. B. Bapat, and S. Goel. “On Resistance Matrices of Weighted Balanced Digraphs”. In: *Linear and Multilinear Algebra* 71.13 (2023), pp. 2222–2248. DOI: 10.1080/03081087.2022.2058298.
- [77] A. Pérez Riascos and J. L. Mateos. “Fractional Dynamics on Networks: Emergence of Anomalous Diffusion and Lévy Flights”. In: *Physical Review E* 90.3 (2014), p. 032809. DOI: 10.1103/PhysRevE.90.032809.
- [78] A. Pérez Riascos et al. “Random Walks with Long-Range Steps Generated by Functions of Laplacian Matrices”. In: *Journal of Statistical Mechanics: Theory and Experiment* 2018.4 (2018), p. 043404. DOI: 10.1088/1742-5468/aab8f2.
- [79] Aditi Saxena, Twinkle Tripathy, and Rajasekhar Anguluri. “Are the flows of complex-valued Laplacians and their pseudoinverses related?” In: *2025 European Control Conference (ECC)*. IEEE, 2025, pp. 653–658. DOI: 10.48550/ARXIV.2411.09254.

- [80] M. A. Shubin. “Discrete Magnetic Laplacian”. In: *Communications in Mathematical Physics* 164.2 (1994), pp. 259–275. DOI: 10.1007/BF02102074.
- [81] D. A. Spielman. “Spectral Graph Theory and Its Applications”. In: *Proc. 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2007, pp. 29–38. DOI: 10.1109/FOCS.2007.58.
- [82] Z. Stanić. *Inequalities for Graph Eigenvalues*. Vol. 423. Cambridge University Press, 2015.
- [83] T. Stephen. “A Majorization Bound for the Eigenvalues of Some Graph Laplacians”. In: *SIAM Journal on Discrete Mathematics* 21.2 (2007), pp. 303–312. DOI: 10.1137/050628176.
- [84] L. R. Varshney et al. “Structural Properties of the *Caenorhabditis elegans* Neuronal Network”. In: *PLoS Computational Biology* 7.2 (2011), e1001066. DOI: 10.1371/journal.pcbi.1001066.
- [85] Y. Colin de Verdière. “Magnetic Interpretation of the Nodal Defect on Graphs”. In: *Analysis & PDE* 6.5 (2013), pp. 1235–1242. DOI: 10.2140/apde.2013.6.1235.
- [86] W. Xiao and I. Gutman. “Resistance Distance and Laplacian Spectrum”. In: *Theoretical Chemistry Accounts* 110 (2003), pp. 284–289. DOI: 10.1007/s00214-003-0480-7.
- [87] Y. Yang and H. Zhang. “Some Rules on Resistance Distance with Applications”. In: *Journal of Physics A: Mathematical and Theoretical* 41.44 (2008), p. 445203. DOI: 10.1088/1751-8113/41/44/445203.
- [88] G. F. Young, L. Scardovi, and N. E. Leonard. “Rearranging Trees for Robust Consensus”. In: *Proc. 50th IEEE Conference on Decision and Control and European Control Conference*. IEEE. 2011, pp. 1000–1005. DOI: 10.1109/CDC.2011.6160761.
- [89] G. F. Young, L. Scardovi, and N. E. Leonard. “Robustness of Noisy Consensus Dynamics with Directed Communication”. In: *Proc. 2010 American Control Conference*. IEEE. 2010, pp. 6312–6317. DOI: 10.1109/ACC.2010.5531276.
- [90] T. Zaslavsky. “Signed Graphs”. In: *Discrete Applied Mathematics* 4.1 (1982), pp. 47–74. DOI: 10.1016/0166-218X(82)90025-5.
- [91] X. Zhang et al. “MAGNET: A Neural Network for Directed Graphs”. In: *Advances in Neural Information Processing Systems* 34 (2021), pp. 27003–27015.