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► **To cite this version:**

Pierre Houedry, Nicolas Courty, Florestan Martin-Baillon, Laetitia Chapel, Titouan Vayer. Bridging Arbitrary and Tree Metrics via Differentiable Gromov Hyperbolicity. 2025. <hal-05088161v1>

HAL Id: hal-05088161

<https://hal.science/hal-05088161v1>

Preprint submitted on 28 May 2025 (v1), last revised 30 Oct 2025 (v2)

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Bridging Arbitrary and Tree Metrics via Differentiable Gromov Hyperbolicity

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Abstract

Trees and the associated shortest-path tree metrics provide a powerful framework for representing hierarchical and combinatorial structures in data. Given an arbitrary metric space, its deviation from a tree metric can be quantified by Gromov's δ -hyperbolicity. Nonetheless, designing algorithms that bridge an arbitrary metric to its closest tree metric is still a vivid subject of interest, as most common approaches are either heuristical and lack guarantees, or perform moderately well. In this work, we introduce a novel differentiable optimization framework, coined DELTAZERO, that solves this problem. Our method leverages a smooth surrogate for Gromov's δ -hyperbolicity which enables a gradient-based optimization, with a tractable complexity. The corresponding optimization procedure is derived from a problem with better worst case guarantees than existing bounds, and is justified statistically. Experiments on synthetic and real-world datasets demonstrate that our method consistently achieves state-of-the-art distortion.

1 Introduction

The notion of δ -hyperbolic metric spaces, introduced by Gromov [19], captures a large-scale analog of negative curvature and has found significant applications in both mathematics and computer science. A prominent example in machine learning is the success of representation learning in hyperbolic spaces [6], which leverages their natural ability to learn on data with hierarchical structure. A key measure of characterizing how well a hyperbolic space fits a given dataset is the Gromov hyperbolicity, which quantifies the extent to which a metric space (X, d_X) deviates from a tree metric, with tree structures achieving a value of zero. As such, it has often been used as a proxy for graph dataset suitability to hyperbolic space embedding (*e.g.* [7, 34]), which has also been recently analyzed in [21].

Metrics that can be exactly realized as shortest-path distances in a tree play a central role in many domains: hierarchical clustering in machine learning [25], phylogenetic tree construction in bioin-

formatics [13], and modeling real-world networks [1, 29], to cite a few. When data is not exactly tree-structured, a natural problem arises: how to approximate an arbitrary metric as faithfully as possible with a tree metric?

Formally, a metric d_X on a finite set X is called a *tree metric* if there exists a tree metric (T, d_T) and an embedding $\Phi : X \rightarrow T$ such that

$$d_X(x, y) = d_T(\Phi(x), \Phi(y)) \quad \text{for all } x, y \in X. \quad (1)$$

That is, pairwise distances in X can be exactly represented as shortest-path distances in a tree. When such a representation is not possible, one seeks a tree metric that approximates d_X as closely as possible under some distortion measure.

This gives rise to the following problem:

Problem (Minimum Distortion Tree Metric Approximation). Given a finite metric space (X, d_X) , our goal is to construct an embedding $\Phi : X \rightarrow T$ into a metric tree (T, d_T) such that the tree metric d_T closely approximates the original metric d_X . We aim to solve

$$\operatorname{argmin}_{(T, d_T) \in \mathcal{T}} \|d_X - d_T\|_\infty := \operatorname{argmin}_{(T, d_T) \in \mathcal{T}} \max_{x, y \in X} |d_X(x, y) - d_T(\phi(x), \phi(y))|,$$

where \mathcal{T} denotes the set of finite tree metrics.

Tree metrics embedding methods. In response, several methods have been developed to embed data into tree-like metrics with minimal distortion, e.g. [8, 31, 33]. Among them, the Neighbor Joining algorithm (NJ) [31] has long been a standard algorithm for constructing tree metrics from distance data, which proceeds by iteratively joining neighboring nodes. An alternative, the TREEREP algorithm [33], offers improved scalability. Despite their algorithmic efficiency and empirical success, both methods lack theoretical guarantees regarding the distortion incurred during the embedding process. This limitation affects their reliability in applications where preserving metric fidelity is critical. In contrast, based on a theoretical characterization of the distortion, the GROMOV algorithm [19], described in more details later in the paper, constructs a tree metric by iteratively selecting pairs of points. It serves as a key component of our approach. Restricted to unweighted graph, the LAYERINGTREE method [8] first chooses a root node, then join iteratively neighbors with a bounded additive distortion. The more recent HCCROOTEDTREEFIT algorithm [38] starts by centering the distance matrix around a chosen root, then fits an ultrametric distance to the centered distance, minimizing an average ℓ_1 -distortion. However, its accuracy is generally lower than that of the methods presented above and remains sensitive to the choice of root. Those various approaches are illustrated in Figure 1, along with our new method described subsequently. Additional illustrations on toy graphs are also provided in Appendix A with details on how the figure is generated.

Contributions. In this paper, we propose DELTAZERO, a new method for solving the minimum distortion tree metric approximation problem. It formulates the problem as a constrained optimization task, where the objective is to minimize the distortion between the input metric and a surrogate metric space $(X', d_{X'})$, while enforcing that the resulting space is close to a true tree metric, as quantified by a low Gromov hyperbolicity. This surrogate is eventually embedded into an actual tree (T, d_T) . One additional key feature of DELTAZERO is its differentiable formulation: it leverages smooth approximations of both the Gromov hyperbolicity and the distortion to enable gradient-based optimization. Computing the exact Gromov hyperbolicity for a dataset of even moderate size is computationally demanding, both in terms of time and memory consumption. The best known theoretical algorithm operates in $O(n^{3.69})$ time [16], which is impractical for large graphs. Although faster empirical approximations have been proposed, such as [9], scalability remains a bottleneck. To address this, we propose an alternative strategy based on batch-wise sampling. Rather than analyzing the entire dataset at once, DELTAZERO estimates hyperbolicity and distortion on subsets of the distance matrix, which significantly reduces computational overhead while preserving accuracy. We demonstrate that this mini-batching approach enables scalable Gromov hyperbolicity approximation without sacrificing fidelity to the exact value.

The rest of the paper is organized as follows. Section 2 introduces the notion of δ - and Gromov hyperbolicity and gives intuition on its geometric interpretation. In Section 3, we formulate our optimization problem and show how it improves the worst-case distortion of Gromov. We then provide a differentiable and efficient algorithm to solve the problem. In the experimental Section 4,

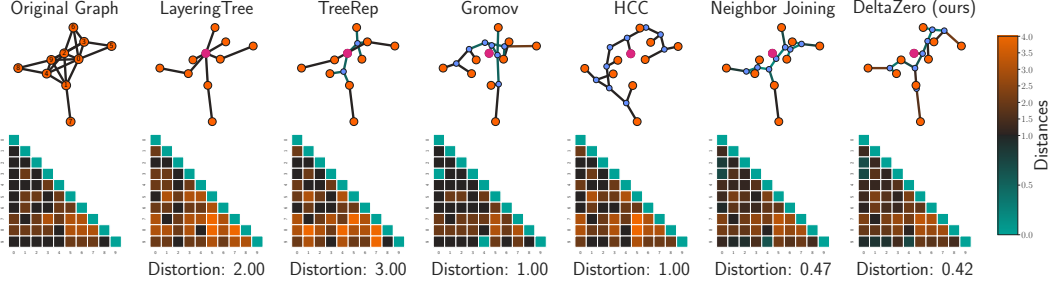


Figure 1: Illustration of the tree metric embedding problem (best viewed with colors). Given an original graph (first column) and the corresponding shortest path distances between nodes (represented on the bottom row as a lower-triangular matrix), we aim at finding a tree and the corresponding tree metric that best approximates the original distances. Competing state-of-the-art methods and our method DELTAZERO results are presented along with the corresponding distortion.

we first evaluate DELTAZERO in a controlled setting to assess its ability to provide hierarchical clusters. We then measure its ability to generate low distortion tree metric approximations in two contexts, where unweighted and weighted graphs are at stake. Finally, we draw some conclusions and perspectives.

2 Background on δ -hyperbolicity

At the heart of our approach lies the concept of δ -hyperbolicity. In this section, we present the foundational principles of this notion and offer intuitive insights into its geometric interpretation. We also discuss relevant computational aspects.

2.1 From Gromov product to δ -hyperbolicity

A key concept used to define δ -hyperbolicity is the *Gromov product*, denoted $(x|y)_w$, which intuitively measures the overlap between geodesic paths from a base point w to the points x and y . It is defined as follows.

Definition 2.1 (Gromov Product). Let (X, d_X) be a metric space and let $x, y, w \in X$. The *Gromov product* of x and y with respect to the basepoint w is defined as

$$(x|y)_w = \frac{1}{2} (d_X(x, w) + d_X(y, w) - d_X(x, y)).$$

With the Gromov product in hand, we now define δ -hyperbolicity.

Definition 2.2 (δ -hyperbolicity and Gromov hyperbolicity). A metric space (X, d_X) is said to be *δ -hyperbolic* if there exists $\delta \geq 0$ such that for all $x, y, z, w \in X$, the Gromov product satisfies

$$(x|z)_w \geq \min \{(x|y)_w, (y|z)_w\} - \delta.$$

The *Gromov hyperbolicity*, denoted by δ_X , is the smallest value of δ that satisfies the above property. Consequently, every finite metric space (X, d_X) has a Gromov hyperbolicity equal to

$$\delta_X = \max_{x, y, z, w \in X} (\min \{(x|y)_w, (y|z)_w\} - (x|z)_w). \quad (2)$$

The concept of δ -hyperbolicity may initially appear abstract, but it has deep and elegant connections to tree metrics. In fact, a metric d_X is a tree metric if and only if, for every four points $x, y, z, w \in X$, two largest among the following three sums

$$d_X(x, y) + d_X(z, w), \quad d_X(x, z) + d_X(y, w), \quad d_X(x, w) + d_X(y, z)$$

are equal [5].

The δ -hyperbolicity generalizes this condition by allowing a bounded deviation. Specifically, a metric space (X, d_X) is δ -hyperbolic if, for all $x, y, z, w \in X$, the following inequality holds

$$d_X(x, y) + d_X(z, w) \leq \max \{d_X(x, z) + d_X(y, w), d_X(x, w) + d_X(y, z)\} + 2\delta.$$

This is known as the *four-point condition*. It offers an alternative characterization of δ -hyperbolicity (see [18, Ch. 2, §1]), providing a quantitative measure of how far a given metric deviates from being a tree metric, and thereby capturing an intrinsic notion of negative curvature in the space. An alternative and more geometrically intuitive way to understand δ -hyperbolicity is through the concept of *δ -slim geodesic triangles*. In this perspective, a geodesic metric space (X, d_X) is δ -hyperbolic if and only if every geodesic triangle in X is δ -slim (up to a multiplicative constant on δ , see [18, Ch.2, §3, Proposition 21]), meaning that each side of the triangle is contained within the δ -neighbourhood of the union of the two other sides (see Figure 2).

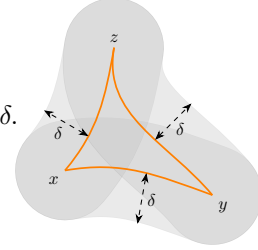


Figure 2: δ -slim triangle.

This captures the idea that geodesic triangles in hyperbolic spaces are “thin”, more closely resembling tripods than the broad, wide-angled triangles characteristic of Euclidean geometry. In this sense, Gromov hyperbolicity quantifies the extent to which the space deviates from tree-like behaviour: the smaller the value of δ , the more the geodesic triangles resemble those in a tree, where all three sides reduce to a union of two overlapping segments. Thus, the δ -slim triangle condition offers a compelling geometric counterpart to the more algebraic formulations via the Gromov product or the four-point condition.

Computing the Gromov hyperbolicity. Computing the Gromov hyperbolicity of a discrete metric space is computationally demanding, as it requires examining all quadruples of points to evaluate the four-point condition. The naive brute-force approach runs in $O(n^4)$ time for a space with n points, making it impractical for large-scale applications. Then, several works have addressed the computational bottlenecks inherent in computing Gromov hyperbolicity. Notably, Fournier et al. [16] show that the computation of hyperbolicity from a fixed base-point can be reduced to a (\max, \min) matrix product. Leveraging the fast algorithm for this class of matrix products, it leads to an overall $O(n^{3.69})$ time complexity.

In graph settings, the notion of *far-apart* vertex pairs plays a key role in accelerating hyperbolicity computation. This concept underpins a structural result [9] stating that certain far-apart pairs suffice to witness the maximum in the definition of Gromov hyperbolicity. Consequently, one can avoid exhaustive examination of all quadruples and focus on a carefully selected subset; this leads to a pruning approach that significantly reduces computational cost [10].

Contrary to these approaches, which aim to accelerate the exact computation of Gromov hyperbolicity, we take a different route: we introduce a smooth, differentiable surrogate of the hyperbolicity function. This relaxation enables gradient-based optimization, and we further propose a batched approximation scheme to make the computation tractable and independent of the size of the graph.

2.2 Embedding of a δ -hyperbolic space into a tree

The δ -hyperbolicity can play a significant theoretical role in establishing guarantees for embedding arbitrary metric spaces into trees. Indeed, a result by Gromov [19] shows that any δ -hyperbolic metric space on n points admits a tree-metric approximation with additive distortion $O(\delta \log n)$, and that this embedding can be computed in $O(n^2)$ time.

Theorem 2.3 ([18, Ch.2, §2, Theorem 12]). *Let (X, d_X) be a finite δ -hyperbolic metric space over n points. For every $w \in X$, there exists a finite metric tree (T, d_T) , and a map $\Phi : X \rightarrow T$ such that*

1. *The distance to the basepoint is preserved: $d_T(\Phi(x), \Phi(w)) = d_X(x, w)$ for all $x \in X$,*
2. *$d_X(x, y) - 2\delta \log_2(n - 2) \leq d_T(\Phi(x), \Phi(y)) \leq d_X(x, y)$ for all $x, y \in X$.*

The approximation of distances provided in the previous theorem is particularly notable due to the non-expansiveness of the mapping, meaning that the embedded distances in the tree never exceed the original graph distances. This ensures that the distortion introduced is purely additive and one-sided,

which is a stronger guarantee than general metric embeddings. In [38], Yim et al. provide an argument to illustrate the asymptotic sharpness of the result, using a construction in the Poincaré disk.

For a practical and efficient computational procedure to obtain Gromov’s tree embedding of a discrete metric space, one can leverage the *Single Linkage Hierarchical Clustering* (SLHC) algorithm [26]. The SLHC algorithm incrementally merges clusters based on minimal pairwise distances, implicitly constructing an ultrametric tree. While originally developed for clustering tasks, this procedure can be naturally adapted to construct the Gromov’s embedding. Further details are provided in Appendix B.

While this result is notable, it provides a worst-case, coarse approximation that does not seek to minimize the actual distortion for a specific input metric. In contrast, our work focuses on directly learning a tree-like metric that is optimally close to the original space, potentially achieving significantly lower distortion in practice, as shown in the experiments, see Section 4.

3 DELTAZERO

We present below our approach to the Minimum Distortion Tree Metric Approximation problem. In this work, we consider discrete metric spaces and we denote by \mathcal{M}_n the set of finite metric spaces over n points, and fix $(X, d_X) \in \mathcal{M}_n$. Our algorithm is inspired by Theorem 2.3 which states that (X, d_X) can be isometrically embedded into a tree if and only if its Gromov hyperbolicity is 0. Building on this result, we introduce the following optimization problem, parametrized by $\mu > 0$,

$$\min_{\substack{(X', d_{X'}) \in \mathcal{M}_n \\ d_{X'} \leq d_X}} \mathcal{L}_X((X', d_{X'}), \mu) := \mu \|d_X - d_{X'}\|_\infty + \delta_{X'}. \quad (3)$$

By minimizing \mathcal{L}_X , we seek a discrete metric space that is both close to (X, d_X) and exhibits low hyperbolicity. This objective balances two competing goals: proximity to the original metric (via the ℓ_∞ distortion term) and tree-likeness (quantified by $\delta_{X'}$). The parameter μ governs the trade-off between these terms. The constraint $d_{X'} \leq d_X$ accounts for the non-expansiveness of the metric, as in Theorem 2.3 for the Gromov embedding. Importantly, the minimization of \mathcal{L}_X leads to improved worst-case guarantees when followed by the Gromov tree embedding procedure, as described below.

Theorem 3.1. *Let (X^*, d_{X^*}) be any metric space minimizing $\mathcal{L}_X((X', d_{X'}), \mu)$, and let (T^*, d_{T^*}) be a tree metric obtained by applying a Gromov tree embedding denoted by Φ to (X^*, d_{X^*}) . Then*

$$d_X(x, y) - C_{X, \mu} \leq d_{T^*}(\Phi(x), \Phi(y)) \leq d_X(x, y) \text{ for all } x, y \in X,$$

with $C_{X, \mu} = 2\delta_X \log_2(n-2) + (1 - 2\log_2(n-2)\mu) \|d_X - d_{X^*}\|_\infty$. In particular, if $\mu \geq \frac{1}{2\log_2(n-2)}$, the distortion of the final tree embedding is smaller than the worst-case bound achieved by applying the Gromov embedding directly to X .

The proof of all theorems are given in App. C. Theorem 3.1 suggests that improved ℓ_∞ distortions can be achieved by minimizing \mathcal{L}_X . Our solution, which we describe below, adopts this principle.

3.1 Smooth Relaxation and Batched Gromov Hyperbolicity

We propose a differentiable loss function that allows us to optimize the metric so that it closely aligns with the original metric while maintaining a small Gromov hyperbolicity. The primary challenge lies in the non-differentiability of δ_X , as it involves max and min operators, rendering it unsuitable for gradient-based optimization. To address this, we introduce a smoothing of Gromov hyperbolicity based on the log-sum-exp function defined as $\text{LSE}_\lambda(\mathbf{x}) = \frac{1}{\lambda} \log(\sum_i e^{\lambda x_i})$ for a vector \mathbf{x} and $\lambda > 0$. As done in many contexts, $\text{LSE}_{\pm\lambda}$ can be used as a differentiable surrogate for max and min [11, 17, 23], recovering both as $\lambda \rightarrow \infty$. We define a smooth version of the δ -hyperbolicity as

$$\begin{aligned} \delta_X^{(\lambda)} &:= \text{LSE}_\lambda \left(\{ \text{LSE}_{-\lambda}((x|y)_w, (y|z)_w) - (x|z)_w \}_{x, y, z, w \in X} \right) \\ &= \frac{1}{\lambda} \log \sum_{(x, y, z, w) \in X} \frac{e^{-\lambda(x|z)_w}}{e^{-\lambda(x|y)_w} + e^{-\lambda(y|z)_w}}. \end{aligned} \quad (4)$$

The rationale behind this formulation is to replace the min in (2) with a soft-minimum $\text{LSE}_{-\lambda}$ and the max with a soft-maximum LSE_λ , which boils down to computing the last quantity. Interestingly, we have the following bounds between the Gromov hyperbolicity and its smoothed counterpart $\delta_X^{(\lambda)}$.

Proposition 3.2. *Let (X, d_X) be a finite metric space over n points, for $\lambda > 0$ we have*

$$\delta_X - \frac{\log(2)}{\lambda} \leq \delta_X^{(\lambda)} \leq \delta_X + \frac{4 \log(n)}{\lambda}.$$

While the smoothed version $\delta_X^{(\lambda)}$ alleviates the non-differentiability of the classical Gromov hyperbolicity, it still requires evaluating all quadruples of nodes in X , resulting in a computational complexity of $O(n^4)$. This remains prohibitive for large number of points. To address this, we introduce a batched approximation of the Gromov hyperbolicity. We sample K independent subsets $X_m^1, \dots, X_m^K \subset X$, each containing m points, and compute the local smooth hyperbolicity $\delta_{X_m^i}^{(\lambda)}$ within each batch. Since δ_X is defined as the maximum over all quadruples in X , and each $\delta_{X_m^i}^{(\lambda)}$ approximates the maximum over quadruples in a smaller subset $X_m^i \subset X$, we aggregate these local estimates using another log-sum-exp to obtain a differentiable approximation of the global maximum

$$\delta_{X,K,m}^{(\lambda)} = \text{LSE}_\lambda \left(\delta_{X_m^1}^{(\lambda)}, \dots, \delta_{X_m^K}^{(\lambda)} \right). \quad (5)$$

This two-level smoothing approach allows us to preserve differentiability while reducing the computational complexity from $O(n^4)$ to $O(K \cdot m^4)$, making the estimation of Gromov hyperbolicity tractable.

It is important to note that this sampling strategy does not, in general, guarantee closeness to the true Gromov hyperbolicity. For example, one may construct a metric space composed of a large tree with four additional points connected far from the main structure, although these few points may determine the global hyperbolicity, they are unlikely to appear in small random subsets. Thus, rare yet impactful configurations may be missed, leading to underestimation. Nonetheless, we find that in practice, the batched approximation performs well across a variety of graph instances (see Appendix D.2). In particular, we can show that for some random graph models, our estimator closely tracks the true Gromov hyperbolicity, as formalized below.

We say that a random metric space $(X, d_X) \in \mathcal{M}_n$ is *uniform* if the random distances $d_X(x_i, x_j)$ are i.i.d. with some law ℓ , called the *law of distances* (see precise Definition C.2).

Theorem 3.3. *Let ℓ be a probability measure on \mathbb{R}_+ . Let (X, d_X) be a uniform metric space with law of distances given by ℓ . In the regime $K \log(m) \sim \log(n)$ and $\lambda \sim \varepsilon^{-1} \log(n)$ we have*

$$\mathbb{P} \left[\left| \delta_X - \delta_{X,K,m}^{(\lambda)} \right| \leq \varepsilon \right] \geq 1 - \frac{1}{\varepsilon^{2K} n^{1-o(1)}}.$$

This result establishes that, in the asymptotic regime where $n \rightarrow \infty$, our batch estimate closely aligns with the true Gromov hyperbolicity, provided that the number of batches times the logarithm of their size, as well as the smoothing parameter, depend logarithmically on n . The assumptions about metric spaces imply independent distances, which can be restrictive. As shown in Appendix C.3, a random graph model satisfying these assumptions is one where shortest-path distances are randomly drawn from a distribution over $\{r, \dots, 2r\}$, or within a certain regime of Erdős–Rényi graphs. In practice, however, some of our datasets exhibit log-normal distance distributions (see Appendix C.3), suggesting that these assumptions are not overly restrictive. Future work will aim to extend these results to encompass a broader class of random metrics.

3.2 Final Optimization Objective

To formalize our algorithm, we start by noting that the set of discrete metrics over n points can be identified with the following subset of $\mathbb{R}^{n \times n}$

$$\mathcal{D}_n := \left\{ \mathbf{D} = (D_{ij}) \in \mathbb{R}_+^{n \times n} : \mathbf{D} = \mathbf{D}^\top, D_{ij} \leq D_{ik} + D_{kj}, D_{ii} = 0 \ \forall i, j, k \in \{1, \dots, n\} \right\}.$$

Each $\mathbf{D} \in \mathcal{D}_n$ encodes the pairwise distances of a metric space over n points, in particular, any metric d_X on a set $X = \{x_1, \dots, x_n\}$ of size n is fully described by its matrix representation $\mathbf{D}_X = (d_X(x_i, x_j))_{i,j} \in \mathcal{D}_n$. Note that \mathcal{D}_n forms a closed, convex, polyhedral cone generated by the half-spaces imposed by the triangle inequalities.

In practice, to find a tree-like approximation of (X, d_X) , we search for a matrix $\mathbf{D} \in \mathcal{D}_n$ that is close to \mathbf{D}_X while exhibiting low Gromov hyperbolicity. Since Gromov hyperbolicity depends only on the

Algorithm 1 DELTAZERO

Require: A finite metric space (X, d_X) , a designated root $w \in X$, learning rate $\epsilon > 0$, number of batches K , batch size m , log-sum-exp scale $\lambda > 0$, regularization parameter $\mu > 0$, number of steps T

- 1: Initialize $\mathbf{D}_0 = \mathbf{D}_X$
 - 2: **for** $t = 0$ to $T - 1$ **do**
 - 3: Randomly pick K batches of m points and compute $\delta_{\mathbf{D}_t, K, m}^{(\lambda)}$
 - 4: Compute gradient: $\mathbf{G}_t = \nabla L_X(\mathbf{D}_t)$
 - 5: Optimization step: $\tilde{\mathbf{D}}_{t+1} = \text{ADAM STEP}(\mathbf{D}_t, \mathbf{G}_t, \epsilon)$
 - 6: Projection step: $\mathbf{D}_{t+1} = \text{FLOYD-WARSHALL}(\tilde{\mathbf{D}}_{t+1})$
 - 7: **end for**
 - 8: **return** $\text{GROMOV}(\mathbf{D}_t, w)$
-

metric, we write $\delta_{\mathbf{D}}$ to emphasize this dependence. Our final optimization objective is

$$\min_{\substack{\mathbf{D} \in \mathcal{D}_n \\ \mathbf{D} \leq \mathbf{D}_X}} L_X(\mathbf{D}) := \mu \|\mathbf{D}_X - \mathbf{D}\|_2^2 + \delta_{\mathbf{D}, K, m}^{(\lambda)}, \quad (6)$$

where $\mathbf{A} \leq \mathbf{B}$ means $\forall(i, j), A_{ij} \leq B_{ij}$. The first term preserves fidelity to the input metric, acting as a smooth counterpart to the ℓ_∞ norm, and the second term promotes low-stretch, hierarchical structure using our smooth log-sum-exp approximation of Gromov hyperbolicity. The constraints $\mathbf{D} \in \mathcal{D}_n, \mathbf{D} \leq \mathbf{D}_X$ account for both the metric requirements and the non-expansiveness of the metric. While the objective (6) includes a convex fidelity term, the hyperbolicity penalty introduces non-convexity. This comes from the structure of the Gromov hyperbolicity functional.

Proposition 3.4. *The Gromov hyperbolicity $\delta_{\mathbf{D}}$ seen as a functional over \mathcal{D}_n is piecewise affine but not convex.*

Thus, we expect our smooth surrogate $\delta_{\mathbf{D}, K, m}^{(\lambda)}$ to also inherit this non-convexity, making the overall objective in (6) non-convex.

To tackle this optimization problem, we employ Projected Gradient Descent (PGD). Specifically, we alternate between taking a plain gradient step of L_X using ADAM [22] (only over the upper-triangular entries of \mathbf{D} , due to symmetry and zero diagonal), and projecting the result back onto \mathcal{D}_n . Each projection step solves

$$\Pi_{\mathcal{D}_n}(\mathbf{D}') = \underset{\substack{\mathbf{D}' \in \mathcal{D}_n \\ \mathbf{D}' \leq \mathbf{D}}}{\text{argmin}} \|\mathbf{D}' - \mathbf{D}\|_2^2.$$

This projection corresponds to the classical *decrease only metric nearest problem*, which we solve using the FLOYD-WARSHALL algorithm [15]. This algorithm computes all-pairs shortest paths in a graph with edge weights given by \mathbf{D} , and as shown in [3, Lemma 4.1], this is equivalent to finding the closest valid metric matrix no greater than \mathbf{D} in each entry (see Section C.5 for a detailed discussion). The projection runs in $O(n^3)$ time and can be interpreted as iteratively “repairing” triangle inequality violations to reach the nearest point in the metric cone.

At the end of the PGD iterations, we obtain a refined metric \mathbf{D}^* . To complete the tree approximation, we apply the Gromov tree embedding to \mathbf{D}^* , producing a tree metric. The full procedure, which we refer to as DELTAZERO, is summarized in Algorithm 1.

Computational complexity. Each gradient step incurs a computational cost of $O(Km^4 + n^3)$, where the $O(n^3)$ term arises from the use of the FLOYD-WARSHALL algorithm. Consequently, the overall complexity of the optimization procedure over T iterations is $O(T(Km^4 + n^3))$. In practice, the computation is typically dominated by the FLOYD-WARSHALL component, as we operate under the condition $Km^4 \ll n^3$.

4 Experiments

We provide in this Section experiments conducted on synthetic dataset to assess the quality of the optimization method, and on real datasets to measure the distortion obtained by DELTAZERO.

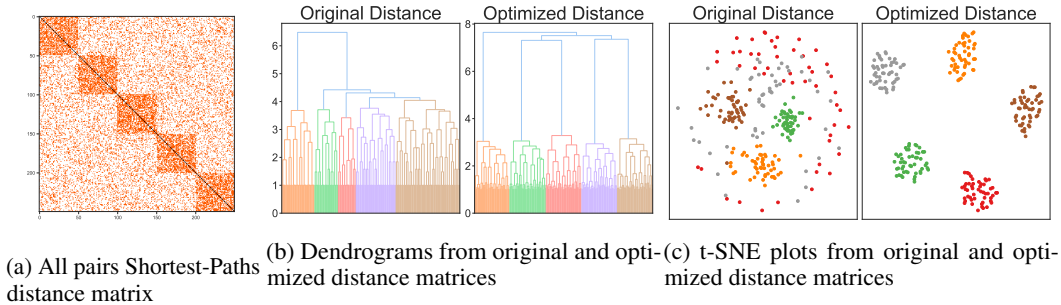


Figure 3: Illustration of the impact of optimizing the distance matrix on a simple Stochastic Block Model with 5 communities (best viewed in colors).

Synthetic Stochastic Block Models dataset. We consider the setting of Hierarchical Clustering (HC) which iteratively merges clusters based on pairwise dissimilarities. The resulting tree-like structure is known as a dendrogram and encodes a hierarchy of nested groupings. Our first objective is to qualitatively and quantitatively analyse the impact of the optimization problem of Eq. (6) on HC within the context of a simple stochastic block model (SBM) graph with a varying number of communities.

In such a probabilistic model of graph structure, we set the intra- and inter-communities connection probabilities to $p_{in} = 0.6$ and $p_{out} = 0.2$. Sizes of communities are fixed to 50. This yields a shortest path distance matrix depicted in Figure 3a for a SBM with 5 communities. We then follow the distance optimization procedure described in Section 3.2 to obtain a new distance matrix (parameters are $\mu = 1$ and $\lambda = 100$). This yields two different dendrograms, shown in Figure 3b, both computed using Ward’s method [37] as implemented by the `linkage` function of Scipy [36]. The color threshold is manually adjusted to magnify 5 clusters. At this point, we can see that the dendrogram corresponding to the optimized distance exhibit a better, more balanced, hierarchical structure with a clear separation between the 5 clusters. This point is also illustrated in Figure 3c where t-SNE [35] is computed from the distance matrices with a similar perplexity (set to 30). One can observe a stronger clustering effect from the optimized distance.

We then compute accuracy scores by comparing the true labels with the optimal permutation of predicted labels, using the `fcluster` function of Scipy. This evaluation is conducted under varying number of communities (3, 5 and 7), introducing increasing levels of difficulty. For each setting, 30 repetitions are performed. The results are summarized as boxplots in Figure 4. As one can observe, our optimization strategy increases drastically the performances in terms of accuracy on this simple example. It suggests that our optimization produces a metric which effectively homogenizes intra- and inter- cluster distances.

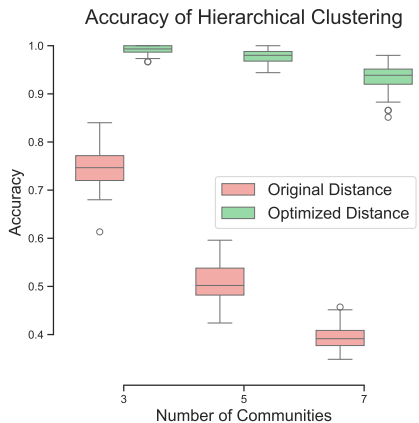


Figure 4: Performances of Hierarchical clustering with varying number of communities

Distortions on real datasets. We evaluate the ability of our method, DELTAZERO, to embed finite metric spaces into tree metrics with low distortion. Our benchmark includes both unweighted graph datasets and general metric datasets to showcase the versatility of our approach. We consider five standard unweighted graphs: C-ELEGAN, CS PHD from [30], CORA, AIRPORT from [32] and WIKI [14]. For graphs with multiple connected components, we extract the largest connected component. We then compute the shortest-path distance matrix, which serves as input to all tree metric fitting algorithms.

Table 1: ℓ_∞ error (lower is better). Best result in bold. The last row reports the relative improvement (%) of DELTAZERO over the second-best method (underlined) for each dataset.

Datasets	Unweighted graphs					Non-graph metrics	
	C-ELEGAN	CS PHD	CORA	AIRPORT	WIKI	ZEISEL	IBD
n	452	1025	2485	3158	2357	3005	396
Diameter	7	28	19	12	9	0.87	0.99
NJ	<u>2.97</u>	16.81	13.42	4.18	6.32	0.51	<u>0.90</u>
TR	5.90 \pm 0.72	21.01 \pm 3.34	16.86 \pm 2.11	10.00 \pm 1.02	9.97 \pm 0.93	0.66 \pm 0.10	1.60 \pm 0.22
HCC	4.31 \pm 0.46	23.35 \pm 2.07	12.28 \pm 0.96	7.71 \pm 0.72	7.20 \pm 0.60	0.53 \pm 0.07	1.25 \pm 0.11
LayeringTree	5.07 \pm 0.25	25.48 \pm 0.60	<u>7.76</u> \pm 0.54	<u>2.97</u> \pm 0.26	<u>4.08</u> \pm 0.27	–	–
Gromov	3.33 \pm 0.45	<u>13.28</u> \pm 0.61	9.34 \pm 0.53	4.08 \pm 0.27	5.54 \pm 0.49	<u>0.43</u> \pm 0.02	1.01 \pm 0.04
DELTAZERO	1.87 \pm 0.08	10.31 \pm 0.62	7.59 \pm 0.38	2.79 \pm 0.15	3.56 \pm 0.20	0.24 \pm 0.00	0.70 \pm 0.03
Improvement (%)	43.8%	22.3%	2.3%	6.0%	12.7%	44.1 %	22.2%

To evaluate DELTAZERO beyond graph-induced metrics, we also include two datasets: ZEISEL [39] and IBD [27]. These datasets are not naturally represented as graphs but instead as high-dimensional feature matrices. We construct a pairwise dissimilarity matrix using cosine distance, a standard choice in bioinformatics. Note that LAYERINGTREE [8], which requires an unweighted graph as input, is not applicable in this setting. We compare DELTAZERO against the following tree fitting methods: TREEREP (TR) [33], NEIGHBORJOIN (NJ) [31], HCCROOTEDTREEFIT (HCC) [38], LAYERINGTREE (LT) [8], and the classical GROMOV (see Algorithm 2). Among these, HCC, LT, and GROMOV are pivot-based methods: for a given distance matrix, they require selecting a root node. We report the average and standard deviation of distortion over 100 runs using the same randomly sampled roots across methods. TREEREP does not require a root but incorporates stochastic elements; we therefore report mean and standard deviation over 100 independent runs. For C-ELEGAN, CS PHD, CORA, and AIRPORT, we report the values of [38]. In contrast, NEIGHBORJOIN is deterministic that neither requires a root nor involves stochasticity, and is thus evaluated once per dataset.

For DELTAZERO, we perform grid search over the following hyperparameters: learning rate $\epsilon \in \{0.1, 0.01, 0.001\}$, distance regularization coefficient $\mu \in \{0.1, 0.01, 1.0\}$, and δ -scaling parameter $\lambda \in \{0.01, 0.1, 1.0, 10.0\}$. We fix the number of training epochs to $T = 1000$, batch size $m = 32$, and vary the number of batches $K \in \{100, 500, 1000, 3000, 5000\}$. For each setting, we select the best configuration which leads to the minimal distortion. To ensure stability, we apply early stopping with a patience of 50 epochs and retain the model with the best training loss. Final results are reported in Table 1. In addition to worst-case distortion defined in eq. (1), we evaluate the average embedding quality using the ℓ_1 distance between the original and tree-fitted distance matrices. We also report the execution time of each method to assess their computational efficiency. Results are presented in Appendix D.1 We also provide an analysis of its robustness and stability with respect to hyperparameter in Appendix D.2. All implementations details are given in Appendix E.

Table 1 demonstrates that DELTAZERO consistently achieves the lowest ℓ_∞ distortion across all datasets, both for unweighted graphs and general metric spaces. Notably, the largest relative improvements are observed on C-ELEGAN and ZEISEL, with reductions in distortion of 43.8% and 44.1% respectively, compared to the second-best methods. On CORA the improvement is more modest (2.3%), yet DELTAZERO still outperforms all baselines. This suggests that while our approach is robust across datasets, gains vary depending on the geometry of the input metric space. Overall, the results validate the effectiveness of our optimization-based method in producing low-distortion tree metric embeddings.

5 Conclusion and Discussion

In this study, we introduced an innovative method for computing a tree metric from a given graph metric or distances derived from an embedding. Our approach leverages a soft and differentiable approximation of Gromov Hyperbolicity, a measure of the tree-likeness of the metric. Initial results demonstrate the competitive performance of our method in generating low-distortion tree metrics across a diverse range of datasets. One potential limitation of our approach is its scalability to very large graphs. This constraint primarily arises from the complexity of projecting onto the cone of metrics, which involves solving all-pairs shortest path problems with a cubic complexity relative

to the number of vertices. To mitigate this challenge, a promising avenue for future research could involve optimizing over an embedding, such as in a hyperbolic space, associated with a natural metric. This strategy would inherently ensure that we remain within the cone of metrics.

The potential for follow-up work is vast and includes the application of soft Gromov hyperbolicity in various machine learning contexts. For instance, its connections to notions of graph curvature, such as Ollivier-Ricci curvature, could help in designing novel re-wiring procedures. Additionally, we envision its utility in tree discovery and embedding contexts, where it could serve as a regularization term in the corresponding optimization procedures.

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A Complementary qualitative and visual results

We provide in Figure 5 complementary qualitative results obtained on simple unweighted graphs, and details about the generation procedure. The original graph is an unweighted graph, with constant unit edge weights, generated following different graph models: Figures 5a and 5b correspond to random connected Erdős–Rényi graph with 10 nodes, Figure 5c is a stochastic block model graph with 2 communities and 40 nodes, generated with an intra-community probability of 0.5 and an inter-community probability of 0.05, and finally Figure 5d is a (4, 4)-grid graph, with a total of 16 nodes. The original nodes are depicted in orange in the figures, whereas additional nodes, when provided by the method, are depicted in blue. The root node used for generating the corresponding tree is depicted in pink, and is set to be the same for all the methods. In every visualizations, the original nodes positions are kept fixed, and the added nodes positions are computed using the `SPRINGLAYOUT` from the `NetworkX` package [20].

In terms of distortion, our method gives the best results in 3 out of 4 cases, and ranks second in the grid-like graph case, where `LAYERINGTREE` performs best.

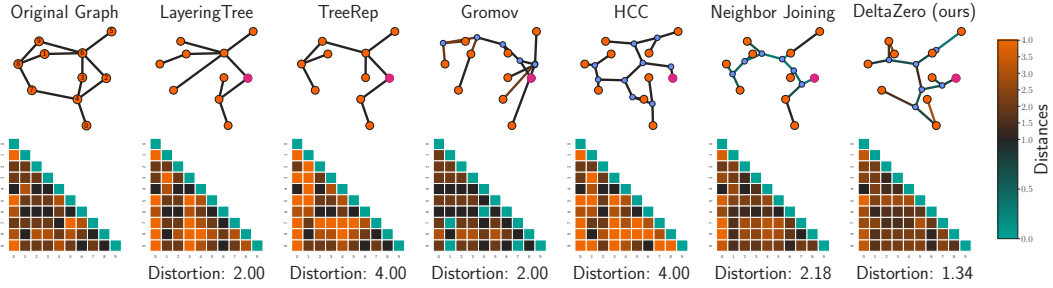
B Gromov’s embedding

The tree metric d_T , obtained by the embedding Φ given by Theorem 2, is characterized via Gromov products with respect to a basepoint w , which we denote by

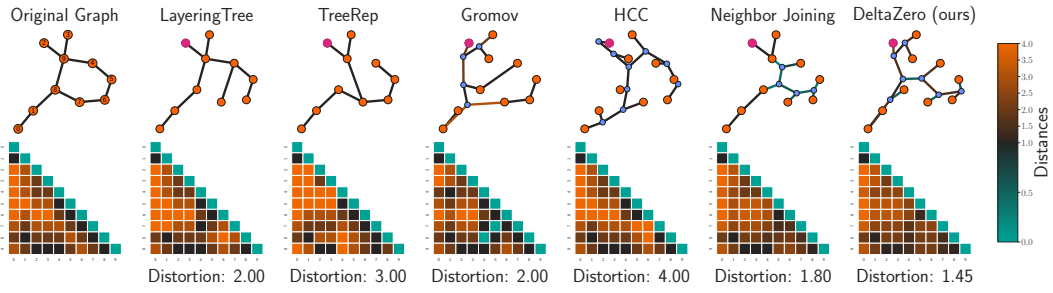
$$(x|y)'_w := \frac{1}{2} (d_T(\Phi(x), \Phi(w)) + d_T(\Phi(y), \Phi(w)) - d_T(\Phi(x), \Phi(y))).$$

This yields the identity

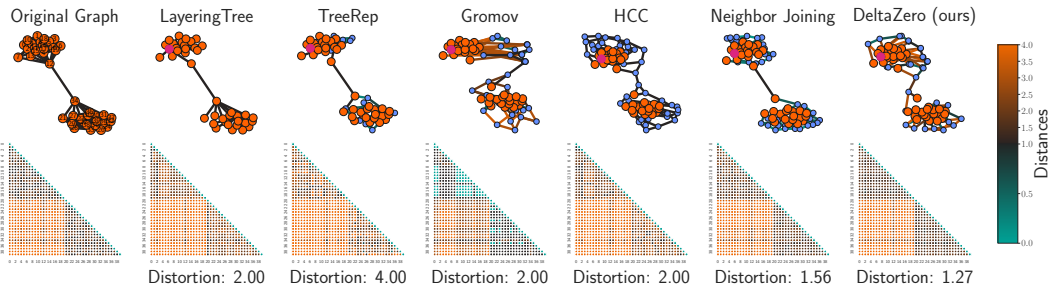
$$d_T(\Phi(x), \Phi(y)) = (x|x)_w + (y|y)_w - 2(x|y)'_w.$$



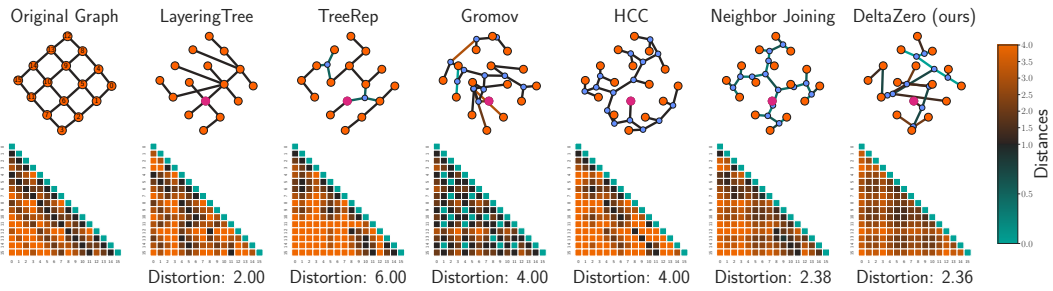
(a) Connected Erdős-Rényi graph with 10 nodes



(b) Another connected Erdős-Rényi graph with 10 nodes



(c) Stochastic block model with 2 components and 40 nodes



(d) (4, 4)-grid graph

Figure 5: Visual illustration of the different methods on different graphs.

Gromov's argument [19, Page 156] further shows that the Gromov product $(x|y)'_w$ can be expressed as

$$(x|y)'_w = \max_{\bar{y} \in P_{x,y}} \left\{ \min_k (y_k|y_{k+1})_w \right\},$$

where $P_{x,y}$ denotes the set of finite sequences $\bar{y} = (y_1, \dots, y_\ell)$ in X with $\ell \geq 2$, $x = y_1$, and $y = y_\ell$.

It is important to note that in Gromov's construction, the resulting tree metric space (T, d_T) may include intermediate points that are not present in the original set X , but are introduced in the tree to satisfy the metric constraints and to preserve approximate distances (see Figure 6). These points serve as internal branching nodes that allow better approximation of the original distances under the tree metric.

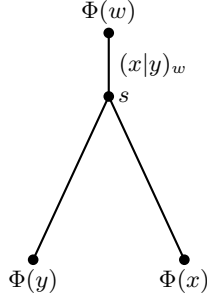


Figure 6: Gromov tree embedding of the points in Euclidean space: $x = (1, 0)$, $y = (-1, 0)$, and $w = (0, 3)$. A intermediate node s is introduced at a position such that it lies at distance $(x|y)_w$ from w .

To approximate an optimal ultrametric tree embedding of a finite metric space (X, d) , one can leverage the *single linkage hierarchical clustering* (SLHC) algorithm, which is known to yield a 2-approximation to the optimal ultrametric tree embedding [26]. The SLHC algorithm returns a new metric u_X on X , defined for all $x, y \in X$ by:

$$u_X(x, y) = \min_{\bar{y} \in P_{x,y}} \left\{ \max_k d(y_k, y_{k+1}) \right\},$$

where $P_{x,y}$ denotes the set of finite sequences $\bar{y} = (y_1, \dots, y_\ell)$ such that $y_1 = x$, $y_\ell = y$, and $y_i \in X$ for all i . To relate this to a tree embedding via Gromov products, fix a base point $w \in X$, and define

$$m := \max_{x \in X} d(x, w).$$

Then the function $d_G : X \times X \rightarrow \mathbb{R}$ given by

$$d_G(x, y) := \begin{cases} m - (x|y)_w & \text{if } x \neq y, \\ 0 & \text{if otherwise} \end{cases}$$

defines a valid metric on X . Applying SLHC to the metric space (X, d_G) yields an ultrametric space (X, u_X) . Finally, we define the tree metric as

$$d_T(x, y) := m - d_G(x, y).$$

One can easily check that this procedure gives the same tree metric than the embedding given by Gromov.

Algorithm 2 GROMOV

Require: Finite metric space (X, d) , base point $w \in X$

Ensure: Tree metric d_T on X

- 1: Compute Gromov products: $(x|y)_w \leftarrow \frac{1}{2} (d(x, w) + d(y, w) - d(x, y))$
 - 2: Set $m \leftarrow \max_{x \in X} d(x, w)$
 - 3: Define $d_G(x, y) \leftarrow m - (x|y)_w$ if $x \neq y$, 0 otherwise
 - 4: Apply SLHC to (X, d_G) to obtain an ultrametric u_X
 - 5: Define tree metric: $d_T(x, y) \leftarrow m - d_G(x, y)$
 - 6: **return** d_T
-

Once the tree metric has been obtained through this procedure, one may then invoke the method described in [5] to reconstruct a tree spanning the set X . This reconstruction yields both the intermediate nodes and the leaves, the latter corresponding to all elements of X except the designated root.

C Proofs of Theoretical Results

C.1 Proof of Theorem 3.1

For a fixed μ , let (X^*, d_{X^*}) be a minimizer of (3). We have, for any (X, d_X) ,

$$\mu \|d_X - d_{X^*}\|_\infty + \delta_{X^*} \leq \delta_X.$$

Applying the Gromov tree embedding Φ to (X^*, d_{X^*}) and denoting the resulting metric tree by (T^*, d_{T^*}) we get the following bound using Theorem 2.3

$$d_{X^*}(x, y) - 2(\delta_X - \mu \|d_X - d_{X^*}\|_\infty) \log_2(n-2) \leq d_{T^*}(\Phi(x), \Phi(y)) \leq d_{X^*}(x, y) \text{ for all } x, y \in X.$$

Combining this with the triangle inequality yields

$$\begin{aligned} \|d_X - d_{T^*}\|_\infty &\leq \|d_X - d_{X^*}\|_\infty + \|d_{X^*} - d_{T^*}\|_\infty \\ &\leq \|d_X - d_{X^*}\|_\infty + 2(\delta_X - \mu \|d_X - d_{X^*}\|_\infty) \log_2(n-2) \\ &\leq 2\delta_X \log_2(n-2) + (1 - 2\log_2(n-2)\mu) \|d_X - d_{X^*}\|_\infty. \end{aligned}$$

Thus, we have

$$d_X(x, y) - (2\delta_X \log_2(n-2) + (1 - 2\log_2(n-2)\mu) \|d_X - d_{X^*}\|_\infty) \leq d_{T^*}(\Phi(x), \Phi(y)).$$

For the second inequality, we have $\forall x, y, d_{T^*}(\Phi(x), \Phi(y)) \leq d_{X^*}(x, y) \leq d_X(x, y)$. The first inequality is true by Gromov theorem and the second due to the constraints in the optimization problem (3).

As long as $\mu \geq \frac{1}{2\log_2(n-2)}$, the worst-case distortion achieved by applying the Gromov embedding directly to X is improved.

C.2 Proof of Proposition 3.2

To control the approximation error introduced by the smooth maximum operator, we first state a standard inequality satisfied by the LSE_λ function.

Proposition C.1. *For $\lambda > 0$, the LSE_λ function satisfies the following inequalities*

$$\begin{aligned} \max \mathbf{x} &\leq \text{LSE}_\lambda(\mathbf{x}) \leq \max \mathbf{x} + \frac{\log(n)}{\lambda}, \\ \min \mathbf{x} - \frac{\log(n)}{\lambda} &\leq \text{LSE}_{-\lambda}(\mathbf{x}) \leq \min \mathbf{x}. \end{aligned}$$

Proof. Let $m = \max_i x_i$. Then we have

$$e^{\lambda m} \leq \sum_{i=1}^n e^{\lambda x_i} \leq n e^{\lambda m}.$$

Taking the logarithm and dividing by $\lambda > 0$, we obtain

$$m \leq \text{LSE}_\lambda(\mathbf{x}) \leq m + \frac{\log(n)}{\lambda}. \quad \square$$

We are now ready to use this bound to control the smooth approximation of Gromov's four-point condition and prove proposition 3.2.

In X , there are n^4 ordered 4-tuples of points with replacement. By proposition C.1, for any fixed x, y, z, w , we have

$$\min\{(x|y)_w, (y|z)_w\} - (x|z)_w - \frac{\log(2)}{\lambda} \leq \text{LSE}_{-\lambda}((x|y)_w, (y|z)_w) - (x|z)_w$$

and

$$\max_{x, y, z, w} \{\text{LSE}_{-\lambda}((x|y)_w, (y|z)_w) - (x|z)_w\} \leq \delta_X^{(\lambda)} \leq \delta_X + \frac{4\log(n)}{\lambda}.$$

Combining those inequalities, we obtain the desired result.

C.3 Proof of Theorem 3.3

For every $n \in \mathbb{N}$, let $X = \{x_1, \dots, x_n\}$ be a fixed n -point set.

We consider a *random metric space* on n points. This means we fix a probability measure ν on \mathcal{M}_n , the set of all metric spaces over n points.

Since a metric space on n points is completely determined by the distances between each pair of points, we can think of ν as a probability law on the set of all possible distance matrices that satisfy the metric conditions (non-negativity, symmetry, triangle inequality, and zero diagonal).

So, when we say that a random variable X has law ν , we mean that we are choosing the distances between the n points at random, according to ν . In this way, the law ν is a *law on the distances* between the points.

Definition C.2. We say that a random metric space (X, d) (with law ν) is *uniform* if the random variables $d(x_i, x_j)$ for $x_i, x_j \in X$ are independent and identically distributed, following a common law ℓ . We say that ℓ is the *law of distances* of the uniform random metric space.

In this class of random metric spaces, we have the following asymptotics for $\delta_X^{(\lambda)}$.

Proposition C.3. *Let ℓ be a probability measure on \mathbb{R}_+ (with compact support). Let ν_n be a probability measure on \mathcal{M}_n such that the associated random metric space is uniform, with law of distances given by ℓ . Let (X, d) be a random metric following the law ν_n . Then, when n goes to infinity and $\lambda = O(\log n)$, we have*

$$\mathbb{P} \left(\left| \delta_X^{(\lambda)} - \frac{1}{\lambda} (\log \mu + 4 \log n) \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2 n^{1-o(1)}},$$

where μ depends on the law ℓ and λ , but not on n and ε .

Proof of Proposition C.3. We use the standard convention that the letter C denotes a universal constant, whose precise value may change from line to line.

Recall from 4 that $\delta_X^{(\lambda)}$ can be written explicitly as

$$\delta_X^{(\lambda)} = \frac{1}{\lambda} \log \sum_{(x,y,z,w) \in X^4} \frac{e^{-\lambda(x|z)_w}}{e^{-\lambda(x|y)_w} + e^{-\lambda(y|z)_w}},$$

where the Gromov product are taken with respect to the metric d . We denote by $f(x, y, z, w)$ the term

$$\frac{e^{-\lambda(x|z)_w}}{e^{-\lambda(x|y)_w} + e^{-\lambda(y|z)_w}}.$$

We start by a preliminary estimate on f . We introduce some notations. We note

$$\delta_{x,y,z,w} = \min((x|y)_w, (y|z)_w) - (x|z)_w.$$

We say that a quadruple $q = (x, y, z, w) \in X^4$ is *generic* if the elements x, y, z, w are pairwise distinct. Observe that if q_1 and q_2 are both generic, then $f(q_1)$ and $f(q_2)$ are identically distributed. In particular,

$$\mathbb{E}[f(q_1)] = \mathbb{E}[f(q_2)],$$

and we denote this common expectation by μ .

Lemma C.4. *For every $p \in \mathbb{N}$, we have the following asymptotics when λ goes to infinity*

$$\mathbb{E}[f(q)^p]^{\frac{1}{p}} = e^{\lambda \max \delta_q} e^{o(\lambda)},$$

and

$$\mathbb{E}[f(q)^p]^{\frac{1}{p}} \leq \mu e^{o(\lambda)}.$$

Proof. Recall that

$$f(x, y, z, w) = \frac{e^{-\lambda(x|z)_w}}{e^{-\lambda(x|y)_w} + e^{-\lambda(y|z)_w}} = \exp(\lambda(\text{LSE}_{-\lambda}((x|y)_w, (y|z)_w) - (x|z)_w)),$$

hence by Proposition C.1:

$$\exp\left(\lambda\left(\delta_{x,y,z,w} - \frac{\log 2}{\lambda}\right)\right) \leq f(x,y,z,w) \leq \exp(\lambda\delta_{x,y,z,w}).$$

We conclude that

$$C_1 e^{\lambda\delta_q} \leq f(q) \leq C_2 e^{\lambda\delta_q},$$

for universal constants C_1, C_2 .

We recall the classical limit

$$\lim_{\lambda \rightarrow \infty} (\mathbb{E}[X^\lambda])^{\frac{1}{\lambda}} = \max X,$$

which holds for a positive, bounded random variable, which we can rewrite

$$\mathbb{E}[X^\lambda] = (\max X)^\lambda e^{o(\lambda)}.$$

Applying that to $X = e^{p\delta_q}$ we obtain

$$(\mathbb{E}[e^{\lambda p\delta_q}])^{\frac{1}{p}} = e^{\lambda \max \delta_q} e^{o(\lambda)},$$

and thus

$$(\mathbb{E}[f(q)^p])^{\frac{1}{p}} = e^{\lambda \max \delta_q} e^{o(\lambda)}.$$

Observe that the law of the random variable δ_q , for $q = (x, y, z, w)$, depends only of the configuration of which x, y, z, w are equal between them. In particular, there is a finite number of possibilities for $\max \delta_q$ (which depend only on the law ℓ). It is tedious but easy to check that $\max \delta_q$ is maximum when q is generic. Recalling that μ is a notation for $\mathbb{E}[f(q)]$ when q is generic, we conclude that:

$$(\mathbb{E}[\bar{f}(q)^p])^{\frac{1}{p}} \leq \mu e^{o(\lambda)}. \quad \square$$

We now study the sum

$$S_n = \sum_{(x,y,z,w) \in X^4} f(x,y,z,w).$$

If the terms of the sum were independent random variables, we could directly apply standard limit theorems. However, this is not the case: each term $f(x, y, z, w)$ depends only on the pairwise distances between the four points x, y, z, w . In particular, two terms $f(x, y, z, w)$ and $f(x', y', z', w')$ are not independent whenever the sets of their arguments intersect, i.e.,

$$\{x, y, z, w\} \cap \{x', y', z', w'\} \neq \emptyset.$$

To formalize this, for two 4-tuples $q_1 = (x, y, z, w)$ and $q_2 = (x', y', z', w')$ in X_n^4 , we write $q_1 \perp q_2$ to denote that they are disjoint

$$\{x, y, z, w\} \cap \{x', y', z', w'\} = \emptyset.$$

Despite the dependencies, the situation remains favorable. The number of dependent pairs,

$$\{(q_1, q_2) \in (X^4)^2 : q_1 \not\perp q_2\},$$

is bounded by Cn^7 , while the total number of pairs in $(X^4)^2$ is n^8 . Thus, most of the terms are approximately independent, which allows for concentration techniques and asymptotic analysis.

We adapt the classical proof of the weak law of large numbers to this weakly dependent setting using Chebyshev's inequality. To that end, we compute the variance of the sum

$$S_n = \sum_{(x,y,z,w) \in X_n^4} f(x,y,z,w).$$

It is convenient to consider the centered version

$$\bar{S}_n = \sum_{(x,y,z,w) \in X_n^4} \bar{f}(x,y,z,w), \quad \text{where } \bar{f}(x,y,z,w) = f(x,y,z,w) - \mathbb{E}[f(x,y,z,w)],$$

so that $\mathbb{E}[\bar{S}_n] = 0$. The variance is then

$$\begin{aligned}\mathbb{E}[\bar{S}_n^2] &= \sum_{(q_1, q_2) \in (X^4)^2} \mathbb{E}[\bar{f}(q_1)\bar{f}(q_2)] \\ &= \sum_{q_1 \perp q_2} \mathbb{E}[\bar{f}(q_1)\bar{f}(q_2)] + \sum_{q_1 \not\perp q_2} \mathbb{E}[\bar{f}(q_1)\bar{f}(q_2)].\end{aligned}$$

Since $q_1 \perp q_2$ implies independence and $\mathbb{E}[\bar{f}(q_1)] = 0$, the first sum vanishes. Thus, we are left with

$$\mathbb{E}[\bar{S}_n^2] = \sum_{q_1 \not\perp q_2} \mathbb{E}[\bar{f}(q_1)\bar{f}(q_2)].$$

By Cauchy-Schwarz inequality we have

$$\mathbb{E}[\bar{f}(q_1)\bar{f}(q_2)] \leq (\mathbb{E}[\bar{f}(q_1)^2] \mathbb{E}[\bar{f}(q_2)^2])^{\frac{1}{2}},$$

and by the triangle inequality

$$(\mathbb{E}[\bar{f}(q)^2])^{\frac{1}{2}} = \mathbb{E}[(f(q) - \mathbb{E}[f(q)])^2]^{\frac{1}{2}} \leq \mathbb{E}[f(q)^2]^{\frac{1}{2}} + \mathbb{E}[f(q)].$$

Applying the Lemma, both terms on the right hand side are bounded by $\mu e^{o(\lambda)}$, hence

$$(\mathbb{E}[\bar{f}(q)^2])^{\frac{1}{2}} \leq \mu e^{o(\lambda)}.$$

Using that the number of pairs with $q_1 \not\perp q_2$ is bounded by Cn^7 we finally obtain the variance bound

$$\mathbb{E}[\bar{S}_n^2] \leq \mu^2 e^{o(\lambda)} n^7.$$

We recall Chebyshev's classical inequality:

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{\varepsilon^2}.$$

Applying this inequality to $X = \bar{S}_n$ and $\varepsilon' = n^4 \varepsilon$ we get

$$\mathbb{P}(|\bar{S}_n| \geq n^4 \varepsilon) \leq \frac{\mathbb{E}[\bar{S}_n^2]}{\varepsilon^2 n^8},$$

and using our previous estimate

$$\mathbb{P}\left(\left|\frac{1}{n^4} \bar{S}_n\right| \geq \varepsilon\right) \leq \frac{\mu^2 e^{o(\lambda)}}{\varepsilon^2 n} = \frac{\mu^2}{\varepsilon^2 n^{1-o(1)}},$$

which we can rewrite as

$$\mathbb{P}\left(\left|\frac{1}{n^4} S_n - \frac{1}{n^4} \mathbb{E}[S_n]\right| \geq \varepsilon\right) \leq \frac{\mu^2}{\varepsilon^2 n^{1-o(1)}}.$$

We now compute the term $\frac{1}{n^4} \mathbb{E}[S_n] = \frac{1}{n^4} \sum_{q \in X^4} \mathbb{E}[f(q)]$. We can write:

$$\begin{aligned}\frac{1}{n^4} \mathbb{E}[S_n] &= \frac{1}{n^4} \sum_{q \in X^4} \mathbb{E}[f(q)] \\ &= \frac{1}{n^4} \sum_{\substack{q \in X^4 \\ \text{generic}}} \mathbb{E}[f(q)] + \frac{1}{n^4} \sum_{\substack{q \in X^4 \\ \text{non-generic}}} \mathbb{E}[f(q)]\end{aligned}$$

Moreover, the number of non-generic quadruples (i.e., $q \in X^4$ where at least two of the entries coincide) is at most Cn^3 for some constant $C > 0$. By the Lemma, $\mathbb{E}[f(q)]$ is bounded by $\mu e^{o(\lambda)}$. Combining these observations, we obtain

$$\frac{1}{n^4} \mathbb{E}[S_n] = \left(1 - O\left(\frac{1}{n}\right)\right) \mu + \frac{\mu e^{o(\lambda)}}{n} = \mu \left(1 + \frac{1}{n^{1-o(1)}}\right)$$

which implies that $\frac{1}{n^4} \mathbb{E}[S_n] \sim \mu$ as $n \rightarrow \infty$.

We can now look at $\delta_X^{(\lambda)} = \frac{1}{\lambda} \log S_n$. We recall the elementary inequality

$$|\log a - \log b| \leq \frac{|a - b|}{b}.$$

If $|S_n - \mathbb{E}[S_n]| \leq n^4 \varepsilon$, then

$$\left| \frac{S_n - \mathbb{E}[S_n]}{\mathbb{E}[S_n]} \right| \leq \frac{n^4 \varepsilon}{\mathbb{E}[S_n]}.$$

Since $\mathbb{E}[S_n] \sim n^4 \mu$, we have for large n ,

$$\frac{1}{\mathbb{E}[S_n]} \leq \frac{2}{n^4 \mu}.$$

Hence, for n large enough,

$$|\log S_n - \log \mathbb{E}[S_n]| \leq \frac{|S_n - \mathbb{E}[S_n]|}{\mathbb{E}[S_n]} \leq \frac{2\varepsilon}{\mu}.$$

Therefore, the log-difference is controlled by the relative deviation scaled by μ^{-1} .

We conclude that when n is large enough,

$$\mathbb{P} \left(|\log S_n - \log \mathbb{E}[S_n]| \geq \frac{2\varepsilon}{\mu} \right) \leq \mathbb{P} \left(\left| \frac{1}{n^4} S_n - \frac{1}{n^4} \mathbb{E}[S_n] \right| \geq \varepsilon \right) \leq \frac{\mu^2}{\varepsilon^2 n^{1-o(1)}},$$

which implies

$$\mathbb{P} \left(\left| \frac{1}{\lambda} \log S_n - \frac{1}{\lambda} \log \mathbb{E}[S_n] \right| \geq \varepsilon \right) \leq \frac{1}{\lambda^2 \varepsilon^2 n^{1-o(1)}} \leq \frac{1}{\varepsilon^2 n^{1-o(1)}}.$$

Using that $\frac{1}{n^4} \mathbb{E}[S_n] \sim \mu$, when n is large enough we then have:

$$\mathbb{P} \left(\left| \frac{1}{\lambda} \log S_n - \frac{1}{\lambda} (\log \mu + 4 \log n) \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2 n^{1-o(1)}}. \quad \square$$

We can finally state the result that justify the batching method used to compute the delta.

Given a random metric space on n points, distributed according to a law ν , we can define a random metric space on $m \leq n$ points by restriction. Let (X, d) be a random metric space drawn from ν and let $X_m \subset X$ be a subset of size m , chosen uniformly at random. We equip X_m with the metric $d|_{X_m}$ induced by d . We can iterate this process k -times, choosing the subsets of size m disjoint from each other, and we obtain a sequence of random metric space X_m^1, \dots, X_m^K , each one uniform with law of distances ℓ .

Theorem C.5. *Let ℓ be a probability measure on \mathbb{R}_+ . Let ν be a probability measure on \mathcal{M}_n such that the associated random metric space is uniform, with law of distances given by ℓ . In the regime $m^K \sim n$ and $\lambda \sim \varepsilon^{-1} \log n$ we have*

$$\mathbb{P} \left[\exists i, \left| \delta_X^{(\lambda)} - \delta_{X_m^i}^{(\lambda)} \right| \leq \varepsilon \right] \geq 1 - \frac{1}{\varepsilon^{2K} n^{1-o(1)}}$$

Proof. For a uniform random metric space (X, d) and a sequence of random subspaces (X_m^1, \dots, X_m^K) as constructed above, each $(X_m^i, d|_{X_m^i})$ is still a uniform random metric space with the same law of distances. Because $m^K \sim n$ and $\lambda = O(\log n)$ we have $\lambda = O(\log m)$. We apply the Proposition C.3 to get

$$\begin{aligned} \mathbb{P} \left(\left| \delta_X^{(\lambda)} - \frac{1}{\lambda} (\log \mu + 4 \log n) \right| \geq \varepsilon \right) &\leq \frac{1}{\varepsilon^2 n^{1-o(1)}}, \\ \mathbb{P} \left(\left| \delta_{X_m^i}^{(\lambda)} - \frac{1}{\lambda} (\log \mu + 4 \log m) \right| \geq \varepsilon \right) &\leq \frac{1}{\varepsilon^2 n^{1-o(1)}}. \end{aligned}$$

The (Y_m^i) are independent random variables hence

$$\mathbb{P}\left(\forall i \leq K, \left|\delta_{X_m^i}^{(\lambda)} - \frac{1}{\lambda}(\log \mu + 4 \log m)\right| \leq \varepsilon\right) \leq \frac{1}{\varepsilon^{2K} m^{K-o(1)}},$$

which means that

$$\mathbb{P}\left(\exists i \leq K, \left|\delta_{X_m^i}^{(\lambda)} - \frac{1}{\lambda}(\log \mu + 4 \log m)\right| \leq \varepsilon\right) \geq 1 - \frac{1}{\varepsilon^{2K} m^{K-o(1)}} = 1 - \frac{1}{\varepsilon^{2K} n^{1-o(1)}},$$

When

$$\left|\delta_{X_m^i}^{(\lambda)} - \frac{1}{\lambda}(\log \mu + 4 \log m)\right| \leq \varepsilon,$$

and

$$\left|\delta_X^{(\lambda)} - \frac{1}{\lambda}(\log \mu + 4 \log n)\right| \leq \varepsilon.$$

Thus,

$$\left|\delta_X^{(\lambda)} - \delta_{X_m^i}^{(\lambda)}\right| \leq 2\varepsilon + \frac{1}{\lambda} \log \frac{n}{m} \leq 3\varepsilon.$$

We can conclude that

$$\begin{aligned} \mathbb{P}\left[\exists i, \left|\delta_X^{(\lambda)} - \delta_{X_m^i}^{(\lambda)}\right| \leq \varepsilon\right] &\geq 1 - \mathbb{P}\left[\forall i, \left|\delta_X^{(\lambda)} - \delta_{X_m^i}^{(\lambda)}\right| \geq \varepsilon\right] \\ &\quad - \mathbb{P}\left(\left|\delta_X^{(\lambda)} - \frac{1}{\lambda}(\log \mu + 4 \log n)\right| \geq \varepsilon\right) \\ &\geq 1 - \frac{1}{\varepsilon^{2K} n^{1-o(1)}}. \end{aligned}$$

□

Corollary C.1. *Let ℓ be a probability measure on \mathbb{R}_+ . Let ν be a probability measure on \mathcal{M}_n , the set of metric spaces over n points, such that the associated random metric space is uniform (see definition C.2), with law of distances given by ℓ . In the regime $m^K \sim n$ and $\lambda \sim \varepsilon^{-1} \log n$ we have*

$$\mathbb{P}\left[\left|\delta_X - \delta_{X,K,m}^{(\lambda)}\right| \leq \varepsilon\right] \geq 1 - \frac{1}{\varepsilon^{2K} n^{1-o(1)}}.$$

Proof. This follows directly from triangular inequality, definition of $\delta_{X,K,m}^{(\lambda)}$ (see equation (5)) and Proposition 3.2. □

Models of uniform random metric spaces. We now describe several models of random metric spaces that are (aproximatively) uniform.

In [24] the authors study a model of discrete random metric spaces. For an integer $r \geq 2$, consider the finite subset $\mathcal{D}_n^r \subset \mathcal{D}_n$ of metric spaces for which the distance function takes values in $\{0, 1, \dots, 2r\}$. Let ν_n^r be the uniform probability on this finite set. Let $C_n^r \subset \mathcal{D}_n^r$ the subset of metric spaces whose distance function takes values in the interval $\{r, \dots, 2r\}$. Observe that any collection of numbers in $\{r, \dots, 2r\}$ satisfy the triangle inequality, hence $C_n^r = \{r, \dots, 2r\}^{\binom{n}{2}}$, and the restriction of the measure ν_n^r to C_n^r is simply a product measure. In other words, the random metric space conditioned to be in C_n^r is uniform. The authors prove that, when n is large, the measure ν_n^r is exponentially concentrated on C_n^r .

Theorem C.6. *There exists $\beta > 0$ such that for n large enough we have*

$$\nu_n^r(C_n^r) \geq 1 - e^{-\beta n}.$$

In other words, the uniform random metric space with integer valued distances and diameter bounded by $2r$ is (aproximatively) uniform.

Let $G(n, p)$ denote the Erdős–Rényi random graph: a graph on n vertices where each edge between a pair of vertices is included independently with probability $p \in [0, 1]$.

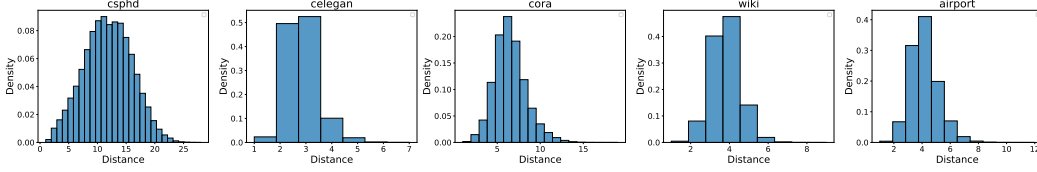


Figure 7: Distribution of the distances of the unweighted datasets of our benchmark.

In the regime

$$\begin{aligned} p^2 n - 2 \log n &\rightarrow \infty, \\ n^2(1 - p) &\rightarrow \infty, \end{aligned}$$

the graph $G(n, p)$ asymptotically almost surely has *diameter* 2, that is, any pair of vertices is either directly connected by an edge or shares a common neighbor, by [2]. According to [12, §4], conditionally to the diameter being 2, the distance matrix of this graph is a function of the incidence matrix; in particular, the associated random metric space is uniform.

We also evaluate the distribution of distances on real datasets as shown in Figure 7. We can see that approximately these distributions are log-normal, suggesting that they could satisfy the hypothesis of our theorem.

C.4 Proof of Proposition 3.4

Following Bridson and Haefliger [4], the four points condition given in (2.1), can be seen in the following way: Let $\mathbf{D} \in \mathcal{D}_n$, for $i, j, k, l \in \{1, \dots, n\}$, let l_1, l_2 and l_3 be defined by

$$\begin{cases} l_1 = D_{ij} + D_{kl} \\ l_2 = D_{ik} + D_{jl} \\ l_3 = D_{il} + D_{jk} \end{cases}$$

and let M_1 and M_2 be the two largest values among l_1, l_2 , and l_3 . We define

$$\delta(i, j, k, l) = M_1 - M_2$$

and the Gromov hyperbolicity $\delta_{\mathbf{D}}$ of the metric space is the maximum of $\delta(i, j, k, l)$ over all possible 4-tuples (i, j, k, l) divided by 2. That is,

$$\delta_{\mathbf{D}} = \frac{1}{2} \max_{i, j, k, l \in \{1, \dots, n\}} \delta(i, j, k, l)$$

The function $\delta_{\mathbf{D}}$, viewed as a function over \mathcal{D}_n is a maximum of piecewise affine functions, and is therefore itself piecewise affine.

We can interpret $\delta_{\mathbf{D}}$ geometrically as follows. Each 4-tuple (i, j, k, l) determines three quantities l_1, l_2 , and l_3 . The differences between these quantities partition the plane into regions corresponding to which of l_1, l_2, l_3 is largest, second largest, and smallest (see Figure 8).

To demonstrate the non-convexity of the set in question, it suffices to exhibit a single counterexample. We do so by constructing two metric spaces, each defined over four points, and represented via their respective distance matrices. These examples are deliberately simple to allow for full transparency of the argument and ease of verification.

While we restrict ourselves here to the case $n = 4$, the same construction strategy can be generalized without difficulty to any number of points $n \geq 4$. The underlying mechanism responsible for the failure of convexity is not peculiar to this specific dimension.

Let us consider

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 1.1 & 1 & 1.2 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \text{ and } \mathbf{D}_2 = \begin{bmatrix} 0 & 1 & 1 & 1.2 \\ & 0 & 1 & 1.1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

Computing the associated triples (l_1, l_2, l_3) for each space, we find

$$L_1 = (2.1, 2.2, 2) \quad \text{for } \mathbf{D}_1, \quad L_2 = (2, 2.2, 2.1) \quad \text{for } \mathbf{D}_2.$$

Now consider the interpolation between \mathbf{D}_1 and \mathbf{D}_2 defined by $t \in [0, 1] \mapsto t\mathbf{D}_1 + (1-t)\mathbf{D}_2$ with associated triple $(l_1(t), l_2(t), l_3(t))$.

Since the values of l_2 are identical for \mathbf{D}_1 and \mathbf{D}_2 , the interpolation affects only l_1 and l_3 . Thus, along this interpolation δ behave like

$$t \mapsto -\max(l_1(t), l_3(t)),$$

which is concave and negates that $\delta_{\mathbf{D}}$ is a convex function over \mathcal{D}_4 .

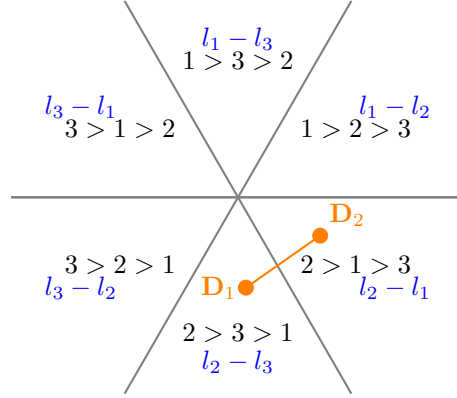


Figure 8: In the diagram, the plane is divided into six sectors, each labeled according to the ordering of l_1, l_2, l_3 (e.g., $1 > 2 > 3$ means l_1 is largest, then l_2 , then l_3). The labels inside each sector indicate the expression that represents $\delta(i, j, k, l)$ in that region.

C.5 Floyd-Warshall as a projection

This section mainly rewrites results described in [3]. Let $\mathbf{W} \in \mathbb{R}_+^{n \times n}$ be the matrix of non-negative edge weights of a (possibly directed) graph with n nodes. Let $\text{SP}(\mathbf{W})$ denote the matrix of shortest path distances between all node pairs. We use the notation $\mathbf{A} \leq \mathbf{B}$ to mean that $A_{ij} \leq B_{ij}$ for all (i, j) , and recall that \mathcal{D}_n denotes the set of $n \times n$ distance matrices (i.e., matrices that satisfy the properties of a metric).

We aim to show that

$$\text{SP}(\mathbf{W}) \in \underset{\mathbf{D} \in \mathcal{D}_n, \mathbf{D} \leq \mathbf{W}}{\text{argmin}} \|\mathbf{D} - \mathbf{W}\|_p,$$

for any ℓ_p norm with $p > 0$.

Note that $\text{SP}(\mathbf{W}) \leq \mathbf{W}$ since the shortest path between two nodes is always less than or equal to the direct edge weight (if it exists). Hence, $\text{SP}(\mathbf{W})$ is a feasible candidate for the minimization problem.

Suppose the following property holds:

$$\forall \mathbf{D} \in \mathcal{D}_n, \mathbf{D} \leq \mathbf{W} \Rightarrow \mathbf{D} \leq \text{SP}(\mathbf{W}).$$

Under this assumption, the optimality of $\text{SP}(\mathbf{W})$ follows immediately. Indeed, for any such \mathbf{D} , we have:

$$0 \leq W_{ij} - \text{SP}(\mathbf{W})_{ij} \leq W_{ij} - D_{ij} = |D_{ij} - W_{ij}|,$$

for all (i, j) , using the facts that $\text{SP}(\mathbf{W}) \leq \mathbf{W}$, $\mathbf{D} \leq \text{SP}(\mathbf{W})$, and $\mathbf{D} \leq \mathbf{W}$. Therefore,

$$|\text{SP}(\mathbf{W})_{ij} - W_{ij}| \leq |D_{ij} - W_{ij}|,$$

which implies

$$\|\text{SP}(\mathbf{W}) - \mathbf{W}\|_p \leq \|\mathbf{D} - \mathbf{W}\|_p.$$

We now prove the property. Assume for contradiction that there exists (i, j) such that $D_{ij} > \text{SP}(\mathbf{W})_{ij}$. Consider the lex smallest such pair under the total ordering induced by increasing values of $\text{SP}(\mathbf{W})_{ij}$. That is, sort all node pairs so that:

$$\text{SP}(\mathbf{W})_{i_1, j_1} \leq \text{SP}(\mathbf{W})_{i_2, j_2} \leq \dots \leq \text{SP}(\mathbf{W})_{i_M, j_M},$$

and let (i, j) be the first pair for which $D_{ij} > \text{SP}(\mathbf{W})_{ij}$.

There are two possibilities: (1) $\text{SP}(\mathbf{W})_{ij} = W_{ij}$. Then $D_{ij} > W_{ij}$, which contradicts the assumption $\mathbf{D} \leq \mathbf{W}$. (2) There exists some node k such that

$$\text{SP}(\mathbf{W})_{ij} = \text{SP}(\mathbf{W})_{ik} + \text{SP}(\mathbf{W})_{kj}.$$

Since \mathbf{D} is a metric, it satisfies the triangle inequality:

$$D_{ij} \leq D_{ik} + D_{kj}.$$

But by minimality of (i, j) in the sorted list, we know:

$$D_{ik} \leq \text{SP}(\mathbf{W})_{ik}, \quad D_{kj} \leq \text{SP}(\mathbf{W})_{kj}.$$

Therefore,

$$D_{ij} \leq D_{ik} + D_{kj} \leq \text{SP}(\mathbf{W})_{ik} + \text{SP}(\mathbf{W})_{kj} = \text{SP}(\mathbf{W})_{ij},$$

which contradicts the assumption that $D_{ij} > \text{SP}(\mathbf{W})_{ij}$.

This contradiction implies that no such (i, j) exists, and thus $\mathbf{D} \leq \text{SP}(\mathbf{W})$, completing the proof.

D Complementary experimental results

D.1 Average ℓ_1 error and Runtime Efficiency

In this section, we complement the ℓ_∞ distortion results (Table 1) with an analysis of the average embedding quality using ℓ_1 error. Table 2 reports the mean ℓ_1 distortion between the input distance matrix and the distance matrix induced by the fitted tree metric. This metric captures the average deviation rather than the worst-case distortion, providing a different perspective on the fidelity of the embedding. The experimental setup is identical to that used for Table 1, including the selection of hyperparameters (based on lowest ℓ_∞ error) and root sampling strategy.

As shown in Table 2, NEIGHBORJOIN (NJ) achieves the lowest average distortion across all datasets, including both graph-based and non-graph metrics. This is expected, as NJ is a greedy algorithm designed to minimize additive tree reconstruction error from a distance matrix. However, it does not guarantee low worst-case distortion, and its performance in terms of ℓ_∞ error is less consistent (see Table 1).

By contrast, our method, DELTAZERO, is not designed to minimize average error directly but rather optimizes a distortion-aware loss function. Despite this, it performs competitively in ℓ_1 distortion, especially on datasets such as C-ELEGAN and ZEISEL. However, on larger datasets like CORA and AIRPORT, it yields higher average distortion than NJ, suggesting that there may be a trade-off between minimizing ℓ_∞ and ℓ_1 error in practice.

Table 3 reports the average runtime (in seconds) over 10 runs for each method and dataset. For DELTAZERO, we report the runtime of the optimization step only, using the best-performing hyperparameter configuration (as selected in Table 1). For NJ and TREEREP, the reported times exclude the $O(n^3)$ time required to construct a full pairwise distance matrix, as these methods operate directly produce a tree structure.

As expected, DELTAZERO incurs higher computational cost than the baselines, particularly on large datasets such as ZEISEL and CORA, where optimization can take several thousand seconds. This is the price of our optimization method although as highlighted in Section 3.2, the major bottleneck comes from the usage of FLOYD-WARSHALL algorithm at each step of our proposed gradient descent. Lightweight methods such as GROMOV and HCC run much faster, though they offer less accuracy in both ℓ_∞ and ℓ_1 metrics.

These results highlight a key trade-off: DELTAZERO offers superior worst-case distortion, but at the cost of increased runtime.

Table 2: ℓ_1 average error (lower is better). Best result in bold.

Data set	Unweighted graphs					Non-graph metrics	
	C-ELEGAN	CS PHD	CORA	AIRPORT	WIKI	ZEISEL	IBD
NJ	0.30	1.35	1.06	0.49	0.51	0.04	0.09
TR	0.83 ± 0.16	2.55 ± 1.34	2.91 ± 0.63	1.28 ± 0.21	1.85 ± 0.26	0.11 ± 0.02	0.13 ± 0.01
HCC	0.88 ± 0.22	2.59 ± 1.11	2.44 ± 0.43	1.10 ± 0.13	1.53 ± 0.17	0.08 ± 0.02	0.27 ± 0.03
LayeringTree	0.73 ± 0.03	4.90 ± 0.07	3.68 ± 0.23	0.62 ± 0.06	0.83 ± 0.04	–	–
Gromov	1.13 ± 0.01	2.80 ± 0.36	3.34 ± 0.12	1.55 ± 0.08	2.12 ± 0.04	0.10 ± 0.01	0.39 ± 0.02
DELTAZERO	0.42 ± 0.01	2.71 ± 0.08	3.20 ± 0.09	1.31 ± 0.03	1.26 ± 0.00	0.08 ± 0.00	0.36 ± 0.02

Table 3: Average running time in seconds over 10 runs.

Data set	Unweighted graphs					Non-graph metrics	
	C-ELEGAN	CS PHD	CORA	AIRPORT	WIKI	ZEISEL	IBD
NJ	0.34	3.42	103.81	244.93	81.89	207.91	0.25
TR	0.78	2.64	25.64	36.37	24.43	94.02	1.23
HCC	0.67	3.54	19.54	31.15	18.28	36.21	0.52
LayeringTree	0.86	8.81	156.54	344.43	140.11	–	–
Gromov	0.03	0.23	1.47	2.20	24.43	2.20	0.02
DELTAZERO	51.00	134.28	3612.35	2268.72	819.90	5664.40	289.07

D.2 Sensitivity analysis

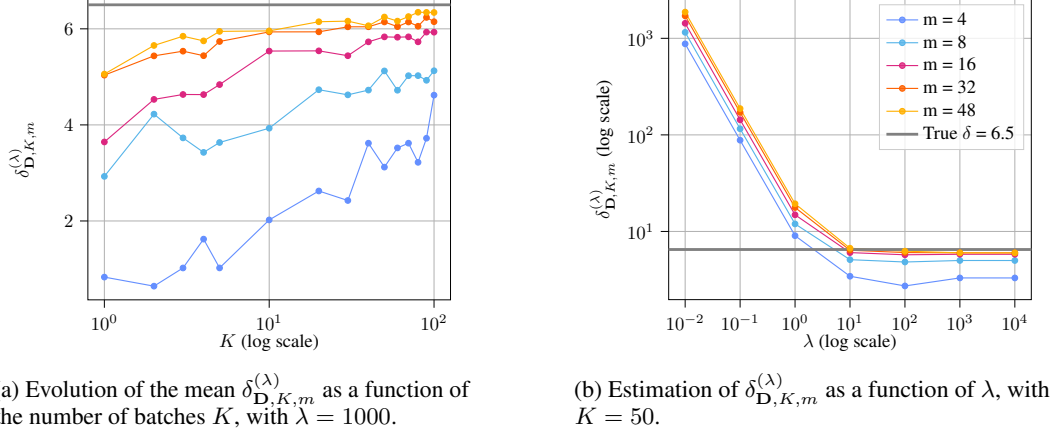
In this section, we provide a sensitivity analysis on the parameters of DELTAZERO, namely the number of batches K , the batch size m , the temperature parameter λ (cf. eq (4)) and the distance regularization coefficient μ on the norm (cf. eq (6)).

Evaluation of the approximation $\delta_{D,K,m}^{(\lambda)}$ of the Gromov hyperbolicity. We first provide a numerical study of the approximation $\delta_{D,K,m}^{(\lambda)}$ (see eq.(5)) of the actual δ -hyperbolicity. We consider the CS-PHD dataset, which has an exact value of $\delta = 6.5$ that can be computed exactly as n is not too large. We evaluate the quality of our approximation with respect to this ground truth.

Figure 9a illustrates the influence of two key hyperparameters: the number of batches K and the batch size m , fixing $\lambda = 1000$. Figure 9b illustrates the influence of the regularization parameter λ and the batch size m , fixing $K = 50$. Both figures also report the true δ . Each point represents the average over 5 independent runs.

We observe that batch size m plays a critical role in the quality of the estimation. When m is very small (e.g., $m = 4$), the estimated values are significantly biased and far from the true δ . As m increases, the estimates become more stable and converge toward the ground truth, even with a moderate number of batches. This pattern is consistent across both panels of the figure. In contrast, the effect of λ is less pronounced once it reaches a sufficient scale, beyond which the estimates flatten out near the true δ value. These results highlight that batch size is the dominant factor influencing the accuracy of our smoothed estimator $\delta_{D,K,m}^{(\lambda)}$. Indeed, computing the exact δ requires evaluating all possible quadruples of nodes, which amounts to $\binom{n}{4} = O(n^4)$ operations. In our batched approximation, we instead sample K batches of size m , resulting in a total of $K \cdot \binom{m}{4}$ sampled quadruples. Since $\binom{m}{4} = O(m^4)$, the number of evaluated quadruples scales as $O(K \cdot m^4)$, highlighting that m has a higher-order impact than K on the estimator’s fidelity. This explains why increasing the batch size leads to more accurate and stable δ estimates, even with a relatively small number of batches.

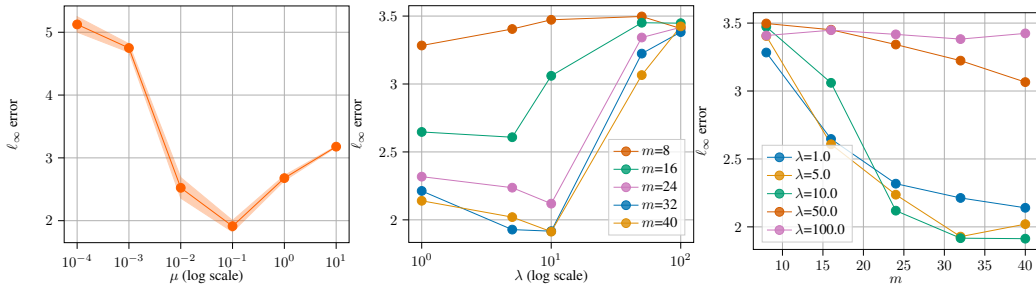
Sensitivity to parameters on the distortion. We now consider the C-ELEGAN dataset and study the impact of the parameter tuning for μ , λ and m . We vary one hyperparameter at a time, while fixing the others to their optimal values obtained from the grid search. We evaluate the mean distortion over



(a) Evolution of the mean $\delta_{D,K,m}^{(\lambda)}$ as a function of the number of batches K , with $\lambda = 1000$.

(b) Estimation of $\delta_{D,K,m}^{(\lambda)}$ as a function of λ , with $K = 50$.

Figure 9: Approximation $\delta_{D,K,m}^{(\lambda)}$ computed on the CS-PHD dataset for different batch sizes m .



(a) Effect of the distance regularization coefficient μ on ℓ_∞ distortion. (b) Impact of the log-sum-exp scale λ on ℓ_∞ error across various batch sizes m . (c) Impact of the batch size m on ℓ_∞ error for multiple λ values.

Figure 10: Sensitivity analysis of optimization hyperparameters on the C-ELEGAN dataset. In each plot, non-varied hyperparameters are set to their optimal values from a prior grid search. Results are averaged over 5 runs, and distortion values averaged over 100 root samples per run.

100 randomly sampled root nodes. This procedure was repeated across 5 independent runs and we plot the mean and standard deviation of the ℓ_∞ distortion across these repetitions.

We first study the influence of the regularization coefficient μ on the ℓ_∞ distortion in Figure 10a. One can notice that, for the lowest μ values, the distortion is high, indicating that the solution is insufficiently driven by the data; for the highest values, the distortion is also high, which suggests that the tree-likeness of the solution is lost. For a good tradeoff between proximity to the original metric and tree-likeness of the solution, an optimum is reached. This further showcases the interest of introducing a term related to the Gromov hyperbolicity into the tree metric approximation problem.

We then study the impact of the λ parameter in Figure 10b, for different batch sizes. Results suggest that a good trade-off between large values (that provide an estimate close to the actual value but at the price of sharp optimization landscape) and small ones (that provides a smooth estimate easy to optimize) should be found to reach the best performances.

Finally, in Figure 10c, we study the impact of the batch size m , for various λ parameters. One can note that, for moderate values of λ , an optimal performance is reached for a moderate size of batch size, which suggests that m could be set with a relatively low value, enhancing the ability of DELTAZERO to tackle moderated-size datasets.

E Implementation details

Our optimization procedure was implemented in Python using PyTorch [28] for automatic differentiation and gradient-based optimization. All experiments were run using PyTorch’s GPU backend with CUDA acceleration when available. We leveraged the Adam optimizer and parallelized computations over batches.

The training objective minimizes a regularized combination of the δ -hyperbolicity loss and the reconstruction error of the distance matrix. The optimization is performed over the upper-triangular weights of a complete undirected graph, which are projected onto shortest-path distances using the Floyd–Warshall algorithm at each iteration. This projection is implemented using a GPU-compatible variant of the algorithm.

To efficiently approximate the Gromov hyperbolicity objective, we randomly sample K batches of quadruples and accumulate gradients across 32 sub-batches in each outer iteration. This simulates a larger batch size while remaining within GPU memory constraints. Gradient accumulation is performed manually by computing the backward pass over each sub-batch before the optimizer step.

Training includes early stopping based on the total loss, with a patience of 50 epochs. We store and return the best-performing weights (in terms of the objective loss (6)) encountered during the optimization.

To ensure reproducibility, we provide all code and experiments at <https://github.com/pierrehouedry/DifferentiableHyperbolicity>.

E.1 Hardware setup

All experiments were conducted using a single NVIDIA TITAN RTX GPU with 24 GB of VRAM. The implementation was written in Python 3.11 and executed on a machine running Ubuntu 22.04. No distributed or multi-GPU training was used.

E.2 Datasets

For C-ELEGAN and CS PHD, we relied on the pre-computed distance matrices provided at <https://github.com/rsonthel/TreeRep>. The C-ELEGAN dataset captures the neural connectivity of the *Caenorhabditis elegans* roundworm, a widely studied model organism in neuroscience. The CS PHD dataset represents a co-authorship network among computer science PhD holders, reflecting patterns of academic collaboration. For CORA and AIRPORT, we used the graph data from <https://github.com/HazyResearch/hgcn> and computed the shortest-path distance matrices on the largest connected components. CORA is a citation network of machine learning publications, while AIRPORT models air traffic routes between airports. The WIKI dataset, representing a hyperlink graph between Wikipedia pages, was obtained from the `torch_geometric` library [14]. ZEISEL is a single-cell RNA sequencing dataset describing the transcriptomic profiles of mouse cortex and hippocampus cells, commonly used to study cell type organization in neurobiology. It was obtained from <https://github.com/solevillar/scGeneFit-python>. The IBD dataset (Inflammatory Bowel Disease) contains 396 metagenomic samples across 606 expressed microbial species. The dataset is publicly available via BioProject accession number PRJEB1220.