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Construction of canonical models for K45 and weaker systems

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Abstract

In this paper we explore the construction of canonical models for K45 and weaker systems by extending Wittgenstein’s picture theory of language to a hierarchy of modal logics. After discussing the philosophical implications of revisiting Wittgenstein’s concept of logical space, especially for K45, we turn to the mathematical side, establishing soundness and completeness results for $K4^2!5^-$. These theorems pave the way for the description of canonical models, which we illustrate for the systems K45, K5 and $K4^2!5^-$ in the case where a single proposition is considered. The purpose of these illustrations is to gain insight into these systems through the combinatorial structure of their canonical models.

Keywords: Modal logic, early Wittgenstein, canonical models, locally tabular logics.

1 Introduction

In this paper we propose a way of thinking that departs from Kripke 63 [4] and returns to Wittgenstein’s picture theory of language presented in the *Tractatus logico-philosophicus* (TLP) [10] before extending it to a hierarchy of modal logics. The idea is to set aside R , the accessibility relation of Kripke 63, and go back to the original concept of “description of the world” (Wittgenstein) via that of “complete assignment” (Kripke 59 [3]). Why this? Mainly to revive the key concept of “logical space” (TLP 2.11 and 3.42), which we believe captures the essence of a logical system.

So let’s start our journey with what Wittgenstein calls the description of the world. We quote: “If all true elementary propositions are given, the result is a complete description of the world. The world is completely described by giving all elementary propositions and adding which of them are true and which false” (TLP 4.26). For n elementary propositions there are therefore 2^n truth-possibilities (complete assignments), which constitute what Wittgenstein calls the logical space generated by these propositions. He goes on to introduce “truth tables” by saying that “a proposition is an expression of agreement and disagreement with the truth-possibilities of elementary propositions” (TLP 4.4). In other words, the truth-possibilities of elementary propositions (shown on the left side of the truth table) are the conditions of the truth and falsity of propositions (TLP 4.41). This means that for n elementary propositions there are 2^{2^n} ways in which a proposition can agree and disagree with their truth-possibilities (TLP 4.42).

When the underlying structure of Wittgenstein’s n -propositional truth tables is reformulated in algebraic terms, one obtains the free Boolean algebra with n generators. Wittgenstein’s method thus provides an elegant and powerful way of constructing these algebras.

The aim of this paper is twofold. The first, mathematical, is to extend Wittgenstein's constructive method to the modal field in order to explicitly represent a number of free modal algebras. The second, philosophical, is to gain insight into well-known modal logics, such as K45, by equipping them with a truth-table semantics, underpinned by Wittgenstein's key concept of logical space.

2 Setting the scene

The language we will use, $\mathcal{L}(N)$, is based on n propositional letters p_1, \dots, p_n , respectively denoting the n propositions of a set N . Well-formed formulas, abbreviated wffs, are built (according to standard grammar rules) from propositional letters using seven logical operators: \neg , \wedge , \vee , \rightarrow , \leftrightarrow , \Box and \Diamond . Nesting of modalities is allowed.

As a subset of $\mathcal{L}(N)$, the general normal modal system we will study is $K(N)$ augmented with the following two axioms:¹

$$4^k! \quad \Box^k \varphi \leftrightarrow \Box^{k+1} \varphi \quad (k \geq 1);$$

$$5^- \quad \Diamond \Diamond \varphi \rightarrow \Box \Diamond \varphi.$$

In the rest of the paper we will denote by α the elements of $\mathcal{P}(N)$, the powerset of N , and by S the elements of $\mathcal{P}(\mathcal{P}(N))$, the powerset of $\mathcal{P}(N)$.

Definition 1. A k -state ($k \geq 1$) generated by N is any element of $\mathcal{P}(N) \times [\mathcal{P}(\mathcal{P}(N))]^k$.

Definition 2. The logical space $\mathbb{S}_k(N)$ is the set of k -states generated by N , $(\alpha, S_1, \dots, S_k)$, where $S_i = \emptyset$ implies $S_j = \emptyset$ ($i < j$).

Definition 3. Let $(\alpha, S_1, \dots, S_k)$ be a state of the logical space $\mathbb{S}_k(N)$. The semantics $\mathcal{S}_k(N)$ has the following truth conditions:

1. $(\alpha, S_1, \dots, S_k) \models p_i$ iff $p_i \in \alpha$;
2. $(\alpha, S_1, \dots, S_k) \models \neg \varphi$ iff not $(\alpha, S_1, \dots, S_k) \models \varphi$;
3. $(\alpha, S_1, \dots, S_k) \models \varphi \wedge \psi$ iff both $(\alpha, S_1, \dots, S_k) \models \varphi$ and $(\alpha, S_1, \dots, S_k) \models \psi$;
4. $(\alpha, S_1, \dots, S_k) \models \varphi \vee \psi$ iff $(\alpha, S_1, \dots, S_k) \models \varphi$ or $(\alpha, S_1, \dots, S_k) \models \psi$, or both;
5. $(\alpha, S_1, \dots, S_k) \models \varphi \rightarrow \psi$ iff if $(\alpha, S_1, \dots, S_k) \models \varphi$ then $(\alpha, S_1, \dots, S_k) \models \psi$;
6. $(\alpha, S_1, \dots, S_k) \models \varphi \leftrightarrow \psi$ iff $(\alpha, S_1, \dots, S_k) \models \varphi$ if and only if $(\alpha, S_1, \dots, S_k) \models \psi$;
7. $(\alpha, S_1, S_2, \dots, S_{k-1}, S_k) \models \Box \varphi$ iff for all β in S_1 , $(\beta, S_2, S_3, \dots, S_k, S_k) \models \varphi$;
8. $(\alpha, S_1, S_2, \dots, S_{k-1}, S_k) \models \Diamond \varphi$ iff for some β in S_1 , $(\beta, S_2, S_3, \dots, S_k, S_k) \models \varphi$.

We will explore the system $K4^k!5^-(N)$ and the semantics given in Definition 3 step by step, starting in the next section with the case $k = 1$. This will culminate in a proof that this system is sound and complete with respect to our proposed alternative to Kripke semantics.

¹ As there are no generally accepted names for these two axioms in the literature, we have borrowed freely from the terminology of Chellas 80 [1] to label them as properly as possible.

3 The modal logic K45

When k is set to 1, the system becomes K4!5⁻ with:

- 4! $\Box\varphi \leftrightarrow \Box\Box\varphi$;
- 5⁻ $\Diamond\Diamond\varphi \rightarrow \Box\Diamond\varphi$.

It is not difficult to see that this system is actually K45. In this case, the states of the logical space can be reduced to the form (α, S) .

Just as *all* possible α are admissible in the propositional calculus (TLP 4.27), *all* possible pairs (α, S) are admissible here, which we express by saying that this logic describes the world in a *purely combinatorial way*. We will see in Section 4 that this property does not generalize to $k > 1$.

Definition 4. Let (α, S) be a state of the logical space $\mathbb{S}_1(N)$. The semantics $\mathcal{S}_1(N)$ has the following truth conditions:

1. $(\alpha, S) \models p_i$ iff $p_i \in \alpha$;
2. $(\alpha, S) \models \neg\varphi$ iff not $(\alpha, S) \models \varphi$;
3. $(\alpha, S) \models \varphi \wedge \psi$ iff both $(\alpha, S) \models \varphi$ and $(\alpha, S) \models \psi$;
4. $(\alpha, S) \models \varphi \vee \psi$ iff $(\alpha, S) \models \varphi$ or $(\alpha, S) \models \psi$, or both;
5. $(\alpha, S) \models \varphi \rightarrow \psi$ iff if $(\alpha, S) \models \varphi$ then $(\alpha, S) \models \psi$;
6. $(\alpha, S) \models \varphi \leftrightarrow \psi$ iff $(\alpha, S) \models \varphi$ if and only if $(\alpha, S) \models \psi$;
7. $(\alpha, S) \models \Box\varphi$ iff for all β in S , $(\beta, S) \models \varphi$;
8. $(\alpha, S) \models \Diamond\varphi$ iff for some β in S , $(\beta, S) \models \varphi$.

Before turning to the mathematical aspects of the problem (soundness and completeness will be proved in Section 4), we would like to emphasize the philosophical importance of presenting semantics in this way.

3.1 Philosophical considerations

We begin by asking what it would mean to extend Wittgenstein's picture theory of language to deontic logic. Imagine that this theory could be adapted to represent the normative landscape. In this case, a normative proposition (e.g. "It is obligatory to do A ") could be understood as a "picture" of a normative state, where the proposition would correspond to a situation framed in terms of what ought to be done. Just as a descriptive proposition represents a fact about the world, a normative proposition could represent a fact about a normative reality or system.

Actually, this is exactly what the above semantics does! The idea of logical form, as found in the *Tractatus*, is extended here to formalize the structure of normative propositions. In the same way that Wittgenstein shows how propositions are syntactically articulated and how they "fill" the logical space, we have developed a framework for dealing with deontic modalities, such as obligation, permission and prohibition. Basically, this means that we have successfully integrated normative reality into the description of the world. But how?

Just as i , the imaginary unit satisfying $i^2 = -1$, is the key to extending \mathbb{R} to \mathbb{C} , we assume that a deontic state is nothing but a set – empty or not – of alethic states. In other words, what is permitted is a set of actions, provided that actions are viewed as subsets of the powerset $\mathcal{P}(N)$, where N is the set of elementary propositions. Thus, while a complex number can be represented as a pair (a, b) composed of a real and an imaginary part, here an alethico-deontic state is defined as a pair (α, S) composed of an alethic and a deontic state. These alethico-deontic states form the logical space $\mathbb{S}_1(N)$, which underpins the semantics proposed for this logic in Definition 4.

Because of its constructive and holistic character (considering the entire logical space at once), this approach allows for a deeper understanding of deontic logic than simply stating that K45 is sound and complete with respect to the class of transitive and Euclidean frames. In our view, it is the essence of the modal system K45 that is captured by the logical space $\mathbb{S}_1(N)$.

In addition to the author’s earlier work in deontic logic (see Sart 09 [8] and Sart 00 [7]), this simplified semantics, rooted in the principles of the *Tractatus*, also appears on several occasions in epistemic logic. Levesque 90 [5] and Halpern-Moses 85 [2] are emblematic papers from this period. Going further back in time, we come across Kripke 59, who imposes $\alpha \in S$ to be complete with respect to S5 (= KT45). But the pioneer of this kind of semantics seems to be von Wright 51 [9], although his decision procedure remains partly axiomatic.

3.2 Example

To make things even clearer, here is an example of what these alethico-deontic states look like. Let p denote the proposition “I close the door”.

This proposition generates two alethic states, namely:

- (α_1) it is the case that p (i.e. I close the door);
- (α_2) it is not the case that p (i.e. I leave the door open);

and four deontic states, namely:

- (S_1) both α_1 and α_2 are permitted;
- (S_2) α_1 is permitted and α_2 is forbidden;
- (S_3) α_1 is forbidden and α_2 is permitted;
- (S_4) both α_1 and α_2 are forbidden.

From this, eight alethico-deontic states emerge.

The first one is:

1. (α_1, S_1) both α_1 and α_2 are permitted and p is the case.

The last two ones are:

7. (α_1, S_4) both α_1 and α_2 are forbidden and p is the case;
8. (α_2, S_4) both α_1 and α_2 are forbidden and p is not the case.

Interestingly, states 7 and 8, where $S = \emptyset$, depict a dilemma, i.e. situations where everything is forbidden and nothing is permitted, which is by no means ethically impossible.

3.3 Limits of language

To conclude this section, we would like to return to Wittgenstein's idea of the limits of language, summed up in his famous dictum: "What we cannot speak about, we must pass over in silence" (TLP 7). One of the consequences of extending his picture theory of language to deontic logic is that, within the boundaries of what can be said, one can make meaningful use of deontic modalities. This is somewhat ironic in the light of TLP 6.421: "It is clear that ethics cannot be put into words", but that's not the point. What really matters is this: just as the meaning of statements about the world lies in their ability to reflect reality, normative statements – such as moral or legal imperatives – derive their meaning from normative reality, i.e. from an underlying structure of norms that define what is considered obligatory or permitted in a given context.

4 An intermediate system

When k is set to 2, the system becomes $K4^2!5^-$ with:

$$\begin{aligned} 4^2! \quad & \Box\Box\varphi \leftrightarrow \Box\Box\Box\varphi; \\ 5^- \quad & \Diamond\Diamond\varphi \rightarrow \Box\Diamond\varphi. \end{aligned}$$

This system occupies a unique position in the hierarchy of those considered here, being general enough to prove completeness without falling into the specifics of $K45$ ($k = 1$) and particular enough not to get bogged down in tedious calculations ($k \geq 3$).

Significantly, for $k = 2$, the states of the logical space can be expressed in the form (α, S, T) , where $S = \emptyset$ implies $T = \emptyset$. Note that this means that some $S - T$ combinations are excluded!

Definition 5. *Let (α, S, T) be a state of the logical space $\mathbb{S}_2(N)$. The semantics $\mathcal{S}_2(N)$ has the following truth conditions:*

1. $(\alpha, S, T) \models p_i$ iff $p_i \in \alpha$;
2. $(\alpha, S, T) \models \neg\varphi$ iff not $(\alpha, S, T) \models \varphi$;
3. $(\alpha, S, T) \models \varphi \wedge \psi$ iff both $(\alpha, S, T) \models \varphi$ and $(\alpha, S, T) \models \psi$;
4. $(\alpha, S, T) \models \varphi \vee \psi$ iff $(\alpha, S, T) \models \varphi$ or $(\alpha, S, T) \models \psi$, or both;
5. $(\alpha, S, T) \models \varphi \rightarrow \psi$ iff if $(\alpha, S, T) \models \varphi$ then $(\alpha, S, T) \models \psi$;
6. $(\alpha, S, T) \models \varphi \leftrightarrow \psi$ iff $(\alpha, S, T) \models \varphi$ if and only if $(\alpha, S, T) \models \psi$;
7. $(\alpha, S, T) \models \Box\varphi$ iff for all β in S , $(\beta, T, T) \models \varphi$;
8. $(\alpha, S, T) \models \Diamond\varphi$ iff for some β in S , $(\beta, T, T) \models \varphi$.

We can now move on to the two central theorems of this paper, namely soundness and completeness.

4.1 Soundness

Theorem 1. *Every theorem of $K4^2!5^-(N)$ is a tautology on the logical space $\mathbb{S}_2(N)$.*

Proof. The only two cases really worth proving concern axioms $4^2!$ and 5^- .

AXIOM 4^{2!}.

Assume that $(\alpha, S, T) \models \Box^2\varphi$.

By the definition of $\models \Box$, for all β in S , $(\beta, T, T) \models \Box\varphi$ or equivalently:

- for all β in S and all γ in T , $(\gamma, T, T) \models \varphi$ – definition of $\models \Box$;
- for all β in S and all γ in T , $(\gamma, T, T) \models \Box\varphi$ – definition of $\models \Box$;
- for all β in S , $(\beta, T, T) \models \Box^2\varphi$ – definition of $\models \Box$;
- $(\alpha, S, T) \models \Box^3\varphi$ – definition of $\models \Box$.

AXIOM 5⁻.

Assume that $(\alpha, S, T) \models \Diamond\Diamond\varphi$.

By the definition of $\models \Diamond$, for some β in S , $(\beta, T, T) \models \Diamond\varphi$ or equivalently:

- for some β in S and some γ in T , $(\gamma, T, T) \models \varphi$ – definition of $\models \Diamond$.

Hence for all β in S and some γ in T , $(\gamma, T, T) \models \varphi$ or equivalently:

- for all β in S , $(\beta, T, T) \models \Diamond\varphi$ – definition of $\models \Diamond$;
- $(\alpha, S, T) \models \Box\Diamond\varphi$ – definition of $\models \Box$. ■

4.2 Completeness

Theorem 2. *Every tautology on the logical space $\mathbb{S}_2(N)$ is a theorem of $K4^2!5^-(N)$.*

Proof. We will prove this theorem using a Kalmár-style method. To this end, we introduce the formulas $p(\alpha, S, T)$, which will play the same role as the disjuncts of the full disjunctive normal form in the propositional calculus:

$$p(\alpha, S, T) \stackrel{\text{def}}{=} p_\alpha \wedge \bigwedge_{\beta \in S} \Diamond p_\beta \wedge \bigwedge_{\beta \notin S} \neg \Diamond p_\beta \wedge \bigwedge_{\gamma \in T} \Diamond\Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond\Diamond p_\gamma, \text{ with}$$

$$p_\alpha \stackrel{\text{def}}{=} \bigwedge_{p_i \in \alpha} p_i \wedge \bigwedge_{p_i \notin \alpha} \neg p_i.^2$$

Lemma 1. *If $(\alpha, S, T) \models \varphi$, then $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \varphi$. Else, $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \neg\varphi$.*

Proof (by induction on φ). The only two cases really worth proving concern the operator \Box .

CASE A.

Let (α, S, T) be a state in $\mathbb{S}_2(N)$ such that $(\alpha, S, T) \models \Box\varphi$.

This means that for all β in S , $(\beta, T, T) \models \varphi$ – definition of $\models \Box$.

Thus for all β in S , $K4^2!5^-(N) \vdash p(\beta, T, T) \rightarrow \varphi$ – induction hypothesis.

Equivalently, $K4^2!5^-(N) \vdash \bigvee_{\beta \in S} p(\beta, T, T) \rightarrow \varphi$ – propositional calculus.

Hence $K4^2!5^-(N) \vdash \Box \bigvee_{\beta \in S} p(\beta, T, T) \rightarrow \Box\varphi$ – $(\varphi \rightarrow \psi) \vdash (\Box\varphi \rightarrow \Box\psi)$ (1).

Now, $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Box \bigvee_{\beta \in S} p(\beta, T, T)$ – Lemma 3 in the Appendix (2).

Combining (1) and (2), $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Box\varphi$.

² The p_α are precisely the disjuncts of the full disjunctive normal form in the propositional calculus.

CASE B.

Let (α, S, T) be a state in $\mathbb{S}_2(N)$ such that $(\alpha, S, T) \not\models \Box\varphi$.

This means that for some β in S , $(\beta, T, T) \not\models \varphi$

– definition of $\models \Box$.

Thus for some β in S , $K4^2!5^-(N) \vdash p(\beta, T, T) \rightarrow \neg\varphi$

– induction hypothesis.

Hence for some β in S , $K4^2!5^-(N) \vdash \Diamond p(\beta, T, T) \rightarrow \Diamond\neg\varphi$

– $(\varphi \rightarrow \psi) \vdash (\Diamond\varphi \rightarrow \Diamond\psi)$ (1).

Now, for all β in S , $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Diamond p(\beta, T, T)$

– Lemma 4 in the Appendix (2).

Combining (1) and (2), $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Diamond\neg\varphi$.

In other words, $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \neg\Box\neg\varphi$

– $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$. ■

Lemma 2. $K4^2!5^-(N) \vdash \bigvee_{\mathbb{S}_2(N)} p(\alpha, S, T)$.

Proof. If all states were taken into account, one could easily conclude by a purely combinatorial argument. However, this is not the case since some states (α^*, S^*, T^*) are excluded from the logical space ($S = \emptyset$ implies $T = \emptyset$). Fortunately, this is not a problem since for them $K4^2!5^-(N) \vdash p(\alpha^*, S^*, T^*) \rightarrow \perp$ (see Lemma 5 in the Appendix). ■

Note that if we consider extensions of this system, such as K5 or K45, other excluded states arise, thereby reducing the size of the corresponding logical space. For example, for K5 we have the condition $S \subseteq T$ and for K45 it is $S = T$.³ Nevertheless, what has just been said about $K4^2!5^-$ regarding excluded states also holds for these two systems (see Lemmas 6 and 7 in the Appendix), so Lemma 2 holds for them as well!

We can now proceed with the proof of Theorem 2 itself.

Let τ be a tautology on the logical space $\mathbb{S}_2(N)$.

By the definition of τ , for all (α, S, T) in $\mathbb{S}_2(N)$, $(\alpha, S, T) \models \tau$.

Thus for all (α, S, T) in $\mathbb{S}_2(N)$, $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \tau$

– Lemma 1.

Equivalently, $K4^2!5^-(N) \vdash \bigvee_{\mathbb{S}_2(N)} p(\alpha, S, T) \rightarrow \tau$

– propositional calculus.

Hence $K4^2!5^-(N) \vdash \tau$

– Lemma 2. ■

4.3 Algebraic considerations

It follows from the soundness and completeness results that the algebra obtained from the classes of equivalent wffs on the logical space $\mathbb{S}_2(N)$ is the Lindenbaum-Tarski algebra of $K4^2!5^-(N)$, i.e. the free modal algebra of $K4^2!5^-(N)$. This means that the states of the logical space $\mathbb{S}_2(N)$ correspond bijectively to the atoms of the free modal algebra of $K4^2!5^-(N)$. Therefore, by Jónsson-Tarski duality, the logical space $\mathbb{S}_2(N)$ equipped with the semantics given in Definition 5 is isomorphic to the canonical model for $K4^2!5^-(N)$. Our approach thus provides an effective way of representing this model, in stark contrast to the non-constructive definition of canonical models (in terms of ultrafilters).

³ The condition $S \subseteq T$ seems to appear for the first time in Nagle 81 [6].

4.4 Illustration

One of the aims of this paper was to provide an explicit representation of free modal algebras or, what amounts to the same thing, logical spaces or canonical models for the corresponding logics. Well, here we are! We will describe the canonical models for the main modal systems we have discussed, in the case where a single proposition is considered.

K45. In the example given in Section 3, the structure of the logical space $\mathbb{S}_1(p)$ is revealed. As we have just said, it is identical to that of the canonical model for K45 generated by a single proposition. Each set S in the example translates into a connected set of reachable nodes in Figure 1 below. Specifically, S_1 maps to the pair $\{1, 2\}$, S_2 maps to the singleton $\{3\}$, S_3 maps to the singleton $\{6\}$ and S_4 maps to the empty set \emptyset .

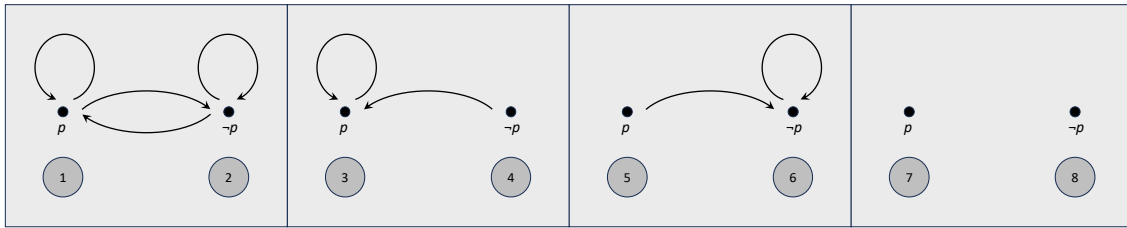


Figure 1. The 8 elements of the canonical model for $K45(\{p\})$.⁴

K5. By dropping axiom 4, a second set T comes into play, with the condition $S \subseteq T$. This leads to two additional possibilities compared to K45. One is $S \cong \{1\}$ and $T \cong \{1, 2\}$. The other is $S \cong \{2\}$ and $T \cong \{1, 2\}$. It follows that the canonical model for K5 generated by a single proposition has 12 elements, the eight for K45 plus four new elements.

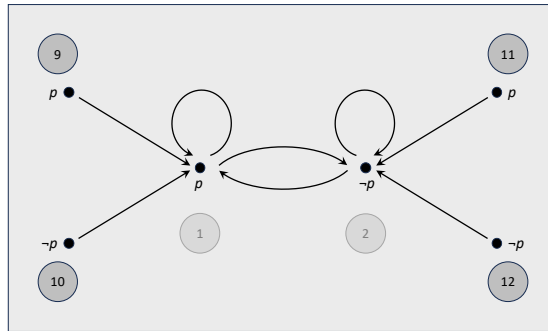


Figure 2. The 4 additional elements compared to K45 (labeled 9 to 12).

It is easy to check that none of the states 9 to 12 satisfy axiom 4. For example, we see that $\Box p$ is true in node 9, but not $\Box\Box p$ (since p is not true in node 2).

K4²!5⁻. Here, S becomes arbitrary with respect to T (except that if S is empty, so is T). This gives rise to several new $S - T$ combinations, from which new states are derived (see Table 1 below). In the end, the canonical model for K4²!5⁻ generated by a single proposition has 26 elements, those for K5 plus fourteen new elements.

⁴ The arrows represent the accessibility relation between the worlds of the canonical model.

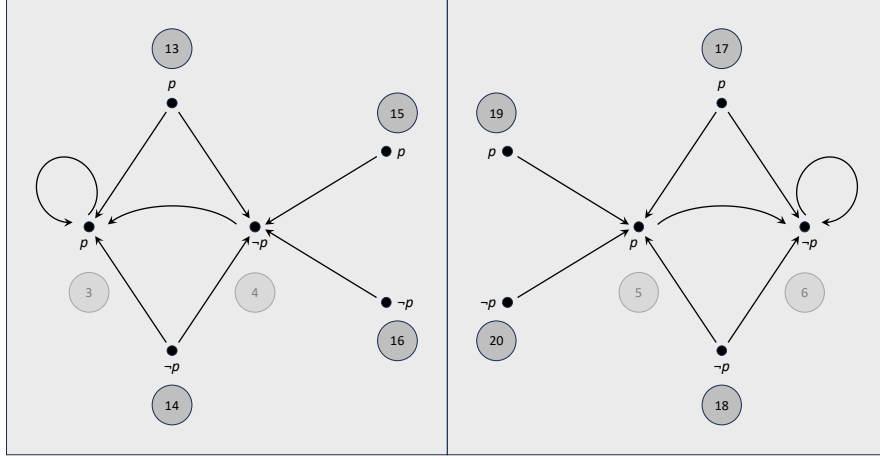


Figure 3. The 8 additional elements compared to K5 for which $T \neq \emptyset$ (labeled 13 to 20).

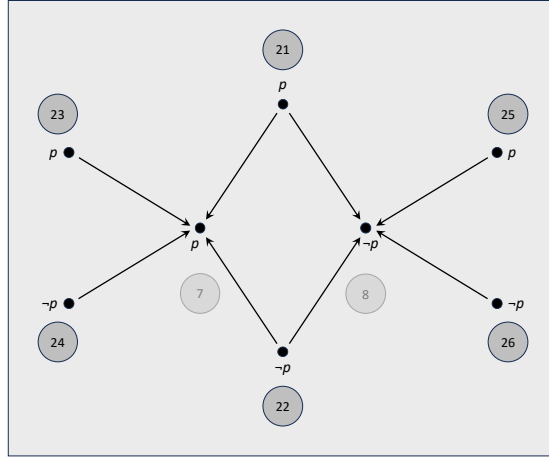


Figure 4. The 6 additional elements compared to K5 for which $T = \emptyset$ (labeled 21 to 26).

It is easy to check that none of the states 13 to 26 satisfy axiom 5. For example, we see that $\Diamond p$ is true in node 20, but not $\Box \Diamond p$ (since p is not true in node 6).

We conclude this section with a summary table of the different $S - T$ combinations involved in the above figures. The states specific to K45 (resp. K5) are highlighted in dark gray (resp. light gray).

State	S	T	State	S	T	State	S	T	State	S	T
{1, 2}	{1, 2}	{1, 2}	{9, 10}	{1}	{1, 2}	{11, 12}	{2}	{1, 2}	Does not exist since $S = \emptyset$ implies $T = \emptyset$		
{13, 14}	{3, 4}	{3}	{3, 4}	{3}	{3}	{15, 16}	{4}	{3}			
{17, 18}	{5, 6}	{6}	{19, 20}	{5}	{6}	{5, 6}	{6}	{6}			
{21, 22}	{7, 8}	\emptyset	{23, 24}	{7}	\emptyset	{25, 26}	{8}	\emptyset	{7, 8}	\emptyset	\emptyset

Table 1. Summary of admissible (non-excluded) $S - T$ combinations.

5 The general case

It is fairly easy to extend the results obtained for $K4^2!5^-(N)$ to $K4^k!5^-(N)$. The proofs, similar to those given in Section 4, do not present any additional pitfalls, except that they require a more careful handling of the notation used.

Theorem 3. *Every theorem of $K4^k!5^-(N)$ is a tautology on the logical space $\mathbb{S}_k(N)$ and vice versa.*

6 Conclusion

In this paper we have explored the construction of canonical models for K45 and weaker systems by extending Wittgenstein's picture theory of language to a hierarchy of modal logics. After discussing the philosophical implications of revisiting Wittgenstein's concept of logical space, especially for K45, we turned to the mathematical side, establishing soundness and completeness results for $K4^2!5^-$. These theorems pave the way for the description of canonical models, which we have illustrated for the systems K45, K5 and $K4^2!5^-$ in the case where a single proposition is considered. The purpose of these illustrations is to gain insight into these systems through the combinatorial structure of their canonical models. Overall, the paper offers a novel perspective on a range of modal logics, reflecting the intersection of philosophical inquiry and mathematical rigor in the field.

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Appendix

For the sake of completeness, we give proofs of the following five technical results, which deal with deduction in modal systems.

Lemma 3. $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Box \bigvee_{\beta \in S} p(\beta, T, T)$.

Proof.

First, $p(\alpha, S, T) \stackrel{\text{def}}{=} p_\alpha \wedge \bigwedge_{\beta \in S} \Diamond p_\beta \wedge \bigwedge_{\beta \notin S} \neg \Diamond p_\beta \wedge \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma$.

Hence $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow (\bigwedge_{\beta \notin S} \neg \Diamond p_\beta \wedge \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma)$
– (1).

On the one hand,

$$\begin{aligned} \bigwedge_{\beta \notin S} \neg \Diamond p_\beta &\rightarrow \neg (\bigvee_{\beta \notin S} \Diamond p_\beta) && \text{– De Morgan's laws;} \\ &\rightarrow \neg \Diamond (\bigvee_{\beta \notin S} p_\beta) && \text{– } \Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond \varphi \vee \Diamond \psi); \\ &\rightarrow \neg \Diamond \neg (\bigvee_{\beta \in S} p_\beta) && \text{– definition of } p_\beta; \\ &\rightarrow \Box (\bigvee_{\beta \in S} p_\beta) && \text{– } \Diamond \varphi \leftrightarrow \neg \Box \neg \varphi \text{ (2).} \end{aligned}$$

On the other hand,

$$\begin{aligned} \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma &\rightarrow \bigwedge_{\gamma \in T} \Box \Diamond p_\gamma && \text{– axiom } 5^- \text{ (3);} \\ \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma &\rightarrow \bigwedge_{\gamma \notin T} \Box \neg \Diamond p_\gamma && \text{– } \Diamond \varphi \leftrightarrow \neg \Box \neg \varphi \text{ (4);} \\ \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma &\rightarrow \bigwedge_{\gamma \in T} \Box \Diamond \Diamond p_\gamma && \text{– axioms } 4^2! \text{ and } 5^- \text{ (5);} \\ \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma &\rightarrow \bigwedge_{\gamma \notin T} \Box \neg \Diamond \Diamond p_\gamma && \text{– axiom } 4^2! \text{ and } \Diamond \varphi \leftrightarrow \neg \Box \neg \varphi \text{ (6).} \end{aligned}$$

Combining (1) to (6), $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow B$, where:

$$B = \Box (\bigvee_{\beta \in S} p_\beta) \wedge \bigwedge_{\gamma \in T} \Box \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \Box \neg \Diamond p_\gamma \wedge \bigwedge_{\gamma \in T} \Box \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \Box \neg \Diamond \Diamond p_\gamma.$$

Since $\Box(\varphi \wedge \psi) \leftrightarrow (\Box \varphi \wedge \Box \psi)$, $K4^2!5^-(N) \vdash B \rightarrow C$, where:

$$C = \Box \left((\bigvee_{\beta \in S} p_\beta) \wedge \bigwedge_{\gamma \in T} \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond p_\gamma \wedge \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma \right).$$

By distributivity, C is equivalent to D , where:

$$D = \Box \left(\bigvee_{\beta \in S} (p_\beta \wedge \bigwedge_{\gamma \in T} \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond p_\gamma \wedge \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma) \right).$$

Rewriting D as $\Box \bigvee_{\beta \in S} p(\beta, T, T)$, it follows by transitivity that

$$K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Box \bigvee_{\beta \in S} p(\beta, T, T). \quad \blacksquare$$

Lemma 4. For all β in S , $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow \Diamond p(\beta, T, T)$.

Proof.

First, $p(\alpha, S, T) \stackrel{\text{def}}{=} p_\alpha \wedge \bigwedge_{\beta \in S} \Diamond p_\beta \wedge \bigwedge_{\beta \notin S} \neg \Diamond p_\beta \wedge \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma$.

Hence for all β in S , $K4^2!5^-(N) \vdash p(\alpha, S, T) \rightarrow (\Diamond p_\beta \wedge \bigwedge_{\gamma \in T} \Diamond \Diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \Diamond \Diamond p_\gamma)$
– (1).

Similarly to the previous lemma,

$$\begin{aligned}
\bigwedge_{\gamma \in T} \diamond \diamond p_\gamma &\rightarrow \bigwedge_{\gamma \in T} \Box \diamond p_\gamma && \text{-- axiom } 5^- \text{ (2);} \\
\bigwedge_{\gamma \notin T} \neg \diamond \diamond p_\gamma &\rightarrow \bigwedge_{\gamma \notin T} \Box \neg \diamond p_\gamma && \text{-- } \diamond \varphi \leftrightarrow \neg \Box \neg \varphi \text{ (3);} \\
\bigwedge_{\gamma \in T} \diamond \diamond p_\gamma &\rightarrow \bigwedge_{\gamma \in T} \Box \diamond \diamond p_\gamma && \text{-- axioms } 4^{2!} \text{ and } 5^- \text{ (4);} \\
\bigwedge_{\gamma \notin T} \neg \diamond \diamond p_\gamma &\rightarrow \bigwedge_{\gamma \notin T} \Box \neg \diamond \diamond p_\gamma && \text{-- axiom } 4^{2!} \text{ and } \diamond \varphi \leftrightarrow \neg \Box \neg \varphi \text{ (5).}
\end{aligned}$$

Combining (1) to (5), for all β in S , $K4^{2!}5^-(N) \vdash p(\alpha, S, T) \rightarrow B$, where:

$$B = \diamond p_\beta \wedge \bigwedge_{\gamma \in T} \Box \diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \Box \neg \diamond p_\gamma \wedge \bigwedge_{\gamma \in T} \Box \diamond \diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \Box \neg \diamond \diamond p_\gamma.$$

Since $(\Box \varphi \wedge \diamond \psi) \rightarrow \diamond(\varphi \wedge \psi)$, for all β in S , $K4^{2!}5^-(N) \vdash B \rightarrow C$, where:

$$C = \diamond(p_\beta \wedge \bigwedge_{\gamma \in T} \diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \diamond p_\gamma \wedge \bigwedge_{\gamma \in T} \diamond \diamond p_\gamma \wedge \bigwedge_{\gamma \notin T} \neg \diamond \diamond p_\gamma).$$

Rewriting C as $\diamond p(\beta, T, T)$, it follows by transitivity that

$$\text{for all } \beta \text{ in } S, K4^{2!}5^-(N) \vdash p(\alpha, S, T) \rightarrow \diamond p(\beta, T, T). \quad \blacksquare$$

Lemma 5. *If $S = \emptyset$ and $T \neq \emptyset$, then $K4^{2!}5^-(N) \vdash p(\alpha, S, T) \rightarrow \perp$.*

Proof.

On the one hand,

$$\begin{aligned}
p(\alpha, S, T) &\rightarrow \bigwedge_{\beta \notin S} \neg \diamond p_\beta && \text{-- definition of } p(\alpha, S, T); \\
&\rightarrow \bigwedge_{\beta \notin S} \Box \neg p_\beta && \text{-- } \diamond \varphi \leftrightarrow \neg \Box \neg \varphi; \\
&\rightarrow \Box(\bigwedge_{\beta \notin S} \neg p_\beta) && \text{-- } \Box(\varphi \wedge \psi) \leftrightarrow (\Box \varphi \wedge \Box \psi); \\
&\rightarrow \Box \neg(\bigvee_{\beta \notin S} p_\beta) && \text{-- De Morgan's laws;} \\
&\rightarrow \Box \neg \top && \text{-- } S = \emptyset; \\
&\rightarrow \Box \perp && \text{-- propositional calculus (1).}
\end{aligned}$$

On the other hand, as $T \neq \emptyset$ by assumption, there is $\gamma \in T$ such that

$$p(\alpha, S, T) \rightarrow \diamond \diamond p_\gamma \quad \text{-- definition of } p(\alpha, S, T) \text{ (2).}$$

Therefore,

$$\begin{aligned}
p(\alpha, S, T) &\rightarrow (\Box \perp \wedge \diamond \diamond p_\gamma) && \text{-- (1) and (2);} \\
&\rightarrow \diamond(\perp \wedge \diamond p_\gamma) && \text{-- } (\Box \varphi \wedge \diamond \psi) \rightarrow \diamond(\varphi \wedge \psi); \\
&\rightarrow \neg \neg \diamond \perp && \text{-- propositional calculus;} \\
&\rightarrow \neg \top && \text{-- } \neg \diamond \perp; \\
&\rightarrow \perp && \text{-- propositional calculus.} \quad \blacksquare
\end{aligned}$$

Lemma 6. *If $S \not\subseteq T$, then $K5(N) \vdash p(\alpha, S, T) \rightarrow \perp$.*

Proof. Assume that $\beta \in S$ and $\beta \notin T$.

$p(\alpha, S, T) \rightarrow \Diamond p_\beta$ and $p(\alpha, S, T) \rightarrow \neg\Diamond\Diamond p_\beta$	– definition of $p(\alpha, S, T)$;
$\rightarrow (\Diamond p_\beta \wedge \neg\Diamond\Diamond p_\beta)$	– propositional calculus;
$\rightarrow (\Diamond p_\beta \wedge \Box\Box\neg p_\beta)$	– $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$;
$\rightarrow \Diamond(p_\beta \wedge \Box\neg p_\beta)$	– $(\Box\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\varphi \wedge \psi)$;
$\rightarrow (\Diamond p_\beta \wedge \Diamond\Box\neg p_\beta)$	– $\Diamond(\varphi \wedge \psi) \rightarrow (\Diamond\varphi \wedge \Diamond\psi)$;
$\rightarrow (\Diamond p_\beta \wedge \neg\Box\Diamond p_\beta)$	– $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$;
$\rightarrow \neg(\Diamond p_\beta \rightarrow \Box\Diamond p_\beta)$	– propositional calculus;
$\rightarrow \neg\top$	– axiom 5;
$\rightarrow \perp$	– propositional calculus. ■

Lemma 7. *If $S \neq T$, then $K45(N) \vdash p(\alpha, S, T) \rightarrow \perp$.*

Proof. By the previous lemma, it suffices to assume that $\beta \notin S$ and $\beta \in T$.

$p(\alpha, S, T) \rightarrow \neg\Diamond p_\beta$ and $p(\alpha, S, T) \rightarrow \Diamond\Diamond p_\beta$	– definition of $p(\alpha, S, T)$;
$\rightarrow \neg(\Diamond\Diamond p_\beta \rightarrow \Diamond p_\beta)$	– propositional calculus;
$\rightarrow \neg\top$	– axiom 4;
$\rightarrow \perp$	– propositional calculus. ■