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# A Robust Dead-Time Compensation Scheme for Lipschitz Nonlinear Systems

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**Abstract:** The presence of delays is common in various real-life dynamic systems and can lead to several complications in terms of control design. In order to address the associated problems, dead-time compensators (DTC) structures are investigated herein. In particular, we propose an Observer-Predictor (O&P) structure, for systems whose variables are not measurable: an **observer** estimates the inaccessible variables, while a **predictor** compensates for the delay based on these estimates. For the case of generic nonlinear systems, applying such a strategy is more challenging, since ensuring stability and performance depends on specific characteristic of each nonlinearity. In this work, we benefit from Lipschitz conditions to develop a robust O&P based on non conservative uncertainty propagation sets. Accordingly, we demonstrate closed-loop stability with our scheme, which is also tested by means of numerical simulation.

*Keywords:* Dead-time compensator; Robust sets; Smith Predictor; Observer-Predictor; Lipschitz Nonlinearities.

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## 1. INTRODUCTION

Since the early 60's, the control of systems subject to time-delay has been investigated. In practice, delays make synthesis quite challenging using conventional techniques - especially when performance requirements are strict. As highlighted by Palmor (1996) and Lima et al. (2015), three primary difficulties arise when dealing with delays: (i) the effects of disturbances are not perceived until the delay has elapsed; (ii) the effect of the control action takes some time to be observed in the controlled variable; (iii) the control action, based on the sampled error signal, attempts to correct a situation that originated some time earlier.

Among the aforementioned body of research, dead-time compensators (DTC) represent a particularly convenient solution - the first DTC is the well-known **Smith Predictor** (SP). Predictors/DTCs use the information available at a given instant to estimate the corresponding output response after the delay, making use of the post-delay prediction error in feedback. In linear systems, the study of predictors is very well-established, as demonstrated by, e.g., Normey-Rico and Camacho (2007) and Santos et al. (2013). In the nonlinear setting - even though sufficient performances can commonly be achieved using local approaches, cf. (Pagano et al., 2001) -, robustness analyses cannot be concretely performed. Furthermore, when performance is critical, the typical SP structure is often insufficient (Haseltine and Rawlings, 2005). Consequently, nonlinear DTCs structures have also been recently investigated, where nonlinearities are accounted for better closed-loop responses (Lima et al., 2015; Morato and Normey-Rico, 2021).

Despite numerous studies on nonlinear predictors guaranteeing improved closed-loop response, robustness and stability studies of nonlinear predictors are more challenging to conduct. Additionally, applying all these concepts to address more general nonlinear settings is indeed a difficult task. Thus, the application of predictors in Lipschitz systems emerged as a proposal to address a wider range of applications in nonlinear systems (Ibrir et al., 2005). Lipschitz nonlinear systems have attracted considerable interest in the control field over the past decades due to their ability to represent, at least locally, a large portion of nonlinear systems (Gao, 2015). Indeed, recent works have discussed how to apply predictors and predictor-based approaches for delay compensation for Lipschitz nonlinear plants (Zuo et al., 2016; Chu et al., 2019).

In practice, the implementation of DTCs often require an observer working in parallel (Mirkin and Raskin, 2003), due to the unavailability of direct access to the system state values, for instance. Accordingly, the so-called *Observer-Predictor* (O&P) structure is typically used (Kleinman, 1969). O&P schemes involve an observer to estimate inaccessible variables, which is directly coupled to a predictor to generate future output predictions (for feedback control). W.r.t. these DTCs, we emphasise that nonlinear observers can be designed using several well-established approaches. Most of these are extensions or generalisations of the *Luenberger* (unknown-input) observer (Luenberger, 1971), such as the Extended Kalman Filter (EKF), among others (Misawa and Hedrick, 1989). However, these techniques are valid only within the neighbourhood of a given operating point and often require heavy computation, especially in the context of real-time

environments (Raghavan and Hedrick, 1994). Thus, several works have proposed Luenberger variations for the Lipschitz nonlinearities, thereby ensuring robustness and implementation ease, cf. (Ibrir et al., 2005; Zemouche and Boutayeb, 2006; Phanomchoeng and Rajamani, 2010).

Besides guaranteeing a functional (internally stable) observer, in the context of generating accurate predictions for delay compensation, the DTC design procedure should also account for the observation error - which is propagated along the predictions and may thus compromise performance. In the linear time-invariant setting, Santos et al. (2013) demonstrate how to synthesise robust uncertainty propagation sets to address these problems. However, for the nonlinear scenario, this topic is not established, up to our best knowledge.

Taking into account the previous discussions, we develop a novel robust O&P DTC scheme for Lipschitz nonlinear systems based on robust uncertainty propagation sets. In particular, we address the O&P synthesis problem extending previous results from Santos et al. (2013) and Ibrir et al. (2005). Accordingly, the **main contributions** of this paper are:

- (1) An O&P DTC structure for Lipschitz nonlinear systems is proposed;
- (2) Novel non-conservative prediction error propagation sets are derived. Furthermore, corresponding closed-loop (robust) stability certificates are developed;
- (3) Simulation results are presented to demonstrate the effectiveness of the scheme in handling Lipschitz nonlinear systems subject to measurement delays.

Our method focuses on nonlinear systems subject to output dead-time. Measurement delays are quite common in practice due to, e.g., sensing issues. Anyhow, we also comment on how the method can be changed to address the case of input delays.

We emphasise that, while Santos et al. (2013) address the robust O&P design problem for linear systems, we handle the more general nonlinear setting in this work - considering that a Lipschitz condition is satisfied. In particular, in order to adapt the O&P to the nonlinear context, we apply the Lipschitz Luenberger observer design detailed in (Ibrir et al., 2005), considering their synthesis by means of Linear Matrix Inequalities (LMIs).

**Organisation:** In Sec. 2, we recap preliminary concepts (model, DTC schemes, etc); In Sec. 3, we detail the proposed scheme; In Sec. 4, we present several simulation results in order to demonstrate the effectiveness of proposed methodology. General conclusions are drawn in Sec. 5.

**Notation.** The predicted value of a given variable  $v(k)$  at time instant  $k + i$ , computed based on the information available at instant  $k$ , is denoted as  $v(k + i | k)$ ; in particular,  $v(k | k) = v(k)$ ; given two sets  $\mathbb{U} \subset \mathbb{R}^m$  and  $\mathbb{V} \subset \mathbb{R}^m$ , the Minkowski sum of this two sets is defined by  $\mathbb{U} \oplus \mathbb{V} \triangleq \{u + v | u \in \mathbb{U}, v \in \mathbb{V}\}$ ; the Pontryagin difference (or Minkowski difference) of this two sets is defined by  $\mathbb{U} \ominus \mathbb{V} \triangleq \{u | u \oplus \mathbb{V} \subset \mathbb{U}, v \in \mathbb{V}\}$ ; given a matrix  $A \in \mathbb{R}^{n \times m}$ ,  $A\mathbb{U} = \{Au | u \in \mathbb{U}\}$ ; the convex hull formed by matrices  $A_1, \dots, A_n$  is given by  $\text{co}\{A_1, \dots, A_n\}$ .

*Definition 1.* (Positively Invariant Set (Blanchini, 1999)). The set  $\mathcal{R}_x \subset \mathbb{R}^{n_x}$  is said positively invariant for a nonlinear system described by  $x(k + 1) = f(x(k))$  if, for all  $x(0) \in \mathcal{R}_x$ , the solution  $x(k) \in \mathcal{R}_x, \forall k \geq 0$ .

*Definition 2.* (Robust Positively Invariant (RPI) set (Rakovic et al., 2005)). The set  $\Omega \subset \mathbb{R}^{n_x}$  is said an RPI set of  $x(k + 1) = f(x(k), w(k))$ , if  $f \in \Omega$  for all  $x \in \Omega$  and  $w \in \mathbb{W}$ .

## 2. PRELIMINARIES

Along the sequel, we consider discrete-time nonlinear system subject to constant process delays, as represented by the following state-space structure:

$$\begin{cases} x(k + 1) = Ax(k) + Bu(k) + \phi(x(k)), \\ y(k) = Cx(k - d), \end{cases} \quad (1)$$

where  $x \in \mathcal{X}^1 \subset \mathbb{R}^{n_x}$  is the state vector,  $y \in \mathcal{Y} \subset \mathbb{R}^{n_y}$  is the output vector,  $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$  is the input vector,  $\phi(x(k))$  is a nonlinear function, and  $d$  represents the process delay (in number of discrete-time samples). We stress that the dead time is present in the system's measurement (i.e., it is an output dead time).

*Assumption 1.* The nonlinear function  $\phi(x(k))$ , as gives Eq. (1) obeys a local Lipschitz condition. That is,  $\phi(x)$  is continuously differentiable with respect to  $x$  and the following inequality holds for all  $x_1, x_2 \in \mathcal{X}$ :

$$\|\phi(x_1) - \phi(x_2)\| \leq \gamma \|x_1 - x_2\|, \quad (2)$$

where the smallest constant  $\gamma$  that satisfied Eq. (2) is named the Lipschitz constant related to  $\phi(\cdot)$ .

To implement state-feedback control for the delayed system described in Eq. (1), we propose the use of a DTC scheme. This approach aims to control the process *as if it were delay-free*. A **predictor** is employed to estimate the state  $x(k + d|k)$ , which is then used in state feedback control as though the system had no delay (Normey-Rico and Camacho, 2007). The prediction can be carried out either in an open-loop or in closed-loop (as with the SP).

## 3. MAIN RESULT: O&P STRUCTURE AND NON-CONSERVATIVE UNCERTAINTY SETS

Taking into account the previous discussions, our problem resides in *how to robustly control discrete-time nonlinear systems subject to measurement delay?* In order to address this issue, we start by considering that the following hypotheses hold:

- (i) The nonlinear system is Lipschitz (Assumption 1);
- (ii) Appropriate corresponding state observers can be designed using LMIs, similar to what was done by Ibrir et al. (2005) (recapped next, Sec. 3.1);
- (iii) The propagation of observer error can be adequately handled by constraints tightening, as done by Santos et al. (2013) for the LTI setting - and herein developed for nonlinear systems (main result, Sec. 3.3).

<sup>1</sup> We consider usual bounded polyhedron constraint sets, as in  $\mathcal{X}$  can be defined as  $\mathcal{X} = \{x \in \mathbb{R}^{n_x} | Hx < h\}$

We consider that the state measurements from Eq. (1) are unavailable. Thus, as previously stated, it becomes necessary to estimate them to implement a state-feedback control policy. For this purpose, we employ an O&P strategy, which is illustrated in Fig. 1. The proposed O&P structure involves the following aspects:

- (i) Given a delay output measurement  $y(k) = Cx(k-d)$ , we implement a state **observer** that estimates the corresponding state  $\hat{x}(k-d)$ ;
- (ii) By using the state estimate  $\hat{x}(k-d)$ , we implement a **predictor** that determines the predicted state  $\hat{x}(k|k-d)$ . Accordingly, this predicted state value is used in feedback to determine the sampled control action, as in standard DTCs. Note that, since an **open-loop prediction** is performed, the observer stability remain unaffected while the feedback loop is perturbed.

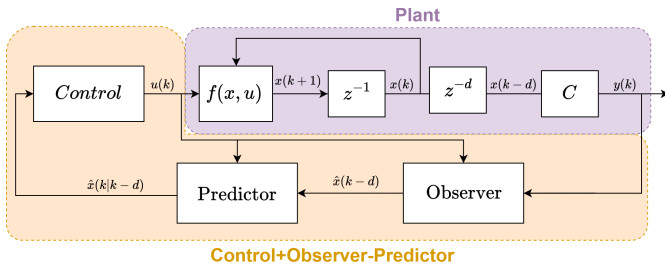


Fig. 1. Diagram of the proposed O&P strategy.

### 3.1 Nonlinear observer design using LMIs

In order to estimate the systems states from Eq. (1), it is implied that the matrix pair  $(A, C)$  is observable. Thus, it is possible to implement a nonlinear state observer of the *Luenberger* type (Ibrir et al., 2005), given under the following representation, using  $\delta(k-d) = x(k-d) - \hat{x}(k-d)$ :

$$\begin{cases} \hat{x}(k-d+1) = A\hat{x}(k-d) + Bu(k-d) + Ly(k) \\ \quad - LC\hat{x}(k-d) + \phi(\hat{x}(k-d)), \\ \delta(k-d+1) = (A-LC)\delta(k-d) \\ \quad + \phi(x(k-d)) - \phi(\hat{x}(k-d)). \end{cases} \quad (3)$$

In Eq. (3), we seek to determine an adequate observer gain  $L \in \mathbb{R}^{n_x \times n_y}$  such that the eigenvalues  $(A-LC)$  are placed as desired. However, to properly observe the system, pole placement is not enough, since nonlinearity  $\phi(x)$  must also be handled. The corresponding rationale is, hence, to guarantee that the observer in Eq. (3) is robustly stable, and, for such, we use the LMI-based synthesis procedure from (Ibrir et al., 2005):

*Theorem 2.* (Ibrir et al., 2005) Consider that the following relation holds for all  $x \in \mathcal{X}$ :

$$\frac{\partial \phi}{\partial x} = MF(x)N. \quad (4)$$

Given that  $M \in \mathbb{R}^{n_x \times n_x}$  and  $N \in \mathbb{R}^{n_x \times n_x}$  are well-defined real matrices, and  $F(x) \in \mathbb{R}^{n_x \times n_x}$  is a norm-bounded matrix satisfying  $F(x)^\top F(x) \leq I_{n_x} \forall x \in \mathcal{X}$ . Then, the Lipschitz constant related to Assumption 1 can be retrieved as  $\gamma = \|MF(x)N\|_\infty$ . Furthermore, if there

exists a positive definite matrix  $P \in \mathbb{S}^{n_x \times n_x}$ , a matrix  $W \in \mathbb{R}^{n_u \times n_x}$  and a constant  $\varepsilon > 0$  such that the following matrix inequality in Eq. (5) holds, the observed state trajectory  $\hat{x}(k-d)$  in Eq. (3) asymptotically converges to the real system state trajectory  $x(k-d)$  from Eq. (1), under  $L = P^{-1}W$ .

$$\begin{bmatrix} -P + \varepsilon N'N & A'P - C'W' & 0 \\ \star & -P & PM \\ \star & \star & -\varepsilon I \end{bmatrix} < 0. \quad (5)$$

### 3.2 Predictor design

For the prediction design in Fig. 1, we iterate the output of the observer,  $\hat{x}(k-d)$ ,  $d$  discrete-time steps using the original (open-loop) model given in Eq. (1). Accordingly, we obtain the future state prediction  $\hat{x}(k|k-d)$ , analytically given by:

$$\hat{x}(k|k-d) = A^d \hat{x}(k-d) + \sum_{j=1}^d A^{j-1} (Bu(k-j) + \phi(\hat{x}(k-j|k-d))). \quad (6)$$

By plugging Eq. (6) to the feedback loop as in Fig. 1, we obtain the following corresponding *delay-free* equivalent closed-loop model:

$$\begin{cases} \hat{x}(k+1|k-d+1) = A\hat{x}(k|k-d) + Bu(k) \\ \quad + \phi(\hat{x}(k|k-d)) + \tilde{w}(k), \\ y(k) = C\hat{x}(k|k-d), \end{cases} \quad (7)$$

where  $\tilde{w}(k)$  is employed to account for the mismatch between the predicted value of  $x(k+1)$  based on the information at time  $k-d+1$  ( $\hat{x}(k+1|k-d+1)$ ) and the same prediction with the information at time  $k-d$  ( $\hat{x}(k+1|k-d)$ ). This discrepancy arises due to the error between the real state  $x$  and the observer state  $\hat{x}$ . We stress that it is implied that  $\hat{x}(k+1|k-d+1) = \hat{x}(k+1|k-d) + \tilde{w}(k)$ .

*Remark 1.* In this paper, we assumed that the model perfectly represents the process, making the observer error the only uncertainty during the predictions. Situations in which the model does not accurately capture the process - due to noise, modeling errors, or external disturbances - would increase the impact of the variable  $\tilde{w}(k)$  in the closed-loop dynamics. This topic will be addressed in future works; the present paper focuses only on the nominal scenario.

*Remark 2.* The case with input delay can also be addressed. Instead of estimating  $\hat{x}(k-d)$  the observer should then estimate  $\hat{x}(k)$ , which would be used to predict  $\hat{x}(k+d)$  in order to obtain a delay-free model.

### 3.3 Control and robust sets

Consider the O&P structure presented in Fig. 1, the Luenberger Lipschitz observer design and the open-loop predictor - altogether unified in the equivalent closed-loop model in Eq. (7). Then, we develop how to construct non-conservative uncertainty propagation sets that bound the propagated error  $\tilde{w}$ . These sets can be directly used for control synthesis.

In particular, to establish a robust control framework under the proposed DTC - considering the equivalent system in Eq. (7) -, we propose a solution based on two steps: first, we introduce two positively invariant sets and, then, we verify if the solution is admissible (i.e. constraints satisfaction). Along the following developments, regarding the proposed DTC scheme, we consider that indeed exists a stabilising robust feedback law (cf. Assumption 3).

*Assumption 3.* Suppose there exists two closed and known polytopic sets  $\tilde{W}$  and  $E$ . Moreover, consider there exists a control law  $u = \kappa(\hat{x}(k|k-d))$  such that Eq. (7) is stable and the state trajectories are admissible, for every value of  $\tilde{w} \in \tilde{W}$ . Then, the original system in Eq. (1) is asymptotically stable and  $x(k) \in \mathcal{X}, \forall k \geq 0$ . Furthermore, suppose that such a control law is associated to an estimated attraction basin  $\mathcal{R}_{\tilde{x}}$  and also that the error prediction  $e(k) = x(k) - \hat{x}(k|k-d)$  belongs to the set  $E$ . Then, if  $\hat{x}(k|k-d) \in \mathcal{R}_{\tilde{x}} \subset \mathcal{X} \ominus E, \forall k > 0$ , it is directly implied that  $x(k) \in \mathcal{R}_x \subset \mathcal{X}, \forall k > 0$ , where  $\mathcal{R}_x = \mathcal{R}_{\tilde{x}} \oplus E$  <sup>2</sup>.

*Remark 3.* Since the uncertainties in the equivalent model from Eq. (7) are only related to the observer error, and the observer is asymptotically stable (Theo. 2), a robustly stabilising law  $u = \kappa(\hat{x}(k|k-d))$  applied to Eq. (1) indeed stabilises the (real) state trajectories towards the origin.

*Remark 4.* As discussed, e.g., Kheloufi et al. (2014); Saad et al. (2019), there are several possible approaches to design robust control laws for Lipschitz systems subject to bounded uncertainties. Indeed, if the uncertainty is constrained to a particular (known) set, i.e  $\tilde{W}$ , we can benefit from such knowledge (set format and bounds) to handle the delayed nonlinear system under the Assumption 3. In the present paper, the uncertainty in Eq. 7 arises from the observer error that propagates through the prediction. Santos et al. (2013) shows how to determine the sets  $\tilde{W}$  and  $E$  for linear systems with delays by analysation the dynamic behaviour. However, identifying these sets in the nonlinear case is more challenging due to the complex dynamics involved.

*Remark 5.* The estimated attraction basin  $\mathcal{R}_{\tilde{x}}$  is defined as the ellipsoid  $\hat{x}^\top P \hat{x} \leq 1$ , where  $P$  is a positive definite matrix and  $\mathcal{R}_{\tilde{x}} \subset \mathcal{X} \ominus E$  is a positively invariant set. Moreover, as stated in the Assumption 3, since the control law guarantees the admissibility of the sates for every value of  $\tilde{w} \in \tilde{W}$ , the attraction basin  $\mathcal{R}_{\tilde{x}}$  is, in fact, an RPI set of the dynamics of Eq. (7).

In this work, we also benefit from the Lipschitz condition to construct the aforementioned uncertainty-related sets ( $\tilde{W}$  and  $E$ ) in a non-conservative manner. Such a result is included in the following novel theorem.

*Theorem 4.* Suppose there exist a closed set  $\Delta$  which bounds the observer estimation error, i.e.,  $\delta(k) \in \Delta, \forall k \geq 0$ . Then, corresponding uncertainty sets  $\tilde{W}$  and  $E$  are direct from:

$$\tilde{W} \triangleq A^d LC \Delta \oplus \bigoplus_{j=0}^{d-1} A^j \gamma (\|A\| + \gamma)^{d-j-1} \|LC\| \Delta, \quad (8)$$

$$E \triangleq A^d \Delta \oplus \bigoplus_{j=0}^{d-1} A^j \gamma (\|A\| + \gamma)^{d-j-1} \Delta, \quad (9)$$

where  $\tilde{W}$  and  $E$  are positively invariant set for the dynamics of  $\tilde{w}(k)$  and  $e(k)$ , respectively, and constraints  $\delta \in \Delta$  and  $x \in \mathcal{X}$ .

*Proof.* This proof is provided separately, w.r.t. each robust. Refer to Appendixes B and C, respectively. ■

*Remark 6.* The proposed sets are non-conservatives in the sense that there can exist a positively invariant set  $\Delta$  such that  $\gamma \Delta$ , which forms the basis for the construction of the sets  $\tilde{W}$  and  $E$  (c.f. proof in Appendix), smaller than a worst-case bound for the nonlinear difference  $\phi(x) - \phi(\hat{x})$ .

*Remark 7.* Consider the matrix  $A$  in Eq. (1) is Hurwitz. Then the following statement holds: if  $\gamma + \|A\| \leq 1$ , the sets  $\tilde{W}$  and  $E$  converge as  $d$  increases. In contrast, if a worst-case bound for the nonlinearity  $\phi(x) - \phi(\hat{x})$  is used, then the sets do not converge w.r.t.  $d$ .

#### 4. NUMERICAL EXAMPLE

Next, in order to illustrate the effectiveness of the detailed DTC approach with the novel corresponding uncertainty propagation sets, we consider the benchmark electromechanical servo process described in (Gao, 2015). In particular, this system is described by:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0.0468 & 0.1564 \\ 0.2083 & 0.8154 \end{bmatrix} x(k) + \phi(x) \\ \quad + \begin{bmatrix} 39.2076 \\ 11.5299 \end{bmatrix} u(k), \\ y(k) = x_2(k-d), \end{cases} \quad (10)$$

where the nonlinearity is given by  $\phi(x(k)) = [0 \ \alpha \sin(x_1(k))]^\top$ , the delay is  $d = 2$  samples of delay. In Eq. (10),  $x_1$  gives the angular position of the load, while  $x_2$  denotes the servo shaft speed. The system's constraints are given by:

$$x \in \mathcal{X} \triangleq \{x : -500 \leq x_1 \leq 500, -1000 \leq x_2 \leq 1000\}.$$

In Eq. (10), the Lipschitz constant related to  $\phi(\cdot)$  actually coincides with the constant  $\alpha$ , whose original value is  $\alpha = 0.005$ . Next, seeking to test the conservativeness of our set computation, we increase the nominal Lipschitz constant forty times, that is, our sets are synthesised using  $\gamma = 0.2$ . The numerical results presented next were obtained using Matlab and Yalmip, with a 2.1 GHz, 8GB RAM PC. Sets were computed using zonoLAB (Koeln et al., 2024).

First, we solve the LMI problem related to Eq. (5) and thus obtain  $L = [0.157 \ 0.8167]^\top$ . For our simulations, we consider  $\Delta = \{\delta \in \mathbb{R}^{n_x} \mid \|\delta\| \leq 100\}$ , which is a positively invariant set for the dynamics of the observer error. Then, to construct  $\tilde{W}$  and  $E$ , we apply our novel set definitions from Theo. 4. In particular, by using the Eq. (8), we determine  $\tilde{W}$  and using Eq. (9) we derive the set for  $E$ . Both these sets are illustrated in Fig. 2.

Due to the spherical geometry  $\Delta$ , it is approximated by a tight exterior polytope to enable the application of the

<sup>2</sup> For the input-delay case, the condition becomes  $\hat{x}(k+d|k) \in \mathcal{R}_{\tilde{x}} \subset \mathcal{X} \ominus E, \forall k > 0 \rightarrow x(k) \in \mathcal{R}_x \subset \mathcal{X}, \forall k > d$

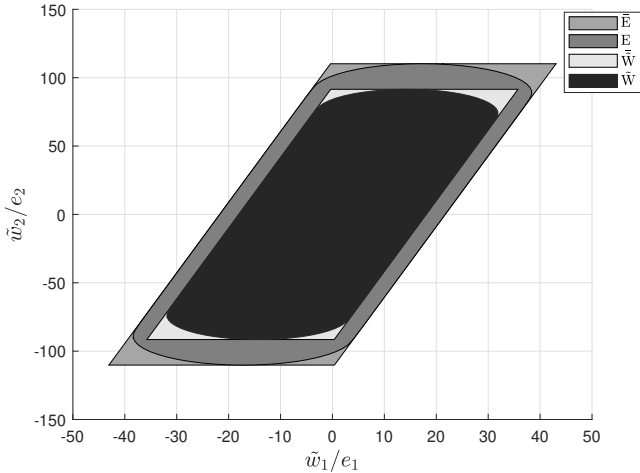


Fig. 2. Computed robust sets  $\tilde{W}$ ,  $E$ ,  $\bar{W}$  and  $\bar{E}$ .

Minkowski sum. Consequently, the robust sets were also represented as convex polytopes with a large number of vertices. The computation of these sets primarily relies on the Minkowski sum, which has a computational complexity of  $\mathcal{O}(n_x)$ , i.e. it depends on the state dimension rather than on the number of polytope vertices.

However, when performing the Pontryagin difference, the number of polytope vertices significantly affects computational effort (Koehn et al., 2024). To mitigate this complexity, we approximate the polytope  $E$  with a simpler convex polytope  $\bar{E}$ , with only four vertices, such that  $E \subset \bar{E}$ . While increasing the number of vertices could yield a tighter approximation, it would also amplify the computational burden of the Pontryagin difference. A similar simplification was applied to the set  $\tilde{W}$ .

In scenarios where robust control is implemented using LMIs based on the polytopic approach, approximating the set  $\tilde{W}$  by a larger set  $\bar{W}$  such that  $\tilde{W} \subset \bar{W}$  can simplify the control problem. The approximated sets  $\bar{W}$  and  $\bar{E}$ , depicted in Fig 2, are given by:

$$\bar{W} = \text{co} \left\{ \begin{bmatrix} -0.335 \\ 91.52 \end{bmatrix}, \begin{bmatrix} -35.74 \\ -91.52 \end{bmatrix}, \begin{bmatrix} 0.335 \\ -91.52 \end{bmatrix}, \begin{bmatrix} 35.74 \\ 91.52 \end{bmatrix} \right\},$$

$$\bar{E} = \text{co} \left\{ \begin{bmatrix} -0.378 \\ 110.2 \end{bmatrix}, \begin{bmatrix} -43.07 \\ -110.2 \end{bmatrix}, \begin{bmatrix} 0.378 \\ -110.2 \end{bmatrix}, \begin{bmatrix} 43.07 \\ 110.2 \end{bmatrix} \right\}.$$

Then, by systematically developing the O&P structure, the state-feedback control law  $u(k) = [0.0031 \ 0.0113]\hat{x}(k|k-d)$ , which satisfies Assumption 3, is employed.

*Remark 8.* Since the simulation starts at  $t = 0$  s, and the system is considered time-invariant, the simulation was conducted with the system described in Eq. (10) shifted  $d$  samples forward.

Indeed, in Fig. 3, we show that the robustness of the state constraints is ensured as long as  $\hat{x}(d|0) \in \mathcal{R}_{\tilde{x}}$ . This occurs due the fact that, since  $\hat{x}(d|0) \in \mathcal{R}_{\tilde{x}}$  and  $x(k+d) \in \hat{x}(k+d|k) \oplus \bar{E}$ , as seen in the figure, it follows that  $x(k+d) \in \mathcal{R}_x, \forall k \geq 0$ . In this same figure, we also show

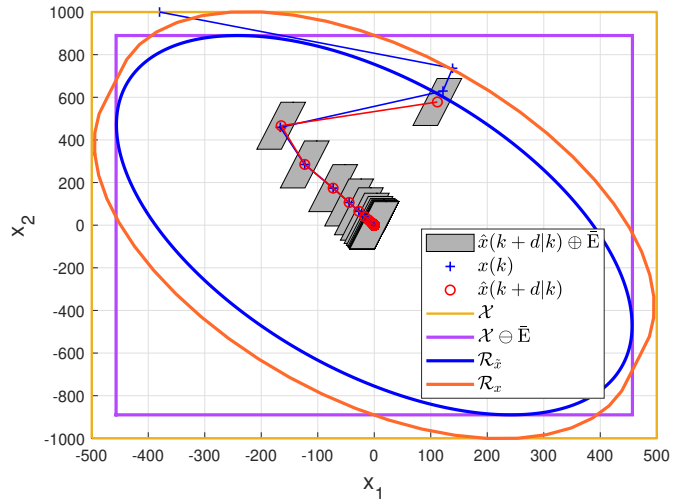


Fig. 3. State plane  $x_1 \times x_2$  with  $x(0) = [-380 \ 1000]$  and  $\hat{x}(0) = [-330 \ 913.4]$ .

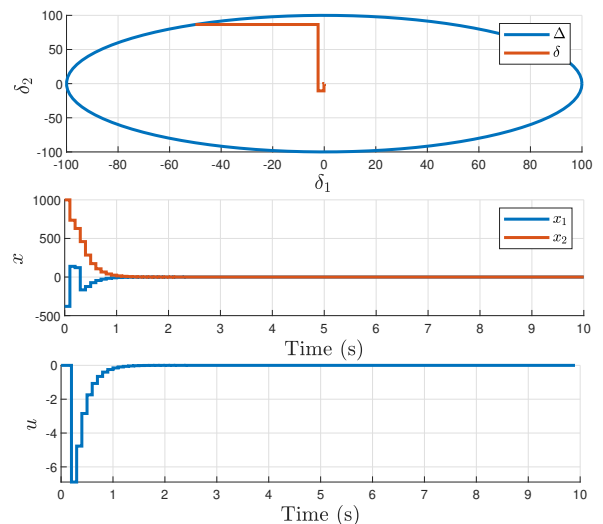


Fig. 4. (a) Observer error  $\delta$ ; (b) State trajectories  $x$ ; (c) Control signal  $u$ .

that the predicted states  $\hat{x}(k+d|k)$  converge to the actual states  $x(k+d)$ , given that there are no load disturbances<sup>3</sup>.

Since the predicted states converge to the actual states, we can infer that the observer states  $\hat{x}(k)$  also converge. This fact can be seen in Fig. 4, where we show how  $\delta$  converges to zero. We can also notice, in this figure, that the limit  $\bar{\delta}$  is not exceeded, as specified. Furthermore, this figure also shows that the actual state trajectories in function of time - it becomes evident that the control law satisfying Assumption 3 is able to stabilise the states towards the origin.

## 5. CONCLUSION

In this work, we develop a method for robust control of Lipschitz nonlinear systems subject to delays, using an observer-predictor structure. To achieve this and to

<sup>3</sup> The case with load disturbances is out of the scope of this work; however, Lima et al. (2015) discusses how uncertainty affects delayed systems and how predictors propagate uncertainty.

thereby ensure an admissible control approach, we develop non-conservative robust sets both for the observer error propagated in the predictions and for the prediction error. For future work, we intend to explore the proposed method in the context of uncertain (non-Lipschitz) systems, as well as to demonstrate how uncertainty propagation affects these systems and how it impacts the robust sets addressed in this paper.

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#### Appendix A. COMPLEMENTARY LEMMAS

*Lemma 5.* Let  $\phi(\cdot)$  be any real function that satisfies the Lipschitz condition (Assumption 1). Then, the difference  $z = \phi(\hat{x}(k-j|k-d+1)) - \phi(\hat{x}(k-j|k-d))$ ,  $\forall j \in \mathbb{N}_{[0,d-1]}$  belongs to a bounding set  $\Psi_{k-j}$  represented by:

$$\Psi_{k-j} = \gamma (\|A\| + \gamma)^{d-j-1} \|LC\| \Delta \quad (\text{A.1})$$

*Proof.* Consider the following nominal bound for  $z$ :

$$\{z \mid \|z\| \leq \gamma \|\hat{x}(k-j|k-d+1) - \hat{x}(k-j|k-d)\|, \forall j \in \mathbb{N}_{[0,d-1]}\}. \quad (\text{A.2})$$

From Eq. (7), it is implied that  $\tilde{w}(k-j-1) = \hat{x}(k-j|k-d+1) - \hat{x}(k-j|k-d)$ . Therefore, we can develop:

$$\begin{cases} \|\tilde{w}(k-d)\| = \|\hat{x}(k-d+1) - \hat{x}(k-d+1|k-d)\|, \\ \quad = \|LC\delta(k-d)\|, \\ \leq \|LC\| \cdot \|\delta(k-d)\|. \end{cases} \quad (\text{A.3})$$

Then, by replacing Eq. (A.3) in Eq. (A.2), considering the set  $\Delta = \{\delta \mid \|\delta\| \leq \bar{\delta}\}$ , we obtain the relation:  $\Psi_{k-d+1} \triangleq \gamma \|LC\| \Delta$ . Furthermore, we obtain  $\|\tilde{w}(k-d+1)\| = \|ALC\delta(k-d) + \phi(\hat{x}(k-d+1)) - \phi(\hat{x}(k-d+1|k))\| \leq \|A\| \|LC\| \|\delta(k-d)\| + \|\phi(\hat{x}(k-d+1)) - \phi(\hat{x}(k-d+1|k))\|$ . Recursively, it is also possible to show that:

$$\begin{cases} \Psi_{k-d+2} \triangleq \gamma \|A\| \|LC\| \Delta \oplus \gamma \Psi_{k-d+1}, \\ \quad \triangleq \gamma (\|A\| + \gamma) \|LC\| \Delta, \\ \quad \vdots \\ \Psi_{k-d+3} \triangleq \gamma (\|A^2\| + 2\gamma \|A\| + \gamma^2) \|LC\| \Delta, \\ \quad \vdots \end{cases} \quad (\text{A.4})$$

By exploiting the properties of Newton's binomial, we obtain the relation in Eq. (8), which completes this proof. ■

*Lemma 6.* Let  $\phi(\cdot)$  be any real function that satisfies the Lipschitz condition (Assumption 1). Thus, the difference  $\phi(x(k-j)) - \phi(\hat{x}(k-j|k-d))$ ,  $\forall j = 1, \dots, d$  belongs to a bounding set  $\Upsilon_{k-j}$  represented by:

$$\Upsilon_{k-j} = \gamma (\|A\| + \gamma)^{d-j} \Delta. \quad (\text{A.5})$$

*Proof.* To demonstrate the bounding set for  $\Upsilon_{k-j}$  shown in Eq. (A.5), we start by exploiting the Lipschitz condition. Accordingly, consider the following nominal bound for  $z$ :

$$\{z \mid \|z\| \leq \gamma \|x(k-j) - \hat{x}(k-j|k-d)\|, \forall j = 1, \dots, d\}.$$

It is implied that  $e(k-j) = x(k-j) - \hat{x}(k-j|k-d)$ . Thus, we have:  $\|e(k-d)\| = \|x(k-d) - \hat{x}(k-d)\| = \|\delta(k-d)\|$ . By replacing this relation in Eq. (A.6), considering that the set  $\Delta = \{\delta \mid \|\delta\| \leq \bar{\delta}\}$ , we arrive at the relation:  $\Upsilon_{k-d} \triangleq \gamma \Delta$ . Recursively, we obtain:

$$\begin{cases} \|e(k-d+1)\| = \|A\delta(k-d) + \phi(x(k-d)) - \phi(\hat{x}(k-d))\| \\ \leq \|A\| \|\delta\| + \|\phi(x(k-d)) - \phi(\hat{x}(k-d))\|, \\ \Upsilon_{k-d+1} \triangleq \gamma \|A\| \Delta \oplus \gamma \Upsilon_{k-d}, \\ \quad \triangleq \gamma (\|A\| + \gamma) \Delta, \\ \quad \vdots \\ \Upsilon_{k-d+2} \triangleq \gamma (\|A^2\| + 2\|A\|\gamma + \gamma^2) \Delta, \\ \Upsilon_{k-d+3} \triangleq \gamma (\|A^3\| + 3\gamma \|A^2\| + 3\gamma^2 \|A\| + \gamma^3) \Delta, \\ \quad \vdots \end{cases} \quad (\text{A.6})$$

Directly from the binomial expansion, we obtain the relation shown in Eq. (A.5). Thus, the proof is complete. ■

## Appendix B. PROOF - ROBUST SET $\tilde{W}$

From Eq. (7), we derive the following relation:

$$\begin{aligned} \tilde{w}(k) &= \hat{x}(k+1|k-d+1) - A\hat{x}(k|k-d) \\ &\quad - Bu(k) - \phi(\hat{x}(k|k-d)). \end{aligned} \quad (\text{B.1})$$

Replacing Eq. (6) in Eq. (B.1) leads to:

$$\begin{cases} \tilde{w}(k) = A^d LC\delta(k-d) + \sum_{j=0}^{d-1} A^j (\phi(\hat{x}(k-j|k-d+1)) \\ \quad - \phi(\hat{x}(k-j|k-d))), \end{cases}$$

which implies in the following bounding set for  $\tilde{w}$ :

$$\tilde{w} \in (\Psi_k \oplus A\Psi_{k-1} \oplus A^2\Psi_{k-2} \oplus \dots \oplus A^{d-1}\Psi_{k-d+1} \oplus A^d LC\Delta), \quad (\text{B.2})$$

where  $\Psi_{k-j}$  represents the bounding set for the difference  $\phi(\hat{x}(k-j|k-d+1)) - \phi(\hat{x}(k-j|k-d))$ ,  $\forall j \in \mathbb{N}_{[0,d-1]}$ .

From Lemma 5, by substituting each respective  $\Psi_{k+j}$  in Eq. (B.2), we obtain the bounding set demonstrated by Eq. (8), which completes this proof. □

## Appendix C. PROOF - ROBUST SET E

The state  $x(k)$  can be analytically obtained by iterating Eq. (1) recursively from  $k-d$ , which yields:

$$\begin{aligned} x(k) &= A^d x(k-d) + \sum_{j=1}^d A^{j-1} (Bu(k-j) \\ &\quad + \phi(x(k-j))). \end{aligned} \quad (\text{C.1})$$

Thus, by subtracting Eq. (C.1) from Eq. (6), we find the following relation for the prediction error:

$$\begin{aligned} e(k) &= A^d \delta(k-d) + \sum_{j=1}^d A^{j-1} (\phi(x(k-j)) \\ &\quad - \phi(\hat{x}(k-j|k-d))). \end{aligned} \quad (\text{C.2})$$

With the prediction error from Eq. (C.2), the following bounding set is defined for the case with an observer:

$$e \in \overbrace{(A^d \Delta \oplus \Upsilon_{k+d-1} \oplus \dots \oplus A^{d-1} \Upsilon_k)}^E, \quad (\text{C.3})$$

where  $\Upsilon_{k-j}$  is the bounding set for the difference  $z = \phi(x(k-j)) - \phi(\hat{x}(k-j|k-d))$ ,  $\forall j = 1, \dots, d$ .

From Lemma 6, by replacing each respective  $\Upsilon_{k+j}$  in Eq. (C.3), we obtain the bounding set demonstrated by Eq. (9), which completes this proof. □