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# Stability and convergence of the $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$ discretization<sup>\*</sup>

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## Abstract

The nonconforming Crouzeix-Raviart mixed finite element is popular to solve Stokes Problem. The discrete velocity is nonconforming (i.e. the moments on the facets between the elements are equal), piecewise linear, say  $\mathbf{P}_{nc}^1$ , and the discrete pressure is piecewise constant, say  $P^0$ . This method introduces some consistency error that may lead to a non-physical solution when solving the Navier-Stokes equations. There exists different cures, such that building a divergence free subspace of the discrete velocity space, or projecting the test function in the source term on some  $\mathbf{H}(\text{div})$ -conforming space *à la Linke*. One can also enrich the discrete pressure space, which is then generated by  $P^1$ -Lagrange plus  $P^0$  basis functions. This leads to the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  mixed finite element method, which allows to reduce the consistency error. Our goal here is to prove the stability of this scheme and to compute error estimates. We exhibit the Fortin operator and we demonstrate that this finite element method yields accurate results. Numerical examples are provided to highlight its efficiency.

*Keywords:* Stokes problem,  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element method, Fortin operator, TrioCFD code  
*2008 MSC:* 65N30, 35J57, 76D07

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## 1. Introduction

The TrioCFD code is a computational fluid mechanics (CFD) simulation software developed at the CEA. It is a C++ open source and massively parallel code. It is dedicated to the numerical simulation of turbulent flows for scientific and industrial applications, particularly in the nuclear field [3].

Let  $\Omega$ , the domain of study, be a connected bounded domain of  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a polygonal ( $d = 2$ ) or Lipschitz polyhedral ( $d = 3$ ) boundary  $\partial\Omega$ , and  $T > 0$  be the final time of the simulation. The TrioCFD code solves the incompressible Navier-Stokes equations, which read:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)) \text{ such that } \forall (\mathbf{x}, t) \in \Omega \times (0, T) \\ \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \text{grad}) \mathbf{u} + \text{grad } p = \mathbf{f}, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \end{array} \right. \quad (1)$$

endowed with suitable boundary conditions. The vector field  $\mathbf{u}$  represents the velocity of the fluid and the scalar field  $p$  represents its pressure divided by the fluid density which is supposed to be uniform. The vector field  $\mathbf{u}_0$  represents the initial velocity of the fluid. The first equation of Problem (1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass.

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<sup>\*</sup> $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization in TrioCFD

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The constant parameter  $\nu > 0$  is the kinematic viscosity of the fluid. The vector field  $\mathbf{f}$  represents the resultant body force divided by the fluid density.

Consider homogeneous Dirichlet boundary conditions. After having discretized the space variables, we obtain the following time-dependent problem, which is of finite dimension in space:

$$\left\{ \begin{array}{l} \text{Find } (\underline{U}(t), \underline{P}(t)) \in \mathbb{R}^{N_U \times N_P} \mid \forall t \in (0, T) \\ M \frac{d}{dt} \underline{U}(t) + \nu K \underline{U}(t) + L(\underline{U}(t)) \underline{U}(t) + {}^t B \underline{P}(t) = \underline{F}(t), \\ B \underline{U}(t) = 0, \\ \underline{U}(0) = \underline{U}_0 \in \mathbb{R}^{N_U}, \end{array} \right. \quad (2)$$

where  $N_U$  (resp.  $N_P$ ) is the dimension of the space of discrete velocities (resp. pressures). The vector  $\underline{U} \in \mathbb{R}^{N_U}$  (resp.  $\underline{P} \in \mathbb{R}^{N_P}$ ) represents the discrete velocity (resp. pressure). The vector  $\underline{F} \in \mathbb{R}^{N_U}$  represents the discrete vector field and the vector  $\underline{U}_0 \in \mathbb{R}^{N_U}$  represents the discrete initial condition. The matrix  $M \in \mathbb{R}^{N_U \times N_U}$  (resp.  $K \in \mathbb{R}^{N_U \times N_U}$ ) is the mass (resp. stiffness) matrix and the matrix  $L(\underline{U}) \in \mathbb{R}^{N_U \times N_U}$  corresponds to the discretization of the operator  $\mathbf{v} \rightarrow (\mathbf{u} \cdot \mathbf{grad})\mathbf{v}$ . The matrix  $B \in \mathbb{R}^{N_P \times N_U}$  is the velocity-pressure coupling matrix. It is well known that the discrete velocity and pressure spaces must satisfy the so called discrete inf-sup condition [12, §5.1].

There exist different time integration schemes, based on the prediction-correction time scheme [21]. Let us recall the explicit time scheme. The time step  $\delta t_n$  is computed at each iteration  $n \in \mathbb{N}$  to keep the time scheme stable. Let  $\underline{P}^0 \in \mathbb{R}^{N_P}$  and  $\delta t_0$  be given. At each iteration  $n \in \mathbb{N}$ , the pair  $(\underline{U}^{n+1}, \underline{P}^{n+1}) \in \mathbb{R}^{N_U \times N_P}$  is evaluated in three main steps, with  $K_P := B M^{-1} {}^t B$  and  $\underline{F}^{n+1} = \underline{F}(t_{n+1})$

where  $t_{n+1} = \sum_{k=0}^n \delta t_k$ :

$$\left\{ \begin{array}{l} \textbf{Step 1 (prediction).} \quad \text{Find } \underline{U}^* \text{ such that:} \\ \delta t_n^{-1} M \underline{U}^* + \nu K \underline{U}^n + L(\underline{U}^n) \underline{U}^n + {}^t B \underline{P}^n = \underline{F}^{n+1} + \delta t_n^{-1} M \underline{U}^n. \\ \textbf{Step 2 (solver pressure).} \quad \text{Find } \delta \underline{P} \text{ such that: } K_P \delta \underline{P} = \delta t_n^{-1} B \underline{U}^*. \\ \textbf{Step 3 (correction).} \quad \text{Find } \delta \underline{U} \text{ such that: } \delta t_n^{-1} M \delta \underline{U} + {}^t B \delta \underline{P} = 0. \\ \textbf{Update:} \quad \underline{U}^{n+1} = \underline{U}^* + \delta \underline{U}, \quad \underline{P}^{n+1} = \underline{P}^n + \delta \underline{P}, \quad \delta t_{n+1}. \end{array} \right. \quad (3)$$

In practise,  $\underline{U}^0$  is evaluated from the user data and  $\underline{P}^0$  solves  $K_P \underline{P}^0 = B M^{-1} \underline{F}^0$ . Notice that this choice will not affect  $(\underline{U}^{n+1}, \underline{P}^{n+1})$ .

The velocity is discretized with linear nonconforming finite elements. Hence, the mass matrix  $M$  is diagonal for  $d = 2$  and it can be lumped when  $d = 3$ : steps 1 and 3 are virtually free. Thus, the matrix  $K_P$  is a sparse matrix and it can be expressed explicitly. If the discrete inf-sup condition is satisfied, the matrix  $K_P$  is a symmetric positive definite matrix.

In the seminal paper [24], the discrete pressure is piecewise constant, which leads to the  $\mathbf{P}_{nc}^1 \times P^0$  finite element. It is well-known that this discretization introduces some consistency error, which can lead to an inaccurate solution when the viscosity is low [27, 39]. There exist different cures, such that building a divergence free subspace of the discrete velocity space [35], or projecting the test functions on some  $\mathbf{H}(\text{div})$ -conforming space [40, 14, 39, 41, 4]. One can also use higher order polynomials [43, 18, 38], Scott-Vogelius finite element [44, 34, 33, 22], or stabilization in the vorticity equation [1].

In the TrioCFD code [27, 36, 31], the discrete pressure space is built with  $P^1$ -Lagrange plus  $P^0$  basis functions. This leads to the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element. This discretization shows interesting numerical results. However, only an incomplete proof of the discrete inf-sup condition is available in [36, 31]. Our goal here is to gather the evidence and to write it as clearly as possible, to derive error

estimates and to prove some super-convergence results obtained in  $2D$ . We try to exhibit the dependency on the shape-regularity parameter. We will show that this finite element method gives accurate results. We will provide numerical results computed with the TrioCFD code to illustrate its efficiency.

## 2. Stokes problem

We consider Stokes problem:

$$\text{Find } (\mathbf{u}, p) \text{ such that } -\nu \Delta \mathbf{u} + \mathbf{grad} p = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u} = 0. \quad (4)$$

with homogeneous Dirichlet boundary conditions for the velocity:  $\mathbf{u}|_{\partial\Omega} = 0$  and a normalisation condition for the pressure  $p$ :  $\int_{\Omega} p = 0$ .

Before stating the variational formulation of Problem (4), we provide some definitions and reminders.

Let  $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$ ,  $\mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d$  its dual space and  $L_{zmv}^2(\Omega) = \{q \in L^2(\Omega) \mid \int_{\Omega} q = 0\}$ . To solve Problem (4), we suppose that  $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega)$  and  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ .

Let  $h_{\Omega}$  be the diameter of  $\Omega$ . We recall the Poincaré-Steklov inequality:

$$\exists C_{PS} > 0 \mid \forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_{PS} h_{\Omega} \|\mathbf{grad} v\|_{L^2(\Omega)}. \quad (5)$$

Thanks to this result, in  $H_0^1(\Omega)$ , the semi-norm is equivalent to the natural norm, so that the scalar product reads  $(v, w)_{H_0^1(\Omega)} = (\mathbf{grad} v, \mathbf{grad} w)_{L^2(\Omega)}$  and the norm is  $\|v\|_{H_0^1(\Omega)} = \|\mathbf{grad} v\|_{L^2(\Omega)}$ .

Let  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ . We denote by  $(v_i)_{i=1}^d$  the components of  $\mathbf{v}$ , and we set  $\mathbf{Grad} \mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$ , where  $\mathbb{L}^2(\Omega) = (L^2(\Omega))^{d \times d}$ . For all  $\mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)$ , we have:  $(\mathbf{Grad} \mathbf{v}, \mathbf{Grad} \mathbf{w})_{\mathbb{L}^2(\Omega)} = (\mathbf{v}, \mathbf{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H_0^1(\Omega)}$  and  $\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \left( \sum_{j=1}^d \|v_j\|_{H_0^1(\Omega)}^2 \right)^{1/2} = \|\mathbf{Grad} \mathbf{v}\|_{\mathbb{L}^2(\Omega)}$ . Let us set  $\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$ . The space  $\mathbf{V}_0$  is a closed subset of  $\mathbf{H}_0^1(\Omega)$ . We denote by  $\mathbf{V}_0^{\perp}$  the orthogonal of  $\mathbf{V}_0$  in  $\mathbf{H}_0^1(\Omega)$ . We recall that:

**Proposition 1** ([2, Theorem 3.1 (e)]). *The operator  $\operatorname{div} : \mathbf{H}_0^1(\Omega) \rightarrow L^2(\Omega)$  is an isomorphism of  $\mathbf{V}_0^{\perp}$  onto  $L_{zmv}^2(\Omega)$ . We call  $C_{\operatorname{div}}$  the constant such that:*

$$\forall p \in L_{zmv}^2(\Omega), \exists! \mathbf{v}_p \in \mathbf{V}_0^{\perp} \mid \operatorname{div} \mathbf{v}_p = p \text{ and } \|\mathbf{v}_p\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p\|_{L^2(\Omega)}. \quad (6)$$

Notice that  $\|\mathbf{v}_p\|_{\mathbf{H}_0^1(\Omega)}^2 = \|\nabla \times \mathbf{v}\|_{L^2(\Omega)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2$ , hence  $C_{\operatorname{div}} \geq 1$ .

The variational formulation of Problem (4) reads:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega) \text{ such that } \forall (\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega): \\ \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} - (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)}. \end{array} \right. \quad (7)$$

We recall the following regularity property [16]:

**Proposition 2** ([32, Theorem I-5.4]). *Suppose that  $\Omega$  is convex. Then, the mapping  $(\mathbf{v}, q) \mapsto -\nu \Delta \mathbf{v} + \mathbf{grad} q$  is an isomorphism of  $(\mathbf{V}_0 \cap \mathbf{H}^2(\Omega)) \times (L_{zmv}^2(\Omega) \cap H^1(\Omega))$  onto  $L^2(\Omega)$ , and it holds:*

$$\nu \|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq \|\mathbf{f}\|_{L^2(\Omega)} \text{ where } \mathbf{f} := -\nu \Delta \mathbf{v} + \mathbf{grad} q.$$

Classically, one proves that Problem (7) is well-posed using Poincaré-Steklov inequality (5) and Proposition 1. Check for instance the proof of [32, Theorem I.5.1]. Notice that one can also use basic T-coercivity [13, 8, 38, 22].

**Remark 1.** Consider inhomogeneous Dirichlet boundary conditions, i.e.  $\mathbf{u} = \mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ , where  $\mathbf{g}$  is some boundary data such that  $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ . One can build a velocity field  $\mathbf{u}_{\mathbf{g}} \in \mathbf{H}^1(\Omega)$  such that:  $-\Delta \mathbf{u}_{\mathbf{g}} = 0$  in  $\Omega$  and  $\mathbf{u} = \mathbf{g}$  on  $\partial\Omega$ . Then  $\mathbf{u}_0 := \mathbf{u} - \mathbf{u}_{\mathbf{g}}$  is such that  $\mathbf{u}_0 = 0$  on  $\partial\Omega$  and it solves the following problem, cf. [22, Appendix B]:

$$\text{Find } (\mathbf{u}_0, p) \text{ such that } -\nu \Delta \mathbf{u}_0 + \mathbf{grad} p = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{u}_{\mathbf{g}}. \quad (8)$$

### 3. Discretization notations

Consider  $(\mathcal{T}_h)_h$  a triangulation sequence of  $\Omega$ . We use the notation  $A \lesssim B$  for  $A \lesssim C B$  where  $A$  and  $B$  are scalar quantities and  $C$  is a generic positive constant which is independent of the sequence  $(\mathcal{T}_h)_h$  and the quantities of interest. For a given triangulation  $\mathcal{T}_h$ , we use the following index sets:

- $\mathcal{I}_K$  denotes the index set of the elements, such that  $\mathcal{T}_h := \bigcup_{\ell \in \mathcal{I}_K} K_\ell$  is the set of elements. For all  $\ell \in \mathcal{I}_K$ , we call  $h_\ell$  and  $\rho_\ell$  the diameters of  $K_\ell$  and its inscribed sphere respectively, and we let:  $\sigma_\ell = \frac{h_\ell}{\rho_\ell}$ . When the  $(\mathcal{T}_h)_h$  is a shape-regular triangulation sequence, cf. [29, def. 11.2], there exists a constant  $\sigma > 1$ , called the shape regularity parameter, such that for all  $h$ , for all  $\ell \in \mathcal{I}_K$ ,  $\sigma_\ell \leq \sigma$ .

- $\mathcal{I}_F$  denotes the index set of the faces, such that  $\mathcal{F}_h := \bigcup_{f \in \mathcal{I}_F} F_f$  is the set of faces.

Let  $\mathcal{I}_F = \mathcal{I}_F^i \cup \mathcal{I}_F^b$ , where  $\forall f \in \mathcal{I}_F^i$ ,  $F_f \in \Omega$  and  $\forall f \in \mathcal{I}_F^b$ ,  $F_f \subset \partial\Omega$ .

For all  $\ell \in \mathcal{I}_K$ , we set  $\mathcal{I}_{F,\ell}^{(i,b)} = \{f \in \mathcal{I}_F^{(i,b)} \mid F_f \in K_\ell\}$ .

- $\mathcal{I}_S$  denotes the index set of the vertices, such that  $(S_j)_{j \in \mathcal{I}_S}$  is the set of vertices. For all  $\ell \in \mathcal{I}_K$ , we set  $\mathcal{I}_{S,\ell} = \{i \in \mathcal{I}_S \mid S_i \in K_\ell\}$ .

For all  $f \in \mathcal{I}_F$ ,  $M_f$  denotes the barycentre of  $F_f$ , and  $\mathbf{n}_f$  its unit normal (outward oriented if  $F_f \in \partial\Omega$ ). We set  $\mathcal{S}_f = |F_f| \mathbf{n}_f$ , and we call  $h_f$  the diameter of  $F_f$ . For all  $\ell \in \mathcal{I}_K$ , for all  $f \in \mathcal{I}_{F,\ell}$ , we call  $\mathbf{n}_{f,\ell}$  the normal vector of  $F_f$  outgoing of  $K_\ell$ . We use the following notations:

$$\forall v \in L^2(\Omega), \forall \ell \in \mathcal{I}_K, \quad v_\ell = v|_{K_\ell} \text{ and } \underline{v}_\ell = |K_\ell|^{-1} \int_{K_\ell} v; \quad (9)$$

$$\forall v \in H^1(\Omega), \forall f \in \mathcal{I}_F, \quad v_f = v|_{F_f} \text{ and } \underline{v}_f = |F_f|^{-1} \int_{F_f} v. \quad (10)$$

Let us introduce spaces of piecewise regular elements:

$$\mathcal{P}_h H^1 = \left\{ v \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, v|_{K_\ell} \in H^1(K_\ell) \right\} \text{ and } \mathcal{P}_h \mathbf{H}^1 = (\mathcal{P}_h H^1)^d.$$

We define the element-wise operator  $\mathbf{grad}_h \in \mathcal{L}(\mathcal{P}_h H^1, \mathbf{L}^2(\Omega))$  such that:

$$\forall (v, \mathbf{w}) \in \mathcal{P}_h H^1 \times \mathbf{L}^2(\Omega), (\mathbf{grad}_h v, \mathbf{w})_{\mathbf{L}^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} (\mathbf{grad} v, \mathbf{w})_{\mathbf{L}^2(K_\ell)},$$

We define the broken scalar product and the broken semi-norm:

$$\left\{ \begin{array}{l} \forall (v, w) \in \mathcal{P}_h H^1 \times \mathcal{P}_h H^1, \quad (v, w)_h = (\mathbf{grad}_h v, \mathbf{grad}_h w)_{\mathbf{L}^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} (\mathbf{grad} v_\ell, \mathbf{grad} w_\ell)_{\mathbf{L}^2(K_\ell)}, \\ \forall v \in \mathcal{P}_h H^1, \quad \|v\|_h^2 := \|\mathbf{grad}_h v\|_{\mathbf{L}^2(\Omega)}^2 = \sum_{\ell \in \mathcal{I}_K} \|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)}^2. \end{array} \right.$$

In the same way, we define the element-wise operator  $\mathbf{Grad}_h \in \mathcal{L}(\mathcal{P}_h \mathbf{H}^1, \mathbb{L}^2(\Omega))$ .

Let us recall the Poincaré-Steklov inequality in cell [29, Lemma 12.11]:

$$\forall v \in \mathcal{P}_h H^1, \forall \ell \in \mathcal{I}_K, \quad \|v_\ell - \underline{v}_\ell\|_{L^2(K_\ell)} \leq \pi^{-1} h_\ell \|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)}. \quad (11)$$

Let  $v \in \mathcal{P}_h H^1$ . Let  $f \in \mathcal{I}_F^i$  (resp.  $f \in \mathcal{I}_F^b$ ) such that  $F_f = \partial K_\ell \cap \partial K_{\ell'}$  (resp.  $F_f \subset \partial K_\ell \cap \partial\Omega$ ) and  $\mathbf{n}_f = \mathbf{n}_{f,\ell}$ . The jump between the facets is defined by:  $[v]_f := v_{\ell|F_f} - v_{\ell'|F_f}$ , (resp.  $[v]_f = v_{\ell|F_f}$ ).

We set  $\mathcal{P}_h \mathbf{H}(\text{div}) = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, \mathbf{v}_\ell \in \mathbf{H}(\text{div}; K_\ell)\}$ . We define the element-wise operator  $\text{div}_h \in \mathcal{L}(\mathcal{P}_h \mathbf{H}(\text{div}), L^2(\Omega))$  such that:

$$\left\{ \begin{array}{l} \forall (\mathbf{v}, q) \in \mathcal{P}_h \mathbf{H}(\text{div}) \times L^2(\Omega), \quad (\text{div}_h \mathbf{v}, q)_{L^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} (\text{div}_h \mathbf{v}_\ell, q)_{L^2(K_\ell)}, \\ \forall \mathbf{v} \in \mathcal{P}_h \mathbf{H}(\text{div}), \quad \|\text{div}_h \mathbf{v}\|_{L^2(\Omega)}^2 = \sum_{\ell \in \mathcal{I}_K} \|\text{div} \mathbf{v}_\ell\|_{L^2(K_\ell)}^2. \end{array} \right.$$

We call  $\hat{K}$  the reference simplex. For all  $\ell \in \mathcal{I}_K$ , we denote by  $T_\ell : \hat{K} \rightarrow K_\ell$  the geometric mapping such that  $\forall \hat{\mathbf{x}} \in \hat{K}$ ,  $\mathbf{x}_{|K_\ell} = T_\ell(\hat{\mathbf{x}}) = \mathbb{B}_\ell \hat{\mathbf{x}} + \overrightarrow{OS}_i$ , where  $\mathbb{B}_\ell \in \mathbb{R}^{d \times d}$ ,  $\overrightarrow{OS}_i \in \mathbb{R}^d$  with  $i \in \mathcal{I}_{S,\ell}$ . Let us set  $J_\ell = \det(\mathbb{B}_\ell)$ . Recall that we have [29, Lemma 11.1]:

$$|J_\ell| = |K_\ell| |\hat{K}|^{-1}, \quad \|\mathbb{B}_\ell\| = h_\ell (\rho_{\hat{K}})^{-1}, \quad \|\mathbb{B}_\ell^{-1}\| = h_{\hat{K}} (\rho_\ell)^{-1}. \quad (12)$$

For all  $v \in L^2(\Omega)$ ,  $\ell \in \mathcal{I}_K$ , we set:  $\hat{v}_\ell = v_\ell \circ T_\ell$ . Changing the variable, we get:

$$\|v_\ell\|_{L^2(K_\ell)}^2 = |J_\ell| \|\hat{v}_\ell\|_{L^2(\hat{K})}^2. \quad (13)$$

Again, changing the variable, we have:

$$\forall v \in \mathcal{P}_h H^1, \forall \ell \in \mathcal{I}_K, \quad \mathbf{grad} v_\ell = (\mathbb{B}_\ell^{-1})^T \mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell \text{ and } \mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell = (\mathbb{B}_\ell)^T \mathbf{grad} v_\ell. \quad (14)$$

Hence, for all  $v \in \mathcal{P}_h H^1$ , for all  $\ell \in \mathcal{I}_K$  it holds:

$$\|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)}^2 \leq \|\mathbb{B}_\ell^{-1}\|^2 |J_\ell| \|\mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell\|_{\mathbf{L}^2(\hat{K})}^2, \quad (15)$$

$$\|\mathbf{grad}_{\hat{\mathbf{x}}} \hat{v}_\ell\|_{\mathbf{L}^2(\hat{K})}^2 \leq \|\mathbb{B}_\ell\|^2 |J_\ell|^{-1} \|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)}^2. \quad (16)$$

For all  $\ell \in \mathcal{I}_K$  and  $f \in \mathcal{I}_{F,\ell}$ , we denote by  $T_{f,\ell}$  the geometric mapping such that for all  $\mathbf{x} \in F_f$ ,  $T_{f,\ell}(T_\ell^{-1}(\mathbf{x})) = \mathbf{x}$ . We let  $\hat{F}_{f,\ell} = T_{f,\ell}(F_f)$  and  $J_{f,\ell} = \det \mathbb{B}_{f,\ell}$  where  $\mathbb{B}_{f,\ell}$  is the Jacobian matrix of  $T_{f,\ell}|_{F_f}$ . Hence:  $|J_{f,\ell}| = |F_f| |\hat{F}_{f,\ell}|^{-1}$ . We recall that [24, Inequality (3.17)]:

$$\forall \ell \in \mathcal{I}_K, \forall f \in \mathcal{I}_{F,\ell}, \quad |J_{f,\ell}| |J_\ell|^{-1} \lesssim (\rho_\ell)^{-1}. \quad (17)$$

Changing the variable, we get:

$$\forall \ell \in \mathcal{I}_K, \quad \forall v_\ell \in H^1(K_\ell), \quad \forall f \in \mathcal{I}_{F,\ell}, \quad \|v_\ell\|_{L^2(F_f)}^2 = |J_{f,\ell}| \|\hat{v}_\ell\|_{L^2(\hat{F}_{f,\ell})}^2. \quad (18)$$

Let  $\mathbf{n}_{\hat{F}_{f,\ell}}$  be the unit normal vector of the face  $\hat{F}_{f,\ell}$  outgoing from  $\hat{K}$ . Changing the variable, we get:

$$\mathbf{n}_{f,\ell} = |J_\ell| |J_{f,\ell}|^{-1} (\mathbb{B}_\ell^{-1})^T \mathbf{n}_{\hat{F}_{f,\ell}}. \quad (19)$$

For all  $D \subset \mathbb{R}^d$ , for  $k \geq 0$ , we call  $P^k(D)$  the set of order  $k$  polynomials on  $D$ ,  $\mathbf{P}^k(D) = (P^k(D))^d$ . Let  $k > 0$  and  $C_{k,d} = ((k+1)(k+d)/d)^{\frac{1}{2}}$ . The following inequality holds [49, Theorem 5]:

$$\forall \ell \in \mathcal{I}_K, \forall v_\ell \in P^k(K_\ell), \quad \|v_\ell\|_{L^2(\partial K_\ell)} \leq C_{k,d} \left( |\partial K_\ell| |K_\ell|^{-1} \right)^{\frac{1}{2}} \|v_\ell\|_{L^2(K_\ell)}. \quad (20)$$

Using successively (20) and (17), we obtain:

$$\forall \ell \in \mathcal{I}_K, \quad \forall v_\ell \in P^1(K_\ell), \forall f \in \mathcal{I}_{F,\ell}, \quad \|v_\ell - \underline{v}_\ell\|_{L^2(F_f)} \lesssim (\rho_\ell)^{-\frac{1}{2}} \|v_\ell - \underline{v}_\ell\|_{L^2(K_\ell)}. \quad (21)$$

**Proposition 3.** *The following inverse inequality holds:*

$$\forall \ell \in \mathcal{I}_K, \forall v_\ell \in P^1(K_\ell), \quad \|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)} \lesssim (\rho_\ell)^{-1} \|v_\ell - \underline{v}_\ell\|_{L^2(K_\ell)}. \quad (22)$$

*Proof.* Let  $v_\ell \in P^1(K_\ell)$ . Since  $\underline{v}_\ell$  is a constant and  $\mathbf{grad} v_\ell$  is a constant vector, it holds:

$$\|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)}^2 = \|\mathbf{grad}(v_\ell - \underline{v}_\ell)\|_{\mathbf{L}^2(K_\ell)}^2 = |K_\ell| |\mathbf{grad}(v_\ell - \underline{v}_\ell)|^2 = |K_\ell|^{-1} \left| \int_{K_\ell} \mathbf{grad}(v_\ell - \underline{v}_\ell) \right|^2.$$

We deduce that:

$$\begin{aligned} \|\mathbf{grad} v_\ell\|_{\mathbf{L}^2(K_\ell)}^2 &= |K_\ell|^{-1} \left| \int_{\partial K_\ell} (v_\ell - \underline{v}_\ell) \mathbf{n}_{|\partial K_\ell} \right|^2 && \text{using an integration by parts,} \\ &\leq |K_\ell|^{-1} \left( \int_{\partial K_\ell} |v_\ell - \underline{v}_\ell| \right)^2 && \text{since } |\mathbf{n}_{|\partial K_\ell}| \leq 1. \\ &\leq |K_\ell|^{-1} |\partial K_\ell| \|v_\ell - \underline{v}_\ell\|_{L^2(\partial K_\ell)}^2 && \text{using Cauchy-Schwarz,} \\ &\lesssim (\rho_\ell)^{-2} \|v_\ell - \underline{v}_\ell\|_{L^2(K_\ell)}^2 && \text{using (17) and (21).} \end{aligned}$$

□

We define the following spaces of piecewise polynomials:

$$\begin{aligned} \forall k \geq 0, \quad \mathcal{P}_{disc}^k(\mathcal{T}_h) &= \left\{ v \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, v_\ell \in P^k(K_\ell) \right\}. \\ \forall k > 0, \quad \mathcal{P}^k(\mathcal{T}_h) &= \left\{ v \in H^1(\Omega); \quad \forall \ell \in \mathcal{I}_K, v_\ell \in P^k(K_\ell) \right\}. \end{aligned}$$

We call  $\pi_0 \in \mathcal{L}(L^2(\Omega), \mathcal{P}_{disc}^0(\mathcal{T}_h))$  the  $L^2$ -projection onto  $\mathcal{P}_{disc}^0(\mathcal{T}_h)$ :

$$\forall v \in L^2(\Omega), \quad \pi_0(v) = \sum_{\ell \in \mathcal{I}_K} \underline{v}_\ell \mathbb{1}_{K_\ell}.$$

By summation over  $\ell \in \mathcal{I}_K$ , we deduce from (11) that:

$$\forall v \in \mathcal{P}_h H^1, \quad \|v - \pi_0(v)\|_{L^2(\Omega)} \leq \pi^{-1} h \|\mathbf{grad}_h v\|_{\mathbf{L}^2(\Omega)}. \quad (23)$$

For all  $\ell \in \mathcal{I}_K$ , for all  $i \in \mathcal{I}_{S,\ell}$ ,  $\lambda_{i,\ell}$  denotes the barycentric coordinate related to the vertex  $S_i$  in  $K_\ell$ . We endow  $\mathcal{P}^1(\mathcal{T}_h)$  with the basis  $(\phi_i)_{i \in \mathcal{I}_S}$  such that:

$$\forall \ell \in \mathcal{I}_K, \phi_{i|K_\ell} = \begin{cases} \lambda_{i,\ell} & \text{if } i \in \mathcal{I}_{S,\ell} \\ 0 & \text{otherwise} \end{cases}, \text{ so that } \mathcal{P}^1(\mathcal{T}_h) = \text{vect}((\phi_i)_{i \in \mathcal{I}_S}).$$

We need the following estimate inspired by [29, Lemma 12.15]:

**Proposition 4.** *For all  $\ell \in \mathcal{I}_K$ , for all  $v \in H^1(K_\ell)$ , for all  $f \in \mathcal{I}_{F,\ell}$ , it holds:*

$$\|v - \underline{v}_\ell\|_{L^2(F_f)} \lesssim (\sigma_\ell h_\ell)^{\frac{1}{2}} |v|_{H^1(K_\ell)}, \quad (24)$$

$$\|v - \underline{v}_f\|_{L^2(F_f)} \lesssim (\sigma_\ell h_\ell)^{\frac{1}{2}} |v|_{H^1(K_\ell)}. \quad (25)$$

*Proof.* Let  $\pi_0^f \in \mathcal{L}(L^2(F_f), \mathbb{R})$  be the operator such that for all  $v \in L^2(F_f)$ ,  $\pi_0^f(v) = \underline{v}_f$ . It holds:  $\|\pi_0^f\|_{\mathcal{L}(L^2(F_f), \mathbb{R})} \leq 1$ . Notice that  $v - \underline{v}_f = (Id - \pi_0^f)(v - \underline{v}_\ell)$ , hence  $\|v - \underline{v}_f\|_{L^2(F_f)}^2 \leq 2\|v - \underline{v}_\ell\|_{L^2(F_f)}^2$ : inequality (25) is a consequence of (24). It holds:

$$\begin{aligned} \|v - \underline{v}_\ell\|_{L^2(F_f)}^2 &\lesssim |F_f| \|\hat{v}_\ell - \underline{v}_\ell\|_{L^2(\hat{F}_{f,\ell})}^2 && \text{from (18),} \\ &\lesssim |F_f| \left( \|\hat{v}_\ell - \underline{v}_\ell\|_{L^2(\hat{K})}^2 + \|\mathbf{grad} \hat{v}_\ell\|_{\mathbf{L}^2(\hat{K})}^2 \right) && \text{by continuity of the trace,} \\ &\lesssim |F_f| |K_\ell|^{-1} (\|v - \underline{v}_\ell\|_{L^2(K_\ell)}^2 + (h_\ell)^2 \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)}^2) && \text{from (13) and (16).} \end{aligned}$$

We conclude using the discrete Poincaré-Steklov inequality (11) and (17).  $\square$

We recall a classical approximation estimate: changing the variable, using Bramble-Hilbert/Deny-Lions Lemma [29, Lemma 11.9] in the reference element, changing the variable again, we have that for  $k \geq 0$ , for all  $m \in \{0, \dots, k+1\}$ ,  $\ell \in \mathcal{I}_K$ , for all  $v \in H^{k+1}(K_\ell)$  (see also [23, Proposition 4.1]):

$$\inf_{v_h \in P^k(K_\ell)} |v - v_h|_{H^m(K_\ell)} \lesssim \frac{h^{k+1}}{\rho^m} |v|_{H^{k+1}(K_\ell)}. \quad (26)$$

In the next section, we recall the definition and properties of the nonconforming finite elements.

#### 4. Nonconforming finite elements

Let us consider  $\mathbf{X}_h = (X_h)^d$  (resp.  $\mathbf{X}_{0,h} = (X_{0,h})^d$ ), the space of nonconforming approximation of  $\mathbf{H}^1(\Omega)$  (resp.  $\mathbf{H}_0^1(\Omega)$ ) where, for  $k > 0$ , we have:

$$\begin{cases} X_h &= \left\{ v_h \in \mathcal{P}_{disc}^k(\mathcal{T}_h); \quad \forall f \in \mathcal{I}_F^i, \forall q \in P^{k-1}(F_f), \int_{F_f} [v]_f q = 0 \right\} \\ X_{0,h} &= \left\{ v \in X_h; \quad \forall f \in \mathcal{I}_F^b, \forall q \in P^{k-1}(F_f), \int_{F_f} v_f q = 0 \right\}. \end{cases} \quad (27)$$

The condition of zero jump across faces is known as the "patch test". Using it, we can show that the mapping  $v \rightarrow \|v\|_h$  is a norm over  $X_{0,h}$ , cf. [24, Lemma 2]. Notice that  $X_h \not\subset H^1(\Omega)$ . Nevertheless, for all  $v \in X_h$ , for all  $f \in \mathcal{I}_F^i$  such that  $\bar{F}_f = \partial K_\ell \cap \partial K_{\ell'}$ ,  $\underline{v}_f := |F_f|^{-1} \int_{F_f} v_\ell = |F_f|^{-1} \int_{F_f} v_{\ell'}$ . It proves the following results, which are useful for the numerical analysis of nonconforming finite elements.

**Proposition 5.** *Let  $(\mathbf{v}, q) \in (\mathbf{X}_{0,h} + \mathbf{H}_0^1(\Omega)) \times H^1(\Omega)$ . We have:*

$$\left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_\ell \cdot \mathbf{n}_{f,\ell}, q)_{L^2(F_f)} \right| \lesssim \sigma h \|\mathbf{v}\|_h \|\mathbf{grad} q\|_{\mathbf{L}^2(\Omega)}. \quad (28)$$

Let  $(\mathbf{v}, q) \in (\mathbf{X}_{0,h} + \mathbf{H}_0^1(\Omega)) \times \mathcal{P}^1(\mathcal{T}_h)$ . We have:

$$\left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_\ell \cdot \mathbf{n}_{f,\ell}, q)_{L^2(F_f)} \right| \lesssim \sigma \|\mathbf{v}\|_h \|q - \pi_0(q)\|_{L^2(\Omega)}. \quad (29)$$

*Proof.* For  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ , it is obvious since:

$$\sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_\ell \cdot \mathbf{n}_{f,\ell}, q)_{L^2(F_f)} = (\mathbf{v} \cdot \mathbf{n}_{|\partial\Omega}, q)_{L^2(\partial\Omega)} = 0.$$

Let  $(\mathbf{v}, q) \in \mathbf{X}_{0,h} \times H^1(\Omega)$ . We are inspired by the proof of [10, Lemma 3.1]. It holds:

$$\sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\underline{\mathbf{v}}_f \cdot \mathbf{n}_{f,\ell}, q)_{L^2(F_f)} = \sum_{f \in \mathcal{I}_F^i} \int_{L^2(F_f)} [\underline{\mathbf{v}}_f q]_f \cdot \mathbf{n}_f = 0, \quad \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} ((\mathbf{v}_\ell - \underline{\mathbf{v}}_f) \cdot \mathbf{n}_{f,\ell}, \underline{q}_\ell)_{L^2(F_f)} = 0.$$

By summation and using Cauchy-Schwarz, we deduce that:

$$\begin{aligned} \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_\ell \cdot \mathbf{n}_{f,\ell}, q)_{L^2(F_f)} &= \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} ((\mathbf{v}_\ell - \underline{\mathbf{v}}_f) \cdot \mathbf{n}_{f,\ell}, q - \underline{q}_\ell)_{L^2(F_f)}, \\ &\leq \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \|(\mathbf{v}_\ell - \underline{\mathbf{v}}_f) \cdot \mathbf{n}_{f,\ell}\|_{L^2(F_f)} \|q - \underline{q}_\ell\|_{L^2(F_f)}. \end{aligned}$$

According to (25), we have:

$$\|(\mathbf{v}_\ell - \underline{\mathbf{v}}_f) \cdot \mathbf{n}_{f,\ell}\|_{L^2(F_f)} \lesssim (\sigma_\ell h_\ell)^{\frac{1}{2}} \|\mathbf{Grad} \mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)}. \quad (30)$$

Let  $q \in H^1(\Omega)$ . Using (24), we have:  $\|q - \underline{q}_\ell\|_{L^2(F_f)} \lesssim (\sigma_\ell h_\ell)^{\frac{1}{2}} \|\mathbf{grad} q\|_{\mathbf{L}^2(K_\ell)}$ . Hence:

$$\|(\mathbf{v}_\ell - \underline{\mathbf{v}}_f) \cdot \mathbf{n}_{f,\ell}\|_{L^2(F_f)} \|q - \underline{q}_\ell\|_{L^2(F_f)} \leq \sigma_\ell h_\ell \|\mathbf{Grad} \mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)} \|\mathbf{grad} q\|_{\mathbf{L}^2(K_\ell)}.$$

Let  $q \in \mathcal{P}^1(\mathcal{T}_h)$ . Using (21), we have:  $\|q - \underline{q}_\ell\|_{L^2(F_f)} \lesssim (\rho_\ell)^{-\frac{1}{2}} \|q - \underline{q}_\ell\|_{L^2(K_\ell)}$ . Using (30), it holds:

$$\|(\mathbf{v}_\ell - \underline{\mathbf{v}}_f) \cdot \mathbf{n}_{f,\ell}\|_{L^2(F_f)} \|q - \underline{q}_\ell\|_{L^2(F_f)} \leq \sigma_\ell \|\mathbf{Grad} \mathbf{v}_\ell\|_{\mathbb{L}^2(K_\ell)} \|q - \pi_0(q)\|_{L^2(K_\ell)}.$$

We conclude using discrete Cauchy-Schwarz inequality.  $\square$

Proposition 5 helps to prove the following discrete Poincaré-Steklov inequality [38, Proposition 10]:

$$\forall v_h \in X_{0,h}, \quad \|v_h\|_{L^2(\Omega)} \lesssim \sigma C_\Omega h_\Omega \|v_h\|_h, \quad (31)$$

where the dimensionless constant  $C_\Omega$  depends on  $C_{\text{div}}$  and  $C_{PS}$  (cf. Equations (6) and (5)).

We now consider the case  $k = 1$  in (27). For all  $\ell \in \mathcal{I}_K$ , for all  $f \in \mathcal{I}_{F,\ell}$ ,  $\lambda_{f,\ell}$  denotes the barycentric coordinate related to the vertex  $S_{f,\ell}$  opposite to face  $F_f$  in  $K_\ell$ . We endow  $X_h$  with the basis  $(\psi_f)_{f \in \mathcal{I}_F}$  such that:

$$\forall \ell \in \mathcal{I}_K, \psi_f|_{K_\ell} = \begin{cases} 1 - d\lambda_{f,\ell} & \text{if } f \in \mathcal{I}_{F,\ell} \\ 0 & \text{otherwise} \end{cases}, \text{ so that } X_h = \text{vect}((\phi_f)_{f \in \mathcal{I}_F}).$$

We have:  $\int_{F_f} \psi_f = |F_f| \delta_{f,f'}$ ,  $[\psi_f]_f = 0$  if  $f \in \mathcal{I}_F^i$  and  $X_{0,h} = \text{vect}((\psi_f)_{f \in \mathcal{I}_F^i})$ .

For all  $\ell \in \mathcal{I}_K$ , we define  $\pi_{cr}^\ell : H^1(K_\ell) \rightarrow P^1(K_\ell)$  such that:

$$\forall \ell \in \mathcal{I}_K, \forall v \in H^1(K_\ell), \quad \pi_{cr}^\ell v = \sum_{f \in \mathcal{I}_{F,\ell}} \underline{v}_f \psi_f|_{K_\ell}. \quad (32)$$

Notice that for all  $\ell \in \mathcal{I}_K$ , for all  $v \in P^1(K_\ell)$ ,  $\pi_{cr}^\ell v = v$ . We recall the following result.

**Proposition 6** ([5, Lemma 2]). *The norm of the operator  $\pi_{cr}^\ell$  is independent of  $K_\ell$ . We have indeed:*

$$\forall \ell \in \mathcal{I}_K, \forall v \in H^1(K_\ell), \quad \|\mathbf{grad} \pi_{cr}^\ell v\|_{\mathbf{L}^2(K_\ell)} \leq \|\mathbf{grad} v\|_{\mathbf{L}^2(K_\ell)}. \quad (33)$$

Hence:  $\|\pi_{cr}^\ell\|_{\mathcal{L}(H^1(K_\ell), P^1(K_\ell))} \leq 1$ .

*Proof.* By construction, using integration by parts, it holds:

$$|K_\ell| \mathbf{grad} \pi_{cr}^\ell v = \int_{K_\ell} \mathbf{grad} \pi_{cr}^\ell v = \int_{\partial K_\ell} \pi_{cr}^\ell v \mathbf{vn}|_{\partial K_\ell} = \int_{\partial K_\ell} v \mathbf{vn}|_{\partial K_\ell} = \int_{K_\ell} \mathbf{grad} v. \quad \square$$

In order to prove convergence estimates, we need the following result.

**Proposition 7.** *The following estimates hold, for all  $\ell \in \mathcal{I}_K$ , for all  $s \in [0, 1]$ , for all  $v \in H^{1+s}(K_\ell)$ :*

$$\|v - \pi_{cr}^\ell v\|_{L^2(K_\ell)} \leq 2 (h_\ell)^{1+s} |v|_{H^{1+s}(K_\ell)}, \quad (34)$$

$$\|\mathbf{grad}(v - \pi_{cr}^\ell v)\|_{\mathbf{L}^2(K_\ell)} \leq 2 \sigma_\ell (h_\ell)^s |v|_{H^{1+s}(K_\ell)}, \quad (35)$$

$$\|v - \pi_{cr}^\ell v\|_{L^2(\partial K_\ell)} \leq 2 (\sigma_\ell h_\ell)^{\frac{1}{2}} (h_\ell)^s |v|_{H^{1+s}(K_\ell)}. \quad (36)$$

*Proof.* Since for all  $v_h \in P^1(K_\ell)$ ,  $v - \pi_{cr}^\ell v = (Id - \pi_{cr}^\ell)(v - v_h)$ , we have:

$$\begin{aligned} \|v - \pi_{cr}^\ell v\|_{H^1(K_\ell)} &\leq \|Id - \pi_{cr}^\ell\|_{\mathcal{L}(H^1(K_\ell), P^1(K_\ell))} \inf_{v_h \in P^1(K_\ell)} \|v - v_h\|_{H^1(K_\ell)}, \\ &\leq 2 \inf_{v_h \in P^1(K_\ell)} \|v - v_h\|_{H^1(K_\ell)}. \end{aligned}$$

Using estimate (26), we deduce that, for  $m = 0, 1$ :

$$\begin{aligned} \|v - \pi_{cr}^\ell v\|_{L^2(K_\ell)} &\leq 2 (h_\ell)^{1+m} |v|_{H^{1+m}(K_\ell)}. \\ \|\mathbf{grad}(v - \pi_{cr}^\ell v)\|_{\mathbf{L}^2(K_\ell)} &\leq 2 \sigma_\ell (h_\ell)^m |v|_{H^{1+m}(K_\ell)}. \end{aligned}$$

We obtain (34) and (35), using interpolation property [46, Lemma 22.2]. We estimate  $\|v - \pi_{cr}^\ell v\|_{L^2(\partial K_\ell)}$  with the same tools, changing the variable, using the trace theorem in the reference element, and changing the variable again.  $\square$

Let  $\pi_{cr} : H^1(\Omega) \rightarrow X_h$  (resp.  $\Pi_{cr} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{X}_h$ ) be the Crouzeix-Raviart interpolation operator:

$$\forall v \in H^1(\Omega), \quad \pi_{cr} v = \sum_{f \in \mathcal{I}_F} \underline{v}_f \psi_f \text{ with } \underline{v}_f = |F_f|^{-1} \int_{F_f} v, \quad (37)$$

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \Pi_{cr}(\mathbf{v}) = (\pi_{cr}(v_i))_{i=1}^d. \quad (38)$$

We can now describe the  $\mathbf{P}_{nc}^1 \times P^0$  finite element [24, Example 4].

## 5. The $\mathbf{P}_{nc}^1 \times P^0$ finite element

The space of the discrete velocities of the Crouzeix-Raviart finite element is  $\mathbf{X}_{0,h}$ . The space of the discrete pressure is  $Q_{0,h}$ , such that:

$$Q_{0,h} := \mathcal{P}_{disc}^0(\mathcal{T}_h) \cap L_{zmv}^2(\Omega) \quad (39)$$

Let  $\tilde{\mathbf{V}}_{0,h}$  be the subspace of  $\mathbf{X}_{0,h}$  functions which divergence is discretely vanishing:

$$\tilde{\mathbf{V}}_{0,h} := \left\{ \mathbf{v}_h \in \mathbf{X}_{0,h} \mid \forall q_{0,h} \in Q_{0,h}, (\operatorname{div}_h \mathbf{v}_h, q_{0,h})_{L^2(\Omega)} = 0 \right\}. \quad (40)$$

This space is such that its velocities are piecewise divergence free [15, Lemma 3.1]:

$$\forall \mathbf{v}_h \in \tilde{\mathbf{V}}_{0,h}, \quad \forall \ell \in \mathcal{I}_K, \quad \operatorname{div} \mathbf{v}_{h,\ell} = 0. \quad (41)$$

We have the tools to establish the following result.

**Proposition 8.** *The Crouzeix-Raviart interpolation operator  $\Pi_{cr}$  can play the role of a Fortin operator for the  $\mathbf{P}_{nc}^1 \times P^0$  finite element:*

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (\operatorname{div}_h \Pi_{cr} \mathbf{v}, q_h)_{L^2(\Omega)} = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}, \quad \forall q_h \in \mathcal{P}_{disc}^0(\mathcal{T}_h). \quad (42)$$

$$\forall \mathbf{v} \in \mathbf{H}^{s+1}(\Omega) \quad \|\mathbf{v} - \Pi_{cr} \mathbf{v}\|_h \leq 2\sigma h^s |\mathbf{v}|_{\mathbf{H}^{s+1}(\Omega)}, \quad s \in [0, 1]. \quad (43)$$

Moreover, we have the following estimates:

$$\forall v \in H^1(\Omega), \quad \|\Pi_{cr}(\mathbf{v})\|_h \leq \|\mathbf{Grad} \mathbf{v}\|_{L^2(\Omega)}. \quad (44)$$

$$\forall v \in H^{1+s}(\Omega), \forall \ell \in \mathcal{I}_K, \quad \|\mathbf{v} - \Pi_{cr} \mathbf{v}\|_{L^2(\partial K_\ell)} \leq 2(\sigma_\ell h_\ell)^{\frac{1}{2}} (h_\ell)^s |\mathbf{v}|_{\mathbf{H}^{1+s}(K_\ell)}, \quad s \in [0, 1]. \quad (45)$$

*Proof.* By definition of  $\Pi_{cr}$  and using an integration by parts, we obtain that for all  $\ell \in \mathcal{I}_K$ :  $(\operatorname{div}(\Pi_{cr} \mathbf{v} - \mathbf{v}), q_h)_{L^2(K_\ell)} = \sum_{f \in \mathcal{I}_{f,\ell}} -q_{h|K_\ell} \int_{F_f} (\Pi_{cr} \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_f = 0$ . We deduce (42) by summation. For all  $v \in H^1(\Omega)$ ,  $\pi_{cr} v|_{K_\ell} = \pi_{cr}^\ell v$ . Then, we deduce (43) from (35), (44) from (33) and (45) from (36).  $\square$

Let  $\mathcal{J}_h : X_{0,h} \rightarrow Y_{0,h}$ , with  $Y_{0,h} = \{v_h \in H_0^1(\Omega) \mid \forall \ell \in \mathcal{I}_K, v_{h|K_\ell} \in P^k(K_\ell)\}$  be the averaging operator described in [29, §22.4.1]. There exists a constant  $C_{\mathcal{J}_h}$  independent of  $h$  and  $\sigma$  such that :

$$\forall v_h \in X_{0,h}, \quad \|\mathcal{J}_h v_h\|_{H_0^1(\Omega)} \leq \sigma C_{\mathcal{J}_h} \|v_h\|_h. \quad (46)$$

Let  $\ell_f \in \mathcal{L}(\mathbf{X}_{0,h}, \mathbb{R})$  be such that:

$$\forall \mathbf{v}_h \in \mathbf{X}_{0,h}, \quad \ell_f(\mathbf{v}_h) = \begin{cases} (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)} & \text{if } \mathbf{f} \in \mathbf{L}^2(\Omega) \\ \langle \mathbf{f}, \mathcal{J}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)} & \text{if } \mathbf{f} \in \mathbf{H}^{-1}(\Omega) \end{cases}. \quad (47)$$

The discretization of Problem (4) with the Crouzeix-Raviart finite element reads:

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{X}_{0,h} \times Q_{0,h} \text{ such that } \forall (\mathbf{v}_h, v_h) \in \mathbf{X}_{0,h} \times Q_{0,h} : \\ \nu(\mathbf{u}_h, \mathbf{v}_h)_h - (\operatorname{div}_h \mathbf{v}_h, p_h)_{L^2(\Omega)} - (\operatorname{div}_h \mathbf{u}_h, v_h)_{L^2(\Omega)} = \ell_f(\mathbf{v}_h). \end{cases} \quad (48)$$

We can build an explicit right inverse of the operator  $\operatorname{div}_h$  from  $Q_{0,h}$  to  $\mathbf{X}_{0,h}$ :

**Lemma 1.** *For all  $q_{0,h} \in Q_{0,h}$ , there exists  $\mathbf{v}_{q_{0,h}} \in \mathbf{X}_{0,h}$  such that:*

$$-(\operatorname{div}_h \mathbf{v}_{q_{0,h}}, q_{0,h})_{L^2(\Omega)} = \nu^{-1} \|q_{0,h}\|_{L^2(\Omega)}^2, \quad (49)$$

$$\|\mathbf{v}_{q_{0,h}}\|_h \leq \nu^{-1} C_{\operatorname{div}} \|q_{0,h}\|_{L^2(\Omega)}. \quad (50)$$

*Proof.* Let  $q_{0,h} \in Q_{0,h}$ . According to Proposition 1, it exists  $\tilde{\mathbf{v}}_{q_{0,h}} \in \mathbf{H}_0^1(\Omega)$  such that  $\operatorname{div} \tilde{\mathbf{v}}_{q_{0,h}} = \pi_0(q_{0,h})$  and  $\|\tilde{\mathbf{v}}_{q_{0,h}}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|q_{0,h}\|_{L^2(\Omega)}$ . Consider  $\mathbf{v}_{q_{0,h}} = -\nu^{-1} \Pi_{cr}(\tilde{\mathbf{v}}_{q_{0,h}}) \in \mathbf{X}_{0,h}$ . We have  $\forall \ell \in \mathcal{I}_K$ :

$$\begin{aligned} |K_\ell| \operatorname{div} \mathbf{v}_{q_{0,h}|K_\ell} &= \int_{K_\ell} \operatorname{div} \mathbf{v}_{q_{0,h}} = \sum_{f \in \mathcal{I}_{F,\ell}^i} \int_{F_f} \mathbf{v}_{q_{0,h}} \cdot \mathbf{n}_{f,\ell} && \text{integrating by parts,} \\ &= -\nu^{-1} \sum_{f \in \mathcal{I}_{F,\ell}^i} \int_{F_f} \tilde{\mathbf{v}}_{q_{0,h}} \cdot \mathbf{n}_{f,\ell} && \text{by definition of } \mathbf{v}_{q_{0,h}}, \\ &= -\nu^{-1} \int_{K_\ell} \operatorname{div} \tilde{\mathbf{v}}_{q_{0,h}} && \text{integrating by parts,} \\ &= -\nu^{-1} |K_\ell| q_{0,h}|K_\ell && \text{by definition of } \tilde{\mathbf{v}}_{q_{0,h}}. \end{aligned}$$

We deduce that  $\operatorname{div} \mathbf{v}_{q_{0,h}|K_\ell} = -\nu^{-1} q_{0,h}|K_\ell$ , hence (49). Equation (50) is obtained using (44):

$$\|\mathbf{v}_{q_{0,h}}\|_h = \nu^{-1} \|\Pi_{cr} \tilde{\mathbf{v}}_{q_{0,h}}\|_h \leq \nu^{-1} \|\tilde{\mathbf{v}}_{q_{0,h}}\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} C_{\operatorname{div}} \|q_{0,h}\|_{L^2(\Omega)}.$$

□

Hence, Problem (48) is well posed and we can derive the following error estimates.

**Theorem 2** ([24, Theorems 3, 4, 6]). *Suppose that the solution to Problem (4) is such that  $(\mathbf{u}, p) \in (\mathbf{V}_0 \cap \mathbf{H}^2(\Omega)) \times (L_{zmv}^2(\Omega) \cap H^1(\Omega))$ . Let us set  $\|(\mathbf{u}, p)\|_\nu = \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \nu^{-1} \|p\|_{H^1(\Omega)}$ . Let  $(\mathbf{u}_h, p_h) \in \tilde{\mathbf{V}}_{0,h} \times Q_{0,h}$  be the solution to Problem (48). We have the following error estimates:*

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \nu^{-1} \|p - p_h\|_{L^2(\Omega)} \lesssim \sigma h \|(\mathbf{u}, p)\|_\nu, \quad (51)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma^2 h^2 \|(\mathbf{u}, p)\|_\nu \quad \text{if } \Omega \text{ is convex.} \quad (52)$$

Consider Problem (4) with data  $\mathbf{f} = \mathbf{grad} \phi$ , where  $\phi \in H^1(\Omega) \cap L_{zmv}^2(\Omega)$ . The unique solution is then  $(\mathbf{u}, p) := (0, \phi)$ . By integrating by parts, the source term in (7) reads:

$$\forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (\mathbf{f}, \mathbf{v})_{\mathbf{L}^2(\Omega)} = -(\operatorname{div} \mathbf{v}, \phi)_{L^2(\Omega)}. \quad (53)$$

Using a nonconforming finite element method, the integration by parts must be done element by element:

$$\forall \mathbf{v}_h \in \mathcal{P}_h \mathbf{H}^1, \quad (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = -(\operatorname{div}_h \mathbf{v}_h, \phi)_{L^2(\Omega)} + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_h \cdot \mathbf{n}_{f,\ell}, \phi)_{L^2(F_f)}. \quad (54)$$

Applying (54) to the right-hand side of (48), choosing  $\mathbf{v}_h = \mathbf{u}_h \in \tilde{\mathbf{V}}_{0,h}$ , we have, using (41) and (28):

$$\nu \|\mathbf{u}_h\|_h \lesssim \sigma h \|\mathbf{grad} \phi\|_{\mathbf{L}^2(\Omega)},$$

The second term in (54) acts as a numerical source for the discrete velocity, whose numerical influence is proportional to  $h/\nu$ . Thus, we cannot obtain exactly  $\mathbf{u}_h = 0$  as expected. Linke proposed in [40] to project the test function  $\mathbf{v}_h \in \mathbf{X}_h$  on a discrete subspace of  $\mathbf{H}(\operatorname{div}; \Omega)$ , like Raviart-Thomas or Brezzi-Douglas-Marini finite elements [12]. Alternatively, one can use a multi-points flux approximation to discretize the pressure gradient term in (4) [19, 20].

In the next Section, we study an alternative solution which consists in reinforcing the constraint by using  $P^0$  plus  $P^1$  discrete pressures.

## 6. Stability of the $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$ finite element

In this Section, we show that the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element satisfies a uniform inf-sup condition [37], exhibiting the dependency on the shape regularity parameter  $\sigma$ . We omit the subscript  $h$  to refer to finite element functions. In order to improve the accuracy of the  $\mathbf{P}_{nc}^1 \times P^0$  finite element, Bernardi and Hecht propose in [10] to replace  $Q_{0,h}$  by the space of  $P^1$ -Lagrange pressure (called  $Q_{1,h}$  below) plus local bubble functions, defined in each element as the product of the barycentric coordinates. However, for this discretization, the convergence of the discrete pressure is not guaranteed, as shown in Heib's thesis [36]. In this thesis, a new finite element is introduced, for which the space of the discrete pressures is the following:

$$Q_h = Q_{0,h} \oplus Q_{1,h}, \text{ where for all } k > 0, Q_{k,h} := \mathcal{P}^k(\mathcal{T}_h) \cap L_{zmv}^2(\Omega). \quad (55)$$

The sum is direct: Let  $(q_0, q_1) \in Q_{0,h} \times Q_{1,h}$  be a couple such that  $q_0 + q_1 = 0$ . Then, we have  $q_1 = -q_0$  over  $\mathcal{T}_h$ . Since  $q_1$  is continuous over  $\mathcal{T}_h$ , it means that  $q_1 = -q_0$  are constant. They are equal to zero since  $Q_{1,h} \subset L_{zmv}^2(\Omega)$ . Let us set:

$$\tilde{Q}_{1,h} = \left\{ q \in Q_h \mid \forall \ell \in \mathcal{I}_K, \int_{K_\ell} q = 0 \right\}. \quad (56)$$

**Proposition 9.** *The space  $\tilde{Q}_{1,h}$  is such that  $Q_h = Q_{0,h} \dot{\oplus} \tilde{Q}_{1,h}$ , and we have:*

$$\forall q \in Q_h, \quad \|q\|_{L^2(\Omega)}^2 = \|\pi_0(q)\|_{L^2(\Omega)}^2 + \|q - \pi_0(q)\|_{L^2(\Omega)}^2. \quad (57)$$

*Proof.* Let  $q \in Q_h$  such that  $q = q_0 + q_1$ , where for  $k \in \{0, 1\}$ ,  $q_k \in Q_{k,h}$ . Notice that:  $q - \pi_0(q) = q_1 - \pi_0(q_1) + q_0 - \pi_0(q_0) = q_1 - \pi_0(q_1)$ , since  $q_0 = \pi_0(q_0)$ . Setting  $\tilde{q} = q_1 - \pi_0(q_1)$ , we have:  $(\pi_0(q), \tilde{q})_{L^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} \pi_0(q)(K_\ell) \int_{K_\ell} \tilde{q} = 0$ , so that:  $\|q\|_{L^2(\Omega)}^2 = \|\pi_0(q) + \tilde{q}\|_{L^2(\Omega)}^2 = \|\pi_0(q)\|_{L^2(\Omega)}^2 + \|\tilde{q}\|_{L^2(\Omega)}^2$ .  $\square$

In order to prove the inf-sup condition of the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element, we need inequality (29), Lemma 1, and another result stated in the next Lemma, which requires the following Hypothesis [10, Hypothesis 4.1]:

**Hypothesis 1.** *We suppose that the triangulation  $\mathcal{T}_h$  is such that the boundary  $\partial\Omega$  contains at most one edge in dimension  $d = 2$  and at most two faces in dimension  $d = 3$ , of the same element  $K_\ell$ ,  $\ell \in \mathcal{I}_K$ .*

Let us set:  $\mathbf{X}_{0,h}^T := \{\mathbf{v} \in \mathbf{X}_{0,h} \mid \forall f \in \mathcal{I}_F, \mathbf{v}_f \cdot \mathbf{n}_f = 0\}$ . Notice that by construction,  $\mathbf{X}_{0,h}^T \subset \tilde{\mathbf{V}}_{0,h}$  since:

$$\forall \mathbf{v} \in \mathbf{X}_{0,h}^T, \forall \ell \in \mathcal{I}_K, \quad \int_{K_\ell} \operatorname{div} \mathbf{v} = - \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} \mathbf{v} \cdot \mathbf{n}_{f,\ell} = \sum_{f \in \mathcal{I}_{F,\ell}} |F_f| \mathbf{v}_f \cdot \mathbf{n}_{f,\ell} = 0. \quad (58)$$

**Lemma 3.** *Under Hypothesis 1, for all  $q_1 \in Q_{1,h}$ , there exists  $\mathbf{v}_{q_1} \in \mathbf{X}_{0,h}^T$  such that:*

$$(\operatorname{div} \mathbf{v}_{q_1}, \pi_0(q))_{L^2(K_\ell)} = 0 \quad \forall q \in Q_h, \quad (59)$$

$$(\mathbf{grad} q_1, \mathbf{v}_{q_1})_{\mathbf{L}^2(\Omega)} \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|q_1 - \pi_0(q_1)\|_{L^2(\Omega)}^2, \quad (60)$$

$$\|\mathbf{v}_{q_1}\|_h \lesssim \nu^{-1} \sigma^2 \|q_1 - \pi_0(q_1)\|_{L^2(\Omega)}. \quad (61)$$

*Proof.* Let us consider  $q_1 \in Q_{1,h}$ . For all  $\ell \in \mathcal{I}_K$ , we set  $q_{1,\ell} = q_1|_{K_\ell}$  and  $\hat{q}_{1,\ell} = q_1 \circ T_\ell$ . Since  $q_1 \in H^1(\Omega)$  we have [6, Proposition 2.2.10]: for all  $f \in \mathcal{I}_F^i$  such that  $F_f = K_\ell \cap K_{\ell'}$ ,  $\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f = \mathbf{grad} q_{1,\ell'} \times \mathbf{n}_f$ . We can then construct  $\mathbf{v}_{q_1} \in \mathbf{X}_{0,h}^T$  such that<sup>1</sup>:

$$\forall f \in \mathcal{I}_F^i \quad \mathbf{v}_{q_1}(M_f) = \nu^{-1} (h_f)^2 \mathbf{n}_f \times (\mathbf{grad} q_1 \times \mathbf{n}_f). \quad (62)$$

<sup>1</sup>In 2D,  $\mathbf{grad} q_1 \times \mathbf{n}_f$  is a scalar, hence we make the following abuse of notation:  $\mathbf{n}_f \times (\mathbf{grad} q_1 \times \mathbf{n}_f) = (\mathbf{grad} q_1 \times \mathbf{n}_f) \boldsymbol{\tau}_f$  where  $(\boldsymbol{\tau}_f, \mathbf{n}_f)$  is an orthogonal basis.

Since  $\mathbf{v}_{q_1} \in \mathbf{X}_{0,h}^T$ , Equation (59) holds, hence (58).

Let us prove (60). Noticing that  $\forall \ell \in \mathcal{I}_K, \forall f \in \mathcal{I}_{F,\ell}, \int_{K_\ell} \psi_f = \frac{|K_\ell|}{d+1}$ , we have:

$$\begin{aligned}
(\mathbf{grad} q_1, \mathbf{v}_{q_1})_{\mathbf{L}^2(K_\ell)} &= (d+1)^{-1} |K_\ell| \sum_{f \in \mathcal{I}_{F,\ell}^i} \mathbf{grad} q_{1,\ell} \cdot \mathbf{v}_{q_1}(M_f), \\
&= \nu^{-1} (d+1)^{-1} |K_\ell| \sum_{f \in \mathcal{I}_{F,\ell}^i} (h_f)^2 \mathbf{grad} q_{1,\ell} \cdot \mathbf{n}_f \times (\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f), \\
&= \nu^{-1} (d+1)^{-1} |K_\ell| \sum_{f \in \mathcal{I}_{F,\ell}^i} (h_f)^2 |\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f|^2.
\end{aligned} \tag{63}$$

Using Equations (14) and (19), we have:

$$\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f = |J_\ell| |J_{f,\ell}|^{-1} (\mathbb{B}_\ell^{-1})^T \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times (\mathbb{B}_\ell^{-1})^T \mathbf{n}_{\hat{F}_{f,\ell}}.$$

By a direct computation, we show that:

$$(\mathbb{B}_\ell^{-1})^T \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times (\mathbb{B}_\ell^{-1})^T \mathbf{n}_{\hat{F}_{f,\ell}} = (J_\ell)^{-1} (\mathbb{B}_\ell)^{d-2} \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}}.$$

Hence:

$$|\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f| = |J_{f,\ell}|^{-1} \left| (\mathbb{B}_\ell)^{d-2} \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right|. \tag{64}$$

Notice that we have:

$$\begin{aligned}
\left| (\mathbb{B}_\ell)^{d-2} \left( \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right) \right| &\geq \left\| \mathbb{B}_\ell^{-1} \right\|^{-(d-2)} \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right| \\
&\gtrsim (\rho_\ell)^{d-2} \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right|.
\end{aligned} \tag{65}$$

Using Equation (65) in (64) times  $h_f$ , squared, we obtain:

$$(h_f)^2 |\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f|^2 \gtrsim (|J_{f,\ell}|^{-1} h_f)^2 (\rho_\ell)^{2(d-2)} \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right|^2.$$

Notice that  $(|J_{f,\ell}|^{-1} h_f)^2 \gtrsim (h_f)^{-2(d-2)} \geq (h_\ell)^{-2(d-2)}$ , hence:

$$(|J_{f,\ell}|^{-1} h_f)^2 (\rho_\ell)^{2(d-2)} \gtrsim \left( \frac{h_\ell}{\rho_\ell} \right)^{-2(d-2)} \geq (\sigma_\ell)^{-2(d-2)}.$$

It follows that:

$$(h_f)^2 |\mathbf{grad} q_{1,\ell} \times \mathbf{n}_f|^2 \gtrsim (\sigma_\ell)^{-2(d-2)} \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right|^2. \tag{66}$$

According to Hypothesis 1, we have for all  $\ell \in \mathcal{I}_K$ :

$$\sum_{f \in \mathcal{I}_{F,\ell}^i} \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \times \mathbf{n}_{\hat{F}_{f,\ell}} \right|^2 \gtrsim \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \right|^2. \tag{67}$$

We can now bound the left-hand term in (63) using Equations (66) and (67):

$$(\mathbf{grad} q_1, \mathbf{v}_{q_1})_{\mathbf{L}^2(K_\ell)} \gtrsim \nu^{-1} (\sigma_\ell)^{-2(d-2)} |K_\ell| \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \right|^2. \tag{68}$$

Using inequality (11) in the reference element and the change of variable (13), we have:

$$\begin{aligned}
|K_\ell| \left| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \right|^2 &= |K_\ell| |\hat{K}|^{-1} \left\| \mathbf{grad}_{\hat{\mathbf{x}}} \hat{q}_{1,\ell} \right\|_{\mathbf{L}^2(\hat{K})}^2, \\
&\gtrsim |K_\ell| \left\| \hat{q}_{1,\ell} - \underline{q}_{1,\ell} \right\|_{L^2(\hat{K})}^2, \\
&\gtrsim \left\| q_{1,\ell} - \underline{q}_{1,\ell} \right\|_{L^2(K_\ell)}^2 \text{ where } \underline{q}_{1,\ell} = |K_\ell|^{-1} \int_{K_\ell} q_1.
\end{aligned} \tag{69}$$

Thus, using (69) in (68), and summing over  $\ell \in \mathcal{I}_K$ , we obtain (60).

Let us prove (61). For all  $\ell \in \mathcal{I}_K$ ,  $\mathbf{Grad} \mathbf{v}_{q_1|_{K_\ell}} = \sum_{f \in \mathcal{I}_{F,\ell}^1} \mathbf{v}_{q_1}(M_f) \otimes \mathbf{grad} \psi_{f|_{K_\ell}}$ . We deduce that:

$$\begin{aligned}
\| \mathbf{Grad} \mathbf{v}_{q_1} \|_{\mathbb{L}^2(K_\ell)}^2 &\lesssim |K_\ell| \sum_{f \in \mathcal{I}_{F,\ell}^1} |\mathbf{v}_{q_1}(M_f)|^2 |\mathbf{grad} \psi_{f|_{K_\ell}}|^2, \\
&\lesssim \nu^{-2} |K_\ell| \sum_{f \in \mathcal{I}_{F,\ell}^1} (h_f)^4 |\mathbf{n}_f \times \mathbf{grad} q_1 \times \mathbf{n}_f|^2 (|K_\ell|^{-2} |F_f|^2) \quad \text{since } |\mathbf{grad} \psi_{f|_{K_\ell}}|^2 = |K_\ell|^{-2} |F_f|^2, \\
&\lesssim \nu^{-2} |K_\ell| (h_\ell)^4 (\rho_\ell)^{-2} \sum_{f \in \mathcal{I}_{F,\ell}^1} |\mathbf{n}_f \times \mathbf{grad} q_1 \times \mathbf{n}_f|^2 \quad \text{using (17),} \\
&\lesssim \nu^{-2} (\sigma_\ell h_\ell)^2 |K_\ell| |\mathbf{grad} q_1|^2 = \nu^{-2} (\sigma_\ell h_\ell)^2 \| \mathbf{grad} q_1 \|_{\mathbb{L}^2(K_\ell)}^2 \quad \text{using Hypothesis 1,} \\
&\lesssim \nu^{-2} (\sigma_\ell)^4 \|q_{1,\ell} - \underline{q}_{1,\ell}\|_{L^2(K_\ell)}^2 \quad \text{using (22).}
\end{aligned}$$

□

Let  $b_h(\cdot, \cdot)$  be the bilinear form such that:

$$b_h : \begin{cases} \mathbf{X}_h \times Q_h & \mapsto \mathbb{R} \\ (\mathbf{v}, q) & \rightarrow -(\operatorname{div}_h \mathbf{v}, q_0)_{L^2(\Omega)} + (\mathbf{v}, \mathbf{grad} q_1)_{\mathbb{L}^2(\Omega)}, \end{cases} \quad (70)$$

with  $q = q_0 + q_1$  such that  $(q_0, q_1) \in Q_{0,h} \times Q_{1,h}$ . Under Hypothesis 1, the bilinear form  $b_h(\cdot, \cdot)$  is continuous and is satisfies a discrete inf-sup condition:

**Theorem 4** (Stability of the coupling bilinear form). *The following continuity property holds:*

$$\forall \mathbf{v} \in \mathbf{X}_{0,h}, \forall q \in Q_h, \quad |b_h(\mathbf{v}, q)| \lesssim \sigma \| \mathbf{v} \|_h \|q\|_{L^2(\Omega)}, \quad (71)$$

Suppose that the triangulation satisfies Hypothesis 1, then the following inf-sup condition holds:

$$\forall q \in Q_h, \exists \mathbf{v} \in \mathbf{X}_{0,h}, \quad b_h(\mathbf{v}, q) \gtrsim (C_{\operatorname{div}})^{-2} \sigma^{-2d} \| \mathbf{v} \|_h \|q\|_{L^2(\Omega)}, \quad (72)$$

*Proof.* Let us prove (71). Consider  $(\mathbf{v}, q) \in \mathbf{X}_{0,h} \times Q_h$ , where  $q = q_0 + q_1$ ,  $(q_0, q_1) \in Q_{0,h} \times Q_{1,h}$ . Integrating by parts the last term of (70), we obtain that:

$$b_h(\mathbf{v}, q) = -(\operatorname{div}_h \mathbf{v}, \pi_0(q))_{L^2(\Omega)} + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v} \cdot \mathbf{n}_{f,\ell} q_1)_{L^2(F_f)}. \quad (73)$$

On the one hand, using Cauchy-Schwarz inequality, we have:

$$|(\operatorname{div}_h \mathbf{v}, \pi_0(q))_{L^2(\Omega)}| \leq \sqrt{d} \| \mathbf{v} \|_h \| \pi_0(q) \|_{L^2(\Omega)} \quad (74)$$

On the other hand, using inequality (29), it holds:

$$\left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v} \cdot \mathbf{n}_{f,\ell} q_1)_{L^2(F_f)} \right| \lesssim \sigma \| \mathbf{v} \|_h \|q_1 - \pi_0(q_1)\|_{L^2(\Omega)} = \sigma \| \mathbf{v} \|_h \|q - \pi_0(q)\|_{L^2(\Omega)}. \quad (75)$$

We obtain (71) summing (74) and (75), and using (57).

Let us prove (72). Consider  $q \neq 0$ . Let  $\mathbf{v}_{\pi_0(q)}$  be built like  $\mathbf{v}_{h,q_0}$  in Lemma 1 with  $q_0 = \pi_0(q)$  here and  $\mathbf{v}_{q_1}$  be built like in Lemma 3. We have:

$$-\left( \operatorname{div}_h \mathbf{v}_{\pi_0(q)}, \pi_0(q) \right)_{L^2(\Omega)} = \nu^{-1} \| \pi_0(q) \|_{L^2(\Omega)}^2, \quad (76)$$

$$\| \mathbf{v}_{\pi_0(q)} \|_h \leq \nu^{-1} C_{\operatorname{div}} \| \pi_0(q) \|_{L^2(\Omega)}, \quad (77)$$

$$(\operatorname{div}_h \mathbf{v}_{q_1}, \pi_0(q))_{L^2(\Omega)} = 0, \quad (78)$$

$$(\mathbf{grad} q_1, \mathbf{v}_{q_1})_{\mathbb{L}^2(\Omega)} \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|q - \pi_0(q)\|_{L^2(\Omega)}^2, \quad (79)$$

$$\| \mathbf{v}_{q_1} \|_h \lesssim \nu^{-1} \sigma^2 \|q - \pi_0(q)\|_{L^2(\Omega)}. \quad (80)$$

Using equality (73) with  $\mathbf{v} = \mathbf{v}_{\pi_0(q)}$ , we have, due to (76):

$$b_h(\mathbf{v}_{\pi_0(q)}, q) = \nu^{-1} \|\pi_0(q)\|_{L^2(\Omega)}^2 + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_{\pi_0(q)} \cdot \mathbf{n}_{f,\ell} q_1)_{L^2(F_f)}.$$

Using inequality (75) with  $\mathbf{v} = \mathbf{v}_{\pi_0(q)}$ , and applying (77), it holds:

$$b_h(\mathbf{v}_{\pi_0(q)}, q) \gtrsim \nu^{-1} \left( \|\pi_0(q)\|_{L^2(\Omega)}^2 - C_{\text{div}}^\sigma \|\pi_0(q)\|_{L^2(\Omega)} \|q - \pi_0(q)\|_{L^2(\Omega)} \right) \text{ with } C_{\text{div}}^\sigma = C_{\text{div}} \sigma.$$

Using Young inequality, we have for all  $\varepsilon > 0$ :

$$b_h(\mathbf{v}_{\pi_0(q)}, q) \geq \nu^{-1} \left( 1 - \frac{\varepsilon}{2} C_{\text{div}}^\sigma \right) \|\pi_0(q)\|_{L^2(\Omega)}^2 - \nu^{-1} \frac{\varepsilon^{-1}}{2} C_{\text{div}}^\sigma \|q - \pi_0(q)\|_{L^2(\Omega)}^2. \quad (81)$$

Now, using Equations (78) and (79), we have:

$$b_h(\mathbf{v}_{q_1}, q) = (\mathbf{grad} q_1, \mathbf{v}_{q_1})_{L^2(\Omega)} \gtrsim \nu^{-1} \sigma^{-2(d-2)} \|q - \pi_0(q)\|_{L^2(\Omega)}^2. \quad (82)$$

Let us set  $\mathbf{v}^* = \mu \mathbf{v}_{\pi_0(q)} + \mathbf{v}_{q_1}$ , where  $\mu > 0$  is a constant. We have:  $b_h(\mathbf{v}^*, q) = \mu b_h(\mathbf{v}_{\pi_0(q)}, q) + b_h(\mathbf{v}_{q_1}, q)$ . Summing Equation (81) times  $\mu$  with Equation (82), we obtain:

$$b_h(\mathbf{v}^*, q) \gtrsim \mu \nu^{-1} \left( 1 - \frac{\varepsilon}{2} C_{\text{div}}^\sigma \right) \|\pi_0(q)\|_{L^2(\Omega)}^2 + \nu^{-1} \left( \sigma^{-2(d-2)} - \frac{\mu \varepsilon^{-1}}{2} C_{\text{div}}^\sigma \right) \|q - \pi_0(q)\|_{L^2(\Omega)}^2.$$

We need to enforce that  $1 - \frac{\varepsilon}{2} C_{\text{div}}^\sigma > 0$  and  $\sigma^{-2(d-2)} - \frac{\mu \varepsilon^{-1}}{2} C_{\text{div}}^\sigma > 0$ , which amounts to requiring  $\varepsilon < 2 (C_{\text{div}}^\sigma)^{-1}$  and  $\mu < 2 (C_{\text{div}}^\sigma)^{-1} \sigma^{-2(d-2)} \varepsilon$ . Setting for instance  $\varepsilon = (C_{\text{div}}^\sigma)^{-1}$  and  $\mu = (C_{\text{div}}^\sigma)^{-1} \varepsilon \sigma^{-2(d-2)}$ , we obtain:

$$b_h(\mathbf{v}^*, q) \gtrsim \frac{1}{2} \left( \mu \nu^{-1} \|\pi_0(q)\|_{L^2(\Omega)}^2 + \nu^{-1} \sigma^{-2(d-2)} \|q - \pi_0(q)\|_{L^2(\Omega)}^2 \right).$$

Since  $C_{\text{div}} \geq 1$  and  $\sigma \geq 1$ , we have:  $(C_{\text{div}}^\sigma)^{-2} \leq 1$  and  $\mu = (C_{\text{div}}^\sigma)^{-2} \sigma^{-2(d-2)} \leq \sigma^{-2(d-2)}$ . Hence:

$$b_h(\mathbf{v}^*, q) \gtrsim \frac{1}{2} \nu^{-1} (C_{\text{div}}^\sigma)^{-2} \sigma^{-2(d-1)} \left( \|\pi_0(q)\|_{L^2(\Omega)}^2 + \|q - \pi_0(q)\|_{L^2(\Omega)}^2 \right). \quad (83)$$

We will now bound  $\|\mathbf{v}^*\|_h$  by  $\|q\|_{L^2(\Omega)}$ . Using Equations (77) and (80), we obtain:

$$\begin{aligned} \|\mathbf{v}^*\|_h^2 &\leq 2 \left( \mu^2 \|\mathbf{v}_{\pi_0(q)}\|_h^2 + \|\mathbf{v}_{q_1}\|_h^2 \right), \\ &\leq 2 \nu^{-2} \left( \sigma^{-4(d-1)} (C_{\text{div}}^\sigma)^{-2} \|\pi_0(q)\|_{L^2(\Omega)}^2 + \sigma^4 \|q - \pi_0(q)\|_{L^2(\Omega)}^2 \right), \\ &\leq 2 \nu^{-2} \sigma^4 \|q\|_{L^2(\Omega)}^2. \end{aligned}$$

Hence:  $\nu^{-1} \|q\|_{L^2(\Omega)} \geq \frac{1}{\sqrt{2}} \sigma^{-2} \|\mathbf{v}^*\|_h$ . Using this bound in (83), we obtain (72).  $\square$

**Remark 2.** We exhibit that the bound of the discrete inf-sup condition depends on  $C_{\text{div}}$  and  $\sigma$ . We could further optimize the choice of  $\varepsilon$  and  $\mu$  in the proof above to obtain a sharper bound.

Let  $\mathbf{V}_{0,h} \subset \tilde{\mathbf{V}}_{0,h}$  be the kernel of the bilinear form  $b_h(\cdot, \cdot)$ :

$$\mathbf{V}_{0,h} := \{ \mathbf{v} \in \mathbf{X}_{0,h} \mid \forall q \in Q_h, b_h(\mathbf{v}, q) = 0 \}. \quad (84)$$

The  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element method allows to improve accuracy compared to the  $\mathbf{P}_{nc}^1 \times P^0$  finite element method:

**Proposition 10.** *Let  $\mathbf{v} \in \mathbf{V}_{0,h}$ . Then for all  $q_1 \in Q_{1,h}$ :  $\sum_{f \in \mathcal{I}_F} (q_1, [\mathbf{v}]_f \cdot \mathbf{n}_f)_{L^2(F_f)} = 0$ .*

*Proof.* Let  $\mathbf{v} \in \mathbf{V}_{0,h}$  and  $q_1 \in Q_{1,h}$ . We then have:

$$-(\operatorname{div}_h \mathbf{v}, \pi_0(q_1))_{L^2(\Omega)} = 0 \quad \text{and} \quad (\mathbf{grad} q_1, \mathbf{v})_{\mathbf{L}^2(\Omega)} = 0.$$

Using integration by parts on all the elements, we get that:

$$0 = (\mathbf{grad} q_1, \mathbf{v})_{\mathbf{L}^2(\Omega)} = -(\operatorname{div}_h \mathbf{v}, \pi_0(q_1))_{L^2(\Omega)} + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (q_1, \mathbf{v} \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} = \sum_{f \in \mathcal{I}_F} (q_1, [\mathbf{v}]_f \cdot \mathbf{n}_f)_{L^2(F_f)}.$$

□

We proved the stability of the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element, cf. Theorem 4 and we showed that the accuracy is improved with respect to the  $\mathbf{P}_{nc}^1 \times P^0$  finite element method, cf. Proposition 10. Hence, we have the tools to prove that the discretization of Problem (4) with the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element is well posed.

## 7. Well-posedness

Let us set  $\mathcal{X}_h := \mathbf{X}_{0,h} \times Q_h$ . We endow  $\mathcal{X}_h$  with the norm:  $\|(\mathbf{v}, q)\|_{\mathcal{X}_h} := (\|\mathbf{v}\|_h^2 + \nu^{-2} \|q\|_{L^2(\Omega)}^2)^{1/2}$ , so that it is a Hilbert space. Consider the symmetric and continuous bilinear form  $a_h : \mathcal{X}_h \times \mathcal{X}_h \rightarrow \mathbb{R}$  such that for all  $((\mathbf{u}'_h, p'_h), (\mathbf{v}_h, q_h)) \in \mathcal{X}_h \times \mathcal{X}_h$ :

$$a_h((\mathbf{u}'_h, p'_h), (\mathbf{v}_h, q_h)) = \nu(\mathbf{u}'_h, \mathbf{v}_h)_h + b_h(\mathbf{v}_h, p'_h) + b_h(\mathbf{u}'_h, q_h). \quad (85)$$

The discrete variational formulation of Problem (4) with the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element reads:

$$\text{Find } (\mathbf{u}_h, p_h) \in \mathcal{X}_h \text{ such that } \forall (\mathbf{v}_h, q_h) \in \mathcal{X}_h, a_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \ell_{\mathbf{f}}(\mathbf{v}_h), \quad (86)$$

The main consequence of the inf-sup condition (72) is that Problem (86) is well posed [12]. We have the following result.

**Theorem 5** (Stability of the solution). *If Hypothesis 1 holds, for any data  $\mathbf{f}$ , Problem (86) has a unique solution  $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$  such that:*

$$\|\mathbf{u}_h\|_h \lesssim \nu^{-1} \sigma C_{\mathbf{f}}, \quad (87)$$

$$\|p_h - \pi_0(p_h)\|_{L^2(\Omega)} \lesssim 2 \sigma^{2d-1} C_{\mathbf{f}}, \quad (88)$$

$$\|\pi_0(p_h)\|_{L^2(\Omega)} \lesssim 2 \sigma^{2d} C_{\operatorname{div}} C_{\mathbf{f}}. \quad (89)$$

where  $C_{\mathbf{f}} = C_{\Omega} h_{\Omega} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}$  if  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ , or  $C_{\mathbf{f}} = C_{\mathcal{J}_h} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$  if  $\mathbf{f} \notin \mathbf{L}^2(\Omega)$ .

*Proof.* We write the proof for  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ . It is very similar when  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ , using inequality (46).

Let  $(\mathbf{u}_h, p_h)$  be the unique solution to Problem (86), with  $p_h = p_{0,h} + p_{1,h}$ ,  $(p_{0,h}, p_{1,h}) \in Q_{0,h} \times Q_{1,h}$ .

Let us prove (87). Choosing  $\mathbf{v}_h = 0$  in (86), we have:  $\mathbf{u}_h \in \mathbf{V}_{0,h}$ . Choosing  $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, 0)$  in (86), we then obtain:  $\nu \|\mathbf{u}_h\|_h^2 = (\mathbf{f}, \mathbf{u}_h)_{\mathbf{L}^2(\Omega)}$ . Using Cauchy-Schwarz inequality and the discrete Poincaré-Steklov inequality (31), we get inequality (87).

Let us prove (88). Choosing  $(\mathbf{v}_h, q_h) = (\mathbf{v}_{p_{1,h}}, 0)$  in (86), where  $\mathbf{v}_{p_{1,h}}$  is constructed like  $\mathbf{v}_{q_1}$  in Lemma 3 with  $q_1 = p_{1,h}$  here, and reshuffling the terms, we obtain:

$$(\mathbf{grad} p_{1,h}, \mathbf{v}_{p_{1,h}})_{\mathbf{L}^2(\Omega)} = (\mathbf{f}, \mathbf{v}_{p_{1,h}})_{\mathbf{L}^2(\Omega)} - \nu(\mathbf{u}_h, \mathbf{v}_{p_{1,h}})_h.$$

Using the the triangular inequality and Cauchy-Schwarz, we get:

$$\left| (\mathbf{v}_{p_{1,h}}, \mathbf{grad} p_{1,h})_{\mathbf{L}^2(\Omega)} \right| \leq \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_{p_{1,h}}\|_{\mathbf{L}^2(\Omega)} + \nu \|\mathbf{u}_h\|_h \|\mathbf{v}_{p_{1,h}}\|_h.$$

Now, using the discrete Poincaré-Steklov inequality (31) and inequality (87), we have:

$$\left| (\mathbf{v}_{p_{1,h}}, \mathbf{grad} p_{1,h})_{\mathbf{L}^2(\Omega)} \right| \lesssim 2 \sigma C_\Omega h_\Omega \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_{p_{1,h}}\|_h.$$

Using inequality (61), we deduce that:

$$\left| (\mathbf{v}_{p_{1,h}}, \mathbf{grad} p_{1,h})_{\mathbf{L}^2(\Omega)} \right| \lesssim 2 \nu^{-1} \sigma^3 C_\Omega h_\Omega \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|p_{1,h} - \pi_0(p_{1,h})\|_{L^2(\Omega)}.$$

Hence, using inequality (60), we get:

$$\nu^{-1} \sigma^{-2(d-2)} \|p_{1,h} - \pi_0(p_{1,h})\|_{L^2(\Omega)}^2 \lesssim 2 \nu^{-1} \sigma^3 C_\Omega h_\Omega \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|p_{1,h} - \pi_0(p_{1,h})\|_{L^2(\Omega)}.$$

We conclude that:

$$\|p_h - \pi_0(p_h)\|_{L^2(\Omega)} = \|p_{1,h} - \pi_0(p_{1,h})\|_{L^2(\Omega)} \lesssim 2 \sigma^{2d-1} C_\Omega h_\Omega \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

Let us prove (89). Consider  $(\mathbf{v}_h, q_h) = (\mathbf{v}_{\pi_0(p_h)}, 0)$  in (86), where  $\mathbf{v}_{\pi_0(p_h)}$  is built like  $\mathbf{v}_{q_{0,h}}$  in Lemma 1 with  $q_{0,h} = \pi_0(p_h)$  here. Using Equation (73) with  $q = p_h$ , Equation (49) and reshuffling the terms, it holds:

$$\nu^{-1} \|\pi_0(q)\|_{L^2(\Omega)}^2 = (\mathbf{f}, \mathbf{v}_{\pi_0(p_h)})_{\mathbf{L}^2(\Omega)} - \nu (\mathbf{u}_h, \mathbf{v}_{\pi_0(p_h)})_h - \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{v}_{\pi_0(p_h)} \cdot \mathbf{n}_{f,\ell}, p_{1,h})_{L^2(F_f)}.$$

Using the Cauchy-Schwarz and inequality (29), we have:

$$\nu^{-1} \|\pi_0(q)\|_{L^2(\Omega)}^2 \leq \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_{\pi_0(p_h)}\|_{\mathbf{L}^2(\Omega)} + \nu \|\mathbf{u}_h\|_h \|\mathbf{v}_{\pi_0(p_h)}\|_h + \sigma \|\mathbf{v}_{\pi_0(p_h)}\|_h \|p_{1,h} - \pi_0(p_{1,h})\|_{L^2(\Omega)}.$$

Using the discrete Poincaré-Steklov inequality (31) and inequality (49), it holds:

$$\|\pi_0(q)\|_{L^2(\Omega)}^2 \leq C_{\text{div}} \left( \sigma C_\Omega h_\Omega \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \nu \|\mathbf{u}_h\|_h + \sigma \|p_{1,h} - \pi_0(p_{1,h})\|_{L^2(\Omega)} \right) \|\pi_0(q)\|_{L^2(\Omega)}.$$

Using inequalities (87) and (88), we deduce (89).  $\square$

In order to compute the error estimates, we now need to exhibit a Fortin operator.

## 8. Fortin operator

In this Section, we construct a Fortin operator, inspired by [10, Proposition 6.3]. Let us split  $\mathbf{X}_{0,h}^T$  this way:  $\mathbf{X}_{0,h}^T = \mathbf{V}_{0,h}^T \oplus \mathbf{W}_{0,h}^T$ , where  $\mathbf{V}_{0,h}^T = \mathbf{V}_{0,h} \cap \mathbf{X}_{0,h}^T$  and  $\mathbf{W}_{0,h}^T$  is the orthogonal of  $\mathbf{V}_{0,h}^T$  in  $\mathbf{X}_{0,h}^T$ . We make the abuse of using the same notation for  $b_h(\cdot, \cdot)$  and its to extension  $b_h(\cdot, \cdot)$  to  $(\mathbf{X}_{0,h} + \mathbf{H}_0^1(\Omega)) \times Q_h$ .

**Proposition 11** (Fortin operator). *Consider the projection operator  $\Pi_h : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{X}_h$  such that: for all  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ ,  $\Pi_h \mathbf{w} = \Pi_{cr} \mathbf{w} + \mathbf{w}_T$ , where  $\mathbf{w}_T \in \mathbf{W}_{0,h}^T$  satisfies:*

$$\forall q_1 \in Q_{1,h}, \quad (\mathbf{w}_T, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)} = (\mathbf{w} - \Pi_{cr} \mathbf{w}, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)}. \quad (90)$$

Then, the projection operator  $\Pi_h$  is such that:

$$\forall \mathbf{w} \in \mathbf{H}_0^1(\Omega) \quad b_h(\Pi_h \mathbf{w}, q_h) = b_h(\mathbf{w}, q_h), \quad \forall q \in Q_h. \quad (91)$$

$$\forall \mathbf{w} \in \mathbf{H}^{s+1}(\Omega) \cap \mathbf{H}_0^1(\Omega) \quad \|\mathbf{w} - \Pi_h \mathbf{w}\|_h \lesssim \sigma^{2d} h^s |\mathbf{w}|_{\mathbf{H}^{s+1}(\Omega)}, \quad s \in [0, 1]. \quad (92)$$

*Proof.* Let  $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ . Let  $q = (q_0, q_1) \in Q_{0,h} \times Q_{1,h}$ . It holds:

$$-(\operatorname{div}_h \Pi_h \mathbf{w}, q_0)_{L^2(\Omega)} = -(\operatorname{div}_h \Pi_{cr} \mathbf{w}, q_0)_{L^2(\Omega)} = -(\operatorname{div} \mathbf{w}, q_0)_{L^2(\Omega)}.$$

Due to Lemma 3,  $\mathbf{w}_T$  exists and is unique. Using (90), it holds:

$$\begin{aligned} b_h(\Pi_h \mathbf{w}, q) &= -(\operatorname{div} \mathbf{w}, q_0)_{L^2(\Omega)} + (\Pi_{cr} \mathbf{w} + \mathbf{w}_T, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)}, \\ &= -(\operatorname{div} \mathbf{w}, q_0)_{L^2(\Omega)} + (\mathbf{w}, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)}, \\ &= b_h(\mathbf{w}, q). \end{aligned}$$

Let  $\mathbf{w} \in \mathbf{H}^{1+s}(\Omega) \cap \mathbf{H}_0^1(\Omega)$ . Integrating by parts, and noting that  $(\operatorname{div}(\mathbf{w} - \Pi_{cr} \mathbf{w}), \pi_0(q_1))_{L^2(K_\ell)} = 0$ , we have, for all  $q_1 \in Q_{1,h}$ , using the proof of (29):

$$\begin{aligned} (\mathbf{w}_T, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)} &= (\mathbf{w} - \Pi_{cr} \mathbf{w}, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)}, \\ &= - \sum_{\ell \in \mathcal{I}_K} (\operatorname{div}(\mathbf{w} - \Pi_{cr} \mathbf{w}), q_1 - \pi_0(q_1))_{L^2(K_\ell)} \\ &\quad + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} ((\mathbf{w} - \Pi_{cr} \mathbf{v}) \cdot \mathbf{n}_{f,\ell}, q_1 - \pi_0(q_1))_{L^2(F_f)}, \\ &\lesssim (\sqrt{d} + \sigma) \|\mathbf{w} - \Pi_{cr} \mathbf{w}\|_h \|q_1 - \pi_0(q_1)\|_{L^2(\Omega)}. \end{aligned}$$

Using estimate (43), we obtain that:

$$|(\mathbf{w}_T, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)}| \lesssim \sigma^2 h^s |\mathbf{w}|_{\mathbf{H}^{1+s}(\Omega)} \|q_1 - \pi_0(q_1)\|_{L^2(\Omega)}. \quad (93)$$

According to [32, Chapter I, Lemma 4.1 (iii)], thanks to Lemma 3, we have the following estimate:

$$\|\mathbf{w}_T\|_h \leq \sigma^{2(d-1)} \sup_{q_1 \in Q_{1,h} \setminus \{0\}} \frac{|(\mathbf{w}_T, \mathbf{grad} q_1)_{\mathbf{L}^2(\Omega)}|}{\|q_1 - \pi_0(q_1)\|_{L^2(\Omega)}}. \quad (94)$$

Using (93) in (94), we deduce that:

$$\|\mathbf{w}_T\|_h \lesssim \sigma^{2d} h^s |\mathbf{w}|_{\mathbf{H}^{1+s}(\Omega)}. \quad (95)$$

We obtain (92) from (43) and (95), since  $\|\mathbf{w} - \Pi_h \mathbf{w}\|_h \leq \|\mathbf{w} - \Pi_{cr} \mathbf{w}\|_h + \|\mathbf{w}_T\|_h$ .  $\square$

We exhibited a Fortin operator. Let us now study the convergence of the discretization of Problem (4) with the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element.

## 9. Classical error estimates

**Definition 1.** Let  $\pi_1$  be the  $L^2$ -projection to  $Q_{1,h}$ . Let  $q \in Q_{1,h}$ . One can set  $\pi_1(q) = \tilde{\pi}_1(q) - |\Omega|^{-1} \int_\Omega \tilde{\pi}_1(q)$ , where  $\tilde{\pi}_1$  is the  $L^2$ -projection to  $\mathcal{P}^1(\mathcal{T}_h)$ . It holds [28, Lemma 1.131]<sup>2</sup>:

$$|\pi_1(q)|_{H^1(\Omega)} \lesssim \sigma |q|_{H^1(\Omega)}. \quad (96)$$

Using the triangular inequality and (96), it holds:

$$|\pi_1(q) - q|_{H^1(\Omega)} \leq |q|_{H^1(\Omega)} + |\pi_1(q)|_{H^1(\Omega)} \lesssim (\sigma + 1) |q|_{H^1(\Omega)}. \quad (97)$$

Let  $q \in L_{zmv}^2(\Omega) \cap H^1(\Omega)$ . Using inequality (23), and (97), it holds:

$$\|(\pi_1(q) - q) - \pi_0(\pi_1(q) - q)\|_{L^2(\Omega)} \leq \pi^{-1} h (\sigma + 1) |q|_{H^1(\Omega)}. \quad (98)$$

We have the tools to prove the convergence of the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization. In what follows, we recall that:  $\|(\mathbf{u}, p)\|_\nu = |\mathbf{u}|_{\mathbf{H}^2(\Omega)} + \nu^{-1} |p|_{H^1(\Omega)}$ .

<sup>2</sup>cf. the proof of [50, Lemma 5.1.1], which uses Scott-Zhang interpolation [45]. We can exhibit  $\sigma$  from [45, Theorem 1].

**Theorem 6** (Convergence of the discrete velocity). *Suppose that Hypothesis 1 holds and that the solution to Problem (4) is such that  $(\mathbf{u}, p) \in (\mathbf{V}_0 \cap \mathbf{H}^2(\Omega)) \times (L_{zmv}^2(\Omega) \cap H^1(\Omega))$ . Let  $(\mathbf{u}_h, p_h) \in \mathbf{V}_{0,h} \times Q_{0,h}$  be the solution to Problem (86). We have the following error estimates for the discrete velocity:*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \sigma^{2d} h \|(\mathbf{u}, p)\|_\nu, \quad (99)$$

$$\text{if } \Omega \text{ is convex } \quad \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma^{4d} h^2 \|(\mathbf{u}, p)\|_\nu. \quad (100)$$

**Remark 3.** Suppose that the solution to Problem (4) is such that  $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ . When  $\Omega$  is nonconvex, one can prove that  $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma^{4d} h^{1+s} \|(\mathbf{u}, p)\|_\nu$  for all  $s \in ]\frac{1}{2}, s_\Omega[$ , where  $s_\Omega \in ]\frac{1}{2}, 1[$  depends on the reentrant faces, cf. [25, 1.A].

Suppose that the solution to Problem (4) is such that  $(\mathbf{u}, p) \in \mathbf{H}^{1+s}(\Omega) \times H^s(\Omega)$  for all  $s \in ]\frac{1}{2}, s_f[$ , with  $s_f \in ]\frac{1}{2}, 1[$ . One can prove that for all  $s \in ]\frac{1}{2}, s_f[$ ,  $\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \sigma^{2d} h^s \|(\mathbf{u}, p)\|_\nu$ . If  $\Omega$  is convex, one can prove that  $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma^{4d} h^{1+s} \|(\mathbf{u}, p)\|_\nu$ . If  $\Omega$  is nonconvex, one can prove that  $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma^{4d} h^{s+s'} \|(\mathbf{u}, p)\|_\nu$ ,  $s \in ]\frac{1}{2}, s_f[$  and  $s' \in ]\frac{1}{2}, s_\Omega[$ . This remark applies also to Theorem 2.

The proof of Theorem 6 is similar to that of [24, Theorems 3, 4] since we exhibited a Fortin operator. We enclose it in Appendix A. We can now prove the convergence of the discrete pressure.

**Theorem 7** (Convergence of the discrete pressure). *Suppose that Hypothesis 1 holds and that the solution to Problem (4) is such that  $(\mathbf{u}, p) \in (\mathbf{V}_0 \cap \mathbf{H}^2(\Omega)) \times (L_{zmv}^2(\Omega) \cap H^1(\Omega))$ . Let  $(\mathbf{u}_h, p_h) \in \mathbf{V}_{0,h} \times Q_{0,h}$  be the solution to Problem (86). We have the following error estimates for the discrete pressure:*

$$\|(p - p_h) - \pi_0(p - p_h)\|_{L^2(\Omega)} \lesssim \nu \sigma^{(4d-2)} h \|(\mathbf{u}, p)\|_\nu, \quad (101)$$

$$\|\pi_0(p - p_h)\|_{L^2(\Omega)} \lesssim \nu C_{\text{div}} \sigma^{(4d+1)} h \|(\mathbf{u}, p)\|_\nu, \quad (102)$$

**Remark 4.** Using Proposition 9, Equations (101) and (102) imply that

$$\|p - p_h\|_{L^2(\Omega)} \lesssim \nu C_{\text{div}} \sigma^{4d+1} h \|(\mathbf{u}, p)\|_\nu. \quad (103)$$

Let us prove Theorem 7.

*Proof.* Let  $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$  be the solution to Problem (86) with  $p_h = p_{0,h} + p_{1,h}$ ,  $(p_{0,h}, p_{1,h}) \in Q_{0,h} \times Q_{1,h}$ . Integrating by parts the right-hand side in variational formulation (86), we obtain that for all  $\mathbf{v}_h \in \mathbf{X}_{0,h}$ :

$$\begin{aligned} (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} &= -\nu(\Delta \mathbf{u}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} + (\mathbf{grad} p, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}, \\ &= \nu(\mathbf{u}, \mathbf{v}_h)_h - \nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{v}_h)_{\mathbf{L}^2(F_f)} + (\mathbf{grad} p, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}. \end{aligned} \quad (104)$$

Developping (86) with  $q_h = 0$ , noticing that the discrete velocity  $\mathbf{u}_h \in \mathbf{V}_{0,h}$ , and using (104), we deduce that:

$$\begin{aligned} &\nu(\mathbf{u}_h, \mathbf{v}_h)_h - (\text{div}_h \mathbf{v}_h, p_{0,h})_{L^2(\Omega)} + (\mathbf{grad} p_{1,h}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} \\ &= \nu(\mathbf{u}, \mathbf{v}_h)_h - \nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{v}_h)_{\mathbf{L}^2(F_f)} + (\mathbf{grad} p, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}. \end{aligned} \quad (105)$$

Reshuffling the terms in (105), we get:

$$-(\text{div}_h \mathbf{v}_h, p_{0,h})_{L^2(\Omega)} - (\mathbf{grad}(p - p_{1,h}), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = \varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h), \quad (106)$$

where:

$$\varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h) := \nu(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)_h - \nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}^i, \mathbf{v}_h)_{\mathbf{L}^2(F_f)}. \quad (107)$$

We can bound  $\varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h)$  using (99) and applying (28) to all the components of  $\mathbf{Grad} \mathbf{u} \in \mathbb{H}^1(\Omega)$  and  $\mathbf{v}_h$ , obtaining:

$$\varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h) \lesssim \sigma^{2d} h \nu \|(\mathbf{u}, p)\|_\nu \|\mathbf{v}_h\|_h. \quad (108)$$

We first prove (101). Let  $\mathbf{v}_h \in \tilde{\mathbf{V}}_{0,h}$  cf. (40). Equation (106) now reads:

$$-(\mathbf{grad}(p - p_{1,h}), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = \varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h). \quad (109)$$

Recall that  $\pi_1$  is the  $L^2$ -orthogonal projection on  $Q_{1,h}$ , cf. Definition 1. Let us write (109) this way:

$$-(\mathbf{grad}(\pi_1(p) - p_{1,h}), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = \varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h) + (\mathbf{grad}(p - \pi_1(p)), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}. \quad (110)$$

Notice that:

$$(\mathbf{grad}(p - \pi_1(p)), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (p - \pi_1(p), \mathbf{v}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)}.$$

Applying (28) to all the components of  $\mathbf{v}_h$  with  $q = p - \pi_1(p)$ , and using (97), we get:

$$\left| (\mathbf{grad}(p - \pi_1(p)), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} \right| \lesssim (\sigma + 1) h |p|_{H^1(\Omega)} \|\mathbf{v}_h\|_h. \quad (111)$$

Using (108) and (111) to bound (110), we deduce that:

$$\left| (\mathbf{grad}(\pi_1(p) - p_{1,h}), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} \right| \lesssim \sigma^{2d} h \nu \|(\mathbf{u}, p)\|_\nu \|\mathbf{v}_h\|_h. \quad (112)$$

Consider  $\mathbf{v}_{q_{1,h}} \in \mathbf{X}_{0,h}^T$  built like  $\mathbf{v}_{q_1}$  in Lemma 3 with here  $q_1 = q_{1,h} := \pi_1(p) - p_{1,h}$ . Using (60) and (61), it holds:

$$\|q_{1,h} - \pi_0(q_{1,h})\|_{L^2(\Omega)} \|\mathbf{v}_{q_{1,h}}\|_h \lesssim \sigma^{2(d-1)} (\mathbf{grad} q_{1,h}, \mathbf{v}_{q_{1,h}})_{\mathbf{L}^2(\Omega)}.$$

Using now (112) with  $\mathbf{v}_h = \mathbf{v}_{q_{1,h}}$ , we deduce from the above inequality that

$$\|q_{1,h} - \pi_0(q_{1,h})\|_{L^2(\Omega)} \leq \sigma^{(4d-2)} h \nu \|(\mathbf{u}, p)\|_\nu. \quad (113)$$

Using the triangular inequality, we have:

$$\|(p - p_h) - \pi_0(p - p_h)\|_{L^2(\Omega)} \leq \|q_{1,h} - \pi_0(q_{1,h})\|_{L^2(\Omega)} + \|(p - \pi_1(p)) - \pi_0(p - \pi_1(p))\|_{L^2(\Omega)}.$$

Hence, using (113) and (98) with  $q = p$ , we obtain (101).

Let us prove (102). Integrating by parts the second term in (106) and moving it to the right-hand side, we get:

$$(\operatorname{div}_h \mathbf{v}_h, p - p_h)_{L^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (p - p_{1,h}, \mathbf{v}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} + \varepsilon(\mathbf{u}, \mathbf{u}_h, \mathbf{v}_h). \quad (114)$$

We write:  $p - p_{1,h} = (p - \pi_1(p)) + (\pi_1(p) - p_{1,h})$  where  $\pi_1(p) - p_{1,h} \in \mathcal{P}^1(\mathcal{T}_h)$ . Hence, using (28) and (29), it holds, for all  $\mathbf{v}_h \in \mathbf{X}_{0,h}$ :

$$\begin{aligned} & \left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (p - p_{1,h}, \mathbf{v}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} \right| \\ & \leq \sigma \left( h \|\mathbf{grad}(p - \pi_1(p))\|_{\mathbf{L}^2(\Omega)} + \|(Id - \pi_0)(\pi_1(p) - p_{1,h})\|_{L^2(\Omega)} \right) \|\mathbf{v}_h\|_h. \end{aligned} \quad (115)$$

According to (96), we have:  $\|\mathbf{grad}(p - \pi_1(p))\|_{\mathbf{L}^2(\Omega)} \leq \sigma \nu \|(\mathbf{u}, p)\|_\nu$ . To bound the second term of (115), notice first that:

$$(Id - \pi_0)(\pi_1(p) - p_{1,h}) = (Id - \pi_0)(\pi_1(p) - p_h) = (Id - \pi_0)(\pi_1(p) - p) + (Id - \pi_0)(p - p_h).$$

Using a triangular inequality, we deduce that:

$$\|(Id - \pi_0)(\pi_1(p) - p_{1,h})\|_{L^2(\Omega)} \leq \|(Id - \pi_0)(\pi_1(p) - p)\|_{L^2(\Omega)} + \|(Id - \pi_0)(p - p_h)\|_{L^2(\Omega)}$$

We now use (98) and (101) to bound the first and the second term in the above equation:

$$\|(Id - \pi_0)(\pi_1(p) - p_{1,h})\|_{L^2(\Omega)} \lesssim \nu \sigma h \|(\mathbf{u}, p)\|_\nu + \nu C_{\text{div}} \sigma^{4d} \|(\mathbf{u}, p)\|_\nu.$$

Finally, we have:

$$\left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (p - p_{1,h}, \mathbf{v}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} \right| \lesssim \nu \sigma^{4d+1} h \|(\mathbf{u}, p)\|_\nu \|\mathbf{v}_h\|_h. \quad (116)$$

Using (116) and (108) in (114), it holds, for all  $\mathbf{v}_h \in \mathbf{X}_{0,h}$ :

$$\left| (\text{div}_h \mathbf{v}_h, p - p_h)_{L^2(\Omega)} \right| \lesssim \nu \sigma^{4d+1} h \|(\mathbf{u}, p)\|_\nu \|\mathbf{v}_h\|_h. \quad (117)$$

Let  $q_{0,h} = -\pi_0(p - p_h)$ . Consider Equation (117) with  $\mathbf{v}_{q_{0,h}} \in \mathbf{X}_{0,h}$  built like  $\mathbf{v}_{q_{0,h}}$  in Lemma 1. We have:

$$\nu^{-1} \|\pi_0(p - p_h)\|_{L^2(\Omega)}^2 = (\text{div}_h \mathbf{v}_{q_{0,h}}, p - p_h)_{L^2(\Omega)} \text{ and } \|\mathbf{v}_{q_{0,h}}\|_h \leq \nu^{-1} C_{\text{div}} \|\pi_0(p - p_h)\|_{L^2(\Omega)}.$$

Using this in (117), we deduce that:

$$\nu^{-1} \|\pi_0(p - p_h)\|_{L^2(\Omega)}^2 \lesssim \nu \sigma^{(4d+1)} h \|(\mathbf{u}, p)\|_\nu \|\mathbf{v}_{\pi_0(q)}\|_h \lesssim \sigma^{(4d+1)} C_{\text{div}} h \|(\mathbf{u}, p)\|_\nu \|\pi_0(p - p_h)\|_{L^2(\Omega)}.$$

Simplifying by  $\nu^{-1} \|\pi_0(p - p_h)\|_{L^2(\Omega)}$ , we obtain (102).  $\square$

**Remark 5.** Consider inhomogeneous boundary conditions:  $\mathbf{u}|_{\partial\Omega} = \mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$  such that  $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ . One can proceed as explained in [24, Section 7]. Let us set  $\gamma_{\partial\Omega}(\mathbf{X}_h) := \{\mathbf{v}_h|_{\partial\Omega} \text{ such that } \mathbf{v}_h \in \mathbf{X}_h\}$  and:

$$\mathbf{G}_h := \left\{ \mathbf{v}_h \in \gamma_{\partial\Omega}(\mathbf{X}_h) \text{ and } \int_{\partial\Omega} \mathbf{v}_h \cdot \mathbf{n}|_{\partial\Omega} = 0 \right\}, \quad \tilde{\mathbf{G}}_h := \left\{ \mathbf{v}_h \in \gamma_{\partial\Omega}(\mathbf{X}_h) \text{ and } \forall q_h \in Q_h, b_h(\mathbf{v}_h, q_h) = 0 \right\}.$$

According to [24, Lemma 9],  $\mathbf{G}_h = \{\mathbf{v}_h \in \gamma_{\partial\Omega}(\mathbf{X}_h) \text{ and } \forall q_h \in Q_{0,h}, (\text{div}_h \mathbf{v}_h, q_h) = 0\}$ . Hence,  $\tilde{\mathbf{G}}_h \subset \mathbf{G}_h$ . Notice that we can extend the definition of our Fortin operator  $\Pi_h$  from  $\mathbf{H}^1(\Omega)$  to  $\mathbf{X}_h$  (cf. Proposition 11). Alternatively, one can prove discrete T-coercivity of the bilinear form  $a_h(\cdot, \cdot)$  defined in (85), as proposed in [38] for  $\mathbf{P}_{nc}^{k+1} \times P_{disc}^k$  finite element, and use the second Strang's Lemma [30, §27.4.2] to compute error estimates. We won't go into more detail here, as it's fairly technical.

We proved the convergence of the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element. In the next section, we prove that we can obtain optimal error estimates in  $2D$ .

## 10. Optimal error estimates in $2D$

We consider here  $\Omega \subset \mathbb{R}^2$ . In order to exhibit some convergence properties of the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element method, we need the two next Lemmas:

**Lemma 8.** *Let  $\mathbf{v}_h \in \mathbf{V}_{0,h}$ . Then for all  $q_{2,h} \in Q_{2,h}$ ,  $(\mathbf{grad} q_{2,h}, \mathbf{v}_h)_{L^2(\Omega)} = 0$ .*

*Proof.* Integrating by parts and using Proposition 10, we get that for all  $q_{2,h} \in Q_{2,h}$  and for all  $q_{1,h} \in Q_{1,h}$ :

$$\sum_{\ell \in \mathcal{I}_K} (\mathbf{grad} q_{2,h}, \mathbf{v}_h)_{\mathbf{L}^2(K_\ell)} = - \sum_{f \in \mathcal{I}_F} \int_{F_f} q_{2,h} [\mathbf{v}_h]_f \cdot \mathbf{n}_f = - \sum_{f \in \mathcal{I}_F} \int_{F_f} (q_{2,h} - q_{1,h}) [\mathbf{v}_h]_f \cdot \mathbf{n}_f.$$

Hence, we obtain for all  $q_{2,h} \in Q_{2,h}$  and for all  $q_{1,h} \in Q_{1,h}$ :

$$\begin{aligned} \sum_{\ell \in \mathcal{I}_K} (\mathbf{grad} q_{2,h}, \mathbf{v}_h)_{\mathbf{L}^2(K_\ell)} &= - \sum_{f \in \mathcal{I}_F} \int_{F_f} (q_{2,h} - q_{1,h}) [\mathbf{v}_h - \underline{\mathbf{v}}_{h,f}]_f \cdot \mathbf{n}_f \\ &= - \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \int_{F_f} (q_{2,h} - q_{1,h}) (\mathbf{v}_h - \underline{\mathbf{v}}_{h,f}) \cdot \mathbf{n}_{f,\ell}. \end{aligned}$$

Let us build  $q_{1,h} \in Q_{1,h}$  such that for all  $i \in \mathcal{I}_S$ ,  $q_{1,h}(S_i) = q_{2,h}(S_i)$ . Then, for all  $f \in \mathcal{I}_F$ , the polynomial  $(q_{2,h} - q_{1,h})(\mathbf{v}_h - \underline{\mathbf{v}}_{h,f})$ , which is of order 3, vanishes at the vertices and in the middle of  $F_f$ . Using the Simpson's rule which is exact for order 3 polynomials, we obtain that

$$\int_{F_f} (q_{2,h} - q_{1,h})(\mathbf{v}_h - \underline{\mathbf{v}}_{h,f}) \cdot \mathbf{n}_{f,\ell} = 0.$$

□

**Lemma 9.** *Let  $\phi \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$  such that  $|\phi|_{H^3(\Omega)} \neq 0$ . Then, for all  $\mathbf{v}_h \in \mathbf{V}_{0,h}$ :*

$$(\mathbf{grad} \phi, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)}. \quad (118)$$

*Proof.* Let  $\mathcal{I}_{2,h}$  be the  $P^2$ -interpolation operator on  $\mathcal{T}_h$ . Due to Lemma 8, for all  $\mathbf{v} \in \mathbf{V}_{0,h}$ , we have:  $(\mathbf{grad} \phi, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)} = (\mathbf{grad}(\phi - \mathcal{I}_{2,h}(\phi)), \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}$ . Using Cauchy-Schwarz inequality and [29, Theorem 11.13], we get that:

$$|(\mathbf{grad} \phi, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}| \leq \|\mathbf{grad}(\phi - \mathcal{I}_{2,h}(\phi))\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma h^2 |\phi|_{H^3(\Omega)} \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)}.$$

□

We can now prove the

**Theorem 10** (Pressure robustness). *Suppose that  $\Omega$  is convex. Let  $\phi \in H^3(\Omega) \cap L^2_{zmv}(\Omega)$  such that  $|\phi|_{H^3(\Omega)} \neq 0$ . Consider Problem (4) with  $\mathbf{f} = \mathbf{grad} \phi$ . Then the solution to Problem (4) is such that:  $(\mathbf{u}, p) = (0, \phi)$ . Let  $(\mathbf{u}_h, p_h)$  be the solution to Problem (86). It is such that:*

$$\nu \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |\phi|_{H^3(\Omega)}, \quad (119)$$

$$\nu \|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma^{(2d+1)} h^4 |\phi|_{H^3(\Omega)}, \quad (120)$$

$$\|(\phi - p_h) - \pi_0(\phi - p_h)\|_{L^2(\Omega)} \lesssim \sigma^{2d} h^2 |\phi|_{H^2(\Omega)}, \quad (121)$$

$$\|\pi_0(\phi - p_h)\|_{L^2(\Omega)} \lesssim C_{\text{div}} \sigma^{2d+1} h^2 |\phi|_{H^2(\Omega)}. \quad (122)$$

*Proof.* Setting  $\mathbf{v}_h = \mathbf{u}_h \in \mathbf{V}_{0,h}$  as test-function in Problem (86), using Lemma 8, integrating by parts, we have:

$$\nu \|\mathbf{u}_h\|_h^2 = (\mathbf{grad}(\phi - \mathcal{I}_{2,h}(\phi)), \mathbf{u}_h)_{\mathbf{L}^2(\Omega)} = \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\phi - \mathcal{I}_{2,h}(\phi), \mathbf{u}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)}.$$

Hence, using (28) and [29, Theorem 11.13], we obtain:

$$\nu \|\mathbf{u}_h\|_h^2 \lesssim \sigma h \|\mathbf{grad}(\phi - \mathcal{I}_{2,h}(\phi))\|_{\mathbf{L}^2(\Omega)} \|\mathbf{u}_h\|_h \lesssim \sigma^2 h^3 |\phi|_{H^3(\Omega)} \|\mathbf{u}_h\|_h.$$

We deduce (119). Using this estimate in the proof of (100), we obtain (120). The proofs of estimates (121)-(122) are similar to the proofs of estimates (101)-(102), using (119)-(120) and the fact that since  $\phi \in H^3(\Omega)$ ,  $\|\mathbf{grad}(\phi - \pi_1(\phi))\|_{\mathbf{L}^2(\Omega)} \lesssim \sigma h |\phi|_{H^2(\Omega)}$ . □

Numerical results illustrating the optimal convergence rates in  $2D$  can be found in [37]. To obtain similar results in  $3D$ , one can enrich the  $3D$  pressure space with  $P^2$ -Lagrange degrees of freedom [31, 11]. We now propose three numerical tests.

## 11. Numerical illustration

This section focuses on three numerical tests designed to evaluate the performance of the proposed discretization under different Stokes flow conditions. The resolution algorithm is given in (3), for which the matrices are given in [42, Sections 2 and 3]. We look at two test cases with a regular solution, presented in [42, Section 4]: Bercovier–Engelman flow in  $2D$  [9] and Taylor–Green vortex in  $3D$  [47]. We also study a  $2D$  test case in a nonconvex domain, with a low regular solution [26]. Specifically, we compare the two discretization approaches, the  $\mathbf{P}_{nc}^1 \times P^0$  discretization and the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization. We present first the  $2D$  test cases, which involve a  $2D$  Stokes flow within a convex and a nonconvex domain. Then one expands the problem to  $3D$ . For the test cases with a regular solution, we analyze the influence of viscosity by examining different viscosity values and their impact on the numerical solutions. Additionally, for the three test cases, we explore the effect of mesh topology by testing various mesh configurations, both structured and unstructured. In each case, detailed information on the analytical solutions, viscosity values, mesh setups, and convergence results for velocity gradient, velocity, and pressure is provided. The simulations are conducted using the aforementioned TrioCFD software<sup>3</sup> (version 1.9.4).

The computational domains are discretized with families of successively refined meshes. Let  $d$  be the spatial dimension and  $m \in \mathbb{N}^*$  the label for a level of mesh refinement. Let  $(N_U)_m$  (resp.  $(N_P)_m$ ) the number of discrete velocity (resp. discrete pressure) unknowns of mesh  $m$ . Then  $(N_U)_{m-1} < (N_U)_m$ ,  $(N_P)_{m-1} < (N_P)_m$ . In this Section, we set:

$$\|(\mathbf{u}_{\text{ex}}, p_{\text{ex}})\|_{\nu} = \left( \|\mathbf{Grad} \mathbf{u}_{\text{ex}}\|_{\mathbb{L}^2(\Omega)}^2 + \nu^{-2} \|p_{\text{ex}}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (123)$$

where the subscript "ex" denotes the exact solution of the problem. Let  $\mathcal{I}_{cr,h}$  be the interpolation operator such that for  $v \in \mathcal{C}^0(\bar{\Omega})$ ,  $\mathcal{I}_h(v) = \sum_{f \in \mathcal{I}_F} v(M_f) \psi_f$ . The normalized discrete errors values for the velocity gradient, the velocity and the pressure are respectively computed as in [37]:

$$\varepsilon_1^{\nu}(\mathbf{u}) = \frac{\|\mathbf{u}_h - \mathcal{I}_{cr,h}(\mathbf{u}_{\text{ex}})\|_h}{\|(\mathbf{u}_{\text{ex}}, p_{\text{ex}})\|_{\nu}}, \quad \varepsilon_0^{\nu}(\mathbf{u}) = \frac{\|\mathbf{u}_h - \mathcal{I}_{cr,h}(\mathbf{u}_{\text{ex}})\|_{\mathbb{L}^2(\Omega)}}{\|(\mathbf{u}_{\text{ex}}, p_{\text{ex}})\|_{\nu}}, \quad \varepsilon_0^{\nu}(p) = \nu^{-1} \frac{\|p_h - \pi_0(p_{\text{ex}})\|_{L^2(\Omega)}}{\|(\mathbf{u}_{\text{ex}}, p_{\text{ex}})\|_{\nu}}. \quad (124)$$

Let  $\tau_1(\mathbf{u})$ ,  $\tau_0(\mathbf{u})$  and  $\tau_0(p)$  be the convergence rates of  $\varepsilon_1^{\nu}(\mathbf{u})$ ,  $\varepsilon_0^{\nu}(\mathbf{u})$  and  $\varepsilon_0^{\nu}(p)$  defined as:

$$\tau_{1,0}(\mathbf{u}) = -d \times \frac{\ln [\varepsilon_{1,0}^{\nu}(\mathbf{u})_m] - \ln [\varepsilon_{1,0}^{\nu}(\mathbf{u})_{m-1}]}{\ln [(N_U)_m] - \ln [(N_U)_{m-1}]}, \quad \tau_0(p) = -d \times \frac{\ln [\varepsilon_0^{\nu}(p)_m] - \ln [\varepsilon_0^{\nu}(p)_{m-1}]}{\ln [(N_P)_m] - \ln [(N_P)_{m-1}]}. \quad (125)$$

In the following, the errors are plotted as  $\varepsilon_{1,0}^{\nu}(\mathbf{u})^d$  (resp.  $\varepsilon_0^{\nu}(p)^d$ ) against  $N_U$  (resp.  $N_P$ ).

### 11.1. Stokes problem in a two-dimensional convex domain

The two-dimensional Bercovier–Engelman flow case [9] is studied in the square domain  $\Omega = [0, 1]^2$  with viscosity coefficient  $\nu \in \{10^{-4}; 10^{-2}; 1\}$ . The volumetric source term in (4) is defined as:

$$\mathbf{f}(x, y) = \begin{bmatrix} 256 \nu \left( x^2(x-1)^2(12y-6) + y(y-1)(2y-1)(12x^2-12x+2) \right) + (y-1/2) \\ -256 \nu \left( y^2(y-1)^2(12x-6) + x(x-1)(2x-1)(12y^2-12y+2) \right) + (x-1/2) \end{bmatrix}. \quad (126)$$

The analytical solutions for velocity and pressure read:

$$\mathbf{u}_{\text{ex}}(x, y) = \begin{bmatrix} -256 x^2(x-1)^2 y(y-1)(2y-1) \\ 256 y^2(y-1)^2 x(x-1)(2x-1) \end{bmatrix} \quad \text{and} \quad p_{\text{ex}}(x, y) = \left(x - \frac{1}{2}\right) \left(y - \frac{1}{2}\right). \quad (127)$$

<sup>3</sup><https://github.com/cea-trust-platform/TrioCFD-code>

The analytical velocity is imposed as an inhomogeneous Dirichlet condition on the square boundaries. A triangulation sequence is built based on Delaunay triangulation, as shown on Figure 1. We use a shape-regular triangulation sequence. Some maps of the solution fields are provided on Figure 2 for  $\nu = 10^{-4}$ . We compare the  $\mathbf{P}_{nc}^1 \times P^0$  and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretizations. We observe that the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization is more effective than the  $\mathbf{P}_{nc}^1 \times P^0$  one, especially for coarse meshes. We plot the errors  $\varepsilon_{0,1}^\nu(\mathbf{u})$  (resp.  $\varepsilon_0^\nu(p)$ ) against the number of unknowns  $N_U$  (resp.  $N_P$ ) on Figure 3. Associated convergence rates  $\tau_{0,1}(\mathbf{u})$  and  $\tau_0(p)$  are given in Appendix B, Table B.1. The convergence rates of the two discretization methods are similar. We obtain:  $\tau_1(\mathbf{u}) \approx 1$ ,  $\tau_0(\mathbf{u}) \approx 2$ , and  $\tau_0(p) \approx 1$ , which aligns with Theorems 2, 6 and 7. However, comparing Figure 2 for  $\nu = 10^{-2}$  and  $\nu = 10^{-4}$ , we remark for the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization, the errors on the velocity,  $\varepsilon_{0,1}^\nu(\mathbf{u})$  are weaker than those produced by the  $\mathbf{P}_{nc}^1 \times P^0$  discretization, and this difference is more important for  $\nu = 10^{-4}$  than for  $\nu = 10^{-2}$ . The error on the pressure  $\varepsilon_0^\nu(p)$  is weaker for the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization when  $\nu = 10^{-4}$ .

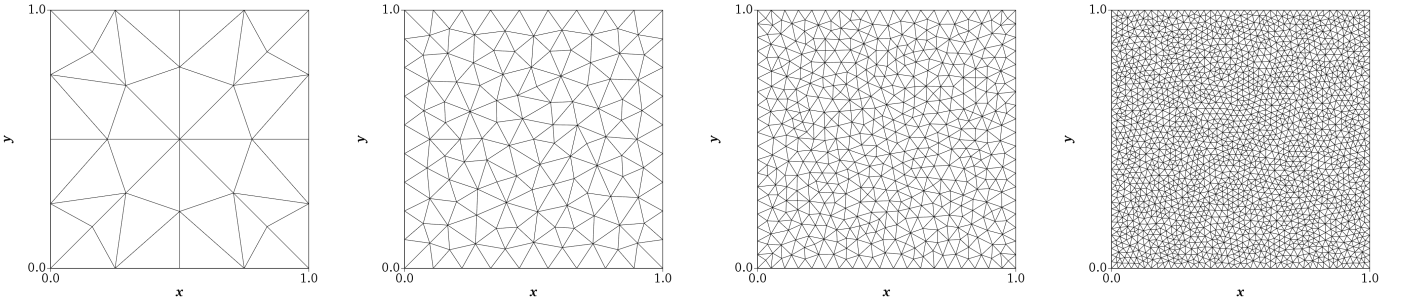


Figure 1: First four grids of the mesh family employed in the Bercovier–Engelman flow.

### 11.2. Stokes problem in a two-dimensional nonconvex domain

In this flow, designed by Dauge in [26], and studied in [48, 7, 17], the Cartesian components of the analytical velocity and the analytical pressure in the nonconvex domain  $\Omega = [0, 1]^2 \setminus [0, 1] \times [-1, 0]$  are given in polar coordinates by:

$$\mathbf{u}_{\text{ex}}(r, \varphi) = r^\kappa \begin{bmatrix} \cos(\varphi)\psi'(\varphi) + (\kappa + 1)\sin(\varphi)\psi(\varphi) \\ \sin(\varphi)\psi'(\varphi) - (\kappa + 1)\cos(\varphi)\psi(\varphi) \end{bmatrix} \quad \text{and} \quad p_{\text{ex}}(r, \varphi) = r^{\kappa-1} \frac{(\kappa + 1)^2\psi'(\varphi) + \psi'''(\varphi)}{\kappa - 1}, \quad (128)$$

with the function  $\psi(\varphi)$  defined as:

$$\psi(\varphi) = \frac{\sin((\kappa + 1)\varphi) \cos(\kappa\omega)}{\kappa + 1} - \frac{\sin((\kappa - 1)\varphi) \cos(\kappa\omega)}{\kappa - 1} - \cos((\kappa + 1)\varphi) + \cos((\kappa - 1)\varphi). \quad (129)$$

The volumetric source term  $\mathbf{f}$  in (4) is set to zero. The parameters are  $\kappa = 856399/1572864 \approx 0.54$  and  $\omega = 3\pi/2$ . The velocity is such that  $\mathbf{u}_{\text{ex}}(0, \varphi) = \mathbf{0}$ . On the other hand, there is a singularity of pressure in the reentrant corner  $(0, 0)$ , since  $\kappa - 1 < 0$ :

$$\begin{cases} \lim_{\substack{r \rightarrow 0 \\ \varphi < \varphi_0}} p_{\text{ex}}(r, \varphi) = -\infty, \\ \lim_{\substack{r \rightarrow 0 \\ \varphi > \varphi_0}} p_{\text{ex}}(r, \varphi) = +\infty. \end{cases} \quad (130)$$

where  $\varphi_0 = -\arctan[\cos(\kappa\omega)/(\kappa - 1)]/(\kappa - 1) = \omega/2$ . The analytic velocity is imposed as an inhomogeneous Dirichlet condition on the domain boundaries. We consider here the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization and viscosity  $\nu = 1$  only. We have, for all  $s \in [0, \kappa[$ :  $(\mathbf{u}, p) \in \mathbf{H}^{1+s}(\Omega) \times H^s(\Omega)$ . We postulate that

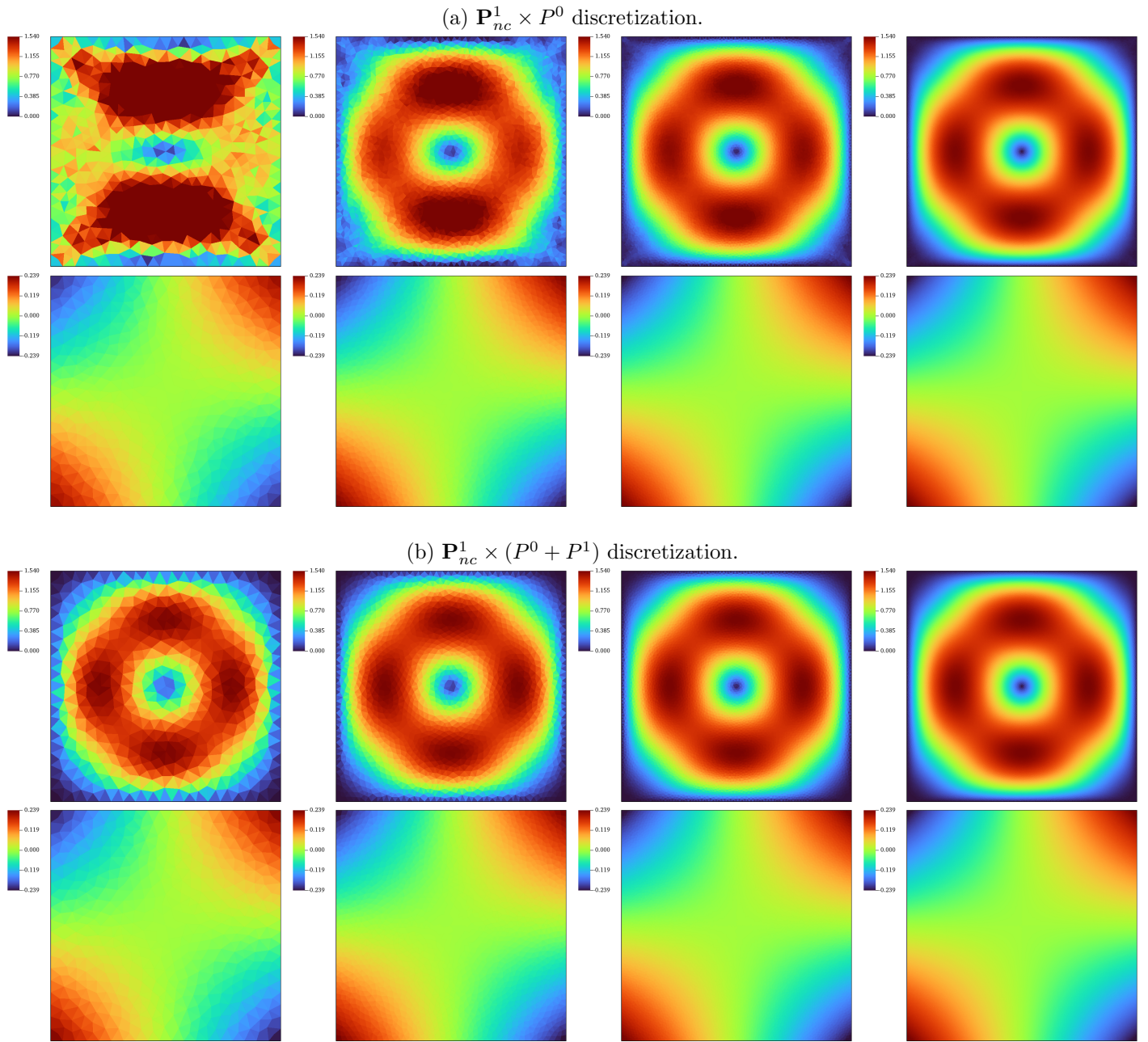


Figure 2:  $\mathbf{P}_{nc}^1 \times P^0$  and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretizations,  $\nu = 10^{-4}$ . From left to right: meshes 3 to 6. From top to bottom:  $P^1$ -Lagrange projection of the discrete velocity norm and of the discrete pressure.

$\tau_1(\mathbf{u}) = \kappa$ ,  $\tau_0(\mathbf{u}) = 2\kappa$ , and  $\tau_0(p) = \kappa$ . The meshes are shown in Figure 4. We use three families of meshes: uniform structured meshes (type 1), uniform unstructured meshes (type 2) and unstructured meshes with a local refinement near the reentrant corner (type 3) to assess the potential gain in accuracy. Maps of the solution fields are provided in Figure 5 for the meshes of type 2. We plot the errors  $\varepsilon_1'(\mathbf{u})$ ,  $\varepsilon_0'(\mathbf{u})$  (resp.  $\varepsilon_0'(p)$ ) against the number of unknowns  $N_U$  (resp.  $N_P$ ) on Figure 6. Associated convergence rates  $\tau_{0,1}(\mathbf{u})$  and  $\tau_0(p)$  are given in Appendix B, Table B.2. It appears that the meshes of types 1 and 2 lead to similar convergence rates, namely  $\tau_1(\mathbf{u}) \approx \kappa$ ,  $\tau_0(\mathbf{u}) \approx 2\kappa$ , and  $\tau_0(p) \approx \kappa$ . They yield very similar results in terms of accuracy. We obtain weaker errors and better convergence rates using the meshes of type 3, which are locally refined. They provide a more accurate approximation of the singularity near the re-entrant corner. It is worth noting that the velocity converges much faster on the initial meshes, before eventually reaching the asymptotic convergence rate.

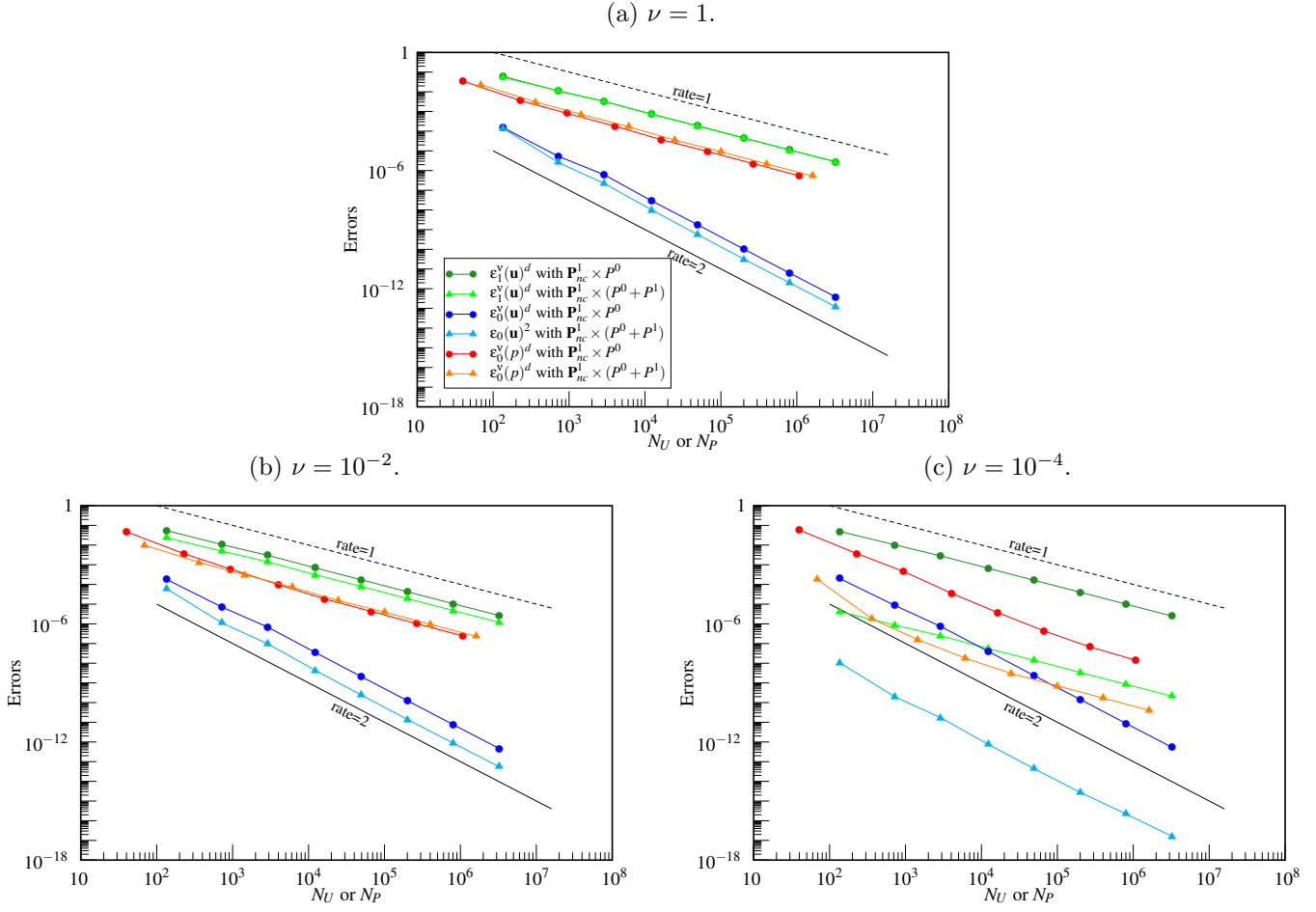


Figure 3: Plots of  $\varepsilon_1^\nu(\mathbf{u})$ ,  $\varepsilon_0^\nu(\mathbf{u})$ , and  $\varepsilon_0^\nu(p)$  for the Bercovier–Engelman flow, for the  $\mathbf{P}_{nc}^1 \times P^0$  and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretizations, for different values of  $\nu$ .

### 11.3. Stokes problem in a three-dimensional convex domain

The three-dimensional Taylor–Green problem [47] is studied in the cubic domain  $\Omega = [0, 1]^3$  with viscosity coefficient  $\nu \in \{10^{-4}, 10^{-2}, 1\}$ . The volumetric source term in (4) is defined as:

$$\mathbf{f}(x, y, z) = \begin{bmatrix} -12\pi^2(2\nu + 1) \cos(2\pi x) \sin(2\pi y) \sin(2\pi z) \\ 12\pi^2(\nu - 1) \sin(2\pi x) \cos(2\pi y) \sin(2\pi z) \\ 12\pi^2(\nu - 1) \sin(2\pi x) \sin(2\pi y) \cos(2\pi z) \end{bmatrix}. \quad (131)$$

The analytical solutions for the velocity and the pressure are:

$$\mathbf{u}_{\text{ex}}(x, y, z) = \begin{bmatrix} -2 \cos(2\pi x) \sin(2\pi y) \sin(2\pi z) \\ \sin(2\pi x) \cos(2\pi y) \sin(2\pi z) \\ \sin(2\pi x) \sin(2\pi y) \cos(2\pi z) \end{bmatrix} \quad \text{and} \quad p_{\text{ex}}(x, y, z) = -6\pi \sin(2\pi x) \sin(2\pi y) \sin(2\pi z). \quad (132)$$

The analytical velocity is imposed as an inhomogeneous Dirichlet condition on the cube boundaries. We use structured and unstructured meshes as shown in Figure 7. The two discretizations,  $\mathbf{P}_{nc}^1 \times P^0$  and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  are compared. For the latter, some solution fields are represented in Figure 8 for  $\nu = 1$ , to illustrate the convergence. The errors  $\varepsilon_{0,1}^\nu(\mathbf{u})$ , (resp.  $\varepsilon_0^\nu(p)$ ) are plotted against the number of unknowns  $N_U$  (resp.  $N_P$ ) on Figure 9. Associated convergence rates  $\tau_{0,1}(\mathbf{u})$  and  $\tau_0(p)$  are given in Appendix B, Table B.3. We obtain:  $\tau_1(\mathbf{u}) \approx 1$ ,  $\tau_0(\mathbf{u}) \approx 2$ , and  $\tau_0(p) \approx 1$ , which aligns with Theorems 2, 6 and 7. It appears that the mesh type has very little impact on the results. On the other hand, the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization provides significantly higher accuracy in most cases than the  $\mathbf{P}_{nc}^1 \times P^0$ ,

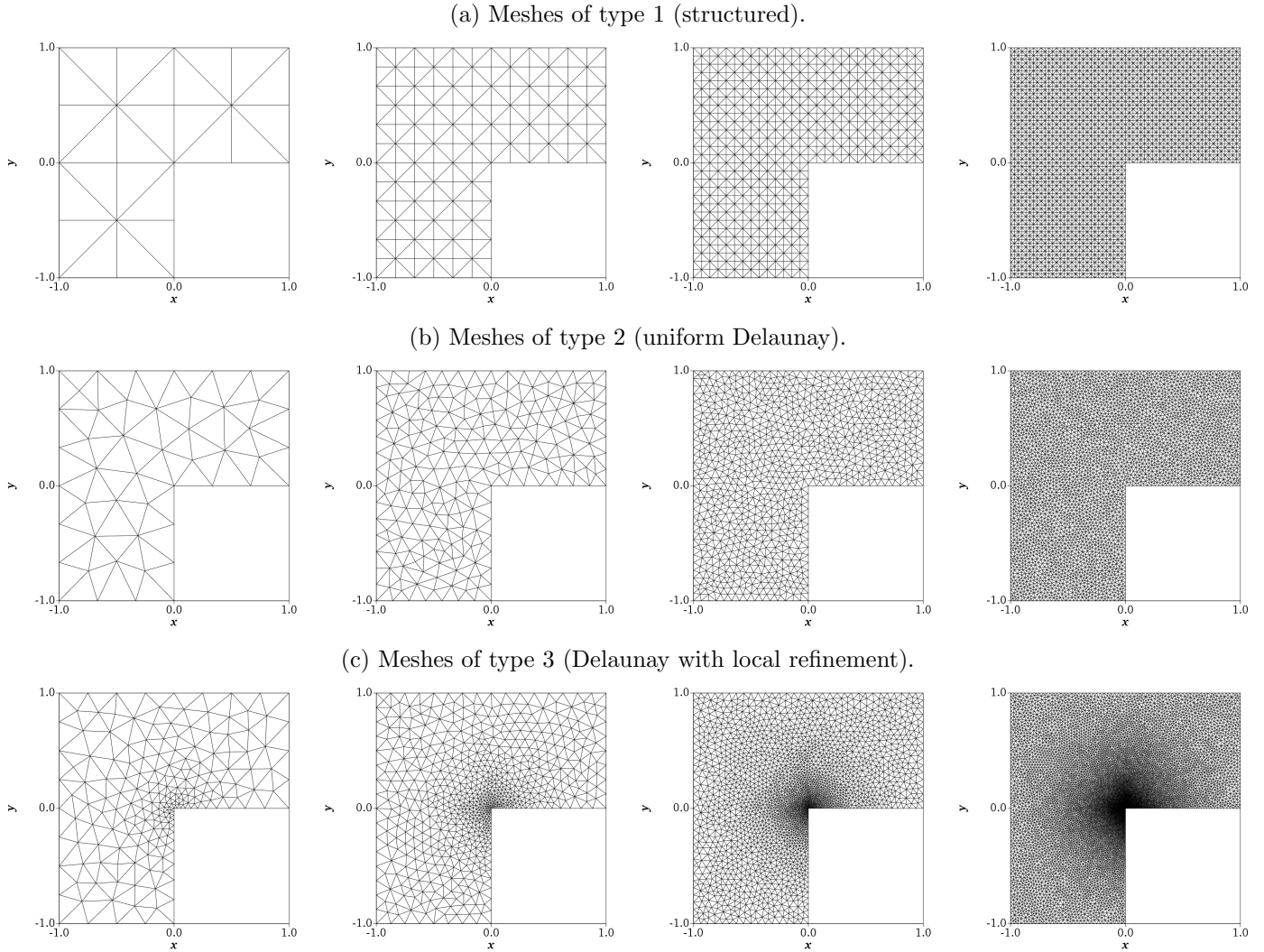


Figure 4: First four grids of the mesh families employed in the two-dimensional nonconvex domain flow.

particularly when  $\nu = 10^{-4}$ . This is consistent with the conclusions from the  $2D$  case (subsection 11.1). It is noted that viscosity has a significant impact on accuracy and convergence rates. For  $\nu = 1$ , we observe that  $\tau_1(\mathbf{u}) \approx 1$ ,  $\tau_0(\mathbf{u}) \approx 2$ , and  $\tau_0(p) \approx 1$ , as in the  $2D$  case. However, the three quantities gain one order of convergence, reaching  $\tau_1(\mathbf{u}) \approx 2$ ,  $\tau_0(\mathbf{u}) \approx 3$ , and  $\tau_0(p) \approx 2$ , respectively, as soon as  $\nu < 1$  and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization is employed, which could indicate that the asymptotic regime convergence is not reach.

## 12. Conclusion

We proved the stability of the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element, cf. Theorem 4 and we showed that the accuracy is improved with respect to the  $\mathbf{P}_{nc}^1 \times P^0$  finite element method, cf. Proposition 10. We proposed a Fortin operator, allowing us to compute error estimates for the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element and we shown optimal error estimates in  $2D$ . We exhibited the dependency of our estimates on the shape regularity parameter  $\sigma$ . We provided numerical illustration in  $2D$  and  $3D$ , including a simulation of a low-regular manufactured solution. We observed that the  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  finite element returns more accurate results than the  $\mathbf{P}_{nc}^1 \times P^0$  finite element, in particular for small values of  $\nu$ , cf. Figure 9 for the  $3D$  case. Further, one could study this strategy to higher order Crouzeix-Raviart finite elements.

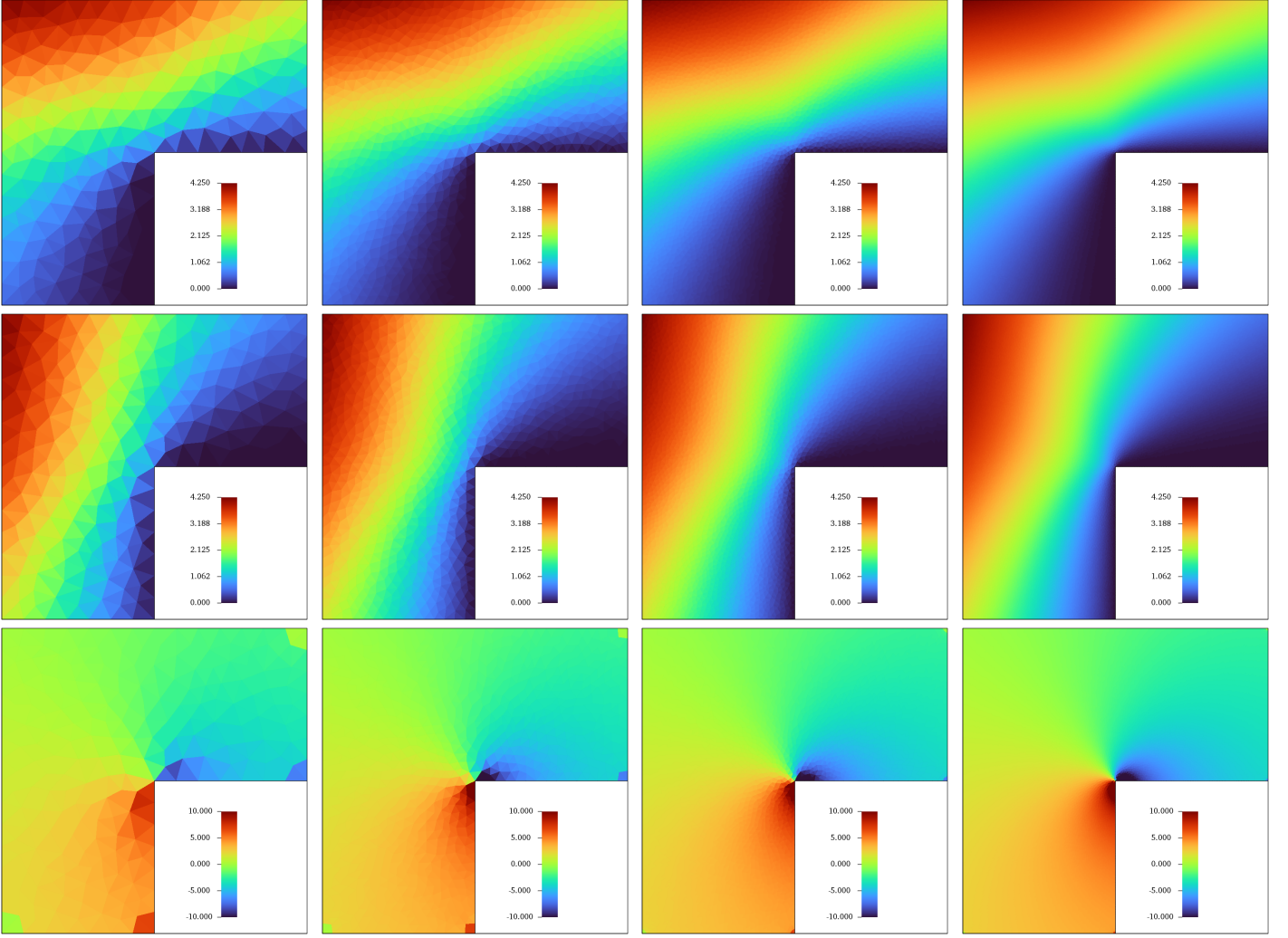


Figure 5:  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization,  $\nu = 1$ . From left to right: meshes 2 to 5. From top to bottom:  $P^1$ -Lagrange projection of the discrete velocity components and of the discrete pressure.

## Appendix A. Proof of Theorem 6

Let  $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$  be the solution to Problem (86) with  $p_h = p_{0,h} + p_{1,h}$ ,  $(p_{0,h}, p_{1,h}) \in Q_{0,h} \times Q_{1,h}$ . We prove (99) and (100), following the proofs of [24, Theorems 3 and 4]. For all  $\mathbf{v}_h \in \mathbf{V}_{0,h}$ :

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq \|\mathbf{u} - \mathbf{v}_h\|_h + \|\mathbf{v}_h - \mathbf{u}_h\|_h, \quad (\text{A.1})$$

$$\|\mathbf{v}_h - \mathbf{u}_h\|_h^2 = -(\mathbf{u} - \mathbf{v}_h, \mathbf{v}_h - \mathbf{u}_h)_h + (\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h)_h \quad (\text{A.2})$$

Let  $\mathbf{v}_h \neq \mathbf{u}_h$ . Using Cauchy-Schwarz in (A.2), it holds:

$$\|\mathbf{v}_h - \mathbf{u}_h\|_h^2 \leq \|\mathbf{u} - \mathbf{v}_h\|_h \|\mathbf{v}_h - \mathbf{u}_h\|_h + \frac{(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h)_h}{\|\mathbf{v}_h - \mathbf{u}_h\|_h} \|\mathbf{v}_h - \mathbf{u}_h\|_h.$$

We obtain that

$$\|\mathbf{v}_h - \mathbf{u}_h\|_h \leq \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{w}_h \in \mathbf{V}_{0,h} \setminus \{0\}} \frac{|(\mathbf{u} - \mathbf{u}_h, \mathbf{w}_h)_h|}{\|\mathbf{w}_h\|_h}. \quad (\text{A.3})$$

We deduce from (A.1) and (A.3) that for all  $\mathbf{v}_h \in \mathbf{V}_{0,h}$ :

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq 2\|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{w}_h \in \mathbf{V}_{0,h} \setminus \{0\}} \frac{|(\mathbf{u} - \mathbf{u}_h, \mathbf{w}_h)_h|}{\|\mathbf{w}_h\|_h}.$$

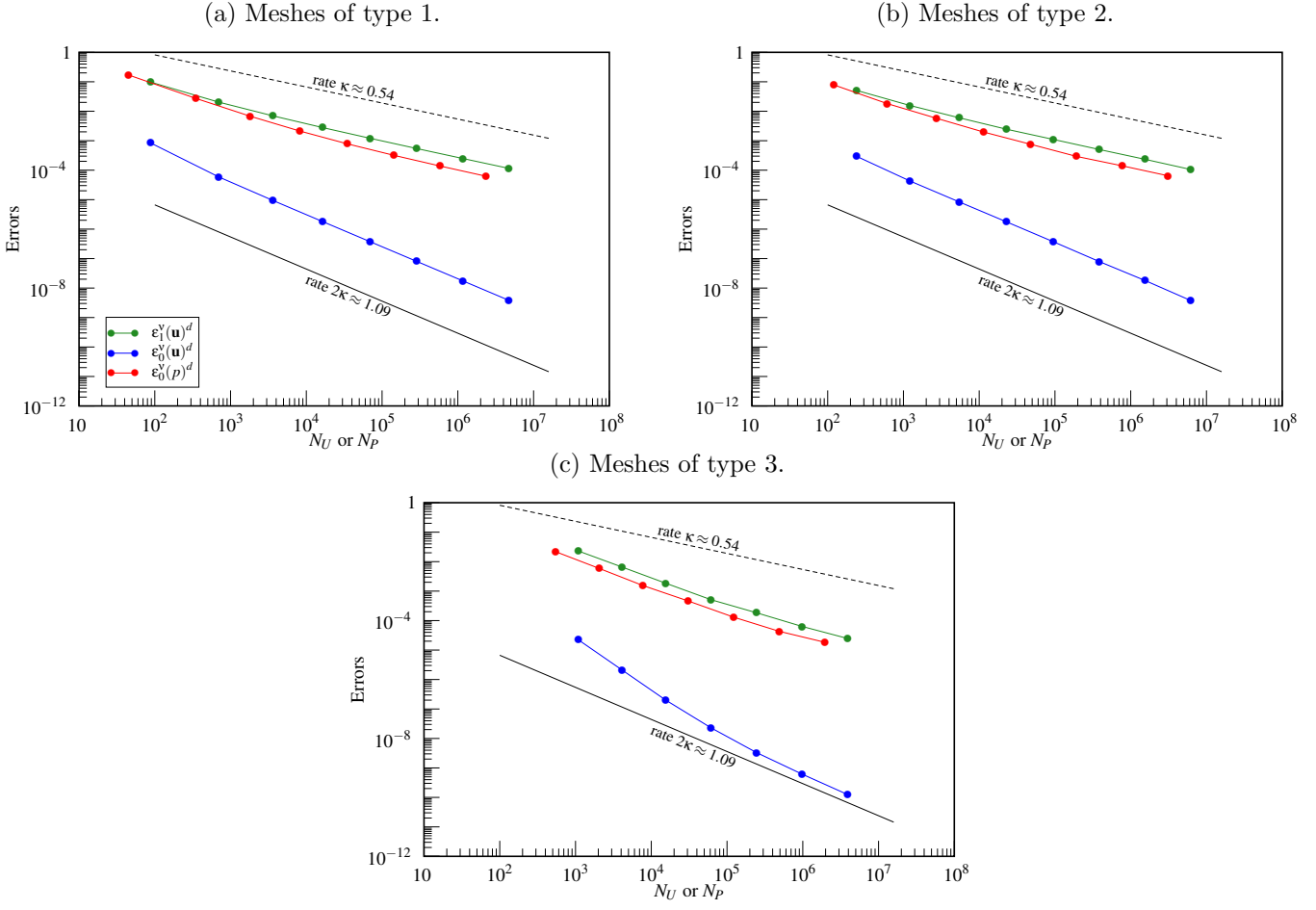


Figure 6: Plots of  $\varepsilon_1^v(\mathbf{u})$ ,  $\varepsilon_0^v(\mathbf{u})$ ,  $\varepsilon_0^v(p)$  in the nonconvex domain flow according to the type of meshes.

Hence:

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq 2 \inf_{\mathbf{w}_h \in \mathbf{V}_{0,h}} \|\mathbf{u} - \mathbf{w}_h\|_h + \sup_{\mathbf{w}_h \in \mathbf{V}_{0,h} \setminus \{0\}} \frac{|(\mathbf{u} - \mathbf{u}_h, \mathbf{w}_h)_h|}{\|\mathbf{w}_h\|_h}. \quad (\text{A.4})$$

To bound the first term in the right-hand side of (A.4), we need the Fortin operator. Using (92), we have indeed:

$$\inf_{\mathbf{w}_h \in \mathbf{V}_{0,h}} \|\mathbf{u} - \mathbf{w}_h\|_h \leq \|\mathbf{u} - \Pi_h \mathbf{u}\|_h \lesssim \sigma^{2d} h |\mathbf{u}|_{\mathbf{H}^2(\Omega)}. \quad (\text{A.5})$$

Let us bound the second term in the right-hand side of (A.4). Consider  $\mathbf{w}_h \in \mathbf{V}_{0,h}$ . It holds:  $\nu(\mathbf{u}_h, \mathbf{w}_h)_h = (\mathbf{f}, \mathbf{w}_h)_{\mathbf{L}^2(\Omega)}$ . Since  $\mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad} p$ , we obtain by integration by parts:

$$\nu(\mathbf{u}_h, \mathbf{w}_h)_h = \nu(\mathbf{u}, \mathbf{w}_h)_h - \nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{w}_h)_{\mathbf{L}^2(F_f)} + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (p, \mathbf{w}_h \cdot \mathbf{n}_{f,\ell})_{\mathbf{L}^2(F_f)}.$$

Hence, for all  $\mathbf{w}_h \in \mathbf{V}_{0,h}$ :

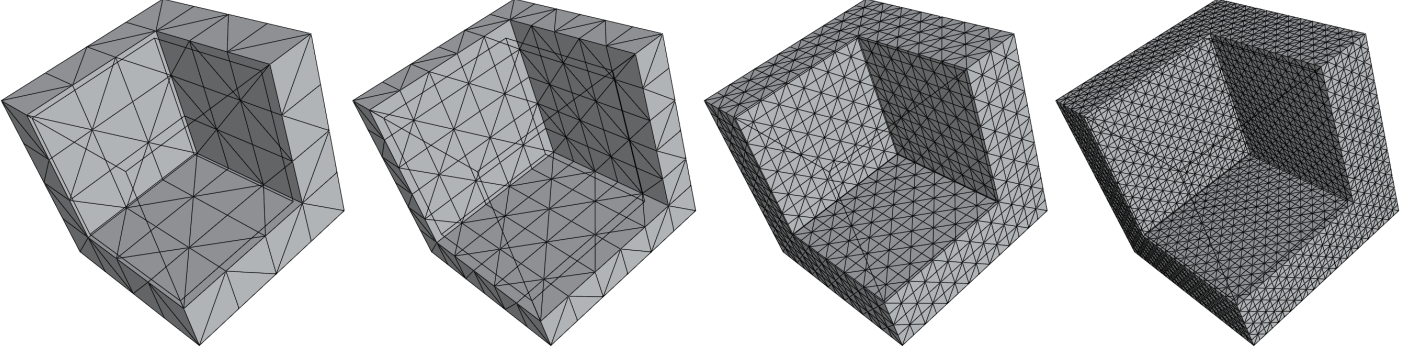
$$(\mathbf{u} - \mathbf{u}_h, \mathbf{w}_h)_h = \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{w}_h)_{\mathbf{L}^2(F_f)} - \nu^{-1} \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (p, \mathbf{w}_h \cdot \mathbf{n}_{f,\ell})_{\mathbf{L}^2(F_f)}. \quad (\text{A.6})$$

We apply (28) to all the components of  $\mathbf{Grad} \mathbf{u} \in \mathbb{H}^1(\Omega)$  and  $\mathbf{w}_h \in \mathbf{X}_{0,h}$  to obtain:

$$|(\mathbf{u} - \mathbf{u}_h, \mathbf{w}_h)_h| \lesssim \sigma h \left( |\mathbf{u}|_{\mathbf{H}^2(\Omega)} + \nu^{-1} |p|_{H^1(\Omega)} \right) \|\mathbf{w}_h\|_h. \quad (\text{A.7})$$

Summing (A.5) and (A.7), we can bound (A.2). We deduce (99).

(a) Meshes of type 1 (structured).



(b) Meshes of type 2 (unstructured).

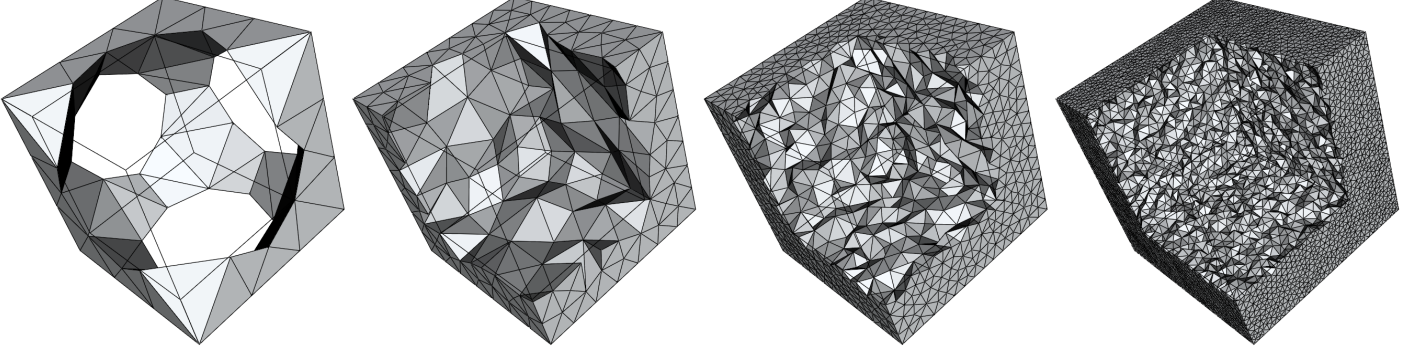


Figure 7: First four grids of the mesh families employed in three-dimensional convex domain. The interior is shown.

Let us prove (100). It holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} = \sup_{\mathbf{w} \setminus \{0\} \in \mathbf{L}^2(\Omega)} \frac{|(\mathbf{u} - \mathbf{u}_h, \mathbf{w})_{\mathbf{L}^2(\Omega)}|}{\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}}. \quad (\text{A.8})$$

Suppose that  $\Omega$  is convex. We obtain (100) using the Aubin-Nitsche trick. Let  $\mathbf{w} \in \mathbf{L}^2(\Omega)$ . According to Proposition 2, it exists  $(\mathbf{v}, q) \in ((\mathbf{V}_0 \cap H^2(\Omega)) \times (L^2_{zmv}(\Omega) \times H^1(\Omega)))$  such that:  $-\nu \Delta \mathbf{v} + \mathbf{grad} q = \mathbf{w}$ . By integration by parts, letting  $\delta \mathbf{u}_h = \mathbf{u} - \mathbf{u}_h$ , we have for all  $\mathbf{w} \in \mathbf{L}^2(\Omega)$ :

$$\begin{aligned} (\mathbf{u} - \mathbf{u}_h, \mathbf{w})_{\mathbf{L}^2(\Omega)} &= \nu (\mathbf{u} - \mathbf{u}_h, \mathbf{v})_h - \nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{v} : \mathbf{n}_{f,\ell}, \mathbf{u} - \mathbf{u}_h)_{\mathbf{L}^2(F_f)} \\ &\quad + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (q, (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)}. \end{aligned} \quad (\text{A.9})$$

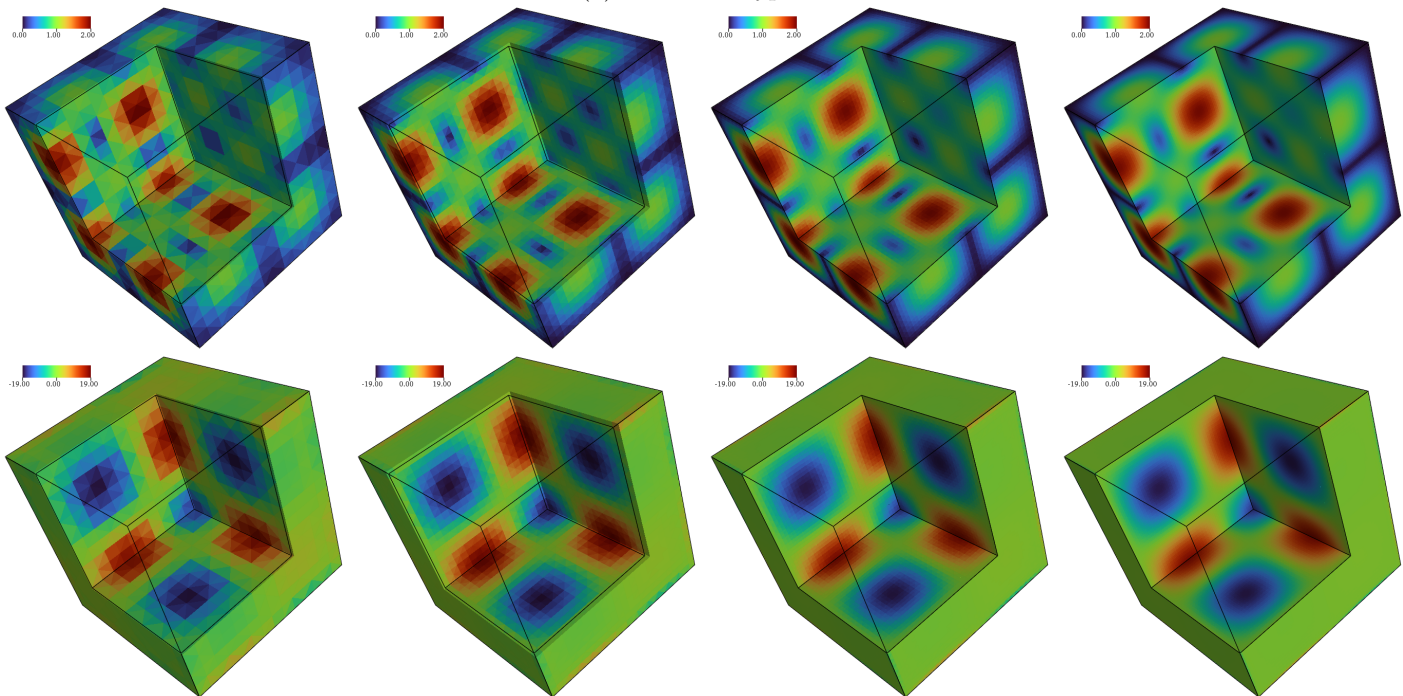
Summing (A.6) and (A.9), letting  $\delta \mathbf{u}_h := \mathbf{u} - \mathbf{u}_h$ , we deduce that for all  $\mathbf{w} \in \mathbf{L}^2(\Omega)$  and for all  $\mathbf{w}_h \in \mathbf{V}_{0,h}$ :

$$\begin{aligned} (\mathbf{u} - \mathbf{u}_h, \mathbf{w})_{\mathbf{L}^2(\Omega)} &= \nu (\mathbf{u} - \mathbf{u}_h, \mathbf{v} - \mathbf{w}_h)_h \\ &\quad + \nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \left( (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{w}_h)_{\mathbf{L}^2(F_f)} - (\mathbf{Grad} \mathbf{v} : \mathbf{n}_{f,\ell}, \delta \mathbf{u}_h)_{\mathbf{L}^2(F_f)} \right) \\ &\quad + \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \left( (q, \delta \mathbf{u}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} + (p, \mathbf{w}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} \right). \end{aligned} \quad (\text{A.10})$$

We now consider  $\mathbf{w}_h = \Pi_h \mathbf{v}$ . By Cauchy-Schwarz, and using (99) and (92), we have:

$$|\nu (\mathbf{u} - \mathbf{u}_h, \mathbf{v} - \Pi_h \mathbf{v})_h| \leq \nu \|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{v} - \Pi_h \mathbf{v}\|_h \lesssim \nu \sigma^{4d} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}.$$

(a) Meshes of type 1.



(b) Meshes of type 2.

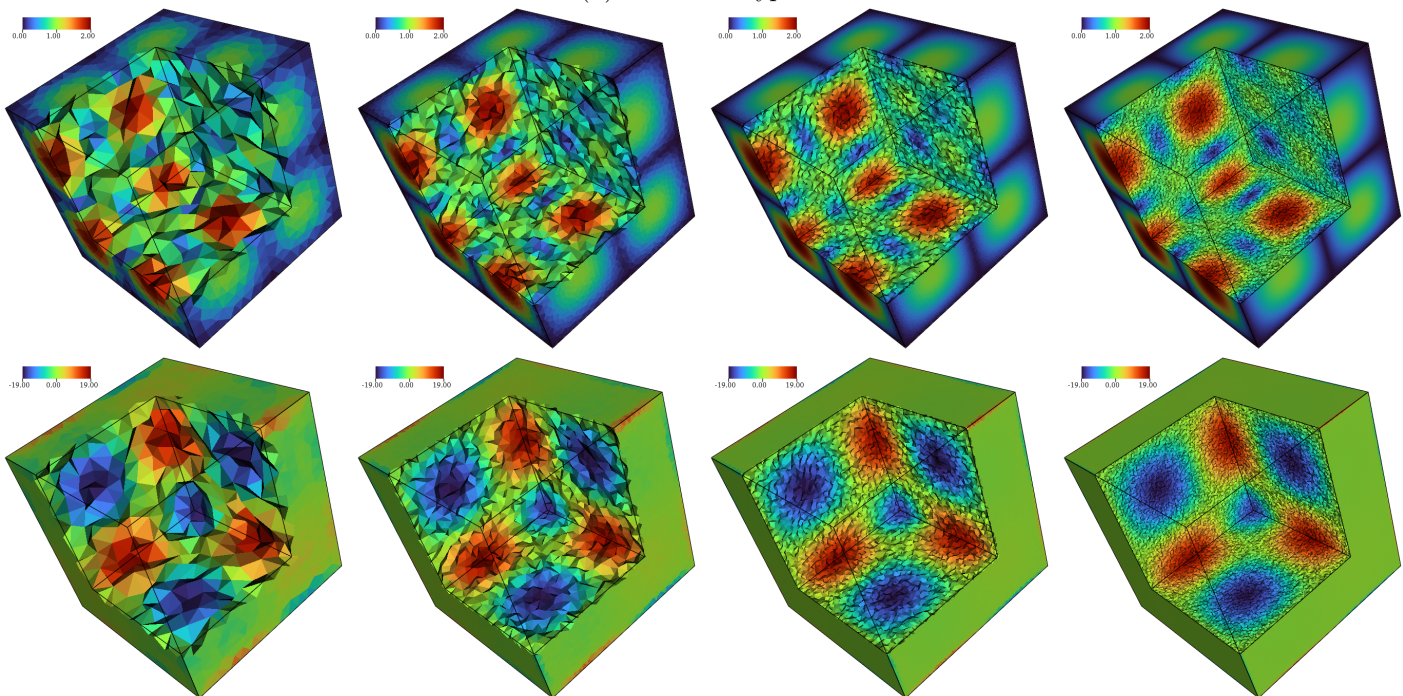
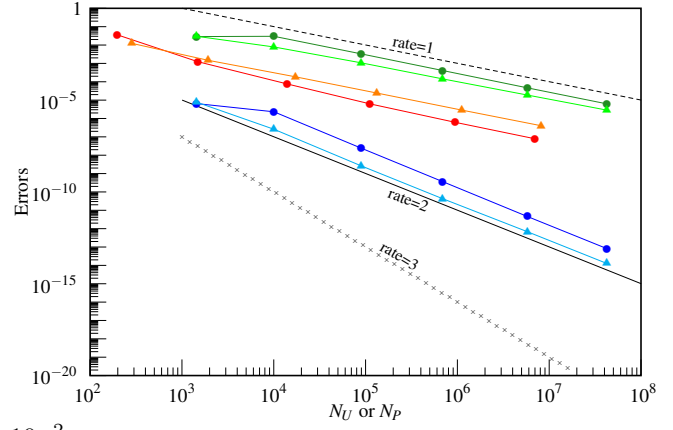
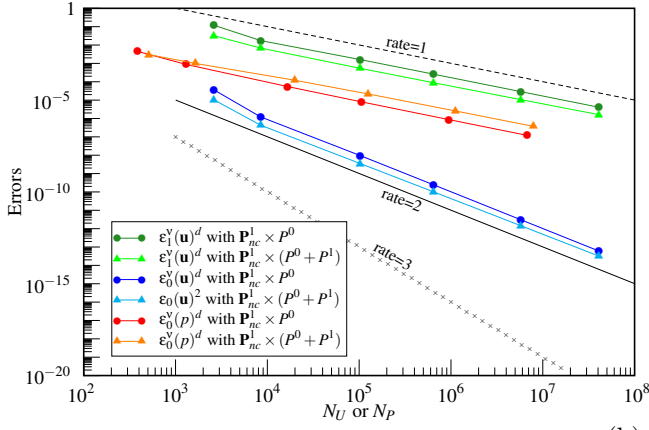


Figure 8:  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization,  $\nu = 1$ . From left to right: meshes 3 to 6. From top to bottom:  $P^1$ -Lagrange projection of the discrete velocity norm and the discrete pressure.

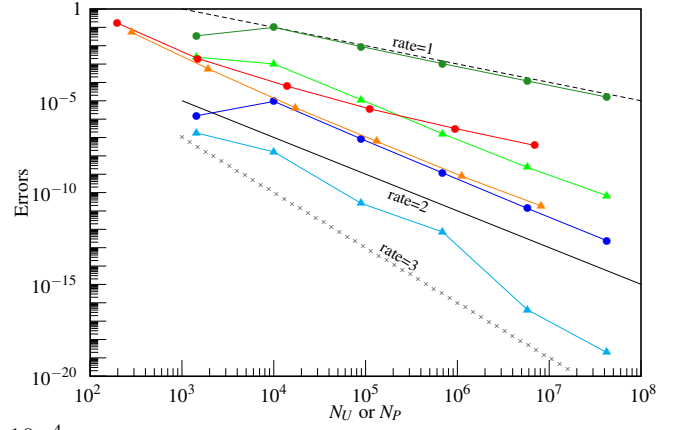
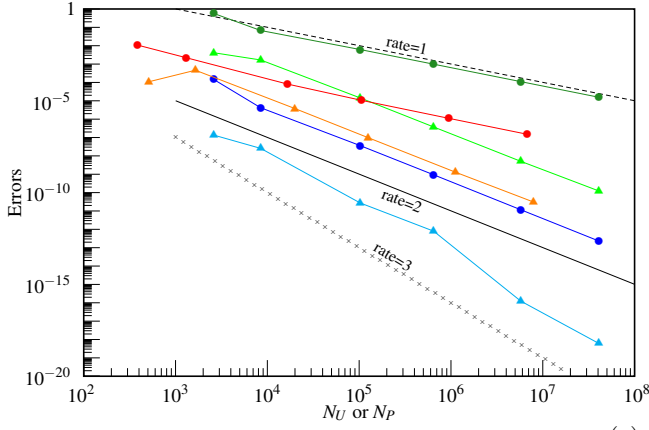
We deduce, using Proposition 2 that:

$$|\nu(\mathbf{u} - \mathbf{u}_h, \mathbf{v} - \Pi_h \mathbf{v})_h| \lesssim \sigma^{2(2d-1)} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \quad (\text{A.11})$$

(a)  $\nu = 1$ .



(b)  $\nu = 10^{-2}$ .



(c)  $\nu = 10^{-4}$ .

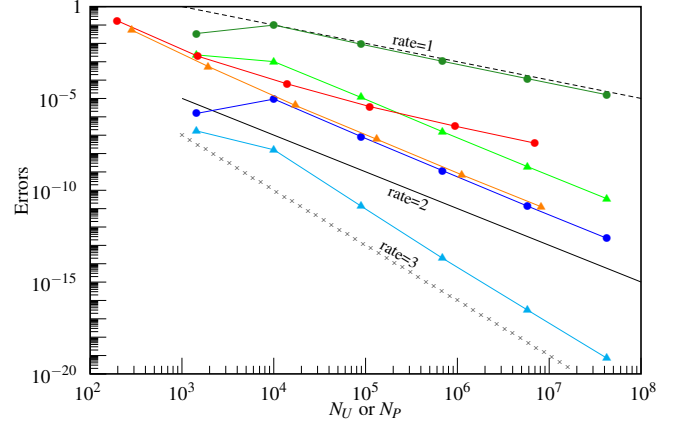
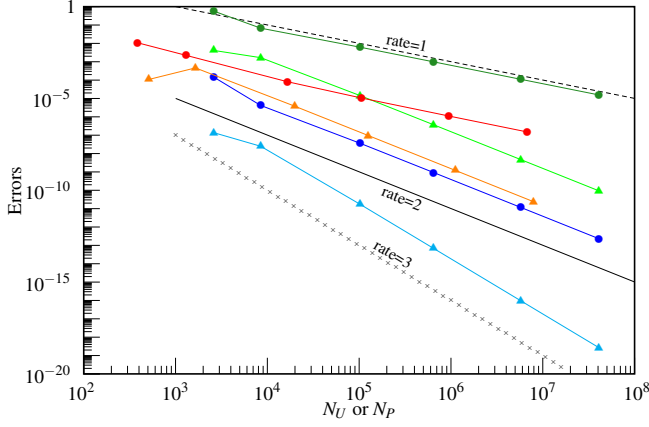


Figure 9:  $\mathbf{P}_{nc}^1 \times P^0$  and  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  discretization, Taylor–Green flow, plots of  $\varepsilon_{0,1}^\nu(\mathbf{u})$  and  $\varepsilon_0^\nu(p)$  for different  $\nu$ . Left columns: meshes of type 1, right columns: meshes of type 2.

We apply (28) to all the components of  $\mathbf{Grad} \mathbf{u} \in \mathbb{H}^1(\Omega)$  and  $\mathbf{w}_h \in \mathbf{X}_{0,h}$ :

$$\begin{aligned} \nu \left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \Pi_h \mathbf{v})_{\mathbf{L}^2(F_f)} \right| &= \nu \left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{v} - \Pi_h \mathbf{v})_{\mathbf{L}^2(F_f)} \right|, \\ &\leq \nu \sigma h |\mathbf{u}|_{\mathbf{H}^2(\Omega)} \|\mathbf{v} - \Pi_h \mathbf{v}\|_h. \end{aligned}$$

Using (92) and Proposition 2, it holds:

$$\nu \left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \Pi_h \mathbf{v})_{\mathbf{L}^2(F_f)} \right| \lesssim \nu \sigma^{2d} h^2 |\mathbf{u}|_{\mathbf{H}^2(\Omega)} |\mathbf{v}|_{\mathbf{H}^2(\Omega)} \lesssim \sigma^{2d} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \quad (\text{A.12})$$

We apply (28) to all the components of  $\mathbf{Grad} \mathbf{v} \in \mathbb{H}^1(\Omega)$  and  $\delta \mathbf{u}_h \in \mathbf{H}_0^1(\Omega) + \mathbf{X}_{0,h}$ . Using (99) and Proposition 2, we obtain:

$$\nu \left| \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} (\mathbf{Grad} \mathbf{v} : \mathbf{n}_{f,\ell}, \delta \mathbf{u}_h)_{\mathbf{L}^2(F_f)} \right| \lesssim \sigma^{2d} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}, \quad (\text{A.13})$$

Summing (A.12) and (A.13), we get:

$$\nu \sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \left( (\mathbf{Grad} \mathbf{u} : \mathbf{n}_{f,\ell}, \mathbf{w}_h)_{\mathbf{L}^2(F_f)} - (\mathbf{Grad} \mathbf{v} : \mathbf{n}_{f,\ell}, \delta \mathbf{u}_h)_{\mathbf{L}^2(F_f)} \right) \lesssim \sigma^{2d} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \quad (\text{A.14})$$

Samely, one proves that the two last terms in the right-hand side of (A.10) are bounded like:

$$\sum_{\ell \in \mathcal{I}_K} \sum_{f \in \mathcal{I}_{F,\ell}} \left( (q, \delta \mathbf{u}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} + (p, \mathbf{w}_h \cdot \mathbf{n}_{f,\ell})_{L^2(F_f)} \right) \lesssim \sigma^{2d} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \quad (\text{A.15})$$

Summing the contributions (A.11), (A.14) and (A.15), we deduce from (A.10) that:

$$|(\mathbf{u} - \mathbf{u}_h, \mathbf{w})_{\mathbf{L}^2(\Omega)}| \lesssim \sigma^{2(2d-1)} h^2 \|(\mathbf{u}, p)\|_\nu \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}.$$

Applying this result to (A.8), we get (100).

## Appendix B. Convergence tables

### Acknowledgements

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Table B.1: Convergence rates in the two-dimensional convex domain (subsection 11.1).

(a) Method  $\mathbf{P}_{nc}^1 \times P^0$ .

$N_U$	$N_P$	$\nu = 10^{-4}$			$\nu = 10^{-2}$			$\nu = 1$		
		$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
136	40	–	–	–	–	–	–	–	–	–
726	230	0.94	1.88	1.57	0.98	1.93	1.47	1.02	2.00	1.31
2,896	940	0.93	1.77	1.48	0.90	1.69	1.27	0.87	1.55	1.06
12,318	4,054	1.00	2.02	1.75	1.02	2.01	1.25	1.04	2.12	1.03
49,438	16,374	1.01	2.04	1.61	1.00	2.02	1.20	0.97	2.00	1.12
201,504	66,956	0.99	1.99	1.52	1.00	2.01	1.03	1.01	2.04	0.98
807,616	268,780	1.00	2.02	1.34	1.00	2.02	1.04	1.01	2.02	1.03
3,241,458	1,079,634	1.00	2.00	1.11	1.00	2.00	1.02	0.99	2.00	1.02

(b) Method  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$ .

$N_U$	$N_P$	$\nu = 10^{-4}$			$\nu = 10^{-2}$			$\nu = 1$		
		$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
136	69	–	–	–	–	–	–	–	–	–
726	364	0.93	2.38	2.77	0.93	2.38	1.22	0.93	2.38	1.21
2,896	1,449	0.90	1.76	1.76	0.90	1.76	1.05	0.90	1.76	1.05
12,318	6,160	1.04	2.16	1.51	1.04	2.15	1.01	1.04	2.15	1.01
49,438	24,720	0.98	2.04	1.29	0.98	2.07	1.11	0.98	2.07	1.11
201,504	100,753	1.01	1.98	1.03	1.01	2.04	0.98	1.01	2.04	0.98
807,616	403,809	1.01	1.82	1.04	1.01	1.99	1.02	1.01	2.00	1.02
3,241,458	1,620,730	0.99	1.87	1.02	0.99	1.98	1.01	0.99	2.01	1.01

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Table B.2: Convergence rates in the two-dimensional nonconvex domain (subsection 11.2).

(a) Meshes of type 1.

$N_U$	$N_P$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
88	45	–	–	–
696	349	0.75	1.28	0.89
3,640	1,821	0.65	1.12	0.84
16,440	8,221	0.61	1.09	0.77
69,688	34,845	0.58	1.09	0.69
286,776	143,389	0.57	1.09	0.63
1,163,320	581,661	0.56	1.09	0.50
4,685,880	2,342,941	0.55	1.09	0.57

(b) Meshes of type 2.

$N_U$	$N_P$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
240	121	–	–	–
1,220	611	0.73	1.19	0.90
5,490	2,746	0.64	1.09	0.77
22,988	11,495	0.62	1.08	0.75
94,962	47,482	0.56	1.09	0.67
385,268	192,635	0.56	1.11	0.68
1,548,786	774,394	0.55	1.06	0.53
6,210,686	3,105,344	0.57	1.13	0.56

(c) Meshes of type 3.

$N_U$	$N_P$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
1,090	546	–	–	–
4,134	2,068	0.94	1.77	0.99
15,532	7,767	0.95	1.77	1.01
61,514	30,758	0.94	1.57	0.88
244,864	122,433	0.74	1.40	0.91
978,944	489,473	0.77	1.22	0.80
3,900,410	1,950,206	0.68	1.15	0.62

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Table B.3: Convergence rates in the three-dimensional convex domain (subsection 11.3).

(a) Method  $\mathbf{P}_{nc}^1 \times P^0$  on meshes of type 1.

$N_U$	$N_P$	$\nu = 10^{-4}$			$\nu = 10^{-2}$			$\nu = 1$		
		$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
2,592	384	–	–	–	–	–	–	–	–	–
8,424	1,296	1.82	3.02	1.30	1.82	3.02	1.30	1.64	2.89	1.34
102,312	16,464	0.96	1.90	1.30	0.96	1.90	1.30	0.95	1.94	1.09
644,904	105,456	0.99	1.99	1.10	1.00	1.99	1.10	0.99	1.99	1.03
5,721,192	944,784	1.00	2.00	1.03	1.00	2.00	1.03	1.00	2.00	1.01
40,689,792	6,749,184	1.00	2.00	1.02	1.00	2.00	1.02	1.00	2.00	1.01

(b) Method  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  on meshes of type 1.

$N_U$	$N_P$	$\nu = 10^{-4}$			$\nu = 10^{-2}$			$\nu = 1$		
		$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
2,592	509	–	–	–	–	–	–	–	–	–
8,424	1,639	0.77	1.45	-1.24	0.77	1.42	-1.24	1.32	2.71	0.94
102,312	19,839	1.91	2.92	1.93	1.91	2.74	1.93	1.02	1.93	0.85
644,904	125,139	1.98	3.00	2.00	1.98	1.92	1.99	0.98	1.93	0.93
5,721,192	1,111,159	2.00	3.00	2.00	1.97	4.01	1.98	0.97	1.94	0.97
40,689,792	7,906,809	2.00	3.00	2.00	1.90	2.71	1.91	0.96	1.95	0.99

(c) Method  $\mathbf{P}_{nc}^1 \times P^0$  on meshes of type 2.

$N_U$	$N_P$	$\nu = 10^{-4}$			$\nu = 10^{-2}$			$\nu = 1$		
		$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
1,428	196	–	–	–	–	–	–	–	–	–
10,008	1,482	-0.54	-0.93	2.20	-0.54	-0.93	2.20	-0.02	0.53	1.67
88,995	13,961	1.10	2.15	1.55	1.10	2.15	1.55	1.01	2.08	1.25
686,958	110,737	1.05	2.09	1.37	1.05	2.09	1.38	1.02	2.05	1.20
5,760,687	944,493	1.02	2.09	1.15	1.02	2.09	1.15	1.02	2.06	1.08
42,186,285	6,967,056	1.01	2.03	1.06	1.01	2.03	1.06	1.01	2.02	1.03

(d) Method  $\mathbf{P}_{nc}^1 \times (P^0 + P^1)$  on meshes of type 2.

$N_U$	$N_P$	$\nu = 10^{-4}$			$\nu = 10^{-2}$			$\nu = 1$		
		$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$	$\tau_1(\mathbf{u})$	$\tau_0(\mathbf{u})$	$\tau_1(p)$
1,428	283	–	–	–	–	–	–	–	–	–
10,008	1,928	0.43	1.22	2.38	0.44	1.22	2.38	0.69	1.75	1.11
88,995	17,125	2.06	3.23	2.24	2.05	2.95	2.23	0.91	2.10	0.94
686,958	132,356	2.10	3.20	2.08	2.08	1.75	2.07	0.99	2.02	1.00
5,760,687	1,111,128	2.06	3.05	2.09	1.99	4.59	2.05	0.95	1.95	1.00
42,186,285	8,142,013	2.02	3.03	2.01	1.78	2.69	1.87	0.95	2.00	0.98

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