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ON A DIVERSITY RANKING OF CHOICE PROFILES

NICOLAS GRAVEL⁺, ERNESTO SAVAGLIO[°], AND STEFANO VANNUCCI[#]

ABSTRACT. The different options people select from a set of *non-rival* alternatives are compared in terms of *singularity*. A criterion for ranking these choices on the basis of the number of other choices from which they differ is introduced and characterized. An axiomatic characterization of the ranking of choice profiles based on the aggregation of the singularities of the chosen alternatives is also provided.

JEL Classification: D63

Keywords: Diversity, Freedom of choice, Orderings, Choice profiles.

“The prospects of freedom in the contemporary world may well lie in the recognition of the plurality of our identities, where personal identity must be understood as an extension of one’s own choice of being someone or doing something” (A. K. Sen (2006))

1. INTRODUCTION

A long-standing tradition (see e.g. Engels and Marx (1844)¹, Mill (1974), Nozick (1974), Kukathas (2003), Sen (2006) or Li (2014)) regards choice diversity as an intrinsically desirable characteristic of human societies that reflects the freedom that their members have to choose the way of life that please them most. In this paper, we propose criteria for evaluating the *singularity* of a particular individual choice in the society in which the individual choice is observed and for comparing different societies on the basis of the diversity of their observed choices. We suppose that a (possibly variable) number of individuals choose alternatives from a common set X , which we interpret as a collection of individual life-plans, or as Sen (1991) put it, alternatives that everyone may claim without preventing others from claiming them as well. Every individual is observed selecting an alternative from that set that - presumably - is the one she most prefers. If a selected alternative is identified as *less-common* than another for a given choice profile, then the former will be classified as more original, or *singular*, than another. We propose to define a chosen alternative as more *singular* than another if it differs from a larger number of choices made by others than the latter or, equivalently, if it is chosen less

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¹“For as soon as the distribution of labour comes into being, each man has a particular, exclusive sphere of activity, which is forced upon him and from which he cannot escape. He is a hunter, a fisherman, a herdsman, or a critical critic, and must remain so if he does not want to lose his means of livelihood; while in communist society, where nobody has one exclusive sphere of activity but each can become accomplished in any branch he wishes, society regulates the general production and thus makes it possible for me to do one thing today and another tomorrow, to hunt in the morning, fish in the afternoon, rear cattle in the evening, criticise after dinner, just as I have a mind, without ever becoming hunter, fisherman, herdsman or critic.” (free English translation).

frequently than the latter. This singularity criterion induces an ordering of all alternatives chosen at least once in a choice profile, that we characterize axiomatically. We remark that the ranking produced by our singularity criterion when applied to a specific choice profile may be interpreted as a ranking of the alternatives based on their aggregate *dissimilarities* from all the other chosen alternatives in the profiles. In particular, it coincides with the ranking of any pair of alternatives on the basis of the sum of their discrete *distances* from all the other chosen alternative in the profile, with the distance defined to be 1 if two compared alternatives are different and 0 if the alternatives are the same. Thus, the model in the present paper belongs to the literature on the measurement of diversity (see e.g. Gravel (2008) and Ciommi, Savaglio, and Vannucci (2018) for surveys) that makes various kinds of assumptions on the structure of the space from which the alternatives are taken, including premetric, semimetric or metric spaces (for example in Pattanaik and Xu (2000a), Pattanaik and Xu (2008), Bossert, Pattanaik, and Xu (2003) and VanHees (2004)), relational systems consisting of preference profiles supplemented with a similarity quaternary relation (Peragine and Romero-Medina (2006)), or a quaternary dissimilarity relation representing a total order of binary dissimilarities (Bervoets and Gravel (2007)) and (subspaces of) suitably ordered mixture spaces (Nehring and Puppe (2002), Nehring and Puppe (2003)).

We also propose to compare alternative *choice profiles* associated to different societies on the basis of the aggregate singularity of their chosen alternatives. Specifically, we assess negatively the diversity of a choice profile as the quadratic average, in the considered profile, of all the alternatives' singularity.

We view the assessment of the diversity of choices observed in a society as an important ingredient in the evaluation of that society's *actual* freedom. Freedom of choice, indeed, involves not only the absence of constraints but, also, the presence of meaningful choices that allow individuals to pursue their own paths and achieve their potential. The extent by which a society recognizes and accommodates the diverse preferences of its member is therefore an indicator of this society's freedom, as recognized some time ago by Suppes (1996), or by Sen (2006) ('heterogeneity of chosen alternatives is important as long as such a dissimilarity comes directly from the value of *freedom of choice*, i.e. it depends on the liberty people have in their choice processes'). For example the political freedom revealed through elections is commonly associated to a significant dispersion of the votes among several candidates and elections in which the winner receives more than 90% of the voter are not usually perceived as being free. While we associate our contribution to the literature on *freedom of choice* surveyed in Barberà, Bossert, and Pattanaik (2004), it differs from this literature in several important aspects. The traditional literature on freedom of choices is concerned with the "size" of the set of options that are available for choice (see e.g. Basili and Vannucci (2013), Pattanaik and Xu (1990), Pattanaik and Xu (1998), Pattanaik and Xu (2000b), Peragine and Romero-Medina (2006) and VanHees (2004)). Hence, one could say that it tends to evaluate freedom of choice *ex ante*, that it is before the freedom is exerted and converted into a final choice. By contrast, our paper evaluates the freedom of choice from an *ex post* perspective by looking at the result of the exercise, for the considered collection of individuals, of their freedom. To that extent, the paper fits also in the somewhat sparse literature on "statistical freedom" (see e.g. Gabor and Gabor (1954) and Suppes (1996)). Admittedly, observing choices that "differ a lot" from each other is more revealing of an exercise of freedom than the observation of very similar choices

which could be evocative of some "conformism". Yet, diversity or dispersions observed in peoples' choices may not always be associated to a greater freedom. In the case of political freedom, if all citizens are really facing a large set of candidates and have the ability to debate freely about which of these candidates is "really" the best, they may well reach a Habermas (1991b) consensus and end up, as a result, to choose the same candidate.² In such a case, non-singularity would actually reveal a large freedom. In the other direction, one could easily think of societies in which people belonging to different religions make diverse choices (e.g. eating pork for low caste Hindus, beef for Muslims and vegetarian for Jains) that are not particularly free, especially if these choices are dictated by religious dogmas. The relationship between freedom and diversity of choice is, for sure, a complex one and the axiomatic characterization of a somewhat specific notion of choice singularity performed herein only captures a very limited aspect of this complexity.

The plan of the rest of the paper is as follows. The next section introduces the notation, definitions and axioms that characterizes the profile-specific ranking of alternatives on the basis of their singularity. Section 3 discusses the use that one could make of this definition of singularity to compare choice profiles on the basis of their aggregate singularity, and makes some additional connections to the literature on freedom of choice. Section 4 draws conclusions and the Appendix provides all the proofs.

2. COMPARING INDIVIDUAL CHOICES IN A PROFILE IN TERMS OF SINGULARITY

Let X be the finite set of feasible alternatives. We interpret these alternatives as bundles of rights or basic liberties, functionings or, simply, public goods that everyone can claim in order to live a life worth living without preventing others from making the same choice.³ We define a *choice profile* as a *multiset* (see e.g. Bender (1974) or Blizard (1989)) described by a (multiplicity) function $m : X \rightarrow \mathbb{N}$ that assigns to any alternative x the number $m(x)$ of individuals who choose x in the profile m . The possibility to observe $m(x) = 0$ for some (typically many if X is large) x is of course not ruled out. We denote by \mathcal{M} the set of all conceivable choice profiles from a finite set of alternatives X . For any choice profile $m : X \rightarrow \mathbb{N}$, we let $s(m) = \sum_{x \in X} m(x)$ denote the number of individuals involved in the profile. We shall limit attention in this paper to choice profiles m for which $s(m) \geq 2$. For any choice profile $m : X \rightarrow \mathbb{N}$, we also denote by $\mathcal{S}(m) = \{x \in X : m(x) > 0\}$ the set of all alternatives chosen at least once in the profile. We will sometimes refer to the set $\mathcal{S}(m)$ as to the *support* of m . For any choice profile m , we say that an alternative x is an *Isolated Choice (IC)* in m if it is such that $m(x) = 1$, i.e. an alternative is an isolated choice in a choice profile if it is chosen by only one individual. Intuitively, an isolated choice may be seen as maximally "original", or singular.

A purpose of this paper is to identify a criterion for comparing the chosen alternatives in various choice profiles on the basis of their singularity. A somewhat natural such criterion would be to define the singularity of an alternative in a choice profile by the *number of chosen alternatives from which*

²"...consensus achieved through communication depends both on the idealizing supposition that an identity in linguistic meaning already prevails and also on the power of negation and the autonomy of unique, non-substitutable subjects – for whom intersubjective consent to criticizable validity claims has to be obtained anew in each case. All consensus achieved within discourse rests on the power of negation held by independent subjects" (Habermas (1991a))

³Alternatively, we may interpret X as a set of private objects, or bundles of the same, that every individual faces.

it differs. This is clearly equivalent to define, negatively, the singularity of a chosen alternative by its multiplicity (the larger the multiplicity, the less singular the choice in the profile). This simple criterion defines, for every choice profile $m : X \rightarrow \mathbb{N}$, the ordering⁴ \succeq_s^m (with asymmetric and symmetric factors \succ_s^m and \sim_s^m , respectively) on its support $\mathcal{S}(m)$ defined formally as follows.

Definition 1 (*Multiplicity-based Singularity Ordering*). *For any choice profile $m : X \rightarrow \mathbb{N}$ and any $x, x' \in \mathcal{S}(m)$,*

$$(2.1) \quad x \succeq_s^m x' \text{ if and only if } m(x) \leq m(x')$$

As noticed above, comparing negatively the chosen alternatives in a choice profile in terms of their multiplicity is equivalent to comparing them positively in terms of the number of chosen alternatives in the profile from which they differ. Hence, one could interpret the proposed singularity-based ranking \succeq_s^m of alternatives as (the opposite of) a ranking of the same alternatives based on the *sum of their pairwise dissimilarities vis-à-vis* all the other chosen alternatives, provided that pairwise dissimilarities between alternatives are measured by the *discrete distance* $d : X \times X \rightarrow \{0, 1\}$ defined as:

$$d(x, x') = \begin{cases} 1 & \text{if } x \neq x' \\ 0 & \text{if } x = x' \end{cases} \quad \text{for any } x, x' \in X,$$

Indeed, it is clear that, for every choice profile $m \in \mathcal{M}$ and two alternatives $x, x' \in \mathcal{S}(m)$, one has $x \succeq_s^m x'$ if and only if:

$$(2.2) \quad \sum_{w \in \mathcal{S}(m)} m(w)d(x, w) \geq \sum_{w \in \mathcal{S}(m)} m(w)d(x', w).$$

In what follows, we characterize \succeq_s^m as the only ordering \succeq^m on $\mathcal{S}(m)$ that satisfies the following three axioms for every choice profile $m : X \rightarrow \mathbb{N}$.

Axiom 1 (*Indifference between Isolated Choices*). *For any choice profile $m : X \rightarrow \mathbb{N}$ and any alternatives $x, x' \in \mathcal{S}(m)$ that are IC in m , $x \sim^m x'$.*

This axiom just says that, in any choice profile, two alternatives that are isolated choices in this profile should be considered equivalent in terms of their singularity. The *Indifference between Isolated Choices* axiom resembles the widely-discussed *Indifference between no-choice situations* axiom of Pattanaik and Xu (1990), requiring that the freedom of choice offered by two singletons be identical. The justification given by Pattanaik and Xu (1990) to indifference between singletons is that a singleton represents the absence of choice possibility and, therefore, of freedom of choice. Hence, all these no-choice situations should be seen as providing the same "extent of freedom (to wit, nil)" (Sen (1991), p. 25). In the current context, *IC* alternatives can be seen indirectly as representing the maximal liberty of individuals of choosing 'to do something or be someone' and to be, therefore, deeply original or singular. Because of this, all these maximally singular alternatives should naturally be considered equivalent.

⁴An ordering is a reflexive, complete and transitive binary relation.

Some scholars (see e.g. Carter (2004) and VanHees (2004)) have proposed notions of similarity among pairs of objects that focus on the objective attributes of these objects. The axiom of Indifference between isolated choices may seem at odd with such notions. Indeed, two distinct isolated choices in a given profile may be perceived to have different "objective" degree of dissimilarity with respect to the other chosen alternatives in the profile and, for this reason, may not be considered equivalent. However, our abstract approach does not endow the objects of choice with a structure that would enable an *a priori* objective differentiation among them in terms of some objective dissimilarity. The only available information about the relative dissimilarity of any chosen alternative *vis-à-vis* the other chosen alternatives of a given profile is the choice pattern of the alternatives by the individuals.

Another possible contentious feature of the *Indifference Between Isolated Choices* axiom concerns the fact that the indifference between two *IC* in a given choice profile could (should ?) depend upon the cardinality of the set $\mathcal{S}(m)$ of alternatives actually chosen in the profile. Namely, isolated choices a and b should not be considered indifferent in the same manner when they are the only two distinct alternatives chosen in a profile with two agents than when they are isolated choices in a large profile in which many other alternatives are chosen several times. However, although the Indifference axiom will impose that a and b have the same relative singularity in a profile where they are the only two chosen alternatives than in any other profiles where they are both *IC*, we believe that one could account for the different contributions of two isolated choices to the overall diversity of a choice profile when defining what this overall diversity may mean. Additional discussion of this issue will be provided in the next section.

The second axiom that a binary relation \succeq^m on $\mathcal{S}(m)$ could satisfy in order to reflect the originality of the choices made in the profile m is the following.

Axiom 2 (Dominance of Isolated Choices). *For any choice profile $m \in \mathcal{M}$ and any x that is *IC* in m and any x' that is not *IC* in m , $x \succ^m x'$.*

In words, the Dominance of Isolated Choices axiom says that isolated choices - the most singular choices that one could think of - must exhibit strictly more singularity than any other chosen alternative that is not an isolated choice. This second axiom is therefore totally in line with the intuition that isolated choices are maximal in terms of the singularity that they express and, therefore, exhibit strictly more singularity than any alternative that is chosen more than once in a given profile. In the current context, the dominance of isolated choices axiom plays a similar role to the preference for choice over no-choice axiom proposed by Pattanaik and Xu (1990).

We now turn to the last axiom imposed on \succeq^m .

Axiom 3 (Sequential Neutrality to Indifference). *Consider any choice profile $m : X \rightarrow \mathbb{N}$, and any list of t choice profiles $m^l : X \rightarrow \mathbb{N}$ (for $l = 1, \dots, t$) satisfying, for any $x \in X$, $\sum_{l=1}^t m^l(x) = m(x)$, and*

such that $a, b \in \bigcap_{l=1}^t \mathcal{S}(m^l)$ for some alternatives a and b . Then:

$$a \sim^{m^l} b \text{ for all } l = 1, \dots, t-1 \text{ and } a \succeq^{m^t} b \implies a \succeq^m b.$$

and

$$a \sim^{m^l} b \text{ for all } l = 1, \dots, t-1 \text{ and } a \succ^{m^t} b \implies a \succ^m b.$$

Observe that this axiom is the only one that imposes a condition on the ranking of alternatives that applies to *different* choice profiles. In order to illustrate the axiom, consider the choice profile m with $s(m) = 7$ defined by $m(a) = 3$, $m(b) = 2$, $m(c) = m(d) = 1$ (with $m(x) = 0$ for all $x \neq a, b, c, d$) and consider the sequence of two choice profiles m^1 and m^2 defined by:

$$m^1(a) = 2, m^1(b) = m^1(c) = 1 \text{ and } m^1(d) = 0$$

and:

$$m^2(a) = m^2(b) = m^2(d) = 1 \text{ and } m^2(c) = 0$$

with $m^1(x) = m^2(x) = 0$ for all $x \neq a, b, c, d$. Observe that the sequence of (two) choice profiles m^1 and m^2 satisfy, for any $x \in X$, $m^1(y) + m^2(y) = m(y)$ (i.e. they form a *partition of the multiset* m). Observe also that in the choice profile m^1 , b would be considered more singular than a by any ranking satisfying the dominance axiom. Indeed, b is an isolated choice in m^1 while a is not an isolated choice in that profile. Observe finally that in the choice profile m^2 , alternatives a and b would be considered as indifferent by any ranking satisfying the indifference axiom (because the two alternatives are isolated choices in m^2). The Sequential Neutrality to Indifference axiom then requires that alternative a be considered less singular than b in the "grand" choice profile m that is the (multiset theoretic) union of the multisets m^1 and m^2 . The intuitive idea is that if one merges an arbitrary number of choice profiles that all exhibit an equivalence of singularity between two alternatives with another choice profile in which the same two alternatives are compared in a specific way, than the latter comparison should be preserved in the grand merged choice profile. The sequence of indifferences between the two alternatives should thus be *neutral* for the appraisal of the overall comparison of the same two alternatives in the grand choice profile.

These three axioms characterize the ordering \succeq_s^m (for any choice profile m) as the following result - proved in the appendix - shows.

Theorem 1. *For any choice profile $m : X \rightarrow \mathbb{N}$, an ordering \succeq^m on $\mathcal{S}(m)$ satisfies Indifference between Isolated Choices, Dominance of Isolated Choices, and Sequential Neutrality to Indifference if and only if $\succeq^m = \succeq_s^m$.*

The following examples show that the axioms of Indifference between Isolated Choices, Dominance of Isolated Choices and Sequential Neutrality to Indifference when applied to an ordering \succeq^m on $\mathcal{S}(m)$ for any choice profile $m : X \rightarrow \mathbb{N}$ are independent.

Example 1. *Consider the ordering \succeq_1^m defined on the support $\mathcal{S}(m)$ of any choice profile $m : X \rightarrow \mathbb{N}$ by:*

$$x \succeq_1^m y \iff r(x)m(x) \leq r(y)m(y)$$

for any distinct x and $y \in \mathcal{S}(m)$ and the one-to-one function $r : X \rightarrow \{-e^{-1}, -e^{-2}, \dots, -e^{-\#X}\}$. This ordering obviously violates the Indifference between Isolated Choices axiom because it ranks any two isolated choices as per the strict ranking induced by comparing the values assigned to those choices by the function r . To see that it satisfies Dominance of Isolated Choices, consider any choice profile $m : X \rightarrow \mathbb{N}$ and any two distinct x and $y \in \mathcal{S}(m)$ such that $m(x) = 1 < m(y)$. We observe that

$$r(x)m(x) = r(x) < r(y)m(y)$$

because $r(x)/r(y) \leq e^{-\#X+1} = \frac{1}{e^{\#X-1}} < 2$. Hence $x \succ_1^m y$. To see that \succeq_1^m satisfies Sequential Neutrality to Indifference, consider any choice profile $m : X \rightarrow \mathbb{N}$, and any list of t choice profiles $m^l : X \rightarrow \mathbb{N}$ (for $l = 1, \dots, t$) satisfying, for any $x \in X$, $\sum_{l=1}^t m^l(x) = m(x)$, and such that $a, b \in \bigcap_{l=1}^t \mathcal{S}(m^l)$ and :

$$a \sim^{m^l} b \text{ for all } l = 1, \dots, t-1$$

for some alternatives a and b . It follows that

$$r(a)m^l(a) = r(b)m^l(b)$$

for all $l = 1, \dots, t-1$. Summing these equalities yield:

$$r(a) \sum_{l=1}^{t-1} m^l(a) = r(b) \sum_{l=1}^{t-1} m^l(b)m^l(b)$$

If $a \succeq_1^{m^t} b \iff r(a)m^t(a) \leq r(b)m^t(b)$ holds, then one has:

$$\begin{aligned} r(a) \sum_{l=1}^t m^l(a) &\leq r(b) \sum_{l=1}^t m^l(b)m^l(b) \\ &\iff \\ &a \succeq_1^m b \end{aligned}$$

as required while if $a \succ_1^{m^t} b \iff r(a)m^t(a) < r(b)m^t(b)$ holds, then one has also the required implication that

$$\begin{aligned} r(a) \sum_{l=1}^t m^l(a) &< r(b) \sum_{l=1}^t m^l(b)m^l(b) \\ &\iff \\ &a \succ_1^m b. \end{aligned}$$

Example 2. Consider the trivial universal indifference relation \sim^m defined, for every choice profile $m : X \rightarrow \mathbb{N}$, by $x \sim^m y$ for any x and $y \in \mathcal{S}(m)$. This universal indifference relation obviously satisfies Indifference Between Isolated Choices and, trivially, Sequential Neutrality to Indifference, but violates Dominance of Isolated choices.

Example 3. Consider the ordering \succeq_2^m defined on the support $\mathcal{S}(m)$ of any choice profile $m : X \rightarrow \mathbb{N}$ by $x \succeq_2^m y$ if and only if either $m(x) = 1$ and $m(x) \leq m(y)$ or $\min\{m(x), m(y)\}$. This ordering obviously satisfies Indifference between Isolated Choices and Dominance of Isolated choices. To see that it violates Sequential Neutrality to Indifference, consider the two choice profiles m' and m'' such

that $m'(a) = 1 < m'(b) = 2 = m''(a) = m''(b)$ for some alternatives a and b . One then has $a \succ_2^{m'} b$ and $a \sim_2^{m''} b$. Consider then the choice profile m defined by $m(x) = m'(x) + m''(x)$ for any $x \in X$. Since $m(a) = 3$ and $m(b) = 4$, it follows that $a \sim_2^m b$, contrary to what Sequential Neutrality to indifference would require.

3. RANKING CHOICE PROFILES IN TERMS OF DIVERSITY

Having defined what it means for an alternative chosen in a *given* choice profile to be more singular than another, it is tempting to use this definition for comparing *different* choice profiles coming from different populations of individuals (or from the same population making different choices) in terms of the *diversity* of these choices. Let \succeq be a binary relation defined on the set of all conceivable choice profiles \mathcal{M} , with the interpretation that $m \succeq m'$ if and only if the profile m offers at least as much *diversity* as the profile m' . A (reasonably) natural criterion for this purpose would be to base the comparisons of choice profiles on a suitable average of the singularities of the alternatives, the greater the average the more diverse the profile. A specific average that could serve for this purpose is the *quadratic mean of the ratios of the multiplicity of the alternatives over the sum of those multiplicity*. Since the multiplicity of an alternative is, thanks to Theorem 1, a negative indicator of its singularity, a diversity ranking should be concerned with reducing this quadratic mean. Hence, a possible ranking \succeq^D of choice profiles based on their diversity could be the following.

Definition 2 (Diversity Order). *For any two choice profiles $m, m' \in \mathcal{M}$,*

$$m \succeq^D m' \text{ if and only if } \frac{\frac{1}{\sqrt{2}} \sqrt{\sum_{x \in X} m^2(x)}}{s(m)} \leq \frac{\frac{1}{\sqrt{2}} \sqrt{\sum_{x \in X} m'^2(x)}}{s'(m)}.$$

We observe that the ordering \succeq^D can be equivalently written as:

$$m \succeq^D m' \iff \sum_{x \in X} \sigma_m^2(x) \leq \sum_{x \in X} \sigma_{m'}^2(x)$$

with $\sigma_m(x) = m(x)/s(m)$ and $\sigma_{m'}(x) = m'(x)/s(m')$ denoting the share of the population who choose alternative x in the choice profiles m and m' , respectively. Viewed in this way, the ranking of choice profiles induced by \succeq^D is nothing else than the ranking of distributions of shares of the population choosing the various alternatives as per their Herfindahl-Hirschman index (Herfindahl (1950), Hirschman (1945), Hirschman (1964)).⁵ Appraising the overall freedom of choice in a society by focusing on the relative frequency with which the various alternatives are chosen has been advocated some time ago by Suppes (1996). While Suppes (1996) advocates - without axiomatic justification - the use of Shannon entropy (see e.g. Shannon (1948)) as the appropriate rule to aggregate those frequencies, we focus instead on another member of the generalized entropy family (see Renyi (1961)) of rankings of (discrete) probability distributions.

⁵This index is also known as the Simpson index in the context of biodiversity measurement (Simpson (1949)) and as the Gini "impurity" index in regression tree analysis.

We now discuss and axiomatically justify the diversity order \succeq^D of Definition 2. The first postulate that we use for this purpose is the well-known Replication Invariance axiom, saying that replicating a choice profile any finite number of times is a matter of indifference. Formally:

Axiom 4 (Replication Invariance). *For any $m, m' \in \mathcal{M}$, $m \sim m'$ whenever $m'(x) = km(x)$ for all $x \in X$ and some strictly positive integer k .*

We then impose a *neutrality condition* saying that the alternatives' names do not matter for appraising the freedom revealed by the relative frequencies of the choices of those alternatives in alternative choice profiles. Only the relative frequencies themselves do. We formulate this axiom as follows.

Axiom 5 (Neutrality). *For any $m, m' \in \mathcal{M}$, $m \sim m'$ whenever $m(x) = m'(\pi(x))$ for all $x \in X$ and some one-to-one (permutation) function $\pi : X \rightarrow X$.*

A third, and last, postulate concerns comparisons of choice profiles with the same number of choosers whose supports are *comparable in terms of set inclusion*. From the viewpoint of freedom of choice assessment, it could be held that a choice profile whose support is a subset of that of another offers less freedom than that other. Indeed, any chosen alternative of the former profile is a chosen alternative of the latter but there are alternatives chosen in the latter that are never chosen in the former. However, such a "monotonicity with respect to set inclusion" (see e.g. Barberà, Bossert, and Pattanaik (2004) or Gravel (2009)), which seems to be a sensible requirement to impose on a freedom-based ranking of opportunity sets, does not have the same appeal when applied to the support of choice profiles that convey information about the relative frequencies with which alternatives are chosen. To see why, assume that $X = \{a, b, c, d, e\}$ and consider the choice profiles m and m' defined by:

$$\begin{aligned} m(a) &= m(b) = m(c) = 5, \\ m(d) &= m(e) = 0 \end{aligned}$$

and

$$\begin{aligned} m'(a) &= 8, m'(b) = 3, m'(c) = 2, \\ m'(d) &= m'(e) = 1 \end{aligned}$$

These two profiles concern fifteen decision makers. The support of m is $\mathcal{S}(m) = \{a, b, c\}$ while the support of m' is $\mathcal{S}(m') = \{a, b, c, d, e\} = X$. Is the diversity (freedom) of choices revealed by m' really larger than that of m ? This is far from clear. Indeed, while all alternatives are chosen in m' (against only three of them in m), a majority of the choosers (8 of them) choose the same alternative (a) and, as such, do not exhibit much originality. On the other hand, while the individuals choose a lower number of distinct alternatives in m , they do so in a much more balanced way by spreading equally their choices among them. Which of m or m' offers the most diversity of choices then ?

A possible criterion that could provide guidance in answering this question is the sum of the distances - defined earlier - between the chosen alternatives and all the other chosen alternatives in the profile

as measured by (2.2). Observe in effect that:

$$\sum_{x \in X} \sum_{z \in X} m(x)m(z)d(x, z) = 15 \times 10 = 150$$

and

$$\sum_{x \in X} \sum_{z \in X} m'(x)(z)d(x, z) = 8 \times 7 + 3 \times 12 + 2 \times 13 + 2 \times 14 = 146$$

Hence, even if the set of alternatives chosen at least once in the profile m is a proper subset of that in m' , the (more limited) choices in m are more distant, in the aggregate, from other choices than they are in m' . While the two choice profiles of this example who involve the same number of individuals are ranked in a way that violates monotonicity with respect to the inclusion of their support, we can also find ranking of such profiles that do satisfy such monotonicity. An example is provided by the choice profile m'' defined by:

$$\begin{aligned} m''(a) &= 6, m''(b) = 4 = m''(c), \\ m''(d) &= 1 \text{ and,} \\ m''(e) &= 0 \end{aligned}$$

The support of this profile, given by $\mathcal{S}(m'') = \{a, b, c, d\}$, contains $\mathcal{S}(m)$ as a proper subset and, as per the requirement of monotonicity with respect to set inclusion, exhibits choices which are more distant from the others than m . Indeed, the aggregate distances between the choices made in this profile is:

$$\sum_{x \in X} \sum_{z \in X} m''(x)m''(z)d(x, z) = 6 \times 9 + 4 \times 11 + 4 \times 11 + 1 \times 14 = 156$$

and is larger than that of the profile m . The last axiom will therefore require that the ranking of choice profiles involving the same number of individuals and whose support can be compared by set inclusion be as per the sum of their distances as measured by (2.2).

This may seem to be an overly determined way to proceed. Couldn't we identify diversity enhancing transformations in a choice profile that would be more elementary than the complex requirement that they increase the sum of pairwise dissimilarities as defined by (2.2) ? An obvious such elementary transformation would be a reduction by one of the multiplicity of a frequently chosen alternative combined with an increase by one of the multiplicity of another less frequently chosen alternative. Such a transformation, which would correspond to a "change of mind" of one agent who would switch from a commonly chosen alternative to another that is less so would obviously increase the diversity of choices in the profile.⁶ It should be observed that an transformation of this kind can only increase - in the sense of set inclusion - the support of the choice profile in which it is performed. Such an increase would happen if the change of mind of the agent goes toward an alternative that was not yet chosen in the profile. But this elementary diversity enhancing operation can never reduce - in the inclusion sense - the support of the profile. We could then consider *combining* such elementary diversity enhancing transformations and their converse - namely diversity reducing transformations - and propose an axiom that would take the form of a monotonicity requirement with respect to such a combination. An example

⁶Such an elementary transformation is nothing but a (particular kind of) Muirhead-Pigou-Dalton transfer of multiplicity (see e.g. Marshall, Olkin, and Arnold (2011), p. 6-7).

of a combination of diversity-enhancing and diversity-reducing transformation discussed in the context of income inequality measurement is the variance-preserving composite transfer discussed by Foster and Shorrocks (1987). Such a transformation combines a Muirhead-Pigou-Dalton transfer made at the bottom of the income distribution with an anti Muirhead-Pigou-Dalton transfer made at the top of the distribution under the constraint that the combination preserves the variance. But one could think of other, more complex and less palatable combinations. The (difficult) problem becomes then to identify what those combinations can be. The following example shows that the precise combination of diversity-enhancing and diversity-reducing elementary transformations that could lead precisely to an increase in the sum of pairwise dissimilarities as per (2.2) may be difficult to identify.

Example 4. *Consider again a population of 15 individuals who choose from a set of alternatives $X = \{a, b, c, d, e, f, g\}$ and consider the choice profiles m and m' defined as follows:*

$$\begin{aligned} m(a) &= m(b) = m(c) = 5, \\ m(d) &= m(e) = m(f) = m(g) = 0 \end{aligned}$$

and

$$\begin{aligned} m'(a) &= 8, m'(b) = 2, \\ m'(c) &= m'(d) = m'(e) = m'(f) = m'(g) = 1 \end{aligned}$$

The support of m is $\mathcal{S}(m) = \{a, b, c\}$ while the support of m' is $\mathcal{S}(m') = \{a, b, c, d, e, f, g\} = X$, hence $\mathcal{S}(m) \subset \mathcal{S}(m')$. It can be easily verified that the aggregate dissimilarity - as defined by (2.2) - is larger in m' than in m . However the difference is somewhat small. How can one go from m to m' by performing specific combinations of diversity-enhancing and diversity-reducing elementary transformations, possibly in a sequential manner? The combinations must be very specific indeed. A combination that works is made of three diversity enhancing elementary transformations from alternative b to alternatives d , e and f respectively, one diversity-enhancing transformation from c to g and three diversity-reducing transformations from c to a . Observe that all these transformations going in opposite directions must be done here "in one shot". It is possible to break them into somewhat more basic combinations of such transformations that can be performed sequentially. For example, one could start by combining two diversity-reducing transformations from c to a with one diversity-enhancing transformation from b to d . This would lead to the choice profile m^1 defined by:

$$\begin{aligned} m^1(a) &= 7, m^1(b) = 4, m^1(c) = 3, m^1(d) = 1, \\ m^1(e) &= m^1(f) = m^1(g) = 0 \end{aligned}$$

that has the same level of aggregate dissimilarity - as defined by (2.2) - as m . One could then move away from m^1 by combining one diversity-reducing transformation from c to a with two diversity-enhancing transformations from b to, respectively, e and f so as to reach the profile m^2 defined by:

$$\begin{aligned} m^2(a) &= 8, m^2(b) = m^2(c) = 2, \\ m^2(d) &= m^2(e) = m^2(f) = 1 \text{ and,} \\ m^2(g) &= 0 \end{aligned}$$

Again this profile has the same aggregate dissimilarity as m^1 (and therefore m). Starting from m^2 , one could then finally reach the profile m' by making one last diversity-enhancing elementary transformation from c to g . Observe that any attempt to do a diversity-enhancing elementary transformation alone from m , whatever this can be, would result in a choice profile that is strictly more dissimilar - in the sense of (2.2) - than m' .

Hence, it does not seem easy to require the ranking of two choice profiles with the same number of choosers and inclusion-related supports to be increasing with respect to a more elementary notion of dissimilarity than that contained in the following axiom.

Axiom 6 (Restricted Dissimilarity Monotonicity). *For any two choice profiles $m, m' \in \mathcal{M}$ such that $s(m) = s(m')$ and either $\mathcal{S}(m) \subset \mathcal{S}(m')$ or $\mathcal{S}(m') \subset \mathcal{S}(m)$,*

$$\sum_{x \in X} \sum_{z \in X} m(x)m(z)d(x, z) \geq \sum_{x \in X} \sum_{z \in X} m'(x)m'(z)d(x, z) \iff m \succeq m'.$$

As it turns out, these three axioms characterize the ranking \succeq^D of Definition 2, as stated in the following proposition.

Proposition 1. *An ordering \succeq of all choice profiles $m : X \rightarrow \mathbb{N}$ satisfies Replication Invariance, Neutrality and Restricted Dissimilarity Monotonicity if and only if $\succeq = \succeq^D$.*

The axioms of *Replication Invariance*, *Neutrality* and *Restricted Dissimilarity Monotonicity* when applied to an ordering of all choice profiles $m : X \rightarrow \mathbb{N}$ are *independent*, i.e. our characterization is *tight*, as shown in the following examples.

Example 5. *Consider the ordering \succeq^1 defined on all choice profiles $m : X \rightarrow \mathbb{N}$ by:*

$$m \succeq^1 m' \iff \sum_{x \in X} m^2(x) \leq \sum_{x \in X} m'^2(x)$$

This ordering satisfies Neutrality, since $\sum_{x \in X} m^2(x) = \sum_{x \in X} m^2(\pi(x))$ for every one-to-one function

$\pi : X \rightarrow X$. *Moreover, for any m and m' such that $s(m) = s(m')$ one has: $\sum_{x \in X} \sum_{z \in X} m(x)d(x, z) \geq$*

$$\sum_{x \in X} \sum_{z \in X} m'(x)d(x, z) \iff \sum_{x \in X} m(x)[s(m) - m(x)] \geq \sum_{x \in X} m'(x)[s(m') - m'(x)] \iff s^2(m) - \sum_{x \in X} m^2(x) \geq$$

$$s^2(m') - \sum_{x \in X} m'^2(x) \iff \sum_{x \in X} m^2(x) \leq \sum_{x \in X} m'^2(x). \text{ Hence } \succeq^1 \text{ satisfies Restricted Dissimilarity Monotonicity}$$

(not only for Support Inclusion Comparable Profiles but for all profiles with the same number of choosers). However, it obviously violates Replication Invariance (because k -plicating a multiplicity function will obviously affect the sum of the squares of those).

Example 6. *Pick any linear ordering⁷ R^* on X and consider the ordering \succeq^2 defined on all choice profiles $m : X \rightarrow \mathbb{N}$ by:*

$$m \succeq^2 m' \iff m \succeq^{R^*} m'$$

⁷An ordering is linear if it never considers two distinct alternatives as equivalent.

if $\max(\#\mathcal{S}(m), \#\mathcal{S}(m')) > 1$ and by

$$m \succeq^2 m' \iff x R x'$$

with $\{x\} = \mathcal{S}(m)$ and $\{x'\} = \mathcal{S}(m')$ otherwise. This ordering satisfies *Restricted Dissimilarity Monotonicity* (trivially for choice profiles with supports containing only one alternatives). It also satisfies *Replication invariance*, as does \succeq^D when applied to choice profiles whose supports are not both singletons) because the ranking of choice profiles whose support are singletons (as per the exogenous ordering R^* of the elements of those singletons) is unaffected by the multiplicities of the profiles. However this ordering obviously violates *Neutrality* when restricted to choice profiles whose supports are singletons.

Example 7. Consider the ordering \succeq^3 defined on all choice profiles $m : X \rightarrow \mathbb{N}$ by:

$$m \succeq^3 m' \iff \frac{\#\mathcal{S}(m)}{s(m)} \geq \frac{\#\mathcal{S}(m')}{s(m')}$$

This ordering satisfies *Replication Invariance* and *Neutrality* (as can be easily checked) but violates *Restricted Dissimilarity Monotonicity* (because it ranks all support inclusion comparable profiles as per set inclusion).

The ranking \succeq^D of choice profiles, which we interpret as reflecting the freedom of choice exerted *ex post* by the agents, can be compared with the ranking of opportunity sets characterized by Pattanaik and Xu (1998). When interpreted in the current setting, the ranking characterized in Pattanaik and Xu (1998) would amount to compare choice profiles in terms of the cardinality of their support (the higher the cardinality, the freer the profile, as the ordering \succeq^3 of Example 7). The analysis of this paper insists instead on the fact that the contribution of the alternatives chosen by people to the freedom of choice they have exerted depends upon the extent to which these alternatives differ from other chosen alternatives. The following two examples illustrate the differences between the comparison of choice profiles provided by \succeq^D and the ranking of those same profiles on the basis of the cardinality of their support.

Example 8. Suppose X is some finite set of alternatives and consider the choice profiles m and m' defined by:

$$\begin{aligned} m(x_1) &= 3 = m(x_2) \\ m(x_3) &= 4 \\ m(x) &= 0 \quad \forall x \in X \setminus \{x_1, x_2, x_3\} \end{aligned}$$

and:

$$\begin{aligned} m'(x_4) &= 7 \\ m'(x_5) &= m'(x_6) = m(x_7) = 1 \\ m(x) &= 0 \quad \forall x \in X \setminus \{x_4, x_5, x_6, x_7\} \end{aligned}$$

Since $\#\mathcal{S}(m) = 3 < 4 = \#\mathcal{S}(m')$, the ranking characterized by Pattanaik and Xu (1998) would consider that m' reveals strictly more freedom of choice from the part of the agents than m , while the opposite

ranking of the two choice profiles would be provided by \succeq^D . Indeed, while more alternatives are actually chosen by people in m' than in m , a vast majority of people in m' choose the very same alternative x_4 and do not, for this reason, express much singularity. By contrast the ten people in the profile m choose "diversely" by dividing almost evenly their choices between the three alternatives x_1 , x_2 and x_3 .

We believe that \succeq^D could avoid some (possibly) *counterintuitive* comparisons of choice profiles to which Pattanaik and Xu (1998) ranking could lead, as illustrated in the following example.

Example 9. Consider the choice profiles m , m' and m'' involving 10 decision makers defined as follows over the set $X = \{a, b, c\}$

$$\begin{aligned} m(a) &= m(b) = 5, m(c) = 0 \\ m'(a) &= 1, m'(b) = 9, m'(c) = 0 \\ m''(a) &= 10, m''(b) = m''(c) = 0 \end{aligned}$$

Pattanaik and Xu (1998) would evaluate equally m and m' (in which two distinct alternatives are chosen by people) and would consider these two choice profiles to reveal strictly more freedom of choice than m'' (in which only one alternative is chosen). The ranking \succeq^D would break quite strongly the tie between m and m' by putting m way above m' but would subscribe to Pattanaik and Xu (1998)'s appraisal of m' as being above m'' .

As mentioned above, the ranking \succeq^D of choice profiles is only one particular member of the class of generalized entropy rankings analyzed in Renyi (1961). Other members of this class are the rankings of choice profiles as per their Shannon (1948) entropy, discussed by Suppes (1996) in the context of freedom appraisal, and the ranking associated to the Berger and Parker (1970) index of diversity (never discussed in this context to the very best of our knowledge). Recall that the Berger and Parker (1970) ranking would compare choice profiles in terms of their most commonly chosen - or less singular - alternative, the less common the alternative, the more diverse the profile. These rankings would compare choice profiles differently than \succeq^D . For example, in the case where $X = \{a, b, c, d, e, f, g\}$ and the choice profiles are m and m' defined in Example 4, we have seen that \succeq^D considers m' to be more diverse than m while both the Berger and Parker (1970) and the Shannon (1948) entropy rankings would lead to the converse conclusion.

4. CONCLUDING NOTE

The paper has explored the problem of evaluating the choices made by free agents on the basis of their singularity. It has provided a characterization of the notion of singularity of a choice as being measured negatively by its multiplicity - e.g. the number of times it is chosen - or positively by the number of chosen alternatives from which it differs. The singularity of an alternative chosen in a choice profile is interpreted, in our setting, as the expression of the freedom of choice of the agent who has made the choice. Endowed with this notion of singularity as a "reflection of the free exercise of choice", we have argued in favour of comparing different choice profiles in terms of their "quadratic" average singularity, interpreted therefore as an aggregate measure of the freedom of choice - or diversity - expressed in the

profile. This aggregate measure of diversity is nothing but the Hirschman (1964)-Herfindahl (1950)-Simpson (1949) index. One could then view Proposition 1 as a first - to our knowledge - characterization of the ordering induced by the comparison of the values of this index. We remark however that one of the axioms used in the characterization, Restricted Dissimilarity Monotonicity, is expressed as the requirement that the ranking of profiles with the same number of choosers and inclusion-comparable supports be monotonic with respect to an aggregate measure of dissimilarity that could need to be decomposed into more elementary diversity-enhancing transformations of the profiles. While we have indicated, notably in Example 4, some of the difficulties raised by such decompositions, we may not have been exhaustive in our attempts. It is our hope that future research could lead to a finer-grain axiomatic characterization of the Hirschman-Herfindahl-Simpson diversity ranking of choice profiles. It could also be worth investigating the possibility of obtaining an axiomatic characterization of the whole family of generalized entropy criteria.

5. APPENDIX: PROOFS

5.1. Theorem 1. It is easy to verify that the collection of orderings \succeq_s^m on $\mathcal{S}(m)$ (for all choice profiles $m : X \rightarrow \mathbb{N}$) of Definition 1 satisfies the axioms of *Indifference between Isolated Choices*, *Dominance of Isolated Choices* and *Sequential Neutrality to Indifference*.

In order to prove the converse implication, consider now any choice profile $m : X \rightarrow \mathbb{N}$ and any two distinct alternatives x and $x' \in \mathcal{S}(m)$ and suppose by contradiction that $x \succeq^m x'$ and $m(x) > m(x')$. We can clearly rule out the case where $m(x') = 1$ because assuming such a case would imply that alternative x' is an isolated choice in m while alternative x is not. But, if this was the case, the statement $x \succeq^m x'$ would contradict the *Dominance of Isolated Choices* axiom. Suppose therefore that $m(x) > m(x') \geq 2$. Consider then a sequence of $m(x')$ choice profiles $\{m^l\}_{l=1}^{m(x')}$ such that $\sum_{l=1}^{m(x')} m^l(y) = m(y)$ for any $y \in X$, x and $x' \in \bigcap_{l=1}^{m(x')} \mathcal{S}(m^l)$, and such that $m^l(x') = m^l(x) = 1$ for $l = 1, \dots, m(x') - 1$, and $m^l(x') = 1 < m^l(x)$. It is clear that such a sequence can be constructed. By the *Indifference between Isolated Choices* axiom, one has $x \sim^{m^l} x'$ for all choice profiles m^l (for $l = 1, \dots, m(x') - 1$). By the *Dominance of Isolated Choice* axiom, one has $x' \succ^{m(x')} x$ (because x' is an isolated choice in the profile $m^{m(x')}$ while x is not). It follows at once from the *Sequential Neutrality to Indifference* axiom that $x' \succ^m x$, which is a contradiction. In the other direction, assume that $m(x) \leq m(x')$ but, by contradiction again, that $x' \succ^m x$ (thanks to the completeness of \succeq^m). If we are in the situation where $m(x) < m(x')$ and $x' \succ^m x$ (and therefore $x' \succeq^m x$), we can establish the contradiction by the argument just made above (modulo a permutation of the alternatives x and x'). Hence, the only case that needs to be considered is $m(x) = m(x')$ and $x' \succ^m x$. We can rule out the possibility that $m(x) = m(x') = 1$ because making this assumption would imply that both x and x' are isolated choices in the profile m and, given the *Indifference between Isolated Choices* axiom, that $x \sim^m x'$, which is a contradiction. If now $m(x) = m(x') = t \geq 2$, we can construct a sequence of t choice profiles $\{m^l\}_{l=1}^t$ such that $\sum_{l=1}^t m^l(y) = m(y)$ for any $y \in X$, x and $x' \in \bigcap_{l=1}^t \mathcal{S}(m^l)$, and such that $m^l(x') = m^l(x) = 1$ for $l = 1, \dots, t$. Since x and x' will be isolated choices in each choice

profiles $\{m^l\}_{l=1}^t$, we will have by the *Indifference between Isolated Choices* axiom, that $x \sim^{m^l} x'$ for all $l = 1, \dots, t$. But then, the required contradictory statement that $x \sim^m x'$ follows at once from application of the neutrality to indifference axiom.

5.2. Proposition 1. We leave to the reader the easy task of verifying that the ordering \succeq_D of the set of all choice profiles $m : X \rightarrow \mathbb{R}$ satisfies *Replication Invariance*, *Neutrality* and *Restricted Dissimilarity Monotonicity*. In the other direction, suppose that \succeq is an ordering that satisfies *Replication Invariance*, *Neutrality* and *Restricted Dissimilarity Monotonicity*. By contradiction, consider first any two choice profiles m and m' such that $m \succeq m'$ but, contrary to what is claimed, such that:

$$(5.1) \quad \frac{\sum_{x \in X} m^2(x)}{[s(m)]^2} > \frac{\sum_{x \in X} m'^2(x)}{[s'(m)]^2}$$

By *Replication Invariance* and transitivity, we can assume without loss of generality that $s(m) = s(m')$ and, therefore, that inequality (5.1) writes:

$$(5.2) \quad \sum_{x \in X} m^2(x) > \sum_{x \in X} m'^2(x)$$

or, equivalently:

$$(5.3) \quad \begin{aligned} \sum_{x \in X} -m^2(x) &< \sum_{x \in X} -m'^2(x) \\ &\iff \\ \sum_{x \in X} m(x)(s(m) - m^2(x)) &< \sum_{x \in X} m'(x)(s(m') - m'^2(x)) \\ &\iff \\ \sum_{x \in X} \sum_{z \in X} m(x)d(x, z) &< \sum_{x \in X} \sum_{z \in X} m'(x)d(x, z) \end{aligned}$$

using expression (2.2). If m and m' have supports that are comparable by set inclusion, this inequality is incompatible with *Restricted Dissimilarity Monotonicity*. If the supports of m and m' are not comparable by set inclusion, then one can find a one-to-one function $\pi : X \rightarrow X$ such that the support of the profile m^π defined, for every $x \in X$, by $m^\pi(x) = m(\pi(x))$ will be comparable with m' by set inclusion. By *Neutrality* and transitivity, one would then have $m^\pi \sim m \succeq m'$ which will again be incompatible with *Restricted Dissimilarity Monotonicity*. Hence $m \succeq m' \implies m \succeq^D m'$. For the other implication of the equivalence, assume $m \succeq^D m' \iff \frac{\sum_{x \in X} m^2(x)}{[s(m)]^2} \leq \frac{\sum_{x \in X} m'^2(x)}{[s'(m)]^2}$ but, by contradiction, assume that $m \succeq m'$ does not hold. The contradiction is then obtained by applying almost exactly the same reasoning as above.

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