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$C^{0,1}$ -Itô chain rules and generalized solutions of parabolic PDEs.

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Abstract

In this paper we first establish an Itô formula for a finite quadratic variation process X expanding $f(t, X_t)$, when f is of class C^2 in space and is absolutely continuous in time.

Second, via a Fukushima-Dirichlet decomposition we obtain an explicit chain rule for $f(t, X_t)$, when X is a continuous semimartingale and f is a “quasi-strong solution” (in the sense of approximation of classical solutions) of a parabolic PDE.

Those results are applied in a companion paper to establish a verification theorem in stochastic differential games.

Key words and phrases: Itô chain rules; Fukushima decomposition; weak Dirichlet processes; quasi-strong solutions of PDEs.

Mathematics Subject Classification 2020: 60H05; 60H10; 60H30; 91A15.

1 Introduction

In this paper we aim to establish two Itô chain rules in view of applications to a stochastic differential game and stochastic control, see e.g. the companion paper [10], expanding $f(t, X_t)$, where $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ and X is a stochastic process. A preliminary version of that paper is the second part of

The first formula, stated in Proposition 2.7, is related to $f \in C_{ac}^{0,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ (see Definition 2.1), when X is a finite quadratic variation process. The second is Corollary 4.4, where $f \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^d)$ is a generalized (quasi-strong) solution of some parabolic PDE and X is an Itô process. This makes use of a weak Dirichlet decomposition that one could call Fukushima-Dirichlet decomposition, see Theorem 4.1.

If X is a semimartingale and $f \in C^{1,2}$, then classical Itô formula tells us that $f(\cdot, X)$ is still a semimartingale. If f has lower than $C^{1,2}$ -regularity, then $f(\cdot, X)$ can still be a semimartingale, via Itô-Tanaka type formulae involving local time for instance. Various generalizations appear in the setting of optimal stopping problems, where one gets expansion of $f(\cdot, X)$ that for f having $C^{1,2}$ -regularity within the continuation (\mathcal{C}) and stopping (\mathcal{D}) regions but the spatial derivatives may not be continuous across the boundary that separates these two sets, see e.g. [26] and [8]. If f has much lower regularity, for instance if $f \in C^1(\mathbb{R}^d)$ then $f(X)$ is not necessarily a semimartingale. If $d = 1$, for instance, if X is a standard Brownian motion then $f(X)$ is a semimartingale if and only if f is difference of convex functions, see e.g. [11]. The first Itô formulae expanding $f(X)$ when $f \in C^1(\mathbb{R}^d)$ and X is a (reversible) semimartingale continuous was performed by [29], see also [18] in the case of Brownian motion under much weaker conditions. In this case $f(X)$ is generally a Dirichlet process i.e. the sum of a local martingale and a zero quadratic variation process, see [17]. In the jump case, at our knowledge the first paper in this direction was [15]. Those papers expressed the Itô formula usual term involving the second order derivatives in term of covariation. For instance if $f \in C^1(\mathbb{R}^d)$ and S is a reversible semimartingale with usual decomposition $S = M + V$, then

$$f(X) = f(X_0) + \int_0^\cdot \partial_x f(X) dS + \frac{1}{2}[\partial_x f(X), X].$$

An alternative approach was the so called Bouleau-Yor formula, which, in dimension 1 expressed the remainder via local time, see [7]. Since then an incredible amount of formulae have been produced, see e.g. [31] for a (even not complete) list of references.

Beyond semimartingales, when $f \in C^{1,2}$, and X is a finite quadratic variation process the first contribution was done by [16] which was followed by a related stochastic calculus in [30]. In the rough case, there were also many chain rules, see e.g. [23] or [19], but this goes beyond the scope of this paper.

The present paper makes use of weak Dirichlet processes as a tool. That notion was introduced in [14], see also Definition 2.5. If a semimartingale is the sum of a local martingale M plus a finite variation process A , a weak Dirichlet process X (generalizing the notion of Dirichlet process of [17]), is the sum of a local martingale and a process which is adapted and martingale orthogonal, in the sense that the covariation $[A, N] = 0$ for every continuous local martingale N . The latter decomposition was also called Fukushima-Dirichlet decomposition in [22] and subsequent papers. It turns out that a $C^{0,1}$ -function of a weak Dirichlet process (with finite quadratic variation) is again a weak Dirichlet process, see Proposition 3.10 of [22]; applications to a verification theorem for stochastic control were performed in [21] in relation with $C^{0,1}$ -(so called) strong solution of the Hamilton-Jacobi-Bellman (HJB) PDE. When X is a càdlàg process that result was generalized

in Theorem 3.37 in [2]. Fukushima type decomposition was introduced in the path-dependent case by [12] in the framework of Banach space valued stochastic calculus. More recently, [5] has extended Proposition 3.2 in [22] to a path-dependent setting, by making use of the Dupire derivative. A subsequent recent paper, generalizing a $C^{0,1}$ -type chain rule (which is indeed a $C^{0,1}$ -Fukushima decomposition in our language) was performed by [6], where the 1 corresponds to a functional derivative in the space and also in the measure in the sense of Definition 5.43 of [9]. That derivative is strongly connected with the Lions-derivative. They have a chain rule expanding $V(t, X_t, \mathcal{L}_{X_t})$ where X is a semimartingale and \mathcal{L}_{X_t} . The resulting process in all cases is a weak Dirichlet process. In [6], the authors obtain a significant application to a verification theorem for a McKean–Vlasov optimal control problem when the value function V is of class $C^{0,1}$. That result of course can be applied to the particular case of classical control problems, when the coefficients do not depend on the law. This is also new, since, in the literature, a comparable verification theorem was the one of [21] which (differently) started with the existence of a $C^{0,1}$ -solution of Hamilton-Jacobi PDE.

As mentioned earlier, the first chain rule Itô formula we prove (Proposition 2.7) holds for functions $f \in C_{ac}^{0,2}$, (see Definition 2.1) and finite quadratic variation processes. In particular f is absolutely continuous in time. The second Itô formula (Corollary 4.4) is based on the Fukushima-Dirichlet decomposition formulated in Theorem 4.1, holds for $f \in C^{0,1}$, being a so called *quasi-strong solution* (see Definition 3.7) of some parabolic PDE and X being a specific weak Dirichlet process with diffusive local martingale component and a finite quadratic variation martingale orthogonal component.

Contrarily to the case of C^1 -transformations of time-reversible semimartingales, we drive the attention on the fact that the $C^{0,1}$ -decompositions mentioned earlier do not provide an explicit decomposition of the martingale orthogonal term. One of the aspects of this paper, see Theorem 4.1 and Corollary 4.4, consists in formulating an explicit expression of the martingale orthogonal process.

A quasi-strong solution to a parabolic PDE problem is defined as the limit of *quasi-strict solutions* (see Definition 3.1) to a sequence of approximating PDEs, which are $C_{ac}^{0,2}$ -functions.

Potential applications of the present paper appear each time we need to apply a chain rule of the type $u(t, S_t)$ where only a non-regular solution u of an underlying PDE is available and S is an underlying stochastic process. One is related to the resolution of the identification problem in a forward-backward stochastic differential equations, where the solution (Y, Z) in the Brownian case (resp. (Y, Z, K) in the jump case), see e.g. [20] (resp. [1]), when one is interested in expressing Z (resp. Z, K) as a “gradient” of a solution of some semilinear Kolmogorov PDE. In this case S is a solution of an underlying forward equation.

In the companion paper [10], we apply Corollary 4.4 to establish a verification theorem in the framework of stochastic control and stochastic differential games. In particular one applies the chain rule to quasi-strong solutions of Hamilton-Jacobi-Bellman (HJB) and Bellman-Isaacs (BI) equations which in general they are not expected to be neither $C^{1,2}$ nor $C_{ac}^{0,2}$.

Indeed, in stochastic control (resp. game theory problems), the driver of the parabolic PDE, which is the HJB (resp. BI equation) is the Hamiltonian, may be highly non-regular. In our formulation, it corresponds to the function h in (3.2). Therefore, a priori, we cannot expect much regularity also from the solution to the PDE. If we now replace h (resp. the terminal function g) by a sequence of smoother (in space) functions, converging to the h (resp. g), we can hope that the existence of solutions to this sequence of PDEs is somehow easier to prove. This is the main idea behind these two notions of solutions. The notion of “quasi-strong” (resp. “quasi-strict”) solution of Definition 3.7 (resp. 3.1) extends the one of “strong” (resp. “strict”) solution of [22],

Definition 4.2 (resp. Definition 4.1). In particular, the quasi-strict solution is not of class C^1 in time. In [22], the coefficients of the PDE differential operators are supposed to be continuous. In this work, we also drop any time regularity assumption on the coefficients of the parabolic operator (3.1).

In [22], the convergence of the approximating functions to the Hamiltonian (characterizing the notion of “strong solution”) is supposed to be uniform in time and space, therefore implying that the Hamiltonian has to be continuous in time. In our formulation of quasi-strong solution, we replace uniform convergence in time by an L^1 -convergence, thus allowing for Hamiltonians which are not continuous in time as well as the coefficients of the PDE. An example of existence of quasi-strict solution is shown in Lemma 3.4. We also show that the notion of quasi-strong solution is related to that of a mild solution.

Our results are organized as follows. In Section 2 we establish Proposition 2.7, which states the Itô formula for $C_{ac}^{0,2}$ functions of finite quadratic variation processes. Section 3 is devoted to various types of solutions of PDEs, i.e. quasi-strict, which generalizes the classical, mild and quasi-strong. Finally Section 4 is devoted to the explicit chain rule type of subclasses of $C^{0,1}$ -functions, see Theorem 4.1 and Corollary 4.4.

2 Complements on stochastic calculus for finite quadratic variation processes

First, we recall some basic definitions and fix some notations, then we recall the definition of the Fukushima-Dirichlet decomposition that plays a key role in this paper.

2.1 Analytical preliminaries

In this section, $0 \leq t < T < \infty$ will be fixed. The definition and conventions of this section will be in force for the whole paper. By convention a continuous function $\phi(s)$, $s \in [t, T]$, is extended to the whole line setting

$$\phi(s) = \begin{cases} \phi(t) & s \leq t, \\ \phi(T) & s \geq T. \end{cases}$$

If E is a (subset of) finite-dimensional space, $C^0(E)$ denotes the space of all continuous functions $f : E \rightarrow \mathbb{R}$. It is a Fréchet space equipped with the topology of the uniform convergence of f on each compact. If $f : E \rightarrow \mathbb{R}$ is a uniformly continuous function, we denote by $\gamma(f, \cdot)$ the modulus of continuity of f .

If k be a non-negative integer, $C^k(\mathbb{R}^d)$ denotes the space of the functions such that all the derivatives up to order k exist and are continuous. It is again a Fréchet space equipped with the topology of the uniform convergence of f and all its derivatives on each compact. Let I be a compact real interval. $C^{1,2}(I \times \mathbb{R}^d)$ (respectively $C^{0,1}(I \times \mathbb{R}^d)$), is the space of continuous functions $f : I \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that $\partial_s f, \partial_x f, \partial_{xx}^2 f$ (respectively $\partial_x f$) are well-defined and continuous. This space is also a Fréchet space. In general \mathbb{R}^d -elements will be considered as column vector, with the exception of $\partial_x f$ which will by default a row vector.

Definition 2.1. $C_{ac}^{0,2}(I \times \mathbb{R}^d)$ will be the linear space of continuous functions $f : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that the following holds.

1. $f(s, \cdot) \in C^2(\mathbb{R}^d)$ for all $s \in I$.

2. For any $x \in \mathbb{R}^d$ the function $s \mapsto f(s, x)$ is absolutely continuous and for almost all $s \in I$, $x \mapsto \partial_s f(s, x)$ is continuous.
3. For any compact $K \subset \mathbb{R}^d$, $\sup_{x \in K} |\partial_s f(\cdot, x)| \in L^1(I)$.
4. Let g be any second order space derivative of f .
 - (a) g is continuous with respect to the space variable x varying on each compact K , uniformly with respect to $s \in I$;
 - (b) for every $x \in \mathbb{R}^d$, $s \mapsto g(s, x)$ is a.e. continuous.

Suppose that f fulfills all previous items except 4.(b) which is replaced by

- 4.(b) Bis: for every $x \in \mathbb{R}^d$, $s \mapsto g(s, x)$ has at most countable discontinuities.

In this case f will be said to belong to $C_{ac, count}^{0,2}(I \times \mathbb{R}^d)$.

In the sequel $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq t}, \mathbb{P})$ will be a given stochastic basis satisfying the usual conditions. $(W_s)_{s \geq t}$ will denote a classical $(\mathcal{F}_s)_{s \geq t}$ -Brownian motion with values in \mathbb{R}^d . A sequence of processes $(X_s^n)_{s \geq t}$ will be said to converge u.c.p. if the convergence holds in probability uniformly on compact intervals.

The space $C_{\mathcal{F}}(I \times \Omega; \mathbb{R})$ of all continuous \mathbb{R}^d -valued $(\mathcal{F}_s)_{s \in [t, T]}$ -progressively measurable processes is a Fréchet space, if equipped with a metric related the u.c.p. convergence.

2.2 Weak Dirichlet processes and Fukushima Dirichlet decomposition

Definition 2.2. Let $(X_s)_{s \in [t, T]} = (X_s^1, \dots, X_s^d)_{s \in [t, T]}$ be a vector of continuous processes and $(Y_s)_{s \in [t, T]} = (Y_s^1, \dots, Y_s^d)_{s \in [t, T]}$ be vector process with integrable paths. We define

$$I^-(\varepsilon, Y, dX)(s) = \int_t^s Y_r \cdot \frac{X_{r+\varepsilon} - X_r}{\varepsilon} dr,$$

where \cdot denotes the inner product in \mathbb{R}^d . If $d = 1$ then, of course, we omit the \cdot . We also say that the **(vector) forward integral** of X with respect to Y exists if

1. $\lim_{\varepsilon \rightarrow 0} I^-(\varepsilon, Y, dX)(s)$ exists in probability for any $s \in [t, T]$,
2. the previous limit admits a continuous version.

That integral limit process, if exist, is denoted by $\int_t^s Y_r d^- X_r$, $s \in [t, T]$.

Definition 2.3. Let $(X_s)_{s \in [t, T]} = (X_s^1, \dots, X_s^d)_{s \in [t, T]}$ and $(Y_s)_{s \in [t, T]} = (Y_s^1, \dots, Y_s^d)_{s \in [t, T]}$ be two vector of continuous processes. We define

$$[X, Y]^\varepsilon(s) = \frac{1}{\varepsilon} \int_t^s (X_{r+\varepsilon} - X_r) \cdot (Y_{r+\varepsilon} - Y_r) dr, s \in [t, T] \quad (2.1)$$

and we say that the **(matrix) covariation or bracket** of X and Y exists if

1. $\lim_{\varepsilon \rightarrow 0} [X, Y]^\varepsilon(s)$ exists in probability for any $s \in [t, T]$,
2. the previous limit admits a continuous version.

That matrix-valued limit process, if it exists, is denoted by $[X, Y]_s$.

If X is a real process, $[X, X]$ will be also called **quadratic variation** of X . The forward integral, the covariations and the finite quadratic variation processes were introduced in [27] and the first steps were performed in [28], see also the recent monograph [31].

- Remark 2.4.** 1. If X is a continuous $(\mathcal{F}_s)_{s \geq t}$ -semimartingale and Y is an $(\mathcal{F}_s)_{s \geq t}$ -progressively measurable càglàd process (resp. an $(\mathcal{F}_s)_{s \geq t}$ -semimartingale) then $\int_s Y d^-X$ (resp. $[Y, X]$) coincides with $\int_s Y dX$ (resp. the usual covariation). In particular if X is bounded variation process then $\int_s Y d^-X$ is the usual Lebesgue integral $\int_s Y dX$, see Proposition 2.1 of [28].
2. If X is a continuous \mathbb{R}^d -valued $(\mathcal{F}_s)_{s \geq t}$ -semimartingale and Y is an \mathbb{R}^d -valued $(\mathcal{F}_s)_{s \geq t}$ -progressively measurable càglàd process then $\int_s Y dX$ will denote the vector Itô integral $\int_s Y \cdot dX$.

We define below a natural extension of the notion of Dirichlet processes. Those processes were defined by H. Föllmer, see Chapter 14 in [17], see also [31]. We remind that an $(\mathcal{F}_s)_{s \geq t}$ -progressively measurable process X is called Dirichlet if it is the sum of $(\mathcal{F}_s)_{s \geq t}$ -local martingale M and a zero quadratic variation process A , i.e. such that $[A, A] = 0$.

- Definition 2.5.** 1. An $(\mathcal{F}_s)_{s \in [t, T]}$ -continuous progressively measurable process real-valued process X is called $(\mathcal{F}_s)_{s \in [t, T]}$ -**weak Dirichlet process** if

$$X = M + A, \tag{2.2}$$

where

- (a) $(M_s)_{s \in [t, T]}$ is a continuous $(\mathcal{F}_s)_{s \in [t, T]}$ -local martingale,
 - (b) $A_0 = 0$ and A is an $(\mathcal{F}_s)_{s \in [t, T]}$ -**martingale orthogonal process**, that is $[N, A] = 0$ for every $(\mathcal{F}_s)_{s \in [t, T]}$ -continuous local martingale N .
2. A continuous \mathbb{R}^d -valued process $X = (X^1, \dots, X^d)$ is said to be a $(\mathcal{F}_s)_{s \in [t, T]}$ -weak Dirichlet (vector) (resp. $(\mathcal{F}_s)_{s \in [t, T]}$ -**martingale orthogonal process**), if each component is $(\mathcal{F}_s)_{s \in [t, T]}$ -weak Dirichlet (resp. martingale orthogonal process).

(2.2) was also called **Fukushima-Dirichlet** decomposition, see [22].

- Remark 2.6.** 1. The decomposition (2.2) is unique, see Proposition 15.1 in [31].
2. In the sequel, when self-explanatory, we will omit the filtration $(\mathcal{F}_s)_{s \in [t, T]}$.
3. A zero quadratic variation process is a martingale orthogonal process, by Kunita-Watanabe theorem, see Proposition 4.7 item 5. in [31].
4. Usual properties of weak Dirichlet processes are given in [13], [14], [22], and in Chapter 15 of [31].

2.3 The Itô formula for finite quadratic variation processes

We state now our first Itô type formula which generalizes the classical Föllmer-Itô type formula, see [17] and [28] in the framework of calculus via regularization.

Proposition 2.7. Let $f \in C_{ac}^{0,2}([t, T] \times \mathbb{R}^d)$ and $(X_s)_{s \in [t, T]}$ be a continuous \mathbb{R}^d -valued process that $[X, X]$ exists.

Suppose that one of the two conditions below is fulfilled.

1. $f \in C_{ac, count}^{0,2}([t, T] \times \mathbb{R}^d)$.
2. The mutual covariations $[X^i, X^j]$ are absolutely continuous, where $[X, X] = ([X^i, X^j])$.

Then the following Itô formula holds.

$$f(s, X_s) = f(t, X_t) + \int_t^s \partial_r f(r, X_r) dr + \int_t^s \partial_x f(r, X_r) d^- X_r + \frac{1}{2} \sum_{i,j=1}^d \int_t^s \partial_{ij}^2 f(r, X_r) d[X^i, X^j]_r. \quad (2.3)$$

Remark 2.8. For shortness we denote the last sum term in (2.3) by $\int_t^s \partial_{xx}^2 f(u, X_u) d[X, X]_u$.

Before the proof we want to say a few words about the preceding assumptions.

Remark 2.9. One consequence of $f \in C_{ac, count}^{0,2}$ (resp. $f \in C_{ac}^{0,2}$) is the following: for every second derivative (in space) g of f there is a countable (resp. Lebesgue null) subset D of $[t, T]$ such that the following holds.

1. $t \mapsto g(t, x)$ is continuous for every $t \in D^c$ for every $x \in \mathbb{R}^d$.
2. For every continuous function $k : [t, T] \rightarrow \mathbb{R}^d$, we have $t \mapsto g(t, k(t))$ is continuous for every $t \in D^c$.

We explain below the reasons. Of course we can substitute \mathbb{R}^d in previous sentence with a fixed compact K of \mathbb{R}^d .

Concerning item 1., let now $\varepsilon > 0$. Since K is compact, it is included in a finite union of balls $B(x_i, \varepsilon)$, where $\{x_i : i = 1, \dots, N\}$ are elements of K . By item 4(b) Bis (resp. item 4(b)) there is a countable (resp. zero Lebesgue measure) set $D \subset [t, T]$ such that $s \mapsto g(s, x_i)$, $i = \{1, \dots, N\}$ is continuous outside D . Let $s_0 \notin D$ and $x \in K$. We show that $g(\cdot, x)$ is continuous in s_0 .

There is $i \in \{1, \dots, N\}$ such that $|x - x_i| < \varepsilon$ so that, for $s \in [t, T]$, we have

$$\begin{aligned} |g(s, x) - g(s_0, x)| &\leq |g(s, x) - g(s, x_i)| + |g(s, x_i) - g(s_0, x_i)| + |g(s_0, x_i) - g(s_0, x)| \\ &\leq \gamma(g(s, \cdot)|_K; \varepsilon) + |g(s, x_i) - g(s_0, x_i)| + \gamma(g|_K(s_0, \cdot); \varepsilon), \end{aligned}$$

where γ is the modulus of continuity. Taking on both sides the $\limsup_{s \rightarrow s_0}$, we get

$$\limsup_{s \rightarrow s_0} |g(s, x) - g(s_0, x)| \leq 2 \sup_{s \in [t, T]} \gamma(g(s, \cdot); \varepsilon).$$

We now take the limit when $\varepsilon \rightarrow 0$ getting

$$\limsup_{s \rightarrow s_0} |g(s, x) - g(s_0, x)| = 0.$$

Concerning item 2., let $s_0 \in D^c$ and (s_n) be a sequence converging to s_0 . We have

$$\begin{aligned} |g(s_n, k(s_n)) - g(s_0, k(s_0))| &\leq |g(s_n, k(s_n)) - g(s_n, k(s_0))| + |g(s_n, k(s_0)) - g(s_0, k(s_0))| \\ &\leq \sup_{s \in [t, T]} \gamma(g(s, \cdot)|_K; |k(s_n) - k(s_0)|) + |g(s_n, k(s_0)) - g(s_0, k(s_0))|. \end{aligned}$$

The first term goes to zero because of item 4. (a) of Definition 2.1 and the second term converges to zero by item 4 (b) Bis (resp. 4. (b)) of Definition 2.1.

Proof of Proposition 2.7. For simplicity of notations we write the proof in the case $d = 1$.

For $r \in [t, T]$ we have

$$\begin{aligned} f(r + \varepsilon, X_{r+\varepsilon}) &= f(r, X_r) + f(r + \varepsilon, X_{r+\varepsilon}) - f(r, X_{r+\varepsilon}) \\ &\quad + \partial_x f(r, X_r)(X_{r+\varepsilon} - X_r) + \partial_{xx}^2 f(r, X_r) \frac{(X_{r+\varepsilon} - X_r)^2}{2} \\ &\quad + \frac{1}{2} \int_0^1 da [\partial_{xx}^2 f(X_r + a(X_{r+\varepsilon} - X_r)) - \partial_{xx}^2 f(X_r)](X_{r+\varepsilon} - X_r)^2. \end{aligned}$$

Integrating from t to s and dividing by ε , we get

$$I_0(s, \varepsilon) := \frac{1}{\varepsilon} \int_t^s f(r + \varepsilon, X_{r+\varepsilon}) - f(r, X_r) dr = I_1(s, \varepsilon) + I_2(s, \varepsilon) + I_3(s, \varepsilon) + I_4(s, \varepsilon),$$

where

$$\begin{aligned} I_1(s, \varepsilon) &= \frac{1}{\varepsilon} \int_t^s f(r + \varepsilon, X_{r+\varepsilon}) - f(r, X_{r+\varepsilon}) dr, \\ I_2(s, \varepsilon) &= \frac{1}{\varepsilon} \int_t^s \partial_x f(r, X_r)(X_{r+\varepsilon} - X_r) dr, \\ I_3(s, \varepsilon) &= \frac{1}{\varepsilon} \int_t^s \partial_{xx}^2 f(r, X_r) \frac{(X_{r+\varepsilon} - X_r)^2}{2} dr, \\ I_4(s, \varepsilon) &= \frac{1}{2\varepsilon} \int_t^s \int_0^1 da [\partial_{xx}^2 f(r, X_r + a(X_{r+\varepsilon} - X_r)) - \partial_{xx}^2 f(r, X_r)] (X_{r+\varepsilon} - X_r)^2 dr. \end{aligned}$$

Of course, as $\varepsilon \rightarrow 0$, since f is jointly continuous,

$$I_0(s, \varepsilon) = \frac{1}{\varepsilon} \int_s^{s+\varepsilon} f(r, X_r) dr - \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(r, X_r) dr \rightarrow f(s, X_s) - f(t, X_t), \quad s \in [t, T], \quad \text{u.c.p.}$$

Let $s \in [t, T]$. We have

$$I_1(s, \varepsilon) = \frac{1}{\varepsilon} \int_t^s dr \int_r^{r+\varepsilon} da \partial_a f(a, X_{r+\varepsilon}) = I_{1,1}(s, \varepsilon) + I_{1,2}(s, \varepsilon),$$

where

$$\begin{aligned} I_{1,1}(s, \varepsilon) &= \frac{1}{\varepsilon} \int_t^{s+\varepsilon} da \int_{(a-\varepsilon) \vee t}^{a \wedge s} dr \partial_a f(a, X_a); \\ I_{1,2}(s, \varepsilon) &= \frac{1}{\varepsilon} \int_t^{s+\varepsilon} da \int_{(a-\varepsilon) \vee t}^{a \wedge s} dr [\partial_a f(a, X_{s+\varepsilon}) - \partial_a f(a, X_a)]. \end{aligned}$$

It is straightforward that

$$I_{1,1}(s, \varepsilon) = \int_t^{s+\varepsilon} da \partial_a f(a, X_a) \frac{(a \wedge s) - [(a - \varepsilon) \vee t]}{\varepsilon},$$

converges to

$$I_{1,1}(s, \varepsilon) \rightarrow \int_t^s da \partial_a f(a, X_a) \text{ a.s.},$$

because of the Lebesgue dominated convergence theorem, as $\varepsilon \rightarrow 0$.

Now we investigate the convergence of the other term

$$I_{1,2}(s, \varepsilon) = \int_t^{s+\varepsilon} J_{1,2}(a, \varepsilon) da,$$

with

$$J_{1,2}(a, \varepsilon) = \frac{1}{\varepsilon} \int_{(a-\varepsilon) \vee t}^{a \wedge s} dr [\partial_a f(a, X_{r+\varepsilon}) - \partial_a f(a, X_a)].$$

We fix $\omega \in \Omega$. We prove the convergence for fixed $a \in [t, s]$ We have

$$J_{1,2}(a, \varepsilon) = \frac{1}{\varepsilon} \int_{(a-\varepsilon) \vee t}^{a \wedge s} dr [\partial_a f(a, \xi(a, \varepsilon)) - \partial_a f(a, X_a)],$$

where $\xi(a, \varepsilon) \in [\min_{r \in [a, a+\varepsilon]} X_r, \max_{r \in [a, a+\varepsilon]} X_r]$ and

$$\text{Leb}([\min_{r \in [a, a+\varepsilon]} X_r, \max_{r \in [a, a+\varepsilon]} X_r]) \leq \gamma(X; \varepsilon).$$

In particular $|X_a - X_r| < \gamma(X; \varepsilon)$ for all $r \in [a, a + \varepsilon]$. Consequently, for almost all $a \in [t, s]$, it holds $|J_{1,2}(a, \varepsilon)| \leq \gamma(\partial_a f(a, \cdot); \gamma(X; \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$ since $\partial_a f(a, \cdot)$ is uniformly continuous on each compact.

Moreover

$$|J_{1,2}(a, \varepsilon)| \leq 2 \sup_{x \in \mathbb{K}_t} |\partial_a f(a, x)|, \quad (2.4)$$

where $\mathbb{K}_t := [\min_{r \in [t, T]} X_r, \max_{r \in [t, T]} X_r]$. Since the right-hand side of (2.4) is integrable by item 3. of Definition 2.1 on $[t, T]$, then

$$\lim_{\varepsilon \rightarrow 0} I_{1,2}(s, \varepsilon) = 0,$$

by Lebesgue dominated convergence theorem. This implies that

$$\lim_{\varepsilon \rightarrow 0} I_1(s, \varepsilon) = \int_t^s da \partial_a f(a, X_a),$$

a.s. and therefore in probability for any $s \in [t, T]$.

We want to prove now that

$$I_3(s, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \int_t^s \partial_{xx}^2 f(r, X_r) d[X, X]_r, \forall s \in [t, T], \quad (2.5)$$

in probability. We will actually prove that the convergence holds u.c.p. By Lemma 3.1 in [28]

$$\frac{1}{\varepsilon} \int_t^s (X_{r+\varepsilon} - X_r)^2 dr \rightarrow [X, X]_s, \quad s \in [t, T], \quad \text{u.c.p.}$$

Let us now (ε_n) be a sequence converging to zero. There exists a subsequence still denoted by (ε_n) such that, setting

$$\mu_{\varepsilon_n}(s) = [X, X]^{\varepsilon_n}(s),$$

where the right-hand side was defined in (2.1), then

$$\lim_{\varepsilon \rightarrow 0} \left[\sup_{s \in [t, T]} |\mu_{\varepsilon_n}(s) - [X, X]_s| \right] = 0, \quad \text{a.s.}$$

Let M be a P -null set such that for $\omega \notin M$ the sequence of real functions $\mu_{\varepsilon_n}(\omega, \cdot) \rightarrow [X, X] \cdot(\omega)$ uniformly on $[t, T]$, which implies that $d\mu_{\varepsilon_n}(\omega, \cdot) \Rightarrow d[X, X] \cdot(\omega)$. Suppose the validity of item 1. (resp. item 2.) Given a continuous function $k : [t, T] \mapsto \mathbb{R}^d$, using Remark 2.9, the set of discontinuities of $r \mapsto \partial_{xx}^2 f(r, k(r))$ is countable (resp. has zero Lebesgue measure). This implies that (for the fixed $\omega \notin M$), for every $s \in [t, T]$,

$$\int_t^s \partial_{xx}^2 f(r, X_r(\omega)) d\mu_{\varepsilon_n}(\omega, r) \rightarrow \int_t^s \partial_{xx}^2 f(r, X_r(\omega)) d[X, X]_r(\omega),$$

making use of Portemanteau theorem. The convergence can be shown to be uniform in s by decomposing the second derivative(s) in difference of positive and negative part, via Dini's theorem. This implies in particular

$$I_3(\cdot, \varepsilon) \rightarrow \int_t^\cdot \partial_{xx}^2 f(r, X_r) d[X, X](r) \quad \text{uniformly a.s.,}$$

consequently u.c.p. and finally (2.5). Now, let us investigate the convergence of the fourth integral I_4 . We have

$$I_4(s, \varepsilon) \leq \frac{1}{2} \int_t^s \eta(\varepsilon, r) d[X, X]^\varepsilon(r),$$

where $[X, X]^\varepsilon$ was introduced in (2.1), and

$$\begin{aligned} \eta(\varepsilon, r) &= \int_0^1 \frac{1}{2} \int_t^s |\partial_{xx}^2 f(r, X_r + a(X_{r+\varepsilon} - X_r)) - \partial_{xx}^2 f(r, X_r)| da \\ &\leq T \sup_{r \in [t, T]} \gamma(\partial_{xx}^2 f K(r, \cdot); \gamma(X, \varepsilon)). \end{aligned}$$

Since $[X, X]^\varepsilon$ converges u.c.p. to $[X, X]$ we obtain that $I_4(\cdot, \varepsilon) \rightarrow 0$ u.c.p.

At this point the limit in probability of $I_2(s, \varepsilon)$ is forced to exist and to be continuous because the other terms are. This will be of course $\int_t^s f(r, X_r) d^- X_r$ and the proof is complete. \square

In the Proposition 2.10 below, item 1. is the so called Fukushima-Dirichlet decomposition and items 2. connects it to the Itô chain rule stated in Proposition 2.7.

Proposition 2.10. Suppose $(X_s)_{s \in [t, T]}$ is an $(\mathcal{F}_s)_{s \in [t, T]}$ -weak Dirichlet continuous process such that $[X, X]$ exists (resp. exists and it is absolutely continuous). Let $X = M + A$ be its decomposition, componentwise.

There is a continuous linear map $\mathcal{B}^X : C^{0,1}([t, T] \times \mathbb{R}^d) \rightarrow C_{\mathcal{F}}([t, T] \times \Omega; \mathbb{R}^d)$ such that the following holds.

1. For every $f \in C^{0,1}([t, T] \times \mathbb{R}^d)$ we have

$$f(s, X_s) = f(t, X_t) + \int_t^s \partial_x f(r, X_r) dM_r + \mathcal{B}^X(f)_s, \quad s \geq t, \quad (2.6)$$

and $\mathcal{B}^X(f)_s$ is a martingale orthogonal process. In particular, for every $f \in C^{0,1}([t, T] \times \mathbb{R}^d)$, $(f(s, X_s), s \in [t, T])$ is a $(\mathcal{F}_s)_{s \in [t, T]}$ -weak Dirichlet process with martingale part $M_s^f = \int_t^s \partial_x f(r, X_r) dM_r, s \in [t, T]$.

2. Suppose $f \in C^{0,1}$. If $f \in C_{ac,count}^{0,2}([t, T] \times \mathbb{R}^d)$ (resp. $f \in C_{ac}^{0,2}([t, T] \times \mathbb{R}^d)$), we have

$$\mathcal{B}^X(f)_s = \int_t^s \partial_r f(r, X_r) dr + \int_t^s \partial_{xx}^2 f(r, X_r) d[M, M]_r + \int_t^s \partial_x f(r, X_r) d^- A_r, s \geq t. \quad (2.7)$$

In particular $\int_0^\cdot \partial_x f(r, X_r) \cdot d^- A_r$ exists and it is a martingale orthogonal process.

Proof. Item 1. was the object of item (c) of Proposition 3.10 in [22]. Concerning item 2., by the Itô formula (2.3), we have first (2.6) with (2.7), taking into account

$$\int_t^s \partial_x f(r, X_r) d^- X_r = \int_t^s \partial_x f(r, X_r) dM_r + \int_t^s \partial_x f(r, X_r) d^- A_r, s \geq t,$$

see Remark 2.4. The last assertion follows since $\mathcal{B}^X(f)$ is a martingale orthogonal process. \square

We remark that item 2. will not be used in the sequel and it has a separate interest. To prove Theorem 4.1, crucial will be item 1. of Proposition 2.10 and Proposition 2.7.

3 Concept of solution of a parabolic PDE

3.1 The framework

Proposition 2.10 above has some significant implications when f is a (generalized) solution of a parabolic partial differential equation.

In the sequel $L(\mathbb{R}^m, \mathbb{R}^d)$ will stand for the linear space of $d \times m$ real matrices. $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^m \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$ will be Borel functions, such that for all $x \in \mathbb{R}^d$, $|\sigma(\cdot, x) + b(\cdot, x)| \in L^1([0, T])$. Let the linear parabolic operator given formally by

$$\mathcal{L}_0 u(s, x) = \partial_s u(s, x) + \partial_x u(s, x) \cdot b(s, x) + \frac{1}{2} Tr[\sigma^\top(s, x) \partial_{xx}^2 u(s, x) \sigma(s, x)]. \quad (3.1)$$

In this section we will fix Borel functions $g : \mathbb{R}^d \rightarrow \mathbb{R}$ and $h : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\int_t^T \sup_{x \in K} |h(s, x)| ds < \infty,$$

for every compact $K \subset \mathbb{R}^d$.

We will consider the inhomogeneous backward parabolic problem

$$\begin{cases} \mathcal{L}_0 u(s, x) = h(s, x), & s \in [t, T], x \in \mathbb{R}^d, \\ u(T, x) = g(x), & x \in \mathbb{R}^d. \end{cases} \quad (3.2)$$

3.2 Quasi-strict solutions

The definition below generalizes the notion of strict solution, that appears in [21], Definition 4.1. For $s \in [t, T]$, we set

$$\mathcal{A}_s f(x) \doteq \partial_x f(x) \cdot b(s, x) + \frac{1}{2} Tr[\sigma^\top(s, x) \partial_{xx}^2 u(s, x) \sigma(s, x)]. \quad (3.3)$$

Definition 3.1. We say that $u : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$, $u \in C_{ac}^{0,2}([t, T] \times \mathbb{R}^d)$ is a *quasi-strict solution* to the backward Cauchy problem (3.2) if

$$u(s, x) = g(x) - \int_s^T h(r, x) dr + \int_s^T (\mathcal{A}_r u)(r, x) dr, \quad \forall x \in \mathbb{R}^d. \quad (3.4)$$

Remark 3.2. In this case, for every $x \in \mathbb{R}^d$,

$$\partial_s u(s, x) = h(s, x) - (\mathcal{A}_s u)(s, x), \quad ds \text{ a.e.} \quad (3.5)$$

where $\partial_s u$ stands for the distributional derivative of u .

The notion of quasi-strict solution allows to consider (somehow classical) solutions of (3.2) even though h, σ, b are not continuous in time.

Definition 3.3. We say that σ to be **non-degenerate** we mean that there is a constant $c > 0$ such that for all $(s, x) \in [0, T] \times \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ we have

$$\xi^\top \sigma(s, x)^\top \sigma(s, x) \xi \geq c |\xi|^2.$$

Lemma 3.4. Suppose the following.

1. σ, b are Hölder continuous in space (uniformly in time) and σ is non-degenerate. Moreover $\sigma, b : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ a.e. continuous in time for all $x \in \mathbb{R}^d$.
2. h has polynomial growth in space (uniformly in time). Moreover $h : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ a.e. continuous in time for all $x \in \mathbb{R}^d$.
3. g is continuous with polynomial growth.
4. For every compact K of \mathbb{R}^d , h is Hölder continuous in space (uniformly in time).

Then there is a quasi-strict solution of (3.2).

Remark 3.5. 1. The proof of Lemma 3.4 makes essentially use of Proposition 4.2 in [24].

2. Indeed, the context of [24] applies to our case, $B = 0$. Therein, one shows existence of a so called strong Lie solution u , which fulfills (3.4). In particular one gets

$$\partial_s u(s, x) = h(s, x) + (\mathcal{A}_s u)(s, x), \quad \forall x \in \mathbb{R}^d, \quad ds \text{ a.e.} \quad (3.6)$$

3. However, the notion of quasi-strict solution requires also u to belong to $C_{ac}^{0,2}([t, T] \times \mathbb{R}^d)$. The spatial first and second order spatial derivatives of u in [24] are (even Hölder) continuous in time, for fixed x . Therefore u fulfills items 1., 3. and 4. of Definition 2.1.

Moreover (3.4) and (3.6) imply that u also fulfills item 2. of Definition 2.1.

Remark 3.6. If $h \in C^{0,\gamma}([t, T] \times \mathbb{R}^d)$ and g is Hölder continuous, the result appears in Theorem 5.1.9 of [25] and in this case we even have a strict solution.

3.3 Quasi-strong solutions

The definition below is a relaxation of the notion of strong solution defined for instance in [22], Definition 4.2., which is based on approximation of strict (classical) solutions. The notion of quasi-strong solution is motivated by the fact that the coefficients σ and b are not supposed to be continuous in time.

Definition 3.7. $u : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a *quasi-strong solution* (with *approximating sequence* u_n) to the backward Cauchy problem (3.2) if there exists a sequence $u_n \in C_{ac}^{0,2}([t, T] \times \mathbb{R}^d)$ and a sequence $h_n : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\int_t^T \sup_{x \in K} |h_n(s, x)| ds < \infty$, for every compact $K \subset \mathbb{R}^d$ realizing the following.

1. $\forall n \in \mathbb{N}$, u_n is a quasi-strict solution of the equation

$$\mathcal{L}_0 u_n(s, x) = h_n(s, x), s \in [t, T], x \in \mathbb{R}^d.$$

2. For each compact $K \subset \mathbb{R}^d$

$$\begin{cases} \sup_{(s,x) \in [t,T] \times K} |u_n - u|(s, x) \rightarrow 0, \\ \sup_{x \in K} |h_n - h|(\cdot, x) \rightarrow 0 \quad \text{in } L^1([t, T]). \end{cases}$$

3. $u(T, \cdot) = g$.

Remark 3.8. According to previous definition any quasi-strong solution is continuous.

3.4 Mild solutions

We define now the notion of a mild solution. We suppose here for simplicity that σ and b are of polynomial growth. Suppose moreover that the SDE

$$\begin{cases} dX_s = b(s, X_s) ds + \sigma(s, X_s) dW_s, \\ X_t = x, \end{cases} \quad (3.7)$$

admits existence and uniqueness in law for every initial condition for any $0 \leq t \leq s \leq T$ and any $x \in \mathbb{R}^d$. Let $\check{\Omega} := C([t, T])$ be the canonical path space equipped with its Borel σ -algebra and let $X = (X_s)_{s \in [t, T]}$ be the canonical process.

For the notion of mild solution we introduce the associated inhomogeneous semigroup $(\mathcal{P}_{t,s})_{0 \leq t \leq s \leq T}$. For this we denote $(\mathbb{P}^{t,x}), 0 \leq t \leq T, x \in \mathbb{R}^d$ the associated Markov canonical class, see e.g. [4, 3], with associated expectation $(\mathbb{E}_s^{t,x})_{0 \leq t \leq s \leq T}$. $\mathbb{P}^{t,x}$ will be the solution of the (unique) solution in law of (3.7). In the sequel $P(t, s; x, dz)$ will denote the (marginal) law of X_s under $\mathbb{P}^{t,x}$.

For $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ continuous with polynomial growth we set

$$\mathcal{P}_{t,s} f(x) = \mathbb{E}^{t,x} f(X_s). \quad (3.8)$$

$(\mathcal{P}_{t,s})_{0 \leq t \leq s \leq T}$ is then the time-inhomogeneous semigroup associated with the generator (3.3).

Definition 3.9. Suppose g and h with polynomial growth. A function $u : [t, T] \times \mathbb{R}^d$ is said to be a *mild solution* to (3.2) if

$$u(s, x) = \mathcal{P}_{s,T} g(x) + \int_s^T \mathcal{P}_{s,r} h(r, x) dr, (s, x) \in [t, T] \times \mathbb{R}^d.$$

Remark 3.10. 1. By the usual techniques of stochastic calculus, see e.g. Burkholder-David-Gundy and Jensen's inequalities we can get

$$\sup_{t \in [0, T]} \mathbb{E}^{t, x} \left(\sup_{t \leq s \leq T} |X_s|^p \right) \leq c|x|^p, \quad \forall x \in \mathbb{R}^d.$$

2. Taking into account item 1., (3.8) is well-defined. Moreover, since h and g have polynomial growth a mild solution has necessarily polynomial growth.
3. A mild solution is not a priori a continuous function. Consider $h = 0$ and g bounded and not continuous. The solution cannot be continuous at $t = T$.

3.5 Relation between the different concepts of solutions

Below, for simplicity of the formulation we set $t = 0$ as initial time of the interval $[t, T]$ in the PDEs. We will need a supplementary time t in the proof.

Proposition 3.11. Let us suppose σ, b, h with polynomial growth. A quasi-strict solution u of (3.2) with polynomial growth is also a mild solution.

Proof. Let u be a quasi-strict solution of (3.2) with polynomial growth. In particular, for every $t \in [0, T]$, $u \in C_{ac}^{0,2}([t, T] \times \mathbb{R}^d)$. We fix $t \in [0, T]$ and we apply the Itô type formula (2.3) in Proposition 2.7, taking into account Remark 2.4. Clearly the canonical process X is a solution of $dX_s = b(s, X_s) ds + \sigma(s, X_s) dW_s$, under $\mathbb{P}^{t, x}$. Explicitly, for $s \in [t, T]$, we get

$$\begin{aligned} u(s, X_s) &= u(t, x) + \int_t^s (\partial_r u(r, X_r) dr + \partial_x u(r, X_r) [b(r, X_r) dr + \sigma(r, X_r) dW_r]) \\ &\quad + \frac{1}{2} \int_t^s Tr[\sigma^\top(r, X_r) \partial_{xx}^2 u(r, X_r) \sigma(r, X_r)] dr \\ &= u(t, x) + \int_t^s \partial_x u(r, X_r) \sigma(r, X_r) dW_r + \int_t^s \partial_r u(r, X_r) dr + \int_t^s (\mathcal{A}_r u)(r, X_r) dr \\ &= u(t, x) + M_s + \int_t^s h(r, X_r) dr, \end{aligned} \tag{3.9}$$

where

$$M_s = \int_t^s \partial_x u(r, X_r) \sigma(r, X_r) dW_r, \quad s \in [t, T].$$

The last equality in (3.9) holds because u is a quasi-strict solution of (3.2) so by (3.4), for all $x \in \mathbb{R}^d$, $u(\cdot, x)$, is absolutely continuous and by Remark 3.2, (3.5) holds. Since u and h have polynomial growth, Remark 3.10 implies that the local martingale vanishing at zero M is such that $\sup_{s \in [t, T]} |M_s|$ is an integrable r.v., which implies that it is a martingale. Being u a solution of (3.2) $u(T, \cdot) = g$ and so (3.9) implies

$$g(X_T) = u(t, x) + M_T + \int_t^T h(r, X_r) dr.$$

Taking the expectation under $\mathbb{P}^{t, x}$, being M is a martingale, therefore it has a null expectation, gives

$$\mathcal{P}_{t, T} g(x) = u(t, x) + \int_t^T \mathcal{P}_{t, r} h(r, x) dr,$$

taking into account (3.8) and Fubini's theorem. Finally u is a mild solution of (3.2). □

Now we prove that, under certain assumptions, a mild solution is a quasi-strong solution. Again we will take $t = 0$ as initial time of the PDE, for simplicity.

Proposition 3.12. Assume item 1., 2. and 3. of Lemma 3.4 and that the restriction of h to each compact is continuous in space, uniformly in $s \in [0, T]$.

Then a mild solution is a quasi-strong solution.

Remark 3.13. • The assumptions of Proposition 3.12 is therefore slightly weaker than those of Lemma 3.4.

- Of course h does not need to be bicontinuous in previous statement.
- The hypothesis on h to be continuous in space is not necessary if we suppose some regularity for the transition semigroup $P(t, r; x, da)$, but it goes beyond the scope of the paper.

Proof of Proposition 3.12. Let $(\mathbb{P}^{t,x})$ be the family of probability measures introduced after (3.7). We write

$$u := v^0 + v,$$

where

$$\begin{aligned} v^0(t, x) &= \mathbb{E}^{t,x}(g(X_T)), \\ v(t, x) &= \mathbb{E}^{t,x} \left(\int_t^T h(r, X_r) dr \right). \end{aligned}$$

Without restriction of generality we can suppose $g = 0$, since the general case can be treated similarly. We prove that there exists a sequence of quasi-strict solutions v_n of (3.2) with $g = 0$, according to Definition 3.7. We proceed in two steps:

1. truncation,
2. regularization.

1. Truncation

This step consists in reducing the problem to the case when h is bounded with compact support. Let h be as in the assumption and $h_n(t, x) = h(t, x)\chi(x)_{[-n,n]^d}$, where $\chi_{[-n,n]^d}$ is a smooth function bounded by 1 which support is included in $[-(n+1), n+1]^d$ and equal to 1 on $[-n, n]^d$. We define

$$v_n(t, x) = \mathbb{E}^{t,x} \left(\int_t^T h_n(r, X_r) dr \right). \tag{3.10}$$

Obviously v_n is a mild solution of (3.2) with h replaced with h_n .

By hypothesis, there exists $p \geq 1$ and a positive constant C such that $|h(r, x)| \leq C(1 + |x|^p)$, for all $(r, x) \in [0, T] \times \mathbb{R}^d$. For every $(t, x) \in [0, T] \times \mathbb{R}^d$, taking $X = (X_s)_{s \geq t}$ as the canonical process solution of (3.7), with $X_t = x$. Its law is of course $\mathbb{P}^{t, x}$ and we have

$$\begin{aligned} |v_n(t, x) - v(t, x)| &\leq \int_t^T \mathbb{E}^{t, x} |h(r, X_r)| 1_{\{X_r \notin [-n, n]^d\}} dr \\ &\leq C \int_t^T \mathbb{E}^{t, x} ((1 + |X_r|^p) 1_{\{X_r \notin [-n, n]^d\}}) dr \\ &\leq 2C \int_t^T (\mathbb{E}^{t, x} ((1 + |X_r|^{2p}))^{\frac{1}{2}} \mathbb{P}^{t, x} \{|X_r| \geq n\}^{\frac{1}{2}}) dr. \end{aligned}$$

By Chebyshev this is smaller than

$$\int_t^T 2C (\mathbb{E}^{t, x} ((1 + |X_r|^{2p}))^{\frac{1}{2}}) \frac{\mathbb{E}^{t, x} (|X_r|^{\frac{1}{2}})}{n^{\frac{1}{2}}} dr \leq \int_0^T 2C (\mathbb{E}^{t, x} ((1 + |X_r|^{2p}))^{\frac{1}{2}}) \frac{\mathbb{E}^{t, x} (|X_r|^{\frac{1}{2}})}{n^{\frac{1}{2}}} dr.$$

By Remark 3.10, for all $p \geq 1$

$$\sup_{0 \leq t \leq s \leq T} \mathbb{E}^{t, x} (|X_s|^p) \leq C(1 + |x|^p).$$

At this point the partial result $\sup_{(t, x) \in [0, T] \times K} |v_n(t, x) - v(t, x)| \rightarrow 0$ is established.

The sequence (h_n) and (v_n) are compatible with 2. of Definition 3.7, which concludes the step 1.

2) Regularization

We have now reduced the problem to the case when h has compact support. We define h_n as

$$h_n(r, y) = \int_{\mathbb{R}^d} h(r, \xi) n \phi(n(y - \xi)) d\xi = \int_{\mathbb{R}^d} \phi(\xi) h(r, y - \frac{\xi}{n}) d\xi, \quad (3.11)$$

where ϕ is a non-negative mollifier on \mathbb{R}^d with compact support. h_n is smooth in space with all the space derivatives being bounded. In particular, each h_n is Hölder continuous in space uniformly time. By Lemma 3.4 there exists a quasi-strict solution (therefore continuous) $u = v_n$ to the problem (3.2) with h replaced by h_n and again $g = 0$. Since a quasi-strict solution is a mild solution by Proposition 3.11, we can represent it as in (3.10), with h_n as in (3.11).

It is now possible to write

$$\begin{aligned} |v_n(t, x) - v(t, x)| &\leq \mathbb{E}^{t, x} \left| \int_{\mathbb{R}^d} d\xi \phi(\xi) \int_t^T \left(h(r, X_r) - h(r, X_r - \frac{\xi}{n}) \right) dr \right| \\ &\leq \int_{\mathbb{R}^d} d\xi \phi(\xi) \int_t^T dr \int_{\mathbb{R}^d} \left| h(r, a) - h(r, a - \frac{\xi}{n}) \right| P(t, r; x, da) \\ &\leq \int_{\mathbb{R}^d} d\xi \phi(\xi) \int_t^T dr \gamma \left(h(r, \cdot); \frac{\text{diam}(\text{supp } \phi)}{n} \right) \int_{\mathbb{R}^d} P(t, r; x, da) \\ &\leq \sup_{r \in [t, T]} \gamma \left(h(r, \cdot); \frac{\text{diam}(\text{supp } \phi)}{n} \right) \cdot \int_{\mathbb{R}^d} d\xi \phi(\xi) \int_t^T dr \int_{\mathbb{R}^d} P(t, r; x, da) \\ &\leq T \sup_{r \in [t, T]} \gamma \left(h(r); \frac{\text{diam}(\text{supp } \phi)}{n} \right) \end{aligned}$$

and this converges to zero by assumption, uniformly in $t \in [0, T]$. We have now proved that the sequence v_n converges to v uniformly on each compact. The convergence of (h_n) is even stronger

than required in item 2. of Definition 3.7. Finally v is a quasi-strong solution to (3.2). This completes the proof. \square

4 The $C^{0,1}$ -type chain rules for weak Dirichlet processes and semimartingales

When $u \in C^{0,1}$ is a strong solution of (3.2), it was the object of Theorem 4.5 of [22].

Theorem 4.1. Let $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$ be Borel functions and locally bounded.

Let $t \in [0, T]$. Let $h : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ as in the lines before (3.2). Consider $u : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ supposed to be of class $C^{0,1}([t, T] \times \mathbb{R}^d)$ be a quasi-strong solution of the Cauchy problem (3.2).

Let $(S_s)_{s \in [t, T]}$ be a process of the form

$$S_s = S_0 + \int_t^s \sigma(r, S_r) dW_r + A_s,$$

where $(A_s)_{s \in [t, T]}$ is an $(\mathcal{F}_s)_{s \in [t, T]}$ -martingale orthogonal process such $[A, A]$ exists.

Let us suppose that the assumptions below are verified.

1. $\int_t^s \partial_x u(r, S_r) d^- A_r$, $s \in [t, T]$ exists and it is a martingale orthogonal process.
- 2.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_t^s (\partial_x u_n(r, S_r) - \partial_x u(r, S_r)) d^- A_r \\ & - \lim_{n \rightarrow \infty} \int_t^s (\partial_x u_n(r, S_r) - \partial_x u(r, S_r)) \cdot b(r, S_r) dr = 0 \quad s \in [t, T] \text{ u.c.p.}, \end{aligned} \quad (4.1)$$

where the sequence u_n is the *approximating sequence* of the quasi-strong solution u .

Then

$$u(s, S_s) = u(t, S_t) + \int_t^s (\partial_x u)(r, S_r) \sigma(r, S_r) dW_r + \mathcal{B}^S(u)_s,$$

where, for $s \in [t, T]$,

$$\mathcal{B}^S(u)_s = \int_t^s h(r, S_r) dr + \int_t^s \partial_x u(r, S_r) d^- A_r - \int_t^s \partial_x u(r, S_r) \cdot b(r, S_r) dr,$$

and $\mathcal{B}^S(u)$ is a martingale orthogonal process.

Remark 4.2. $\int_t^s \partial_x u(r, S_r) d^- A_r$ exists and it is a martingale orthogonal process in the following two cases.

1. A is a bounded variation process so that S is a semimartingale. In this case $\int_t^s \partial_x u(r, S_r) d^- A_r$ is the (componentwise) Lebesgue-Stieltjes integral $\int_t^s \partial_x u(r, S_r) dA_r$, which is in particular a bounded variation process and therefore a martingale orthogonal process.

2. $u \in C_{ac}^{0,2}([t, T], \mathbb{R}^d)$. Indeed, by Proposition 2.7 and Remark 2.4

$$\begin{aligned} \int_t^s \partial_x u(r, S_r) d^- A_r &= u(s, S_s) - u(t, x) - \int_t^s (\partial_x u)(r, S_r) \sigma(r, S_r) dW_r \\ &\quad - \int_t^s \partial_r u(r, S_r) dr - \frac{1}{2} \int_t^s \partial_{xx}^2 u(r, S_r) d[S, S]_r, \quad s \in [t, T]. \end{aligned} \quad (4.2)$$

In particular the aforementioned forward integral exists. It remains to show that it is an $(\mathcal{F}_s)_{s \in [t, T]}$ -martingale orthogonal process. Let us write $M := \int_0^\cdot \sigma(r, X_r) dW_r$. For notational simplicity, we set $d = 1$ and $t = 0$. Since $u \in C^{0,1}$, Proposition 15.3 of [31], $u(\cdot, S) = M^u + A^u$, where $M^u = \int_0^\cdot \partial_x u(r, S_r) dM_r$ and A^u is an $(\mathcal{F}_s)_{s \in [0, T]}$ -martingale orthogonal process. Let N be a continuous local martingale. We get

$$[u(\cdot, S), N] = \int_0^\cdot \partial_x u(r, S_r) d[M, N]_r = \int_0^\cdot \partial_x u(r, S_r) \sigma(r, X_r) d[W, N]_r.$$

The process

$$u(\cdot, S) - \int_0^\cdot \partial_x u(r, S_r) \sigma(r, S_r) dW_r, \quad t \in [0, T],$$

is martingale orthogonal since

$$\left[\int_0^\cdot \partial_x u(r, S_r) \sigma(r, S_r) dW_r, N \right] = \int_0^\cdot \partial_x u(r, S_r) \sigma(r, X_r) d[W, N]_r.$$

Finally the covariation of the right-hand side of (4.2) with N vanishes since all the other processes appear in the right-hand side of (4.2) are martingale orthogonal being of bounded variation.

Remark 4.3. In fact condition (4.1) is a bit obscure: it is for instance verified if A has bounded variation and if there is a quasi-strong solution such that space derivatives of the approximating sequence u_n , i.e. $\partial_x u_n$, converges pointwise (with uniform bound in n on each compact) to the derivative of u .

This can happen even in a possibly degenerate case, if for instance, for all $s \in [0, T]$, $\sigma(s, \cdot), b(s, \cdot)$ are of class C^1 such that $\partial_x \sigma, \partial_x b$ are uniformly bounded. We also suppose $\sigma, b : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ a.e. continuous in time for all $x \in \mathbb{R}^d$. Concerning the other coefficients we suppose g (resp. h) such that $\partial_x g$ (resp. $\partial_x h$) has polynomial growth. In this case (3.2) admits a mild $C^{0,1}$ -solution of the type

$$u(s, x) = \mathbb{E}(g(Y_T^{s,x}) + \int_t^T h(r, Y_T^{r,x}) dr), \quad (s, x) \in [t, T] \times \mathbb{R}^d,$$

where $Y^{s,x}$ is a strong solution of the SDE (3.7) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with related expectation \mathbb{E} . In this case, adopting the formalism with $d = 1$ for simplicity

$$\partial_x u(s, x) = \mathbb{E}(g'(Y_T^{s,x}) Z_T^{s,x}) + \int_t^T \partial_x h(r, Y_T^{r,x}) Z_T^{r,x} dr, \quad (4.3)$$

where $Z^{s,x} := \partial_x Y^{s,x}$, the derivative being taken in $L^2(\Omega)$.

By the same construction as in the proof of Proposition 3.12 (truncation-regularization procedure) u can be shown to be a quasi-strong solution of (3.2) by approximation via solutions u_n , where u_n is a solution of (3.2) replacing g (resp. h) by g_n (resp. h_n) by a truncation-regularization procedure. We apply (4.3) replacing g, h, u with g_n, h_n, u_n . In this way $\partial_x u_n$ suitably converges to $\partial_x u$.

Proof of Theorem 4.1. For simplicity of the formulation we set $d = 1$ and without restriction of generality we set $t = 0$. Let $u_n \rightarrow u$ be a sequence as in Definition 3.7. By Proposition 2.7 and Remark 2.4, we get

$$\begin{aligned} u_n(s, S_s) &= u_n(0, S_0) + \int_0^s \mathcal{L}_0 u_n(r, S_r) dr - \int_0^s \partial_x u_n(r, S_r) b(r, S_r) dr \\ &\quad + \int_0^s \partial_x u_n(r, S_r) d^- A_r + N_s^n, \end{aligned}$$

where

$$N_s^n \doteq \int_0^s \partial_x u_n(r, S_r) \sigma(r, S_r) dW_r,$$

which is in particular a local martingale vanishing at zero. By assumption (4.1), the process N^n converges u.c.p. to

$$N_s \doteq u(s, S_s) - u(0, S_0) - \bar{\mathcal{B}}^S(u)_s, \quad (4.4)$$

where

$$\bar{\mathcal{B}}^S(u)_s = \int_0^s h(r, S_r) dr + \int_0^s \partial_x u(r, S_r) d^- A_r - \int_0^s \partial_x u(r, S_r) b(r, S_r) dr,$$

Now, the space of the space of continuous local martingales vanishing at zero, as a linear subspace of $C_{\mathcal{F}}([0, T] \times \Omega; \mathbb{R})$, is closed (under the u.c.p. convergence topology), see e.g. Proposition 4.4 in [22]. Consequently N is a continuous local martingale vanishing at zero. It remains to prove the following.

1. $\bar{\mathcal{B}}^S(u)$ is a martingale orthogonal process.
- 2.

$$N_s = \int_0^s \partial_x u(r, S_r) \sigma(r, S_r) dW_r. \quad (4.5)$$

1. follows by additivity since $\int_0^\cdot \partial_x u(r, S_r) d^- A_r$ and all bounded variation processes are martingale orthogonal processes.

Concerning 2., setting $M_s = \int_0^s \sigma(r, S_r) dW_r$, besides (4.4), Proposition 2.10 1. provides another weak Dirichlet decomposition

$$u(s, S_s) = u(0, S_0) + \int_0^s \partial_x u(r, S_r) \sigma(r, S_r) dW_r + \mathcal{B}^S(u)_s,$$

where $\mathcal{B}^S(u)$ is a martingale orthogonal process. By the uniqueness of the weak Dirichlet decomposition we get finally (4.5). The proof is now complete. \square

In the next corollary we will apply Theorem 4.1 to a solution S of a non-Markovian SDE. That result extends Corollary 4.6 of [22], where the coefficients σ, b were continuous. Here, we also assume $u \in C^{0,1}([t, T] \times \mathbb{R}^d) \cap C^0([t, T] \times \mathbb{R}^d)$, that is, $u(T, \cdot)$ may not be differentiable in the space variable. This allows lower regularity in the terminal condition of the PDE (3.2).

Corollary 4.4. Let $f : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a progressively measurable field and a.s. locally bounded. Let $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$ be Borel locally bounded functions. Let $(S_s)_{s \in [t, T]}$ be a solution to the SDE

$$dS_s = f(s, S_s)ds + \sigma(s, S_s)dW_s.$$

Let $t \in [0, T]$, $h : [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ as in the lines before (3.2). Let $u \in C^{0,1}([t, T] \times \mathbb{R}^d) \cap C^0([t, T] \times \mathbb{R}^d)$ be a quasi-strong solution of the Cauchy problem (3.2) fulfilling

$$\int_0^T |\partial_x u(s, S_s)|^2 ds < \infty \text{ a.s.} \quad (4.6)$$

Moreover we suppose the validity of one of the following items.

1. The approximating sequence u_n of Definition 3.7 fulfills

$$\lim_{n \rightarrow \infty} \int_t^T (\partial_x u_n(r, S_r) - \partial_x u(r, S_r)) \cdot (f(r, S_r) - b(r, S_r)) dr = 0 \text{ u.c.p.} \quad (4.7)$$

2. $\sigma^\top(r, S_r)$ is invertible for every $r \in [t, T]$ and denote the right pseudo-inverse $\sigma^{-1}(r, S_r) := \sigma^\top(\sigma\sigma^{-1})(r, S_r)$. Moreover the Novikov condition

$$\mathbb{E} \left(\exp \left(\frac{1}{2} \int_t^T |\sigma^{-1}(f - b)|^2(r, S_r) dr \right) \right) < \infty, \quad (4.8)$$

holds.

Then, for $s \in [t, T]$,

$$u(s, S_s) = u(t, S_t) + \int_t^s (\partial_x u)(r, S_r) \sigma(r, S_r) dW_r + \mathcal{B}^S(u)_s, \quad (4.9)$$

where,

$$\mathcal{B}^S(u)_s = \int_t^s h(r, S_r) dr + \int_t^s \partial_x u(r, S_r) \cdot f(r, S_r) dr - \int_t^s \partial_x u(r, S_r) \cdot b(r, S_r) dr. \quad (4.10)$$

Remark 4.5. Clearly, whenever $(\omega, s, x) \mapsto \sigma^{-1}(s, x)(f(\omega, s, x) - b(s, x))$ is bounded, then (4.8) is fulfilled.

Proof of Corollary 4.4. In the proof, we set $t = 0$ and $d = 1$ without loss of generality. We first prove the result assuming that item 1. holds. We set

$$A = \int_0^T f(r, S_r) dr.$$

Let $\varepsilon > 0$. The idea is to apply Theorem 4.1 with T replaced by $T - \varepsilon$. The function u restricted to $[0, T - \varepsilon] \times \mathbb{R}^d$ is trivially a quasi-strong solution to

$$\mathcal{L}_0 u(s, x) = h(s, x), \quad u(T - \varepsilon, \cdot) = g_\varepsilon,$$

where $g_\varepsilon = u(T - \varepsilon, \cdot)$. By Theorem 4.1, taking into account Remark 2.4, we get the decomposition (4.9) and (4.10) for $s \in [0, T - \varepsilon]$. This implies those decompositions for $s \in [0, T[$. The case $s = T$ follows letting $s \rightarrow T$ and using in particular condition (4.6).

Now let us assume that item 2. is verified. Let \mathbb{Q} be the probability measure on (Ω, \mathcal{F}_T) defined

$$d\mathbb{Q} = Z_T(\eta)d\mathbb{P},$$

where

$$Z_s(\eta) = e^{(\int_0^s \eta_r dW_r - \frac{1}{2} \int_0^s |\eta_r|^2 dr)}, \quad s \in [0, T]$$

and

$$\eta_s = \sigma^{-1}(s, S_s)(b(s, S_s) - f(s, S_s)), \quad s \in [0, T].$$

Then, the Novikov condition implies that $Z(\eta)$ is a martingale. Hence by Girsanov's Theorem, the process

$$\tilde{W}_s = W_s - \int_0^s \sigma^{-1}(r, S_r)(b(r, S_r) - f(\omega, r, S_r))dr, \quad s \in [0, T],$$

is an $(\mathcal{F}_s, \mathbb{Q})$ -Brownian motion and $(S_s)_{s \in [0, T]}$ satisfies the stochastic differential equation

$$dS_s = b(s, S_s)ds + \sigma(s, S_s)d\tilde{W}_s.$$

Then, by item 1. in the statement (here \mathbb{Q} replaces \mathbb{P} , $f = b$ and \tilde{W} replaces W), we obtain

$$u(s, S_s) = u(0, S_0) + \int_0^s \partial_x u(r, S_r)\sigma(r, S_r)d\tilde{W}_r + \int_0^s h(r, S_r)dr, \quad s \in [t, T],$$

and the conclusion follows. \square

Remark 4.6. If $\lim_{n \rightarrow \infty} \partial_x u_n = \partial_x u$ in $C^0([t, T] \times \mathbb{R}^d)$, then assumption (4.7) is trivially verified.

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