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On the k -clique graph and the k -clique operator

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ABSTRACT

Given a graph G , a clique is a maximal complete subgraph of G and a biclique is a maximal induced complete bipartite subgraph of G . The intersection graphs of cliques and bicliques, denoted by $K(G)$ and $KB(G)$ respectively, have been studied largely in the last years. Several articles have been made from a structural point of view, characterisations, recognition problem, etc, and about the behaviour of their respective iterated operator.

In this paper, we generalise the clique and biclique graphs as k -clique graphs, that is, the intersection graphs of the family of all maximal induced complete k -partite subgraphs of G . We consider first $k = 3$, i.e. triclique graphs, $KT(G)$, and we then generalise our results to $k > 3$, k -clique graphs, $KC(G)$.

In particular, we study the connectivity relation between G and $KT(G)$, its generalisation to $KC(G)$, and we propose several structural results. We then investigate its characterisation and recognition problem. Next, we consider the iterated k -clique operators KT and KC , giving sufficient conditions for a graph to be convergent or divergent under these operators. Finally, we compare all known results so far between clique, biclique and k -clique graphs, and we propose some general conjectures on the subject.

Keywords: Cliques; Bicliques; Complete k -partite; Iterated graph operators; Graph dynamics

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1 Introduction

Intersection graphs of specific subgraphs in a general graph have been extensively investigated. Notable examples include line graphs (intersection graphs of the edges of a graph), interval graphs (intersection graphs of a family of subpaths of a path), and, notably, clique graphs (intersection graphs of the family of all maximal cliques of a graph) [3, 4, 8, 10, 11, 26, 27].

A *clique* is a maximal complete subgraph. The *clique graph* of a graph G , denoted by $K(G)$, is the intersection graph of the family of all cliques of G . Clique graphs were introduced by Hamelink in [18] and characterised in [33]. It was proved in [1] that the clique graph recognition problem is NP-complete.

A *biclique* is a maximal induced complete bipartite subgraph. The *biclique graph* of a graph G , denoted by $KB(G)$, is the intersection graph of the family of all bicliques of G . It was defined and characterised in [16]. However no polynomial-time algorithm is known for recognizing biclique graphs nor it was proven that the problem is NP-complete.

The clique and biclique graphs can be thought as operators from *Graphs* into *Graphs*. The *iterated clique graph* $K^k(G)$ is the graph obtained by applying the clique operator k successive times. This operator, introduced by Hedetniemi and Slater in [19], has been the subject of extensive research. Numerous studies have focused on exploring the various possible behaviours of this operator. The primary objective is to determine whether a given graph converges, diverges, or exhibits periodic behaviour as the parameter k approaches infinity under the clique operator. This question remains unresolved in the general case, and its computability status is unknown. Nevertheless, partial characterisations have been provided for convergent, divergent, and periodic graphs within certain graph classes. Some of these characterisations have led to the development of polynomial-time algorithms for solving the problem.

For the clique-Helly graph class, graphs which are convergent to the trivial graph have been characterised in [2]. Cographs, P_4 -tidy graphs, and circular-arc graphs are examples of classes where the different behaviours were also characterised [5, 20]. On the other hand, divergent graphs were considered. For example, in [30], families of divergent graphs are given. Periodic graphs were studied in [8, 24]. It has been proved that for every integer i , there are graphs with period i and graphs which converge in i steps. More results about iterated clique graphs can be found in [9, 21, 22, 23, 25, 31].

The *iterated biclique graph* $KB^k(G)$ is the graph obtained by applying to G the biclique operator k times iteratively. This concept was introduced in [13], where all possible behaviours were characterised. It was demonstrated that a graph is either divergent or convergent but never periodic (with a period greater than 1). General characterisations for convergent and divergent graphs were provided. These results were established based on the observation that if a graph G contains a clique of size at least 5, then $KB(G)$ or $KB^2(G)$ contains a clique of a larger size, indicating that G diverges. Similarly, if G contains the *gem* or the *rocket* graphs as induced subgraphs, then $KB(G)$ contains a clique of size 5, and again G diverges. Otherwise, it was shown that after removing false-twin vertices of $KB(G)$, the resulting graph forms a clique with at most 4 vertices, in which case G converges. Furthermore, it was proven that if a graph G converges, then it does so to either the graphs K_1 or K_3 , and it achieves convergence in at most 3 steps. These characterisations led

to an $O(n^4)$ -time algorithm (later improved to an $O(n + m)$ -time one [12]) for recognizing convergent or divergent graphs under the biclique operator.

In this work, we generalise the clique and biclique graphs as k -clique graphs, that is, the intersection graphs of the family of all maximal induced complete k -partite subgraphs of G . For this we will split our study in two cases. The case $k = 3$, triclques and triclique graphs, are first considered, after which our results are generalised for $k > 3$. This division primarily serves to prevent unnecessary technical complications in the proofs while maintaining the core ideas, which remain identical in both cases.

This work is organized as follows. In Section 2 the necessary notation is given. In Section 3 we give connectivity and structural results of triclique and k -clique graphs. In Section 4 we study the characterisation and recognition problem of triclique graphs followed by its generalisation to k -clique graphs. In Section 5 we present some results about divergent and convergent graphs under the KT and KC operators. In Section 6 we compare known results on clique, biclique and k -clique graphs, and finally, we propose some future research directions and suggest several general conjectures.

We remark that this current work stands as a somewhat final stone in the study of the subject, as it generalises, to triclques and beyond, problems and results for the clique and biclique iterators.

2 Preliminaries

Along the paper we restrict to undirected simple graphs. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $n = |V(G)|$ and $m = |E(G)|$. A *subgraph* G' of G is a graph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$ such that all endpoints of the edges of E' are in V' . When E' has all the edges of E whose endpoints belong to the vertex subset V' , we say that V' *induces* the subgraph $G' = (V', E')$, that is, G' is an *induced subgraph* of G . Also, let $G[V']$ denote the induced subgraph of G by the set V' . A graph $G = (V, E)$ is *bipartite* when there exist sets U and W such that $V = U \cup W$, $U \cap W = \emptyset$, $U \neq \emptyset$, $W \neq \emptyset$ and $E \subseteq U \times W$. Say that G is a *complete graph* when every possible edge belongs to E . A complete graph on n vertices is denoted K_n . A bipartite graph is *complete bipartite* when every vertex of the first set is adjacent to every vertex of the second set. A complete bipartite graph on p vertices in one set and q vertices in the other is denoted $K_{p,q}$. A *clique* of G is a maximal complete subgraph, while a *biclique* is a maximal induced complete bipartite subgraph of G . We generalise the concepts of clique and biclique, and say that, for any $k \geq 1$, a *k -clique* of G is a maximal induced complete k -partite subgraph of G . That is, a 1-clique is a clique, a 2-clique a biclique, a 3-clique a triclique and so on.

The *open neighbourhood* of a vertex $v \in V(G)$, denoted $N(v)$, is the set of vertices adjacent to v . The *closed neighbourhood* of a vertex $v \in V(G)$, denoted $N[v]$, is the set $N(v) \cup \{v\}$. Two vertices $u, v \in V(G)$ are called *false-twins* if $N(u) = N(v)$ and *true-twins* if $N[u] = N[v]$. Given a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, we denote by $N_S(v)$ the neighbourhood of the vertex v restricted to the set S . Given a set of vertices $S \subseteq V(G)$, \overline{S} denotes the set $V(G) - S$. The *degree* of a vertex v , denoted by $d(v)$, is defined as $d(v) = |N(v)|$. A *path* (*cycle*) on k vertices ($k \geq 3$), denoted by P_k (C_k), is a sequence of vertices $v_1, v_2, \dots, v_k \in G$ such that $v_i \neq v_j$ for all $1 \leq i \neq j \leq k$ and v_i is adjacent to v_{i+1}

for all $1 \leq i \leq k - 1$ (and v_k is adjacent to v_1). A graph is *connected* if there exists a path between each pair of vertices. The *girth* of G is the length of a shortest induced cycle in the graph. Unless stated otherwise, we assume that all graphs of this paper are connected.

Given a family of sets \mathcal{H} , the *intersection graph* of \mathcal{H} is the graph that has the members of \mathcal{H} as vertices, and there is an edge between two sets $E, F \in \mathcal{H}$ when E and F have non-empty intersection. A graph G is an *intersection graph* if there exists a family of sets \mathcal{H} such that G is the intersection graph of \mathcal{H} . We remark that any graph is an intersection graph [34].

Two main problems arise when studying intersection graphs: the *characterisation problem* and the *recognition problem*. The *characterisation problem* for a class of intersection graphs \mathcal{C} consists in determining necessary and sufficient conditions under which a graph G belongs to \mathcal{C} . This often involves identifying a structural description of the graphs in \mathcal{C} , such as a set of forbidden induced subgraphs, a decomposition theorem, or a specific construction model that all and only graphs in \mathcal{C} satisfy. The *recognition problem* asks whether there exists an efficient (typically polynomial-time) algorithm that, given an arbitrary graph G , decides whether $G \in \mathcal{C}$. Alternatively, in the absence of a polynomial-time algorithm, the problem may be shown to be NP-complete, establishing its computational intractability under standard complexity assumptions ($P \neq NP$).

Let F be any graph operator and let G be a graph. The *iterated graph under the operator F* is defined iteratively as follows: $F^0(G) = G$ and for $k \geq 1$, $F^k(G) = F^{k-1}(F(G))$. We say that G *diverges* (resp. *converges* or *is periodic*) under the operator F whenever $\lim_{k \rightarrow \infty} |V(F^k(G))| = \infty$ (resp. $\lim_{k \rightarrow \infty} F^k(G) = F^m(G)$ for some m or $F^k(G) = F^{k+s}(G)$ for some k and $s \geq 2$). The study of the *behaviour of a graph G* under the operator F consists of deciding if G converges, diverges or is periodic under F .

The *triclique graph* of a graph G , denoted by $KT(G)$, is the intersection graph of the family of all tricliques of G , and in general, for any $k \geq 1$, the *k -clique graph* of a graph G , denoted by $KC(G)$, is the intersection graph of the family of all k -cliques of G . The *iterated triclique graph* $KT^k(G)$ and the *iterated k -clique graph* $KC^k(G)$, are defined analogously as the iterated clique and biclique ones.

We assume that the empty graph is convergent under the operators KT and KC , as it is obtained by applying these operators to a graph that does not contain any tricliques or k -cliques ($k > 3$) respectively.

3 Connectivity, numbers and structure

In this section we first study the connectivity relation between G and $KT(G)$, and its generalisation to $KC(G)$. Later we show that the number of tricliques and k -cliques can be exponential and present some families having those numbers. Finally, we extend some results from the clique's and biclique's world concerning subgraphs, induced subgraphs and false-twin vertices.

3.1 Connectivity

About connectivity, for clique and biclique graphs the relation is very simple:

- G is connected if and only if $K(G)$ is connected.
- G is connected if and only if $KB(G)$ is connected [15].

The main result of this subsection is the following theorem that characterises when $KT(G)$ is connected.

Theorem 3.1. *Let G be a connected graph. $KT(G)$ is connected if and only if the graph G' obtained by removing all edges in G that do not belong to a triangle (and eventually all resulting isolated vertices) is connected.*

Proof. \rightarrow) Let $u, v \in G'$, let T and T' be two triclques containing u and v respectively (they must exist as u and v are endpoints of edges in triangles) and let t and t' be the vertices in $KT(G')$ corresponding to T and T' . Clearly $KT(G') = KT(G)$ since all edges (and all new isolated vertices) removed in G that do not belong to a triangle, will not belong to a triclque as well. Now by contrapositive, if there is no path from u to v in G' , there there is no path from t to t' either in $KT(G)$, that is, $KT(G)$ is not connected.

\leftarrow) Let $t, t' \in KT(G)$ and let T, T' be their corresponding triclques in G . Since G' is connected, there is a path from a vertex in T to a vertex in T' and each edge of this path belongs to some triangle. Therefore, each of these edges belongs to some triclques. Among these triclques, take T_1, T_2, \dots, T_k such that $T = T_1$, $T' = T_k$ and $T_i \cap T_{i+1} \neq \emptyset$, that is, a set of intersecting consecutive triclques from T to T' in G' . Finally if t_1, t_2, \dots, t_k are the vertices in $KT(G)$ corresponding to those triclques, then they form a path from $t_1 = t$ to $t_k = t'$; that is, $KT(G)$ is connected. \square

We can then easily generalise this result for $k > 3$ as follows.

Theorem 3.2. *Let G be a connected graph. $KC(G)$ is connected if and only if the graph G' obtained by removing all edges in G that do not belong to a K_k , for $k > 3$, (and eventually all resulting isolated vertices) is connected.*

Proof. This proves similarly as Theorem 3.1. \square

3.2 Numbers

In this subsection we will present some families of graphs that show that the number of cliques, bicliques, triclques and k -cliques of a graph can be exponential. This might be useful not only from the mathematical point of view, but also to characterise graphs with a polynomial number of such objects. In such cases, we can then obtain the intersection graph in polynomial time (as it will be discussed in the last section).

First of all, for $n \geq 3$, let O_n denote the graph obtained by taking the complement of the union of n complete graphs on 2 vertices (also called m -dimensional octahedron). The resulting graph O_n has $2n$ vertices. It is not difficult to see that $K(O_n) = O_{2^n-1}$ implying that the number of cliques in a graph can be exponential (see Figure 1).

In respect to the number of triclques and k -cliques of a graph, we can extend the three families of graphs presented by Prisner in [32]. See Figure 2.

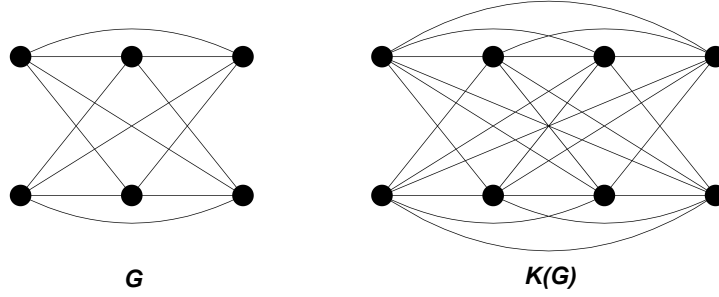


Figure 1: Graphs $G = O_3$ and $K(G) = K(O_3) = O_4$.

- The *Cocktail Party graph* of order j , denoted by $CP(j)$, is a bipartite graph with bipartition $V = V_1 \cup V_2$, where $|V_1| = |V_2| = j$, and each vertex $v_i \in V_1$ is adjacent to every vertex $w_k \in V_2$ for all $i \neq k$. Observe that bicliques of $CP(j)$ are $B = \{v_i \in V_1\}_{i \in I} \cup \{w_k \in V_2\}_{k \in K}$ where $\emptyset \subsetneq I \subsetneq \{1, \dots, j\}$ and $K = \{1, \dots, j\} - I$. Note that once a subset S of vertices of V_1 is fixed, a biclique of $CP(j)$ is fully determined where the subset of vertices of V_2 consists of those vertices in V_2 whose indices are not in S . As $S \neq \emptyset$ and $S \neq \{1, \dots, j\}$, we conclude that $CP(j)$ has $2j$ vertices and $2^j - 2$ bicliques.

Now we can easily generalise Cocktail Party graphs in order to obtain $2^j - 2$ triclques/ k -cliques by simply adding one (for triclques) or $k - 2$ (for k -cliques) universal vertices to the graph $CP(j)$. Clearly, each biclique in $CP(j)$ is now extended to a k -clique in the new graph with these $k - 2$ added universal vertices.

- The graph $H(t, \ell)$ consists of t isolated vertices and ℓ copies of K_2 , where each isolated vertex is adjacent to both vertices of each K_2 . It is clear that all bicliques of $H(t, \ell)$ are of the form $K_{t, \ell}$ and that we have 2^ℓ of them since we obtain a different one when taking all t isolated vertices and choosing one vertex of each of the ℓ copies of K_2 .

Similar to the Cocktail Party graphs' generalisation, we can do exactly the same in this family and add one (for triclques) or $k - 2$ (for k -cliques) universal vertices to the graph $H(t, \ell)$. This extended graph clearly possesses 2^ℓ triclques/ k -cliques.

- For triclques, take the graph consisting of $K_{j,j,j} - jK_3$, that is, remove the edges of j vertex-disjoint triangles from a complete tripartite graph $K_{j,j,j}$. To obtain the triclques of this graph, one need to pick j elements such that there is at least one from each part and no two non-adjacent vertices from different parts are chosen. This can be counted (using the inclusion-exclusion principle) as 3^j , the total number of picks (no constraints), minus 3×2^j , the picks that use only two parts, plus 3 since we removed two times in the previous term the picks that use only one part. Therefore we have in total $3^j - 3 \times 2^j + 3$ triclques. In general, for k -cliques, take the complete k -partite graph $K_{j, \dots, j}$ with k parts, and remove the edges of j vertex-disjoint K_k . Here, k -cliques are obtained in the same way as we did for triclques. Again, by the inclusion-exclusion principle, it can be seen that the number of k -cliques is $\sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^j \in O(k^j)$.

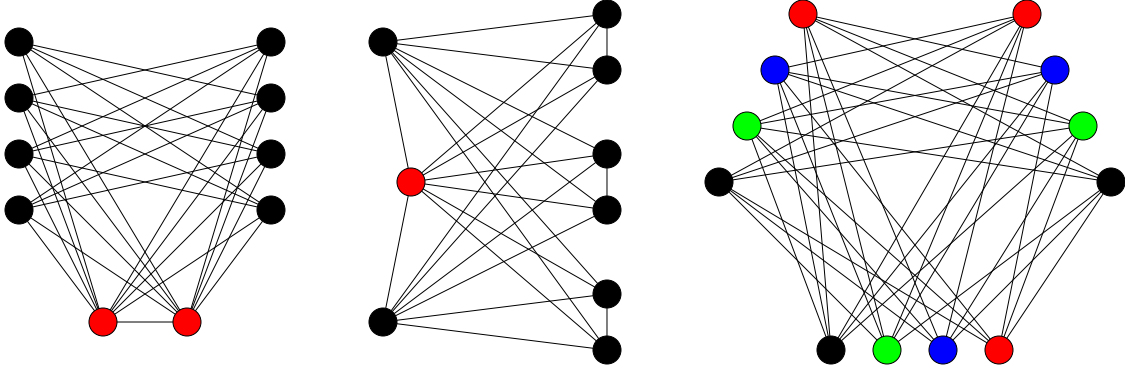


Figure 2: Left: extended $CP(4)$ with two (red) universal vertices. It has $2^4 - 2 = 14$ 4-cliques. Middle: extended $H(2,3)$ with one (red) universal vertex. It has $2^3 = 8$ tricliques. Right: $K_{4,4,4} - 4K_3$. Vertices with the same colour represent the triangles with removed edges. It has $3^4 - 3 \times 2^4 + 3 = 36$ tricliques.

In his article [32], Prisner showed that the number of bicliques of a graph can be exponential and used the first two families of graphs ($CP(j)$ and $H(t, \ell)$) to characterise when a graph has a polynomial number of bicliques. It would be of interest to study in a similar way this question for tricliques and k -cliques. That is, given a graph G such that it does not contain any graph belonging to any of the three families given as an induced subgraph, is it true that the number of tricliques/ k -cliques of G is polynomial?

3.3 Structure

For clique and biclique graphs, it is known that if G is an induced subgraph of H , then $K(G)$ and $KB(G)$ are subgraphs (not necessarily induced) of $K(H)$ and $KB(H)$ respectively. This result holds as well for triclique and k -clique graphs.

Observation 3.3. *If G is an induced subgraph of H , then $KT(G)$ and $KC(G)$ are subgraphs (not necessarily induced) of $KT(H)$ and $KC(H)$ respectively.*

Proof. Just observe that each triclique (resp. k -clique) of G is either a triclique (resp. k -clique) of H or it is contained in a larger triclique (resp. k -clique) of H . Therefore two intersecting tricliques (resp. k -cliques) of G , will be intersecting also in H , that is, $KT(G)$ (resp. $KC(G)$) is a subgraph of $KT(H)$ (resp. $KC(H)$). \square

This result is useful for proving divergence under the operator, as we will see later in Section 5.

Now, on the one hand, it is known for clique graphs, that if $u, v \in V(G)$ are true-twins, then $K(G) = K(G - \{u\}) = K(G - \{v\})$, that is, the clique graph structure remains unchanged when removing true twins. On the other hand, for biclique graphs we have the same situation but considering instead false-twins [13], that is, if $u, v \in V(G)$ are false-twins, then $KB(G) = KB(G - \{u\}) = KB(G - \{v\})$. This last observation for biclique graphs happens to be very important for the characterisation of convergent and divergent graphs

under the KB operator since we can bound the number of bicliques in false-twin-free graphs and therefore obtain specific structures in its biclique graph.

For triclique and k -clique graphs, the scenario is identical to that of biclique graphs. We have the following results.

Proposition 3.4. *Let G be a graph. Then, $KT(G) = KT(G')$, where G' denotes the graph obtained from G by keeping only one vertex in every set of false-twins.*

Proof. This is simply because if T is a triclique of G' containing some vertex u , then, in G , adding to T all false-twins of u results in a triclique of G . \square

Analogously, we have:

Proposition 3.5. *Let G be a graph. Then, $KC(G) = KC(G')$, where G' denotes the graph obtained from G by keeping only one vertex in every set of false-twins.*

These results might be therefore useful to obtain lower bounds on the number of tricliques and k -cliques in false-twin-free graphs as it was done for bicliques in [14] or perhaps to characterise the behaviour of the operators (KT and KC).

4 Characterisation and recognition problem

In this section we study the characterisation and the recognition problem of triclique graphs and its generalisation to k -clique graphs. We recall that clique graphs and biclique graphs were characterised in [33, 16] respectively. In respect to the recognition problem, it was proven NP-complete for clique graphs [1] while the problem is still open for biclique graphs (but suspected to be NP-complete as well).

Surprisingly for triclique and k -clique graphs ($k > 3$) these problems are trivial. We indeed have the following two main results of this section that show that every graph is the triclique and k -clique graph of some graph.

Theorem 4.1. *If G is a connected graph, then $G = KT(H)$ for some graph H such that:*

- (a) H is K_4 -free;
- (b) every edge in H is contained in some triangle;
- (c) for every triclique T in H , there exists a vertex v_T that belongs only to the triclique T .

Proof. The proof is by induction on n . For $n = 1, 2, 3$ we can see in Figure 3 that the result trivially holds.

Now let G be a graph on $n + 1 \geq 4$ vertices and let $G' = G - \{v\}$ for some vertex $v \in G$ such that G' is connected. By inductive hypotheses there exists a graph H' such that $G' = KT(H')$ and H' verifies (a), (b) and (c). Now let $k = d_G(v) \geq 1$, $N_G(v) = \{t_1, t_2, \dots, t_k\}$ and T_1, T_2, \dots, T_k the tricliques in H' corresponding to vertices t_1, t_2, \dots, t_k in G' . Since H' verifies (c), let $t'_i \in T_i$ be a vertex in H' such that it only belongs to the triclique T_i for each $i = 1, \dots, k$. Note that for every $1 \leq i \neq j \leq k$, t'_i cannot be adjacent to t'_j since, otherwise,


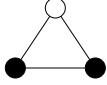
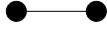
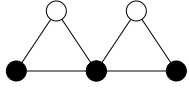
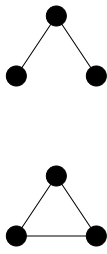
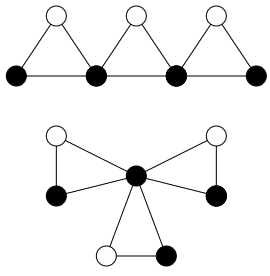
	$G=KT(H)$	H
$n=1$		
$n=2$		
$n=3$		

Figure 3: Base step of induction. Graphs G with order $n = 1, 2, 3$ and possible graphs H such that $G = KT(H)$. Clearly (a) and (b) are verified. White vertices show that (c) is verified.

by (b), the edge $t'_i t'_j \in E(H')$ would be contained in some triangle, therefore in some triclique contradicting how these vertices were chosen.

We construct a graph H as follows: Take H' and add k new vertices s'_1, s'_2, \dots, s'_k such that s'_i is a false-twin of t'_i for $i = 1, \dots, k$. Clearly each s'_i extends the triclique T_i and s'_i only belongs to that triclique (i.e. s'_i, t'_i are false-twins). We remark also that s'_i and s'_j , $i \neq j$, cannot be adjacent either. Finally add two new adjacent vertices v_1, v_2 such that v_1 and v_2 are adjacent to s'_i for every $i = 1, \dots, k$. Observe that H has exactly one triclique more than H' , that is, $T = \{v_1\} \cup \{v_2\} \cup \{s'_1, \dots, s'_k\}$. See Figure 4. We show now that this graph H verifies $G = KT(H)$ and (a), (b), (c).

It is clear that (a) and (b) are verified in H . For (c), observe that in H , each vertex t'_i still belongs only to (now extended with vertex s'_i) triclique T_i , v_1 (or v_2) belongs only to T and the rest of the tricliques did not change. Finally, $G = KT(H)$ since $G' = G - \{v\}$ verifies $G' = KT(H')$, H has exactly one triclique T more than H' , T in H corresponds to vertex v in G and T intersects only (extended) tricliques T_1, T_2, \dots, T_k in H , that is, v is adjacent to corresponding vertices t_1, t_2, \dots, t_k in G' . Therefore $G = KT(H)$ as we wanted to show. \square

Previous Theorem 4.1 generalises as follows.

Theorem 4.2. *If G is a connected graph, then $G = KC(H)$ for some graph H such that:*

- (a) H is K_{k+1} -free;
- (b) every edge in H is contained in some K_k ;
- (c) for every k -clique R in H , there exists a vertex v_R that belongs only to the k -clique R .

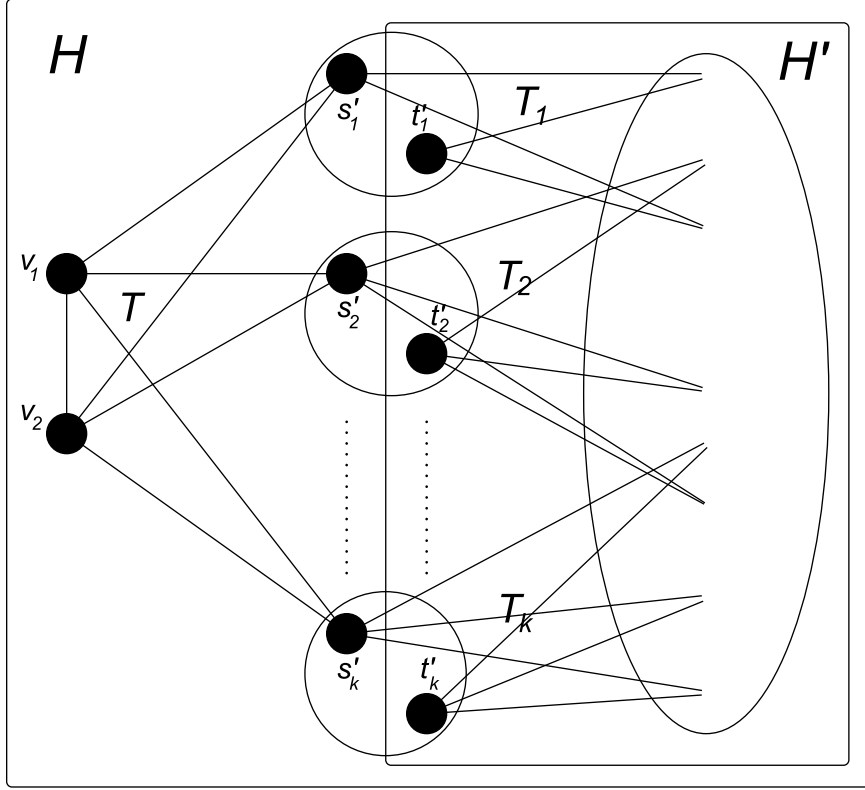


Figure 4: Construction of graph H from H' , adding new vertices v_1, v_2 and s'_1, \dots, s'_k . Each vertex s'_i is a false-twin of t'_i in H' and extends triclique T_i . There is only one new triclique in H : $T = \{v_1\} \cup \{v_2\} \cup \{s'_1, \dots, s'_k\}$.

Note that in Theorems 4.1 and 4.2, graphs K_4 and K_{k+1} are excluded respectively, since these graphs do not verify condition (c).

We remark as well that the presented construction technique does not work for clique and biclique graphs since if we try to add a new clique/biclique in the graph that intersects only some other specific cliques/bicliques, then new ones might slip in between those, therefore breaking the expected structure. See Figure 5 for more details.

Another interesting question that arises from the construction of graphs H in the proof of Theorem 4.1 (and of Theorem 4.2) is the size optimality, that is, whether they produce the smallest H such that $KT(H) = G$ (resp. $KC(H) = G$). The answer to this question is clearly no. For example, consider H_n as the graph obtained by the construction such that $KT(H_n) = K_n$. It can be seen that $V(H_1) = V(K_3) = 3$, and for $n \geq 1$, $V(H_{n+1}) = 2 + n + V(H_n)$ since we obtain H_{n+1} from H_n by adding two vertices (v_1, v_2 in the proof) and n new false-twin vertices (s'_i in the proof). Now, it is easy to see that $V(H_n) = \frac{(n+1)(n+2)}{2}$, so this results in a quadratic number of vertices which is much larger than just taking H_n as the graph formed by n triangles that share a same common vertex. Clearly this graph has $2n + 1$ vertices, so a linear number of vertices.

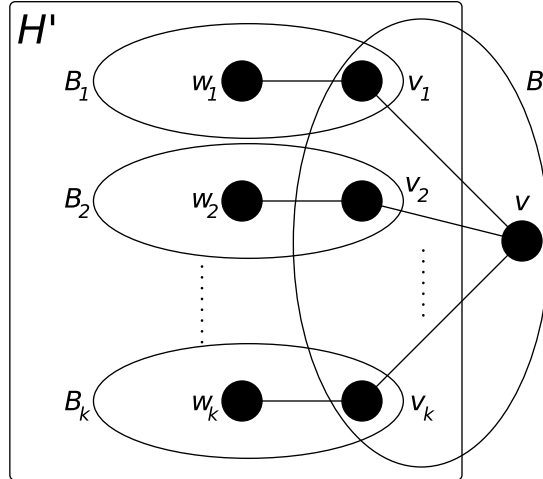


Figure 5: As in the proof of Theorem 4.1, we try to add a new biclique $B = \{v\} \cup \{v_1, \dots, v_k\}$ to the graph H' such that it intersects only with bicliques B_1, B_2, \dots, B_k . Even though these intersections are respected, new bicliques are accidentally created. That is, for each $i = 1, \dots, k$, $\{v_i\} \cup \{v, w_i\}$ will be contained in some new biclique, unless v is also adjacent to some w_i , in which case this new edge vw_i will be contained in a new biclique. In any case, we cannot control the structure of the biclique graph.

5 Iterated k -clique operator

In this section we study the behaviour of the iterated KT and KC operators. We establish several general results (divergence/convergence), and moreover, for various classes of graphs, we fully characterise their behaviour. Note that since the number of k -cliques for $k \geq 3$ can be exponential (as we have shown in Section 3), it might be unfeasible to compute the iterated graphs KT and KC of a graph in order to decide its behaviour under the operators.

5.1 Divergence

We have first the following easy lemma.

Lemma 5.1. *Let G be a complete graph K_n . Then every vertex belongs to $\binom{n-1}{2}$ tricliques.*

Proof. Take a vertex $v \in K_n$. It is clear that v belongs to $\binom{n-1}{2}$ different triangles and since each different pair of vertices in K_n forming a triangle with v is adjacent to each other, each triangle is therefore a triclique. \square

This result can be easily generalised to $k > 3$ as follows.

Lemma 5.2. *Let G be a complete graph K_n . Then every vertex belongs to $\binom{n-1}{k-1}$ k -cliques.*

The main result of this section is the following.

Theorem 5.3. *Let G be a graph. If $K_5 \subseteq G$, then G diverges under the KT operator.*

Proof. Let $v_0 \in G$ such that $v_0 \in K_5$. Clearly v_0 belong to at least $6 = \binom{4}{2}$ different triangles of the K_5 and therefore, by Lemma 5.1, six different triclques. Since all triclques intersect in v_0 , we have that $K_6 \subseteq KT(G)$. Now take $v_1 \in K_6 \subseteq KT(G)$. Using same arguments as before we have that v_1 belongs to at least $10 = \binom{5}{2}$ different triclques so $K_{10} \subseteq KT^2(G)$. In general, let $a_0 = 5$ and $a_{n+1} = \binom{a_n-1}{2}$, then we have $K_{a_n} \subseteq KT^n(G)$. Finally, since $\lim_{n \rightarrow \infty} a_n = \infty$ we can deduce $\lim_{n \rightarrow \infty} |V(KT^n(G))| = \infty$, that is, G is divergent under the KT operator. \square

Theorem 5.3 can be generalised to higher k as follows.

Theorem 5.4. *Let G be a graph. If $K_{k+2} \subseteq G$, then G diverges under the KC operator.*

As a corollary of Theorem 5.3 we have the following result that gives a polynomial upper bound on the number of triclques a graph should have to be convergent.

Corollary 5.5. *Let G be a graph. If G has more than $\frac{4}{3}n$ triclques, then it diverges under the KT operator.*

Proof. If G has more than $\frac{4}{3}n$ triclques, then there must exist a vertex $v \in G$ such that v belongs to at least five different triclques. Otherwise, every vertex could belong to at most four triclques and since each triclque has at least three vertices, G would have at most $\frac{4}{3}n$ triclques contradicting the hypothesis. Finally, since v belongs to five triclques, $K_5 \subseteq KT(G)$; thus by Theorem 5.3, G is divergent under the KT operator. \square

The converse of Corollary 5.5 is false. Indeed, observe that the wheel graph W_5 (obtained by adding a universal vertex to a cycle of length 4) is divergent since there are only five intersecting triclques and $5 \leq \frac{4}{3} \cdot 6 = 8$.

Corollary 5.5 can be generalised to bigger k as follows:

Corollary 5.6. *Let G be a graph. If G has more than $\frac{k+1}{k}n$ triclques, then it diverges under the KC operator.*

5.1.1 Divergent classes

In this subsection we discuss several divergent classes. First of all, it is clear that all complete graphs K_n are divergent under the KT operator for $n \geq 5$ and under the KC operator for $n \geq k + 2$ (Theorems 5.3 and 5.4).

Consider now the three classes of graphs presented in Section 3.2.

- The extended $CP(j)$ graph (with $k - 2 \geq 1$ universal vertices) is divergent under the KC operator for $j \geq \log_2(k + 4)$ since all k -cliques (there are $2^j - 2$ of them) intersect in the universal vertices; therefore we need $2^j - 2 \geq k + 2$ (Theorem 5.4). Otherwise, the graph is convergent. In particular, for the KT operator, this class is divergent for $j \geq 3$ and convergent otherwise.
- Similarly, the extended $H(t, \ell)$ graph (with $k - 2 \geq 1$ universal vertices) is divergent under the KC operator for $\ell \geq \log_2(k + 2)$ as it has 2^ℓ intersecting k -cliques and convergent otherwise.

- For the class $K_{j,j,j} - jK_3$, since there are $3^j - 3 \times 2^j + 3$ triclques, one can check that for $j \geq 4$, each vertex belongs to at least six triclques; therefore it diverges under the KT operator. For $j \leq 3$, it is convergent. Note that this can be generalised to k -cliques, suggesting that this family is, in general, divergent under the KC operator. However, the corresponding counting argument becomes technically intricate.

For any $n \geq 3$, the *wheel graph* on $n + 1$ vertices, denoted by W_n , is the graph obtained by connecting a single universal vertex u to all vertices of a cycle v_0, \dots, v_{n-1}, v_0 .

We provide the following characterisation:

Theorem 5.7. *For $n \geq 3$, under the KT operator, the wheel W_n*

- *converges if $n \in \{3, 4\}$; and*
- *diverges otherwise.*

Proof. Observe that $W_3 = K_4$ and has four intersecting triangles. Therefore since $KT(W_3) = KT(K_4) = K_4$, W_3 is convergent. Regarding W_4 , we have $KT(W_4) = K_1$ since W_4 is itself a triclque. Now, for $n \geq 5$, every triclque of W_n is of the form $\{u, v_i, v_{i+1}, v_{i+2}\}$ (where operations over the subscripts are modulo n). Note that $v_i v_{i+2}$ cannot be an edge, since $n \geq 5$. Thus, W_n has n triclques all containing u . Thus, $KT(W_n)$ contains K_5 , and the second item of the statement follows from Theorem 5.3. \square

We remark that for $n \geq 3$ and $k \geq 4$, W_n is convergent under the KC operator as it contains no k -cliques (unless W_3 and 4-cliques are considered in which case only one exists and clearly converges as well).

Below, we deal with *triangular grids* $T_{n,m}$ under the following terminology. Assuming we have $n \geq 2$ rows and $m \geq 2$ columns, for every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, the vertex on the i th row and j th column is denoted by $u_{i,j}$. Also, $u_{i,j}$ is adjacent to each one of the following (existing) vertices: $u_{i-1,j}$, $u_{i-1,j+1}$, $u_{i,j-1}$, $u_{i,j+1}$, $u_{i+1,j-1}$, and $u_{i+1,j}$.

We first need an auxiliary result, dealing with cubes of paths. Recall that if P_n denotes the path of order n , then P_n^3 denotes the *cube* of P_n , being obtained from P_n by adding an edge joining any two vertices that are (originally) at distance at most 3.

Lemma 5.8. *For $n \geq 3$, under the KT operator, the cubed path P_n^3*

- *converges if $n \in \{3, 4, 5\}$; and*
- *diverges otherwise.*

Proof. For $n \in \{3, 4, 5\}$, one can easily verify that P_n^3 is convergent. Now consider $n \geq 6$ and let us denote by v_1, \dots, v_n the consecutive vertices of P_n . We have (at least) the following five triclques (all inducing diamonds) $T_1 = \{v_1, v_3, v_4, v_5\}$, $T_2 = \{v_1, v_2, v_3, v_5\}$, $T_3 = \{v_1, v_2, v_4, v_5\}$, $T_4 = \{v_1, v_3, v_4, v_6\}$ and $T_5 = \{v_2, v_3, v_4, v_6\}$. Finally, observe that any two of T_1, \dots, T_5 have non-empty intersection. Then $KT(P_n^3)$ contains K_5 , and the result follows from Theorem 5.3. \square

Theorem 5.9. *For $n, m \geq 2$, $n \leq m$, under the KT operator, the triangular grid $T_{n,m}$*

- diverges if $n, m \geq 3$, or $n = 2$ and $m \geq 5$;
- converges to K_4 if $n = 2$ and $m = 4$;
- converges to K_1 if $n = 2$ and $m \in \{2, 3\}$.

Proof. Set $G = T_{n,m}$, where $n \leq m$. If $n, m \geq 3$, then note that the subgraph induced by $N[u_{2,2}]$ is the wheel W_6 , so G diverges under KT by Observation 3.3 and Theorem 5.7. If $n = 2$, then note that every triclique of G is of the form either $\{u_{1,i}, u_{1,i+1}, u_{2,i}, u_{2,i+1}\}$ or $\{u_{1,i}, u_{1,i+1}, u_{2,i-1}, u_{2,i}\}$ (inducing a diamond in both cases). In other words, every edge e that it not on the border of G is the support of a diamond. In particular, every triclique of the form $\{u_{1,i}, u_{1,i+1}, u_{2,i}, u_{2,i+1}\}$ intersects up to three triclques “forwards” ($\{u_{1,i+1}, u_{1,i+2}, u_{2,i}, u_{2,i+1}\}$, $\{u_{1,i+2}, u_{1,i+3}, u_{2,i+2}, u_{2,i+3}\}$, and $\{u_{1,i+2}, u_{1,i+3}, u_{2,i+1}, u_{2,i+2}\}$), and similarly for every triclique of the form $\{u_{1,i}, u_{1,i+1}, u_{2,i-1}, u_{2,i}\}$. Likewise, every triclique intersects up to three triclques “backwards”. From these arguments, remark that $KT(G) = P_{2m-3}^3$. Thus, if $m \geq 5$, then the result follows from Lemma 5.8.

It remains to consider when $n = 2$ and $m \in \{2, 3, 4\}$. One can easily check by arguments above that for $m = 2$, we have $KT(G) = K_1$. For $m = 3$, $KT(G) = K_3$ and then $KT(K_3) = K_1$. For $m = 4$, we have $KT(G) = P_5^3$, then $KT(P_5^3) = K_4$ and finally $KT(K_4) = K_4$. Thus the convergence for all these cases holds. \square

5.2 Convergence

Here we discuss some results about convergence concerning the operators KT and KC . We have the following first easy result that differs from the KB operator.

Theorem 5.10. *For every $i \geq 1$, there is a graph converging in i steps under the KT operator.*

Proof. We prove this by induction on i . As a base step, note that for $G = K_1$, we have that $KT(G)$ is the empty graph, so G converges in one step. Now, for a proof of the general case, let G' be a graph converging under KT in $i - 1$ steps. Then just consider G , a graph such that $KT(G) = G'$. Such a G exists by Theorem 4.1. Then, clearly, G converges under KT in i steps. \square

Quite similarly, but invoking Theorem 4.2 instead of Theorem 4.1, we can easily prove:

Theorem 5.11. *For every $i \geq 1$, there is a graph converging in i steps under the KC operator.*

We finish off with obvious observations.

Observation 5.12. *If G is K_3 -free, then $KT(G)$ is the empty graph. In particular, G converges under the KT operator.*

In particular, this means trees, and more generally bipartite graphs, form a converging class. Observation 5.12 generalises naturally to:

Observation 5.13. *If G is K_k -free, then $KC(G)$ is the empty graph. In particular, G converges under the KC operator.*

6 Final comparison and open problems

In this final section we compare and summarize all known results about clique, biclique and k -clique ($k \geq 3$) graphs and operators. Moreover, we propose several conjectures, open problems and research directions.

	K	KB	KT, KC
Characterisation	✓ [33]	✓ [16]	Trivial (all are)
Recognition	NP-complete [1]	Open (suspected NP-complete)	Trivial (all are)
Convergence	✓ [8]	✓ [13]	✓
Divergence	✓ [30]	✓ [13]	✓
Periodic	✓ [8]	✗ [13]	Open (✗ suspected)
Behavioural characterisation	Open	✓ [13] (linear time [12])	Open (conjecture exists)
Iterations to convergence	Any $i \in \mathbb{N}$ [8]	At most 3 [13]	Any $i \in \mathbb{N}$
Number of self graphs	Infinite [8]	One [13]	Open (one conjectured)
Characterisation of self graphs	Open	✓ [13]	Open (conjecture exists)

Table 1: Combined overview of K , KB , and KT and KC under various structural and dynamic properties.

The first part of Table 1 summarizes results about the characterisation and recognition problems for clique, biclique and k -clique graphs ($k \geq 3$).

Note that the clique graph recognition problem is NP-complete as it is also suspected (but still open) for biclique graphs (some results given in [28] might help). However, for triclique graphs and k -clique graphs ($k > 3$) the problem becomes trivial, since every graph is a k -clique graph ($k \geq 3$). Furthermore, given a graph G , one can (easily) obtain a graph H such that $G = KT(H)$ or $G = KC(H)$.

In the second part of Table 1, we summarize results about the iterated K , KB and KT and KC operators. In particular, we compare two different topics:

- Possible behaviours: Existence of graphs exhibiting convergent, divergent or periodic behaviours.
- Behavioural decidability: Whether the general behaviour of a given graph under the operator is decidable.

This table shows that the clique operator is the only one (as it is suspected not possible for KT and KC) that admits periodic behaviour. About the decidability of the behaviour in

the general case, the biclique operator is the only one that is characterised and even a linear-time algorithm exists. For the clique operator, this problem is suspected to be undecidable. Finally, for the k -clique operator, we propose the following conjectures that might help to solve the general problem.

Conjecture 6.1. *Let G be a graph. If $|V(G)| < |V(KT(G))|$, then we have $K_5 \subseteq KT^2(G)$.*

Conjecture 6.2. *Let G be a graph. If $|V(G)| < |V(KC(G))|$, then we have $K_{k+2} \subseteq KC^2(G)$.*

We remark that if true, those conjectures imply that graphs verifying the hypothesis are actually divergent. Otherwise $|V(G)| \geq |V(KC(G))|$ and therefore we can obtain $KC(G)$ in polynomial time (as one can adapt the algorithm given in [6, 7] to obtain triclques/ k -cliques in polynomial time delay) and repeat the process from $KC(G)$, ending up with a polynomial-time algorithm to decide the behaviour.

In the last part of Table 1, we compare results about convergence and self- F graphs (recall that a graph G is a *self- F graph* under the operator F if $G = F(G)$). We can observe that only self-biclique graphs have been fully characterised and there is only one of them: K_3 . The problem still remains open for self-clique graphs. For self- KT and self- KC graphs, the problem is also open, however, we present here the following conjectures that fully characterise them.

Conjecture 6.3. *Let G be a connected graph. $G = KT(G)$ if and only if $G = K_4$.*

Conjecture 6.4. *Let G be a connected graph. $G = KC(G)$ if and only if $G = K_{k+1}$.*

Among the questions we considered, we believe the convergence/divergence behaviours of the iterated operators are of particular interest. Recall that we exhibited such related results in Section 5. We think we are not far from a full understanding of these questions. Solving the proposed conjectures will probably help in the matter.

In this vein, it would also be interesting to investigate more sufficient and necessary conditions for graphs to converge/diverge upon some iterated operator. For instance, regarding dense graphs, Theorems 5.3 and 5.4, combined with Turán's Theorem [35], imply that, for any $k \geq 3$, graphs with sufficiently many edges diverge under the KC operator. It would be interesting to investigate how to build upon this sufficient condition for divergence.

For more specific classes of graphs, we wonder about k -trees. Recall that, for any $k \geq 2$, a k -tree is obtained from K_{k+1} by repeatedly picking a clique K of size k and adding a new vertex adjacent only to all vertices of K (thereby forming a new copy of K_{k+1}). By construction, k -trees contain relatively many cliques of size $k+1$, and one can thus ask about a characterisation of their divergence/convergence under the iterated $(k+1)$ -clique operator. It can be observed that the $(k+1)$ -cliques of a k -tree G are easy to understand: if $\{v_1, \dots, v_k\}$ is a clique of size k of G , then, for every $i \in \{1, \dots, k\}$, vertices in $\{v_1, \dots, v_k\} \setminus \{v_i\}$ and all vertices attached (during the construction of G) to $\{v_1, \dots, v_k\} \setminus \{v_i\}$ form a $(k+1)$ -clique. Thus, $\{v_1, \dots, v_k\}$ leads to up to k pairwise intersecting $(k+1)$ -cliques. From these observations, one can easily build arbitrarily large k -cliques that are either divergent or convergent under the iterated $(k+1)$ -clique operator. In particular, any k -tree G obtained

from an initial $(k+1)$ -clique $\{v_1, \dots, v_{k+1}\}$ by just repeatedly attaching vertices to k vertices of $\{v_1, \dots, v_{k+1}\}$ verifies $KT(G) = K_3$ or $KC(G) = K_{k+1}$, and thus G converges (essentially, note that, despite G has arbitrarily many vertices, most of these are actually equivalent $(k+1)$ -clique-wise, recall Propositions 3.4 and 3.5). On the other hand, it is not too hard to see that, from an initial $(k+1)$ -clique, one can easily attach vertices to obtain a k -tree G such that $KT(G)$ or $KC(G)$ contains K_{k+2} , and thus G diverges by Theorems 5.3 and 5.4. It might be interesting to investigate further about a full characterisation, and similarly about other graph classes.

To finish, we could define the *edge- k -clique graph*, denoted by $KC_e(G)$, as the edge-intersection graph of the family of all k -cliques of G . We recall that edge-intersection means that $KC_e(G)$ has a vertex for each k -clique of G and two vertices are adjacent in $KC_e(G)$ if their corresponding k -cliques in G share an edge (and not just a vertex as in $KC(G)$).

For $k = 2$, this graph is called the *edge-biclique graph* and denoted $KB_e(G)$. It was defined in [17] and studied in [29]. The characterisation and recognition problems of edge-biclique graphs are still open. About the *iterated edge-biclique graph*, it was defined and studied in [29] where some non-trivial sufficient conditions for a graph to be convergent or divergent under the KB_e operator (based on induced substructures) were given. In particular, induced (n, m) -*necklace* graphs are used (these graphs have $n+m$ vertices and consist of an induced cycle C_n and a complete graph K_m , such that for an edge $e \in C_n$, every vertex of the K_m is only adjacent to both endpoints of e). Several conjectures that might characterise the behaviour of a graph under the KB_e operator are proposed. No periodic graphs have been found and it is therefore suspected that they do not exist (as in biclique graphs). Moreover, for edge-biclique graphs, we have that iterations for convergence are any $i \in \mathbb{N}$, the number of self edge-biclique graphs is infinite and there was given a conjecture that might fully characterise self edge-biclique graphs.

It might be interesting to study edge- k -clique graphs in a general way as in this article. Furthermore, see how known results from edge-biclique graphs (that were just presented in the previous paragraph) might be adapted for other values of k . For example, for $k = 1$, edge-clique graphs (K_e), $k = 3$, edge-triclique graphs (KT_e) and generalise to $k > 3$ edge- k -clique graphs (KC_e).

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