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Low-Gain Integral Control for Average Set-Point Tracking (with proofs) ^{*}

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Abstract: Achieving desired set-point tracking is a fundamental challenge in control theory. Within this context, low-gain integral control methods stand out as an attractive design tool in many applications thanks to their effective and straightforward implementation. We present here a generalization of existing low-gain integral control techniques, which allows for the plant to exhibit complex behavior. For instance, we allow the plant to exhibit limit cycles. In these settings, we guarantee average set-point tracking, in place of “standard” set-point tracking. Our result is unifying in the sense that we recover known results from the literature, and we provide several generalizations. We showcase our theoretical results through a numerical example, where we illustrate how to control the average output of a nonlinear oscillator.

Keywords: nonlinear systems, set-point tracking, integral control, low-gain, singular perturbations, average tracking, average methods.

1. INTRODUCTION

The problem of *set-point tracking*, i.e., of controlling a plant such that its output tracks a constant reference, is ubiquitous in engineering applications. It had been clear since the 1920s that an integral action is needed to guarantee robust set-point tracking (Bennett (2000)). Many techniques have been proposed for integral control of nonlinear systems, see, e.g., Mahmoud and Khalil (1996); Astolfi and Praly (2017); Giaccagli et al. (2022) (and the references therein), yet simple designs, such as low-gain integral control, are particularly attractive in practice.

Low-gain integral control consists in multiplying the integral of the tracking error by a “sufficiently small” constant gain and then feeding it as input to the plant. This control method has been investigated for linear time-invariant (LTI) systems in Davison (1976); Morari (1985) and for nonlinear systems in Desoer and Lin (1985); Simpson-Porco (2020). Thanks to its straightforward design and effectiveness, low-gain integral control is a successful tool in many applications, see, e.g., Khayat et al. (2020); Lorenzetti et al. (2022); Simpson-Porco and Monshizadeh (2022); Simpson-Porco (2021), for power electronics.

The design methods mentioned above rely on singular perturbations theory, see, e.g., Kokotović et al. (1999): By choosing the integrator gain “sufficiently small”, a time-scale separation is induced in the closed-loop system. In this way, the stability properties of the closed-loop system can be inferred from those of two simpler models: (i) A *boundary-layer (fast) system*, coinciding with the plant in open-loop, fed by a constant input; (ii) A *reduced (slow)*

model, consisting of the original closed-loop with the plant replaced by its steady-state input-output map.

Interestingly, most (if not all) low-gain integral control designs assume that the plant possesses an exponentially stable equilibrium point for any constant input, see, e.g., Desoer and Lin (1985); Lorenzetti and Weiss (2022a,b); Simpson-Porco (2020). However, more complex behaviors can emerge in practice. For instance, solutions may converge to general attractors (other than equilibrium points) and convergence may not be exponential. These scenarios fall outside the scope of existing methods, highlighting the need for novel design methods for low-gain integral control.

In this paper, we propose a low-gain integral control design that allows for the plant to exhibit complex behavior. For this, we rely on the singular perturbation framework proposed in Teel et al. (2003). We show that, by taking a sufficiently small gain, practical average set-point tracking can be achieved in a semi-global practical sense, i.e., the average of the tracking error can be driven to a (tunable) neighborhood of zero when initial conditions are chosen in certain compact sets. Our theory generalizes existing results and recovers “standard” set-point tracking as a particular case. Possible applications of our theory include power systems and power electronics, see, e.g., Colombino et al. (2019); Lehman and Bass (1996); Krein et al. (1990); Lu et al. (2019); Lorenzetti et al. (2025b), where limit cycles are ubiquitous and regulation is often performed with respect to averaged signals.

This paper is an extended version of Lorenzetti et al. (2025a) where, for space reasons, proofs were omitted.

Notation. We denote by \mathbb{N} the set of natural numbers, excluding 0, and by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{> 0}$) the set of non-negative (resp. positive) real numbers. For any $n \in \mathbb{N}$, we denote by $\|\cdot\|$ the Euclidean norm. Given two Euclidean

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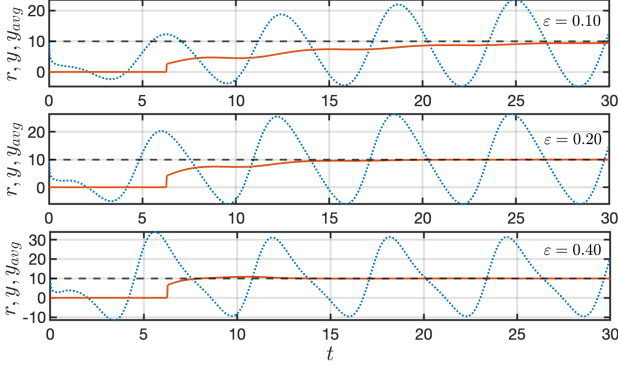


Fig. 1. The output y (in dotted blue), its average y_{avg} (in solid red), and the reference r (in dashed black), for $\varepsilon \in \{0.1, 0.2, 0.4\}$, when \mathbf{P}_{ex} from (1) is in feedback with a low-gain integrator, as in \mathbf{CL}_{ex} from (2).

spaces U and V , we denote by $C(U; V)$ (resp. $C^1(U; V)$) the set of continuous (resp. continuously differentiable) functions from U to V , and by $\text{Lip}(U; V)$ the set of locally Lipschitz functions from U to V . A function $\gamma \in C(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ is of class- \mathcal{K} if it is strictly increasing and $\gamma(0) = 0$. A function $\beta \in C(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ is of class- \mathcal{KL} if $\beta(\cdot, s) \in \mathcal{K}$ for all $s \in \mathbb{R}_{\geq 0}$ and $\beta(r, \cdot)$ is decreasing, with $\lim_{s \rightarrow \infty} \beta(r, s) = 0$. Given a closed set $U \subset \mathbb{R}^p$, for any $z \in \mathbb{R}^p$ we denote $|z|_U := \min_{u \in U} |z - u|$, i.e., $|z|_U$ is the distance of z from U . Given a set $U = \{(v, w) \mid v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w}\}$, we denote by $U|_v \subseteq \mathbb{R}^{n_v}$ (resp. $U|_w \subseteq \mathbb{R}^{n_w}$) the projection of U into \mathbb{R}^{n_v} (resp. \mathbb{R}^{n_w}). We denote by \mathcal{B}_ρ (resp. \mathcal{B}_ρ°), the closed (resp. open) unit ball of radius $\rho > 0$ centered at the origin of the considered set, which will be clear from the context. Given $v \in \mathbb{R}^{n_v}$, $w \in \mathbb{R}^{n_w}$, we write (v, w) in place of $(v^\top, w^\top)^\top$.

2. MOTIVATING EXAMPLE

We start by presenting a motivating example, whose goal is twofold. On the one hand, it highlights the current gaps in the low-gain control literature. On the other hand, it hints at meaningful applications in power electronics and power systems, by reproducing the dynamics found, e.g., in grid-forming inverters, where trajectories often converge to limit cycles parametrized by the control input. These systems can be found, e.g., in Colombino et al. (2019); Lu et al. (2019); Lorenzetti et al. (2022, 2025b).

Consider a nonlinear system \mathbf{P}_{ex} with input $u \in \mathbb{R}$, state $x := (x_1, x_2, x_3) \in \mathbb{R}^3$, and output $y \in \mathbb{R}$, described by

$$\mathbf{P}_{ex} : \begin{cases} \dot{x}_1 = -x_2 + x_1(u^2 - \|(x_1, x_2)\|^2), \\ \dot{x}_2 = +x_1 + x_2(u^2 - \|(x_1, x_2)\|^2), \\ \dot{x}_3 = -x_3 + u, \\ y = x_1 + x_2 + x_3. \end{cases} \quad (1)$$

Given a constant reference signal $r \in \mathbb{R}$, we want to *practically* drive the average of the tracking error $e := r - y$ to zero, i.e., for any $\delta > 0$ we want to guarantee that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \left| \int_0^T e(\tau) d\tau \right| \leq \delta.$$

As we will show, this can be achieved using low-gain integral control, by connecting the plant (1) in feedback with the integrator

$$\dot{u} = \varepsilon e,$$

where $\varepsilon > 0$ “sufficiently small”, obtaining the closed-loop

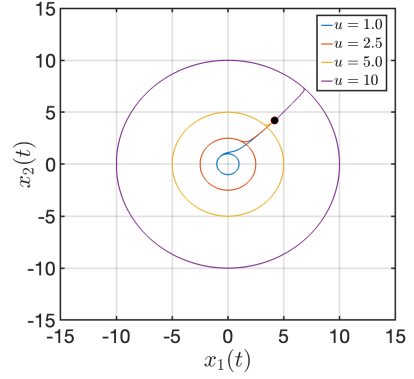


Fig. 2. The states (x_1, x_2) from (1) when \mathbf{P}_{ex} is supplied with a constant input $u \in \{1, 2.5, 5, 10\}$ and $(x_1(0), x_2(0)) = (4.2, 4.2)$, marked in black above.

$$\mathbf{CL}_{ex} : \begin{cases} \dot{x}_1 = -x_2 + x_1(u^2 - \|(x_1, x_2)\|^2), \\ \dot{x}_2 = +x_1 + x_2(u^2 - \|(x_1, x_2)\|^2), \\ \dot{x}_3 = -x_3 + u, \\ \dot{u} = \varepsilon(r - x_1 - x_2 - x_3). \end{cases} \quad (2)$$

The details are worked out in Section 4. Meanwhile, as a teaser, we show in Fig. 1 the output y of \mathbf{P}_{ex} , when the closed-loop system is formed as in \mathbf{CL}_{ex} , with initial conditions $(x(0), u(0)) = (4.2, 4.2, 1, 0.5)$ and reference $r = 10$. As clear from Fig. 1, the closed-loop system \mathbf{CL}_{ex} solves the average set-point tracking problem, when the integrator gain $\varepsilon > 0$ is “sufficiently small”. For larger values of ε , the system \mathbf{CL}_{ex} approaches instability.

Remark 1. Results on low-gain integral control for nonlinear systems, see, e.g., Desoer and Lin (1985); Simpson-Porco (2020); Lorenzetti and Weiss (2022a,b), aim at solving the set-point tracking problem, i.e., $\lim_{t \rightarrow \infty} e(t) = 0$. For this, a crucial assumption is the following: When supplied by constant inputs $u \in U \subset \mathbb{R}^{n_u}$, the trajectories of the plant (to be controlled) converge to an *equilibrium point*. Clearly, this is not the case for \mathbf{P}_{ex} . Indeed, when \mathbf{P}_{ex} is supplied by constant control inputs $u \in \mathbb{R}$, (x_1, x_2) converge to a limit cycle along the circle of radius $|u|$, centered at the origin (as shown in Fig. 2), while x_3 converges to u . (A rigorous proof of this is given in Section 4.) Thus, this fundamental assumption does not hold and, to the best of our knowledge, no results on low-gain integral control are available for this scenario. Yet, as we will show in the following, low-gain integral control techniques can still be used. However, instead of guaranteeing set-point tracking, we will ensure practical *average* set-point tracking.

3. AVERAGE SET-POINT TRACKING

We introduce here the mathematical framework within which we study the problem of practical average set-point tracking, and we present our main result. The last part of this section contains a comparison of our result with those available in the low-gain integral control literature.

3.1 Problem Setting

Consider a nonlinear system \mathbf{P} with input $u \in \mathbb{R}^{n_u}$, output $y \in \mathbb{R}^{n_y}$, and state $x \in \mathbb{R}^{n_x}$, given by

$$\mathbf{P} : \begin{cases} \dot{x} = f(u, x), \\ y = g(x), \end{cases} \quad (3)$$

where $f \in \text{Lip}(\mathbb{R}^{n_u} \times \mathbb{R}^{n_x}; \mathbb{R}^{n_x})$ and $g \in \text{Lip}(\mathbb{R}^{n_x}; \mathbb{R}^{n_y})$, with $n_x, n_u \in \mathbb{N}$. We want to design a simple controller

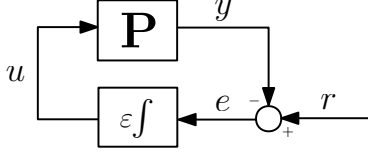


Fig. 3. Closed-loop system formed by the plant \mathbf{P} from (3) in feedback with the integrator \mathbf{I}_ε from (4) with gain $\varepsilon > 0$. The signal r is the reference to be tracked.

such that the average of the plant output y “approximately tracks” a desired constant reference $r \in \mathbb{R}^{n_u}$, and closed-loop stability (in terms of boundedness of the trajectories) is guaranteed. To this aim, we connect \mathbf{P} in feedback with an integrator \mathbf{I}_ε , as shown in Figure 3, described by

$$\mathbf{I}_\varepsilon : \dot{u} = \varepsilon(r - y), \quad (4)$$

where $\varepsilon > 0$ is the gain of the integrator. The resulting closed-loop system \mathbf{CL} is given by

$$\mathbf{CL} : \begin{cases} \dot{r} = 0, \\ \dot{u} = \varepsilon(r - g(x)), \\ \dot{x} = f(u, x). \end{cases} \quad (5)$$

We want to solve the following control problem.

Practical average set-point tracking. For any $\delta > 0$ and a “suitable” set of initial conditions $\mathcal{S} \subset \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x}$, there exists $\varepsilon_\delta^* > 0$ such that the following holds: For any $\varepsilon \in (0, \varepsilon_\delta^*]$, any solution to \mathbf{CL} initialized in \mathcal{S} exists for all positive times and it is bounded¹. Moreover, the corresponding tracking error $e := r - y$ satisfies

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \left| \int_0^T e(\tau) d\tau \right| \leq \delta. \quad (6)$$

Remark 2. In the context of harmonic regulation, Astolfi et al. (2022); Giaccagli et al. (2024) have shown that when the tracking error e converges to a periodic signal, the presence of an integrator guarantees that its average converges to zero. Here, we work under different assumptions, and we do not require that e converges to a periodic signal.

3.2 Plant \mathbf{P} with Constant Inputs

When plant \mathbf{P} from (3) is supplied by constant inputs, its behavior can be equivalently described by

$$\mathbf{P}_f : \begin{cases} \dot{u}_f = 0, \\ \dot{x}_f = f(u_f, x_f), \end{cases} \quad (7)$$

where $(u_f, x_f) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_x}$.

Remark 3. (Boundary-layer system (7)). In the language of singular perturbation theory, \mathbf{P}_f from (7) is related to the *boundary-layer system* associated to \mathbf{CL} , see, e.g., (Kokotović et al., 1999, Chapter 7), (Khalil, 2002, Chapter 11), (Teel et al., 2003, Section III-B). Indeed, \mathbf{P}_f can be obtained from the closed-loop system \mathbf{CL} from (5), by setting $\varepsilon = 0$ (and ignoring \dot{r}). That is, \mathbf{P}_f can be seen as the *open-loop system* obtained by breaking the feedback-loop ($\varepsilon = 0$) with the integrator in Fig. 3, and by supplying the plant \mathbf{P} from (3) with a constant control input u_f . \square

Let $z_f := (u_f, x_f) \in \mathbb{R}^{n_f}$, where $n_f := n_u + n_x$. We make the following assumption on \mathbf{P}_f from (7), which amounts to the existence of an asymptotically stable compact attractor $\mathcal{A}_f \subset \mathbb{R}^{n_f}$, possibly parametrized by u_f .

¹ By *bounded*, we mean that we guarantee an upper bound on the distance between the solutions to \mathbf{CL} and certain compact sets.

Assumption 1. (Stability of \mathbf{P}_f). There exist a set $\mathcal{Z}_f \subset \mathbb{R}^{n_f}$, a compact set $\mathcal{A}_f \subset \mathcal{Z}_f$, and a function $\beta_f \in \mathcal{KL}$ such that the following holds: Any solution z_f to \mathbf{P}_f from (7) initialized in \mathcal{Z}_f exists for all positive times and satisfies

$$|z_f(t)|_{\mathcal{A}_f} \leq \beta_f(|z_f(0)|_{\mathcal{A}_f}, t) \quad \forall t \geq 0, \quad (8)$$

i.e., \mathcal{A}_f is an asymptotically stable set in \mathcal{Z}_f for \mathbf{P}_f . \square

3.3 The Average Closed-Loop System

We introduce here the notion of average closed-loop system, which captures the average behavior of \mathbf{CL} from (5). As we will see, this can be determined using only the knowledge of the trajectories of \mathbf{P}_f from (7).

Recall $\mathcal{A}_f \subset \mathcal{Z}_f \subset \mathbb{R}^{n_f}$ from Assumption 1. We denote by $\mathcal{S}_f(z_f(0))$ the solutions of \mathbf{P}_f from (7), starting from $z_f(0)$.

Assumption 2. (Average input-output map). There exists $G_{av} \in \text{Lip}(\mathbb{R}^{n_u}; \mathbb{R}^{n_u})$ such that the following holds:

For each $\rho > 0$ there exists $T^* > 0$ such that for any $T \geq T^*$, $z_f(0) \in \mathcal{A}_f$, $(u_f, x_f) \in \mathcal{S}_f(z_f(0))$,

we have that

$$\left| \int_0^T [G_{av}(u_f(\tau)) - g(x_f(\tau))] d\tau \right| \leq T\rho, \quad (9)$$

i.e., G_{av} is the *average input-output map* of \mathbf{P} in \mathcal{A}_f . \square

A few comments on Assumption 2 are in place. First, although we write $u_f(\tau)$ in (9), $u_f(\tau) = u_f(0)$ for all $\tau \geq 0$ from (7). Second, we highlight that (9) is computed only along the trajectories of \mathbf{P}_f that are starting from (and, thus, remaining in) the set \mathcal{A}_f . Thanks to the stability of \mathcal{A}_f (see Assumption 1), this is enough to characterize the average “steady-state” input-output map of \mathbf{P} from (3).

Thanks to Assumption 2, we can introduce the *average closed-loop system* associated to (5), described by

$$\mathbf{AVG} : \begin{cases} \dot{r}_s = 0, \\ \dot{u}_s = \varepsilon(r_s - G_{av}(u_s)). \end{cases} \quad (10)$$

where $(r_s, u_s) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$.

Remark 4. (Average model (10)). In the language of singular perturbation theory, \mathbf{AVG} from (10) is the *average (reduced) model* associated to the closed-loop system \mathbf{CL} from (5), see, e.g., (Teel et al., 2003, Section III-C). Indeed, \mathbf{AVG} is obtained by averaging (5) along the trajectories of \mathbf{P}_f from (7). According to “classical” singular perturbation theory, see, e.g., (Kokotović et al., 1999, Chapter 7), (Khalil, 2002, Chapter 11), the reduced model associated to \mathbf{CL} is defined only when \mathbf{P}_f admits equilibrium points. This is the theory adopted, e.g., in Desoer and Lin (1985); Simpson-Porco (2020); Lorenzetti and Weiss (2022a,b), where, for this reason, it is assumed that \mathbf{P} from (3) has a well-defined steady-state input-output map, obtained in correspondence of the *equilibrium points* of (7). Here, instead, we are using a more general singular perturbation framework, presented in Teel et al. (2003), which allows us to consider solutions to \mathbf{P}_f that do not necessarily converge to equilibrium points. \square

Let $z_s := (r_s, u_s) \in \mathbb{R}^{n_s}$, where $n_s := 2n_u$. We make the following assumption on \mathbf{AVG} from (10), which mirrors Assumption 1. In addition to stability, we require that the trajectories of \mathbf{AVG} guarantee set-point tracking, i.e.,

$$\lim_{t \rightarrow \infty} (r_s - G_{av}(u_s(t))) = 0. \quad (11)$$

Assumption 3. (Stability of AVG). There exist a set $\mathcal{Z}_s \subset \mathbb{R}^{n_s}$ and a function $\beta_s \in \mathcal{KL}$ such that the following holds: For all $\varepsilon > 0$, any solution z_s to **AVG** from (10) initialized in \mathcal{Z}_s exists for all positive times and satisfies

$$|z_s(t)|_{\mathcal{A}_s} \leq \beta_s(|z_s(0)|_{\mathcal{A}_s}, \varepsilon t) \quad \forall t \geq 0, \quad (12)$$

with $\mathcal{A}_s := \{(r_s, u_s) \in \mathcal{Z}_s \mid G_{av}(u_s) = r_s\}$ compact. \square

We remark that, since $\dot{r}_s = 0$ in (10), it follows that

$$\mathcal{Z}_s|_{r_s} = \mathcal{A}_s|_{r_s} =: \mathcal{R},$$

where, by Assumption 3, $\mathcal{R} \subset \mathbb{R}^{n_u}$ is a compact set. The set \mathcal{R} can be thought of as a set of *feasible references*, i.e., it contains the (constant) references for which the average closed-loop system **AVG** from (10) solves the set-point tracking problem (11) for any $u_s(0) \in \mathcal{Z}_s|_{u_s}$.

3.4 Main Result

We present here our main result: Under Assumptions 1–3 and additional “compatibility conditions” (see Assumption 4), the solutions of the closed-loop system (5) inherit the stability properties assumed, respectively, for **P_f** from (7) (see Assumption 1) and for **AVG** from (10) (see Assumption 3), in a “practical” sense. Moreover, the tracking error e satisfies (6), i.e., the practical average set-point tracking problem introduced in Section 2 is solved.

Assumption 4. (Compatibility conditions). There exist compact sets $\mathcal{K}_f \subseteq \mathcal{Z}_f$ and $\mathcal{K}_s \subseteq \mathcal{Z}_s$, and $\alpha > 0$ such that

$$\{(r, u) \in \mathcal{Z}_s \mid |(r, u)|_{\mathcal{A}_s} \leq \alpha\} \subseteq \mathcal{K}_s, \quad (13)$$

$$\{(r, u, x) \in \mathbb{R}^{n_u} \times \mathcal{Z}_f \mid |(u, x)|_{\mathcal{A}_f} \leq \alpha, \\ |(r, u)|_{\mathcal{A}_s} \leq \beta_s(c_{s,o}, 0) + \alpha\} \subseteq \mathbb{R}^{n_u} \times \mathcal{K}_f, \quad (14)$$

where $c_{s,o} := \max_{(r,u) \in \mathcal{K}_s} |(r, u)|_{\mathcal{A}_s}$. \square

Intuitively, Assumption 4 requires that there exists $\alpha > 0$ such that $(\mathcal{A}_s + \mathcal{B}_\alpha)|_{\mathcal{Z}_s} \subseteq \mathcal{K}_s$, $(\mathcal{A}_f + \mathcal{B}_\alpha)|_{\mathcal{Z}_f} \subseteq \mathcal{K}_f$, and $(\mathcal{A}_s + \mathcal{B}_\sigma)|_{u_s} \subseteq \mathcal{K}_f|_{u_f}$, with $\sigma := \beta_s(c_{s,o}, 0) + \alpha$. This assumption yields the following chain of *strict* inclusions:

$$\mathcal{A}_s|_{u_s} \subset \mathcal{K}_s|_{u_s} \subset \mathcal{K}_f|_{u_f} = \mathcal{A}_f|_{u_f}. \quad (15)$$

(The last equality comes from $\dot{u}_f = 0$ in (7).) These inclusions guarantee that the “slow” stable attractor \mathcal{A}_s is compatible with the “fast” stable attractor \mathcal{A}_f . This reflects on the choice of initial conditions in Theorem 1 below. Similar assumptions can be found in related works from the literature, as discussed in Section 3.5.

We are now ready to state our main result.

Theorem 1. (Practical average tracking). Under Assumptions 1–4, for any $\delta > 0$ there exists $\varepsilon_\delta^* > 0$ such that the following holds: For any $\varepsilon \in (0, \varepsilon_\delta^*]$, any solution (r, u, x) to **CL** from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, exists for all positive times and satisfies

$$|(r, u(t))|_{\mathcal{A}_s} \leq \beta_s(|(r, u(0))|_{\mathcal{A}_s}, \varepsilon t) + \delta, \quad (16)$$

$$|(u(t), x(t))|_{\mathcal{A}_f} \leq \beta_f(|(u(0), x(0))|_{\mathcal{A}_f}, t) + \delta. \quad (17)$$

for all $t \geq 0$. Moreover, the tracking error satisfies:

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \left| \int_0^T e(\tau) d\tau \right| \leq \delta, \quad (18)$$

i.e., practical average set-point tracking is guaranteed. \square

The proof of Theorem 1 is postponed to Appendix A.

Remark 5. When $\lim_{t \rightarrow \infty} e(t) \in \mathbb{R}^{n_u}$, (18) implies that

$$\lim_{t \rightarrow \infty} |e(t)| \leq \delta, \quad (19)$$

i.e., we recover (practical) set-point tracking. This is the case, e.g., under the assumptions of Desoer and Lin (1985); Simpson-Porco (2020); Lorenzetti and Weiss (2022a,b). \square

3.5 Link with the literature on low-gain integral control

We elaborate here on the link between this work and related results from the low-gain integral control literature.

Assumption 1 in the literature. For LTI systems of the form $\dot{x} = Ax + Bu$ and $y = Cx$, a well-known sufficient condition for low-gain integral control is to assume that the matrix A is Hurwitz, see, e.g., Davison (1976); Morari (1985). In this case, Assumption 1 is satisfied by choosing $\mathcal{Z}_f \subset \mathbb{R}^{n_f}$, with $\mathcal{Z}_f|_{u_f}$ compact, $\beta_f(\sigma_f, t) = \gamma_f \|e^{tA}\| \sigma_f$, for some $\gamma_f \geq 1$, and $\mathcal{A}_f = \{(u_f, x_f) \in \mathcal{Z}_f \mid x_f = -A^{-1}Bu_f\}$. In addition, Assumption 1 allows for A having complex conjugate eigenvalues with zero real part, as long as the eigenvalues on the imaginary axis are *semisimple* (i.e., their algebraic and geometric multiplicity coincide). For nonlinear systems, it is usually assumed that there exists a function $\Xi \in C^1(\mathbb{R}^{n_u}; \mathbb{R}^{n_x})$ mapping constant inputs u to *uniformly exponentially stable equilibrium points* of **P** from (3), see, e.g., Desoer and Lin (1985); Simpson-Porco (2020); Lorenzetti and Weiss (2022a,b). This can be captured by Assumption 1, by choosing a suitable set $\mathcal{Z}_f \subset \mathbb{R}^{n_f}$ (subset of the region of attraction of the equilibria) with $\mathcal{Z}_f|_{u_f}$ compact, $\beta_f(\sigma_f, t) = \gamma_f \sigma_f e^{-\alpha_f t}$, for some $\gamma_f \geq 1$ and $\alpha_f > 0$, and $\mathcal{A}_f = \{(u, x) \in \mathcal{Z}_f \mid x = \Xi(u)\}$.

Assumption 2 in the literature. For LTI systems, since A is typically assumed to be Hurwitz (as mentioned above), $G_{av}(u) = -CA^{-1}Bu$, i.e., G_{av} coincides with the DC-gain matrix of **P**. Additionally, when A contains eigenvalues with zero real part (see the discussion in the previous paragraph), $G_{av}(u)$ can be computed using (9). For nonlinear systems, under the assumptions of, e.g., Desoer and Lin (1985); Simpson-Porco (2020); Lorenzetti and Weiss (2022a,b), $G_{av}(u) = (g \circ \Xi)(u)$, i.e., G_{av} is the *steady-state input-output map* of the plant, obtained by computing the plant output $y = g(x)$ in correspondence of the equilibrium points $x = \Xi(u)$. (The proof follows from (9), since $\mathcal{A}_f = \{(u, x) \in \mathcal{Z}_f \mid x = \Xi(u)\}$ in this case.)

Assumption 3 in the literature. For LTI systems, it is usually assumed that $CA^{-1}B$ (i.e., minus the DC-gain matrix) is Hurwitz, see, e.g., Davison (1976); Morari (1985), which clearly implies Assumption 3. Additionally, when A contains eigenvalues with zero real part, Assumption 3 is satisfied if $-G_{av}$ is Hurwitz. For nonlinear systems, Desoer and Lin (1985); Lorenzetti and Weiss (2022a,b) assume that $G_{av} = g \circ \Xi$ is *strictly monotone*, i.e., there exists $\alpha_s > 0$ such that $\langle g(\Xi(u_1)) - g(\Xi(u_2)), u_1 - u_2 \rangle \geq \alpha_s \|u_1 - u_2\|^2$ for all $u_1, u_2 \in U \subset \mathbb{R}^{n_u}$. It can be shown (e.g., using the Lyapunov function $V(u_s) = 0.5 \|u_s - G_{av}^{-1}(r_s)\|^2$ for (10)) that this implies Assumption 3 with the set $\mathcal{Z}_s \subset \mathbb{R}^{n_s}$ suitably chosen, and $\beta_s(\sigma_s, \varepsilon t) = \gamma_s e^{-\varepsilon t \alpha_s} \sigma_s$, for some $\gamma_s \geq 1$ and $\alpha_s > 0$. In Simpson-Porco (2020), this condition is relaxed: It is assumed that (10) is infinitesimally contracting in the sense of (Sontag, 2010, Definition 1), with rate $\alpha_s > 0$. Again, it can be shown, e.g., using (Sontag, 2010, Theorem 1), that this implies Assumption 3, with \mathcal{Z}_s suitably chosen and β_s as above. We remark that, differently than Desoer and Lin (1985); Lorenzetti and Weiss (2022a,b); Simpson-Porco (2020), given $r_s \in \mathcal{A}_s|_{r_s}$ we allow for multiple $u_s \in \mathbb{R}^{n_s}$ guaranteeing $G_{av}(u_s) = r_s$.

Assumption 4 in the literature. For LTI systems and globally stable nonlinear systems no “compatibility conditions” are needed. However, when *local* stability properties are considered, these are needed to relate the stability of the fast and slow dynamics. This is the case, e.g., in Lorenzetti and Weiss (2022a,b); Simpson-Porco (2020), where, using our notation, it is assumed that $\mathcal{A}_s|_{u_s} \subset \mathcal{A}_f|_{u_f}$. Clearly, this relates to the chain of inclusions (15).

Theorem 1 in the literature. With respect to the results of, e.g., Desoer and Lin (1985); Lorenzetti and Weiss (2022a,b); Simpson-Porco (2020), Theorem 1 provides several generalizations. (i) We guarantee *average* set-point tracking, which naturally generalizes “standard” set-point tracking, see Remark 5. (ii) We characterize the behavior of the closed-loop trajectories, which are guaranteed to satisfy (16)–(17) for all $t \geq 0$. (iii) We identify the sets of initial conditions, i.e., \mathcal{K}_s and \mathcal{K}_f , from which tracking is guaranteed (this is not the case, e.g., in Simpson-Porco (2020)). (iv) We allow for non-exponential convergence. The price to pay for these generalizations is to replace “exact” tracking with “practical” tracking, by including the term $\delta > 0$ in (18),(19). Besides, we require compact sets of initial conditions. We conjecture that, under suitable assumptions (e.g., exponential stability and “suitable” average input-output map G_{av}), we could replace $\delta = 0$ in (16)–(19). This is subject of future work.

4. SOLVING THE MOTIVATING EXAMPLE

In this section, we show that all the assumptions of Theorem 1 hold for the motivating example from Section 2.

Checking Assumption 1: \mathbf{P}_f from (7) for (1) becomes

$$\begin{cases} \dot{u}_f = 0, \\ \dot{x}_{f,1} = -x_{f,2} + x_{f,1}(u_f^2 - \|(x_{f,1}, x_{f,2})\|^2), \\ \dot{x}_{f,2} = +x_{f,1} + x_{f,2}(u_f^2 - \|(x_{f,1}, x_{f,2})\|^2), \\ \dot{x}_{f,3} = -x_{f,3} + u_f. \end{cases} \quad (20)$$

In the following, we show that there exists $\beta_f \in \mathcal{KL}$ such that (20) satisfies Assumption 1 with initial conditions in

$$\mathcal{Z}_f = U_\eta \times (\mathbb{R}^2 \setminus \mathcal{B}_\eta^\circ) \times \mathbb{R} \quad \forall \eta > 0,$$

where $U_\eta \subset \mathbb{R} \setminus \mathcal{B}_\eta^\circ$ compact, and attractor given by

$$\mathcal{A}_f = \{(u_f, x_f) \in \mathcal{Z}_f \mid x_{f,1}^2 + x_{f,2}^2 = u_f^2, x_{f,3} = u_f\}.$$

Let $(r_f, \theta_f) \in \mathbb{R}^2$. Using polar coordinates, we write

$$\begin{cases} x_{f,1} = r_f \cos \theta_f, \\ x_{f,2} = r_f \sin \theta_f. \end{cases} \quad (21)$$

By doing this, we can study the stability of (20) by studying (equivalently) the stability of the system

$$\begin{cases} \dot{u}_f = 0, \\ \dot{\theta}_f = 1, \\ \dot{r}_f = r_f(u_f^2 - r_f^2), \\ \dot{x}_{f,3} = -x_{f,3} + u_f, \end{cases} \quad (22)$$

where $w_f := (u_f, \theta_f, r_f, x_{f,3}) \in U_\eta \times [0; 2\pi) \times (\mathbb{R} \setminus \mathcal{B}_\eta^\circ) \times \mathbb{R} =: \mathcal{W}_f$, which is forward invariant for the flow of (22). We want to study the stability of the compact attractor

$$\mathcal{L}_{u_f} := \{w_f \in \mathcal{W}_f \mid r_f = \pm u_f, x_{f,3} = u_f\}$$

for system (22). Consider the Lyapunov function

$$V(w_f) = \frac{1}{2}(u_f^2 - r_f^2)^2 + \frac{1}{2}(u_f - x_{f,3})^2 \quad \forall w_f \in \mathcal{W}_f. \quad (23)$$

V is defined and differentiable everywhere in \mathcal{W}_f , $V(w_f) = 0$ in \mathcal{W}_f if and only if $w_f \in \mathcal{L}_{u_f}$, and $V(w_f) > 0$ elsewhere. The time-derivative of V along any solution to (22) is

$$\dot{V}(w_f) = -2r_f^2(u_f^2 - r_f^2)^2 - (u_f - x_{f,3})^2. \quad (24)$$

Since the set \mathcal{W}_f is forward invariant for system (22), $r_f^2(t) \geq \eta^2$ for all $t \geq 0$. Hence, using (23) and (24),

$$\dot{V}(w_f) \leq -2 \min\{2\eta^2, 1\}V(w_f) \quad \forall t \geq 0.$$

It follows from, e.g., (Goebel et al., 2012, Theorems 3.18 and 3.40), that there exists a function $\tilde{\beta}_f \in \mathcal{KL}$ such that for any initial conditions $w_f(0) \in \mathcal{W}_f$, the corresponding trajectories of system (22) exist for all $t \geq 0$ and satisfy

$$|w_f(t)|_{\mathcal{L}_{u_f}} \leq \tilde{\beta}_f(|w_f(0)|_{\mathcal{L}_{u_f}}, t) \quad \forall t \geq 0. \quad (25)$$

Since the coordinate change (21) is continuous and bijective on \mathcal{W}_f , (25) implies the existence of a function $\beta_f \in \mathcal{KL}$ such that any solution z_f to (20) initialized in \mathcal{Z}_f (as defined above) exists for all positive times and satisfies

$$|z_f(t)|_{\mathcal{A}_f} \leq \beta_f(|z_f(0)|_{\mathcal{A}_f}, t) \quad \forall t \geq 0, \quad (26)$$

i.e., Assumption 1 is satisfied with \mathcal{Z}_f and \mathcal{A}_f as above. \square

Checking Assumption 2: We show below that

$$G_{av}(u_s) = u_s \quad (27)$$

is an average input-output map for \mathbf{P}_{ex} from (1).

Any solution z_f to (20) initialized in \mathcal{A}_f is given by

$$(u_f, u_f \cos(t + \psi), u_f \sin(t + \psi), u_f),$$

where $\psi = \text{atan}(x_{f,2}(0)/x_{f,1}(0))$. Recall that $g(x) = x_1 + x_2 + x_3$ in (1). Then, using $G_{av}(u_f) = u_f$, (9) becomes

$$\left| \int_0^T u_f [\cos(\tau + \psi) + \sin(\tau + \psi)] d\tau \right| < 2\sqrt{2}\pi \max_{u_f \in U_\eta} |u_f|,$$

for any $T > 0$. Thus, for each $\rho > 0$, we can choose

$$T^* := \frac{2\sqrt{2}\pi}{\rho} \max_{u_f \in U_\eta} |u_f|,$$

so that (9) is indeed satisfied for all $T \geq T^*$. (Note that max is well-defined since the set U_η is compact.) \square

Checking Assumption 3 for \mathbf{P}_{ex} . As shown above, \mathbf{P}_{ex} from (1) satisfies Assumption 2 with $G_{av}(u_s) = u_s$. Thus, the average closed-loop system \mathbf{AVG} associated to (2) is

$$\begin{cases} \dot{r}_s = 0, \\ \dot{u}_s = \varepsilon(r_s - u_s). \end{cases} \quad (28)$$

Assumption 3 is trivially satisfied, with, e.g., initial conditions in $\mathcal{Z}_s = \mathcal{R} \times \mathbb{R}^{n_u}$, where $\mathcal{R} \subset U_\eta$ compact (chosen such that Assumption 4 holds – see below), function $\beta_s(\sigma_s, \varepsilon t) = \gamma_s \sigma_s e^{-\varepsilon t}$, for some $\gamma_s \geq 1$, and attractor

$$\mathcal{A}_s = \{(r_s, u_s) \in \mathcal{Z}_s \mid u_s = r_s\}.$$

Checking Assumption 4 for \mathbf{P}_{ex} . Fix $\eta > 0$ and recall the sets $\mathcal{Z}_f, \mathcal{A}_f, \mathcal{Z}_s, \mathcal{A}_s$ defined above. Define $\mathcal{K}_f \subset \mathcal{Z}_f$ as

$$\mathcal{K}_f := U_\eta \times (\mathcal{B}_{\rho_f + \alpha} \setminus \mathcal{B}_\eta^\circ) \times (U_\eta + \mathcal{B}_\alpha), \quad (29)$$

where $\mathcal{B}_{\rho_f + \alpha} \subset \mathbb{R}^2$ and $\mathcal{B}_\alpha \subset \mathbb{R}$, with $\rho_f := \max_{u_f \in U_\eta} |u_f|$ and $\alpha > 0$. The set \mathcal{K}_f in (29) is compact and satisfies $(\mathcal{A}_f + \mathcal{B}_\alpha)|_{\mathcal{Z}_f} \subseteq \mathcal{K}_f$ for any $\alpha > 0$. Define $\mathcal{K}_s \subset \mathcal{Z}_s$ as

$$\mathcal{K}_s := \mathcal{R} \times (\mathcal{R} + \mathcal{B}_\alpha), \quad (30)$$

where $\mathcal{B}_\alpha \subset \mathbb{R}$, with $\alpha > 0$. The set \mathcal{K}_s in (30) is compact and satisfies $(\mathcal{A}_s + \mathcal{B}_\alpha)|_{\mathcal{Z}_s} \subseteq \mathcal{K}_s$, i.e., (13), for any $\alpha > 0$. Recalling the definition of $c_{s,o}$ from Assumption 4, we have

$$c_{s,o} := \max_{(r,u) \in \mathcal{K}_s} |(r,u)|_{\mathcal{A}_s} = \alpha, \quad (31)$$

by construction. Let $\sigma := \beta_s(c_{s,o}, 0) + \alpha$. Then, from (31),

$$\sigma = \alpha(\gamma_s + 1). \quad (32)$$

Thus, (14) is satisfied as long as the (compact) set $\mathcal{R} \subset U_\eta$ is chosen such that there exists $\alpha > 0$ guaranteeing

$$\mathcal{R} + \mathcal{B}_{\alpha(\gamma_s+1)} \subseteq U_\eta. \quad (33)$$

Intuitively, (33) tells us the following: Given $\gamma_s \geq 1$ (coming from β_s), the larger is U_η , i.e., the set of inputs guaranteeing the stability of (20), the larger will be the admissible set of feasible references $\mathcal{R} := \mathcal{A}_s|_{r_s}$. \square

Applying Theorem 1: As shown above, Assumptions 1–4 are satisfied. Thus, thanks to Theorem 1, we are guaranteed that for any $\delta > 0$ there exists $\varepsilon_\delta^* > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta^*]$ the trajectories of \mathbf{CL}_{ex} from (2) initialized in \mathcal{K}_s and \mathcal{K}_s (defined above) satisfy (16)–(17) and, as shown in Fig. 1 (in Section 2), (18) holds. Interestingly, although we guarantee *practical* tracking, it can be seen from Fig. 1 that the limit (18) holds with $\delta = 0$. We conjecture that this is a consequence of the exponential stability of the attractors $\mathcal{A}_f, \mathcal{A}_s$, combined with the properties of the average (27). Further research is needed to derive sufficient conditions guaranteeing that (18) holds with $\delta = 0$.

5. CONCLUDING REMARKS

We have presented an extension of low-gain integral control methods to plants exhibiting “complex” behavior, guaranteeing practical average set-point tracking. There are plenty of directions for future research: (i) Considering a more involved controller structure, e.g., allowing for a nonlinear gain between the integrator and the plant in Figure 3; (ii) Allowing for time-varying disturbances affecting the plant; (iii) Allowing for the average input-output map (see Assumption 2) to be set-valued, since this may be the case in certain applications. Besides, it would be of great interest to investigate under which conditions we could guarantee “exact” average set-point tracking, in place of “practical” average set-point tracking, i.e., $\delta = 0$.

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Appendix A. PROOF OF THEOREM 1

The proof is divided in two steps, along the lines of (Teel et al., 2003, Theorem 1 and Proposition 2). First, we will show that (Teel et al., 2003, Assumptions 2–4,7,8) are satisfied. This will allow us to use (Teel et al., 2003, Proposition 2) to infer that the trajectories of **CL** from (5) are “close” (in some appropriate sense) to the trajectories of **P_f** from (7) and **AVG** from (10) on arbitrary long compact time intervals. Second, we will use arguments along the lines of (Teel et al., 2003, Theorem 1) to translate the estimates (8) and (12), assumed, respectively, for **P_f** and **AVG**, to the trajectories of **CL** from (5). Thus, deriving (16)–(17). Moreover, we will show that the limit (18) holds, concluding the proof of Theorem 1. To aid the reader, we show the correspondence between our notation and the one used in Teel et al. (2003) in Table A.1.

In the rest of this section, we work under Assumptions 1–4.

Step 1: Closeness of solutions. We prove the following lemmas, which translate, respectively, (Teel et al., 2003, Assumption 5 and Assumption 6) to our settings.

Lemma 1. (Closeness to **P_f**). There exists $T^* > 0$ such that for each $T \geq T^*$ and $\delta > 0$ there exists $\varepsilon_{T,\delta}^* > 0$: For any $\varepsilon \in (0, \varepsilon_{T,\delta}^*)$, any solution (r, u, x) to **CL** from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, there exists a solution (u_f, x_f) to (7), with $(u_f, x_f(0)) \in \mathcal{Z}_f$, so that

$$|||(u(t), x(t))|_{\mathcal{A}_f} - |(u_f, x_f(t))|_{\mathcal{A}_f}| \leq \delta \quad \forall t \in [0, T]. \quad (\text{A.1})$$

Moreover, the following holds:

$$(u(t), x(t)) \in \mathcal{K}_f \quad \forall t \in [T^*, T]. \quad (\text{A.2})$$

□

Lemma 2. (Closeness to **AVG**). There exists $T^* > 0$ such that for each $T \geq T^*$ and $\delta > 0$ there exists $\varepsilon_{T,\delta}^* > 0$: For any $\varepsilon \in (0, \varepsilon_{T,\delta}^*)$, any solution (r, u, x) to **CL** from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, there exists a solution (r_s, u_s) to (10), with $(r_s, u_s(0)) \in \mathcal{Z}_s$, so that

$$|||(r, u(t))|_{\mathcal{A}_s} - |(r_s, u_s(t))|_{\mathcal{A}_s}| \leq \delta \quad \forall t \in [0, T/\varepsilon]. \quad (\text{A.3})$$

Moreover, the following holds:

$$(r, u(t)) \in \mathcal{K}_s \quad \forall t \in [T^*/\varepsilon, T/\varepsilon]. \quad (\text{A.4})$$

□

Proof (Lemma 1 and Lemma 2). To prove Lemma 1 and Lemma 2, it is sufficient to show that all the assumptions of (Teel et al., 2003, Proposition 2) hold in our settings. Again, we recall that Table A.1 shows the correspondence between our notation and the one used in Teel et al. (2003).

(Teel et al., 2003, Assumption 2) holds since $F_s(\cdot) = (0, \varepsilon(r - g(x)))$ and $d_s = 0$ in our settings.

(Teel et al., 2003, Assumption 3) holds thanks to our Assumption 1, with the set $\mathcal{H}_f = \mathbb{R}^{n_u} \times \mathcal{Z}_f$, output measuring function $\omega_{f,o}(r_f, z_f) = |(r_f, z_f)|_{\mathbb{R}^{n_u} \times \mathcal{A}_f}$, and input measuring function $\omega_{f,i} = 0$. (We have extended the state of (7) including $\dot{r}_f = 0$. This is needed to comply with the framework proposed in Teel et al. (2003).)

(Teel et al., 2003, Assumption 4) holds thanks to our Assumptions 2 and 3, as shown in the following. Let

$$F_{av}(r, u) := (0, r - G_{av}(u)),$$

with G_{av} as defined in our Assumption 2. The data

$$(0, 0, F_{av}, \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}, \mathbb{R}^{n_u} \times \mathcal{K}_f)$$

constitutes an admissible average in the sense of (Teel et al., 2003, Definition 1) since, in our settings, (Teel et al., 2003, eq. (40)) reduces to our (9). Under our Assumptions 2 and 3, (Teel et al., 2003, Assumption 4) holds with $\mathcal{H}_s = \mathcal{Z}_s$, output measuring function $\omega_{s,o}(z_s) = |z_s|_{\mathcal{A}_s}$, and input measuring function $\omega_{s,i} = 0$.

(Teel et al., 2003, Assumption 7) holds as shown in the following. (Teel et al., 2003, Assumption 7, items (1),(2)) hold since $\mathcal{K}_s \subseteq \mathcal{Z}_s$, $\mathcal{K}_f \subseteq \mathcal{Z}_f$. (Teel et al., 2003, Assumption 7, items (3)–(6)) hold with $c_{f,o} := \max_{z_f \in \mathcal{K}_f} |z_f|_{\mathcal{A}_f}$, $c_{f,i} = 0$, $c_{s,o} := \max_{z_s \in \mathcal{K}_s} |z_s|_{\mathcal{A}_s}$, and $c_{s,i} = 0$. (The constants $c_{f,o}, c_{s,o} > 0$ exist since the sets $\mathcal{K}_f, \mathcal{K}_s$ are assumed compact, while $c_{f,i} = c_{s,i} = 0$ since $\omega_{f,i} = \omega_{s,i} = 0$ here.) Finally, (Teel et al., 2003, Assumption 7, item (7)) reduces, in our settings, to the existence of $\alpha > 0$ such that (13) and (14) hold, which is the case thanks to Assumption 4.

(Teel et al., 2003, Assumption 8) holds as shown in the following. Recall $c_{s,o}, c_{f,o}$ from above. We define the sets

$$\mathcal{X}_s := \{(r, u) \in \mathcal{Z}_s \mid |(r, u)|_{\mathcal{A}_s} \leq \beta_s(c_{s,o}, 0)\}, \quad (\text{A.5})$$

$$\mathcal{X}_f := \{(r, u, x) \in \mathbb{R}^{n_u} \times \mathcal{Z}_f \mid |(u, x)|_{\mathcal{A}_f} \leq \beta_f(c_{f,o}, 0)\},$$

which correspond, respectively, to (Teel et al., 2003, \mathcal{X}_s from (62), \mathcal{Z}_f from (63)). Similarly, for $\sigma > 0$, we define

$$\mathcal{U}_f(\sigma) := \{(r, u, x) \in \mathbb{R}^{n_u} \times \mathcal{Z}_f \mid (r, u) \in \mathcal{X}_s + \mathcal{B}_\sigma, (r, u, x) \in \mathcal{X}_f + \mathcal{B}_\sigma\},$$

which corresponds to (Teel et al., 2003, $\mathcal{U}_f(\sigma)$ from (64)). We remark that \mathcal{X}_s is compact, since \mathcal{A}_s is assumed compact in Assumption 3, and that, similarly, $\mathcal{X}_f|_{(u,x)}$ is compact, since \mathcal{A}_f is assumed compact in Assumption 1. It follows that $\mathcal{U}_f(\sigma)$ is compact, for any $\sigma > 0$. Thus, the following holds. (Teel et al., 2003, Assumption 8, item (1)) holds true since $\omega_{s,o}(\cdot) = |\cdot|_{\mathcal{A}_s}$ in our settings, which is globally Lipschitz. (Teel et al., 2003, Assumption 8, item (2)) reduces to $\mathcal{X}_s + \mathcal{B}_\sigma \subset \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$ in our settings, which is obviously true. (Teel et al., 2003, Assumption 8, item (3)) holds true since $\omega_{f,o}(\cdot) = |\cdot|_{\mathbb{R}^{n_u} \times \mathcal{A}_f}$ in our settings, which, again, is globally Lipschitz. (Teel et al., 2003, Assumption 8, item (4a)) holds true since $\omega_{f,o}(\cdot) = |\cdot|_{\mathbb{R}^{n_u} \times \mathcal{A}_f}$ in our settings and, thus, we can argue as in (Teel et al., 2003, Remark 29). Finally, (Teel et al., 2003, Assumption 8, item (4b)) holds true since $\mathcal{U}_f(\sigma)$ is compact (as argued above) and the functions f, g, G_{av} are assumed to be locally Lipschitz.

Since we have shown that (Teel et al., 2003, Assumptions 2–4,7,8) hold, we can use (Teel et al., 2003, Proposition 2), which yields Lemma 1 and Lemma 2. ■

Step 2: Practical average set-point tracking. We prove here that the estimates (16)–(17) hold, and we show how to derive (18). Thus, concluding the proof of Theorem 1.

Recall the sets \mathcal{K}_s and \mathcal{K}_f introduced in Assumption 4.

Lemma 3. (Stability of **CL**). For any $\delta > 0$ there exists $\varepsilon_\delta^* > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta^*)$, any solution (r, u, x) to **CL** from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, exists for all $t \geq 0$ and satisfies (16)–(17) for all $t \geq 0$. □

Proof (Lemma 3). To prove Lemma 3, it is sufficient to show that all the assumptions of (Teel et al., 2003, Theorem 1) hold in our settings. Again, we refer the reader to Table A.1, which shows the correspondence between our notation and the one used in Teel et al. (2003).

Table A.1. Correspondence between the notation used in Teel et al. (2003) and ours.

| Teel et al. (2003) | x_s | x_f | $F_s(\cdot)$ | d_s | d_f | $F_f(\cdot)$ | z_{bl} | \mathcal{H}_f | $\omega_{f,o}(\cdot)$ | $\omega_{f,i}$ | $\omega_{s,o}(\cdot)$ |
|--------------------|----------------|-------|------------------------------|-------------|--------------------|--------------|-----------------|--|---|-----------------|---------------------------|
| Section 3 | (r, u) | x | $(0, \varepsilon(r - g(x)))$ | 0 | 0 | $f(\cdot)$ | (r_f, z_f) | $\mathbb{R}^{n_u} \times \mathcal{Z}_f$ | $ \cdot _{\mathbb{R}^{n_u} \times \mathcal{A}_f}$ | 0 | $ \cdot _{\mathcal{A}_s}$ |
| Teel et al. (2003) | $\omega_{s,i}$ | m | $F_{av}(\cdot)$ | ϕ_{bl} | \mathcal{S}_{bl} | e | $\omega_{av,i}$ | \mathcal{R}_s | \mathcal{K}_f | \mathcal{H}_s | |
| Section 3 | 0 | 0 | $(0, r - G_{av}(u))$ | (r, z_f) | \mathcal{S}_f | 0 | 0 | $\mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$ | $\mathbb{R}^{n_u} \times \mathcal{K}_f$ | \mathcal{Z}_s | |

(Teel et al., 2003, Assumption 1) trivially holds in our settings since we have no disturbances, i.e., $d_s = d_f = 0$.

(Teel et al., 2003, Assumptions 3–4) hold in our settings, as shown in the proof of Lemma 1 and Lemma 2.

(Teel et al., 2003, Assumptions 5–6) correspond, respectively, to the results of Lemma 1 and Lemma 2.

Since we have shown that (Teel et al., 2003, Assumptions 1,3–6) hold, we can use (Teel et al., 2003, Theorem 1), which, in our settings, yields Lemma 3. \blacksquare

What is left now is to show that the limit (18) holds.

We start by proving the following auxiliary results: Using the properties of \mathcal{KL} functions, we show that asymptotic stability (AS) is equivalent to practical exponential stability (pES) on compact sets of initial conditions.

Lemma 4. (AS equals pES). Let $\beta \in \mathcal{KL}$. For any $\Delta, \lambda, \nu > 0$, there exists $\gamma_{\Delta, \nu} \geq 0$ such that for all $s \in [0, \Delta]$ we have

$$\beta(s, t) \leq \beta(\Delta, 0)e^{-\lambda(t-\gamma_{\Delta, \nu})} + \nu \quad \forall t \geq 0. \quad (\text{A.6})$$

\square

Proof (Lemma 4). Fix $\Delta, \lambda, \nu > 0$. Choose $\gamma_{\Delta, \nu} \geq 0$ such that $\beta(\Delta, \gamma_{\Delta, \nu}) \leq \nu$. (The existence of $\gamma_{\Delta, \nu}$ follows from the properties of \mathcal{KL} functions.) Then, for all $t \in [0, \gamma_{\Delta, \nu}]$:

$$\beta(s, t) \leq \beta(\Delta, t) \leq \beta(\Delta, 0) \leq \beta(\Delta, 0)e^{-\lambda(t-\gamma_{\Delta, \nu})}.$$

Similarly, by definition of $\gamma_{\Delta, \nu}$, we have

$$\beta(s, t) \leq \beta(\Delta, t) \leq \nu \quad \forall t \geq \gamma_{\Delta, \nu}.$$

The estimate (A.6) follows by combining the two above. \blacksquare

We show next that the estimate (9) remains quantitatively the same when computed along the trajectories of \mathbf{CL} from (5) (instead of using those of \mathbf{P}_f from (7)).

Lemma 5. (Average input-output map of \mathbf{P}). For any $\rho > 0$ there exist $T_\rho^* > 0, \varepsilon_\rho^* > 0$: For any $\varepsilon \in (0, \varepsilon_\rho^*]$, any solution (r, u, x) to \mathbf{CL} from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, exists for all $t \geq 0$ and satisfies

$$\left| \int_0^T [G_{av}(u(\tau)) - g(x(\tau))] d\tau \right| \leq T\rho \quad \forall T \geq T_\rho^*. \quad (\text{A.7})$$

\square

Proof (Lemma 5). Let $\rho > 0$. As shown in the proof of Lemma 1 and Lemma 2, all the assumptions of (Teel et al., 2003, Proposition 2) hold in our settings. Therefore, we can enter the proof of (Teel et al., 2003, Proposition 2) directly in Claim 4 to generate $\tilde{T}_\rho^* > 0$ and $\tilde{\varepsilon}_\rho^* > 0$ corresponding to $\rho > 0$ fixed above. Choose $\delta > 0$ such that $\delta < \tilde{\sigma}$, with $\tilde{\sigma} > 0$ as in the proof of (Teel et al., 2003, Proposition 2). Use Lemma 3 to generate the corresponding $\varepsilon_\delta^* > 0$. Choose $\varepsilon_\rho^* := \min\{\varepsilon_\delta^*, \tilde{\varepsilon}_\rho^*\}$ and $T_\rho^* := \tilde{T}_\rho^*$.

By Lemma 3, for any $\varepsilon \in (0, \varepsilon_\rho^*]$, any solution (r, u, x) to \mathbf{CL} from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, exists for all $t \geq 0$ and, in particular, satisfies

$$|(r, u(t))|_{\mathcal{A}_s} \leq \beta_s(|(r, u(0))|_{\mathcal{A}_s}, 0) + \delta \quad \forall t \geq 0.$$

Since $\delta < \tilde{\sigma}$, recalling the set \mathcal{X}_s from (A.5), we have

$$(r, u(t)) \in \mathcal{X}_s + \tilde{\sigma}\mathcal{B} \quad \forall t \geq 0. \quad (\text{A.8})$$

Then, using (Teel et al., 2003, Claim 4) combined with (A.8), it follows that (A.7) holds for any $T \geq T_\rho^*$. \blacksquare

Lemma 6. (Average tracking on compact time intervals). For any $\delta > 0$ there exist $T_\delta^* > 0$ and $\varepsilon_\delta^* > 0$: For any $\varepsilon \in (0, \varepsilon_\delta^*]$ there exists $\eta_{\delta, \varepsilon} > 0$ such that any solution (r, u, x) to \mathbf{CL} from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, exists for all $t \geq 0$, satisfies (16)–(17) for all $t \geq 0$, and, moreover, satisfies

$$\left| \int_0^T e(\tau) d\tau \right| \leq \eta_{\delta, \varepsilon} + \delta T \quad \forall T \geq T_\delta^*. \quad (\text{A.9})$$

\square

Proof (Lemma 6). Fix $\delta > 0$. Let $\tilde{\delta} = \delta/4$ and use Lemma 3 to generate the corresponding $\varepsilon_\delta^* > 0$. Let $\rho = \tilde{\delta}$ and use Lemma 5 to generate the corresponding $\varepsilon_\rho^* > 0$ and $T_\rho^* > 0$. Choose $T_\delta^* := T_\rho^*$ and $\varepsilon_\delta^* := \min\{\varepsilon_\delta^*, \varepsilon_\rho^*\}$.

Let $\varepsilon \in (0, \varepsilon_\delta^*]$. From Lemmas 3,5: Any solution (r, u, x) to \mathbf{CL} from (5), with $(r, u(0)) \in \mathcal{K}_s$ and $(u(0), x(0)) \in \mathcal{K}_f$, exists for all $t \geq 0$. Fix one such solution (r, u, x) to \mathbf{CL} .

Recalling that $e := r - g(x)$, we can write

$$\left| \int_0^T e(\tau) d\tau \right| = \left| \int_0^T [r - g(x(\tau))] d\tau \right| \quad \forall T \geq 0. \quad (\text{A.10})$$

By adding and subtracting $G_{av}(u(\tau))$ above, we obtain

$$\begin{aligned} \left| \int_0^T [r - g(x(\tau))] d\tau \right| &\leq \left| \int_0^T [r - G_{av}(u(\tau))] d\tau \right| \\ &+ \left| \int_0^T [G_{av}(u(\tau)) - g(x(\tau))] d\tau \right| \quad \forall T \geq 0. \end{aligned} \quad (\text{A.11})$$

We start by studying the first integral in the right-hand side of (A.11). From the definition of \mathcal{A}_s in Assumption 3,

$$|r - G_{av}(u(t))| = |(r, u(t))|_{\mathcal{A}_s}.$$

Thus, using the estimate (16) (with $\tilde{\delta}$) yields

$$\begin{aligned} \left| \int_0^T [r - G_{av}(u(\tau))] d\tau \right| &\leq \int_0^T |r - G_{av}(u(\tau))| d\tau \\ &\leq \int_0^T \beta_s(|(r, u(0))|_{\mathcal{A}_s}, \varepsilon\tau) d\tau + \tilde{\delta}T \quad \forall T \geq 0. \end{aligned} \quad (\text{A.12})$$

Use Lemma 4 with $\Delta = c_{s,o}$ (from Assumption 4), $\lambda = 1$, and $\nu = \tilde{\delta}$, to generate $\gamma_{\tilde{\delta}, \varepsilon} > 0$. Define

$$\eta_{\tilde{\delta}, \varepsilon} := \frac{1}{\varepsilon} \beta_s(c_{s,o}, 0) e^{\varepsilon\gamma_{\tilde{\delta}, \varepsilon}} > 0.$$

Using Lemma 4 as indicated above, for all $T \geq 0$ we have

$$\begin{aligned} \int_0^T \beta_s(|(r, u(0))|_{\mathcal{A}_s}, \varepsilon\tau) d\tau + \tilde{\delta}T \\ \leq \frac{1}{\varepsilon} \beta_s(c_{s,o}, 0) e^{\varepsilon\gamma_{\tilde{\delta}, \varepsilon}} (1 - e^{-\varepsilon T}) + 2\tilde{\delta}T \leq \eta_{\tilde{\delta}, \varepsilon} + 2\tilde{\delta}T. \end{aligned} \quad (\text{A.13})$$

We move to the second integral in the right-hand side of (A.11). From our choice of $T_\delta^*, \varepsilon_\delta^*$, using Lemma 5,

$$\left| \int_0^T [G_{av}(u(\tau)) - g(x(\tau))]d\tau \right| \leq \tilde{\delta}T \quad \forall T \geq T_\delta^*. \quad (\text{A.14})$$

The estimate (A.9) follows combining (A.11)–(A.14). ■

The proof of Theorem 1 follows from Lemma 6, since the limit (18) can be derived from the estimate (A.9).