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# Partitioned Linear Orders and Belief Revision

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**Abstract**—Despite their expressive power in representing alternatives or beliefs tainted by incomparabilities, it is often difficult to use generic partial orders. Hence, various structures have been historically studied in the literature, such as weak orders, semi-orders or interval orders, each capturing a different fragment of partial orders. In this article, we propose and study in detail a fragment rich enough to represent some highly intuitive patterns involving incomparable elements that cannot be captured by the aforementioned fragments: partitioned linear orders (PLOs). These structures can be viewed as split independent linear orders, each one of them representing a local totally ranked situation. Then, we show how they can be used in Katsuno-Mendelzon-style belief revision. We propose a set of postulates, a representation theorem and a family of concrete operators.

**Index Terms**—partitioned linear orders, partial orders, KM belief revision, incomparable beliefs

## I. INTRODUCTION

Belief revision [1] refers to the challenge that arises when an agent receives new information about the world that potentially contradicts their current beliefs. This article takes place in Katsuno Mendelzon (KM) framework [2], where the beliefs of the agent, i.e. its representation of the world (the situation), is captured by a propositional formula  $\varphi$ , and the new information by another propositional formula  $\alpha$ . In this case, the belief revision process change the current agent’s belief and hence grants him a better overview of the current situation. The postulates of Katsuno and Mendelzon lead to totally preordered beliefs, i.e. to a total preorder on interpretations representing the relative plausibility between each situation. This kind of ordered structure is not able to capture 2+1 patterns (i.e. a chain of two comparable elements, themselves incomparable with a third element, see Figure 1).

Nevertheless, it can be difficult for an agent to compare all the situations or alternatives it may face. Let us consider Raz’ example on career choice [3]. Two successful careers are offered to a person: to be either a lawyer or a clarinetist. Saying that one is better than the other seems as wrong as saying that both careers are as good as each other. In the latter case, any slight modification of one of the careers would lead the other one to become worse or better than this altered career, which is counterintuitive. Moreover, two slight improvements, one of each situation, should maintain the incomparability between the two types of careers, while they are locally comparable. This simple situation can be captured in an ordinal way with a 2+2 pattern (i.e. two incomparable chains of two

elements), as represented in Figure 1. In certain other cases, incomparabilities may represent a refusal of choice, either in the expectation of more context (do you prefer an adjustable spanner or a tennis ball?), or in case of a dilemma (for instance in the novel *Sophie’s Choice*) [4]. When we focus on uncertain beliefs, incomparabilities can also occur in case of imprecision, lack of information or incommensurability of the uncertainty measures provided by the sources [5].

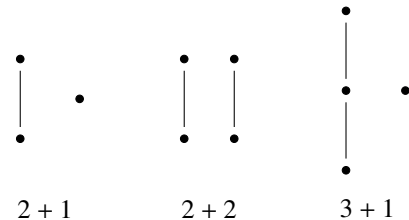


Fig. 1: Some usual forbidden patterns.

In order to manage incomparabilities, another set of postulates has been proposed in [6] and is able to capture partial preorders. Nevertheless, as a trade-off for their expressive power in representing incomparabilities, partial preorders are often difficult to use in practice. Due to their high linear dimension (i.e. the minimal number of linear orders needed to express the partial order as their intersection, see e.g. [7]), they are often hard to graphically represent in a usable way, hard to quantify using numbers and above all can be computationally difficult to use. Among some other proposed postulate sets, still in KM framework, we can also cite the ones of [8], generalized in [9]. They lead to semiorders [10] and hence cannot represent either 2+2 nor 3+1 patterns (see Figure 1).

We propose in this article to consider belief revision in the light of *partitioned linear orders* (PLOs). These order structures can be viewed as split independent linear orders, each of them representing a local totally informed situation. They are able to capture 2+2 without dimensional explosion, indeed, only two linear orders are needed to represent them. We study them in depth and characterize them in terms of intersection of orders, score function representations, and forbidden patterns. We then propose a new belief revision framework able to capture PLOs. In order to show the consistency of our postulates, we also propose an operator based

on Voronoi diagrams and *tie-breaking* functions [11], [12] satisfying our rational postulates.

*Outline:* Section II gives some necessary background on ordered structures and on belief revision. Section III situates our proposition in the landscape of belief revision operators. Next, we deeply investigate a new ordered structure, the partitioned linear orders (PLO) in Section IV. Section V focuses on belief revision using PLOs.

## II. PRELIMINARIES

We provide in this section the needed material used in the sequel of this paper. The first part is a reminder on different notions of orders. The second part treats propositional logic, belief revision and its links with orders.

a) *Binary Relations.*: Given  $\mathcal{R}$  a binary relation over a set  $X$ , it is said to be a *partial preorder*, denoted here by  $\lesssim$ , when it is a reflexive ( $\forall x \in X, x \mathcal{R} x$ ) and transitive relation ( $\forall x, y, z \in X$ , if  $x \mathcal{R} y$  and  $y \mathcal{R} z$ , then  $x \mathcal{R} z$ ). An *order*, denoted by  $\leq$ , is a preorder which is antisymmetric ( $\forall x, y \in X$ , if  $x \mathcal{R} y$  and  $y \mathcal{R} x$ , then  $x = y$ ). A *linear order* (aka a total order), is an order which is total ( $\forall x, y \in X$ ,  $x \mathcal{R} y$  or  $y \mathcal{R} x$ ).

For a given (pre)order  $\lesssim$ , we denote by  $<$  its asymmetric part, by  $\approx$  its symmetric part, and by  $\asymp$  its incomparability relation, i.e.:

- $x < y$  when  $x \lesssim y$  and not  $y \lesssim x$ ,
- $x \approx y$  when both  $x \lesssim y$  and  $y \lesssim x$ ,
- $x \asymp y$  when neither  $x \lesssim y$  nor  $y \lesssim x$ .

b) *Posets:* A set  $X$  associated with a (pre)order  $\mathcal{R}$  is a (pre)ordered set denoted  $P = (X, \mathcal{R})$ . A partially ordered set is often called a *poset*.

The *union of posets* is defined as follows. Given two posets  $P_a = (X_a, \lesssim_a)$  and  $P_b = (X_b, \lesssim_b)$ , we have  $P = (X, \lesssim) = P_a \cup P_b$  such that  $X = X_a \cup X_b$  and  $\forall x, y \in X, x \lesssim y$  iff

$$\begin{cases} x, y \in X_a \text{ and } x \lesssim_a y, & \text{or} \\ x, y \in X_b \text{ and } x \lesssim_b y \end{cases}$$

Finally, given a preordered set  $P = (X, \lesssim)$  and  $Y \subseteq X$ , we use the following definition to denote the set of the *minimal elements* of  $Y$ :  $\text{Min}(Y, \lesssim) = \{x \in Y : \forall y \in Y, y < x \text{ does not hold}\}$

c) *Propositional Logic.*: We denote by  $\mathcal{P}$  a **finite** set of propositional variables and by  $\mathcal{L}$  the set of propositional formulas built over  $\mathcal{P}$ . We will use the usual connectors  $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$  and the symbols  $\top$  and  $\perp$  to respectively represent the always-true formula and the always false formula. We use Greek letters, except  $\Omega$  and  $\omega$ , to represent an element of  $\mathcal{L}$ .  $\Omega$  is the set of all interpretations of  $\mathcal{L}$  and we denote by  $\llbracket \alpha \rrbracket$  the models of  $\alpha$ , that is the set  $\{\omega \in \Omega : \omega \models \alpha\}$ . A formula  $\varphi$  is complete if  $\forall \mu \in \mathcal{L}$ , we have either  $\varphi \vdash \mu$  or  $\varphi \vdash \neg \mu$  but not both. In other words,  $\llbracket \varphi \rrbracket$  is a singleton. The symbols  $\vdash$  and  $\models$  have their usual meanings.

d) *Belief Revision.*: Belief revision corresponds to the problem arising from giving a new piece of information about the world to an agent such that this new information and the agent's current beliefs are potentially inconsistent. The set of rationality postulates proposed in [1] constitutes the base of the AGM framework for belief revision, which is well accepted.

A translation of the AGM postulates into propositional logic introduced in [2] leads to the following ones:

- (**R**<sub>1</sub>)  $\varphi \circ \alpha \vdash \alpha$
- (**R**<sub>2</sub>) if  $\varphi \wedge \alpha \not\vdash \perp$ , then  $\varphi \circ \alpha \equiv \varphi \wedge \alpha$
- (**R**<sub>3</sub>) if  $\alpha \not\vdash \perp$ , then  $\varphi \circ \alpha$  is consistent
- (**R**<sub>4</sub>) if  $\varphi_1 \equiv \varphi_2$  and  $\alpha \equiv \beta$  then  $\varphi_1 \circ \alpha \equiv \varphi_2 \circ \beta$
- (**R**<sub>5</sub>)  $(\varphi \circ \alpha) \wedge \beta \vdash \varphi \circ (\alpha \wedge \beta)$
- (**R**<sub>6</sub>) if  $(\varphi \circ \alpha) \wedge \beta \not\vdash \perp$ , then  $\varphi \circ (\alpha \wedge \beta) \vdash (\varphi \circ \alpha) \wedge \beta$

They also showed that a revision operator  $\circ$  satisfying all these postulates is equivalent to searching the minimum in a specific total preordered set provided by a faithful assignment of  $\circ$ .

**Definition 1** (Faithful assignment, [2]). *A function assigning, to a formula  $\varphi$ , a total preorder  $\lesssim_\varphi$  over  $\Omega$  is a faithful assignment iff:*

- (1) if  $\omega_1, \omega_2 \in \llbracket \varphi \rrbracket$  then  $\omega_1 <_\varphi \omega_2$  does not hold
- (2) if  $\omega_1 \in \llbracket \varphi \rrbracket$  and  $\omega_2 \notin \llbracket \varphi \rrbracket$  then  $\omega_1 <_\varphi \omega_2$  holds
- (3) if  $\varphi \equiv \psi$ , then  $\lesssim_\varphi = \lesssim_\psi$

The first condition states that the models of the current beliefs are equivalent. The second one requires that all models are preferred to all countermodels. The third condition expresses independence from the syntax.

**Theorem 1** ([2]). *A revision operator  $\circ$  satisfies the postulates (**R**<sub>1</sub>)–(**R**<sub>6</sub>) iff there exists a faithful assignment that maps  $\varphi$  to a total preorder  $\lesssim_\varphi$  such that:*

$$\llbracket \varphi \circ \mu \rrbracket = \text{Min}(\llbracket \mu \rrbracket, \lesssim_\varphi).$$

e) *Revision of Partially Ordered Information.*: The usage of the KM postulates implies that the information can be represented by a total preorder. This requirement constitutes a limit in terms of expressiveness in situations where the information is incomplete. Thus, a replacement of (**R**<sub>6</sub>), by two weaker postulates, (**R**<sub>7</sub>) and (**R**<sub>8</sub>), as proposed in [2], can represent partially ordered information. This proposition, though, is not sufficient to allow the representation of any partial order. Therefore, to be able to represent any kind of partially ordered information, a weakening of (**R**<sub>2</sub>), namely (**R**<sub>2</sub>'), has been proposed in [6].

- (**R**<sub>2</sub>')  $\varphi \circ \top \equiv \varphi$
- (**R**<sub>7</sub>) if  $\varphi \circ \alpha \vdash \beta$  and  $\varphi \circ \beta \vdash \alpha$  then  $\varphi \circ \alpha \equiv \varphi \circ \beta$
- (**R**<sub>8</sub>)  $(\varphi \circ \alpha) \wedge (\varphi \circ \beta) \vdash \varphi \circ (\alpha \vee \beta)$

They also defined the notion of p-faithful assignment to capture partially ordered information.

**Definition 2** (p-faithful assignment, [6]). *A function assigning, to a formula  $\varphi$ , a partial preorder  $\lesssim_\varphi$  over  $\Omega$  is a p-faithful assignment iff:*

- (1) if  $\omega_1, \omega_2 \in \llbracket \varphi \rrbracket$  then  $\omega_1 <_\varphi \omega_2$  does not hold
- (2') if  $\omega_2 \notin \llbracket \varphi \rrbracket$  then there exists  $\omega_1$  such that  $\omega_1 \in \llbracket \varphi \rrbracket$  and  $\omega_1 <_\varphi \omega_2$  holds
- (3) if  $\varphi \equiv \psi$ , then  $\lesssim_\varphi = \lesssim_\psi$

The first difference with the definition of a faithful assignment is that the preorder is no longer total. The second difference is the condition (2'). Relaxing condition (2), it solely

TABLE I: A brief panorama of representation theorems in literature.

Reference	Ordered structure
[2]	total preorder
[2]	min-partial preorder
[6]	partial preorder
[8]	min-semiorder
[9]	semiorder
here	partitioned linear order

requires, for each countermodel, the existence of a preferred model. They finally proposed the following representation theorem.

**Theorem 2** ([6]). *A  $p$ -revision operator  $\circ$  satisfies postulates  $(R_1)$ ,  $(R'_2)$ ,  $(R_3)$ – $(R_5)$ ,  $(R_7)$ ,  $(R_8)$  iff there exists a  $p$ -faithful assignment that maps  $\varphi$  to a partial preorder  $\lesssim_\varphi$  such that:*

$$\llbracket \varphi \circ \mu \rrbracket = \text{Min}(\llbracket \mu \rrbracket, \lesssim_\varphi).$$

A detailed proof of this theorem can be found in [9].

### III. RELATED WORKS

Ordered structures play a central role in belief revision, both in its classical [1], [13] and iterated versions [14]. By means of representation theorems, they offer a better understanding of the postulates of rationality and provide semantics for revision operators.

At first, we reintroduce the KM family of operators which covers the revision operators that can be translated into a total preorder. PLO revision operators are slightly less specific, as they deal with *locally* totally ordered situation and thus introduce some degree of incomparability.

A first family dealing with incomparability is the one proposed in [2] obtained by replacing  $(R_6)$  by  $(R_7)$  and  $(R_8)$ . This family has been designed to handle a convenient but specific type of partially ordered information, where the models of the current beliefs are preferred to any other interpretation. In order to capture any partial order, even those where there is no global minima, [6] introduced a new family of operators by further replacing  $(R_2)$  by its weakening  $(R'_2)$ . PLO revision operators are a subclass of this second family of partial revision operators.

A family of operators representing semiorders with a global minimum has been proposed in [8]. To do so, they replace  $(R_6)$  by  $(R_8)$ ,  $(R_9)$  and  $(R_{10})$ .

$(R_9)$  If  $(\varphi \circ \alpha) \wedge \beta \not\prec \varphi \circ \beta$  then  $(\varphi \circ \beta) \wedge \alpha \vdash \varphi \circ \alpha$

$(R_{10})$  If  $(\varphi \circ \alpha) \wedge \beta \vdash \perp$  and  $(\varphi \circ \alpha) \wedge \gamma \not\prec \perp$  then  $(\varphi \wedge \gamma) \wedge (\alpha \wedge \beta) \vdash \varphi \circ (\alpha \wedge \beta)$

Semiorders are well known ordered structures in the economic field for having a more cautious approach with respect to indifference. They have the capability of making indifference intransitive which is analogous to setting thresholds to be able to distinguish between items. This desirable property can be illustrated by the famous coffee example from [15]. With the objective of capturing the whole class of semiorders

another family of operators based on the one from [8] but replacing  $(R_2)$  by  $(R'_2)$  has been proposed in [9]. None of these families handles local totally informed situations, notably because of the forbidden patterns of the ordered structure they are analogous to.

Table I summarizes and situates our contribution in the landscape of belief revision. It completes the one provided in [9]. Each article mentioned as a reference in the left column provides a representation theorem leading to the ordered structure given in the right column.

Finally, PLOs can be found in [16] as the *multi-linear assumption*. They serve as a technical stepping stone for their tableaux method, but there is no study of the properties of such structures. Hence, we propose a characterization (with an axiomatization and forbidden patterns) and a numerical representation (as a bilinear order). Additionally, we provide a characterization of revision operators that lead to beliefs ordered with a PLO. Our result, positioning this structure as a subcategory of bilinear orders, tends to show the name of *multi-linear order* as inaccurate, thus we rather refer to them as partitioned linear orders.

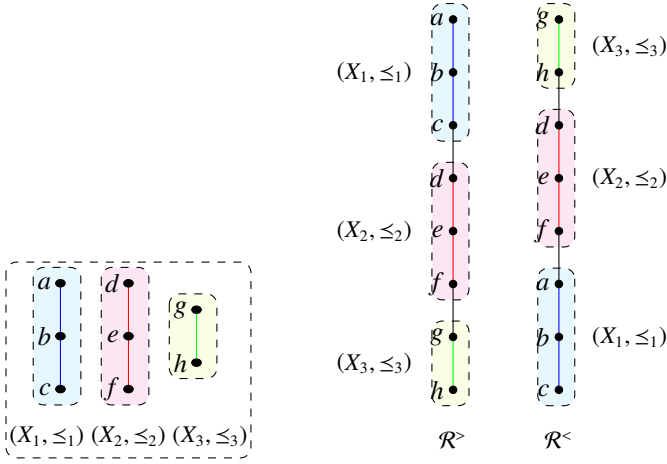
### IV. PARTITIONED LINEAR ORDERS

In belief revision, ordered structures can be considered for modelling beliefs, or at least an equivalent representation of the plausibility that the agent assigns to each state of the world. For these ordered structures, the *total* property is both very demanding when it comes to feeding the model with information, and very assertive when it comes to interpreting the semantics of the model in the real world. We therefore introduce a relaxation of this property, namely *transitive comparability* and the corresponding category of orders satisfying it that we call *partitioned linear orders* (PLO).

#### A. Definition

We introduce the category of *partitioned linear orders* with a detour via the notion of *comparability*: given a binary relation  $\mathcal{R}$  over a set  $X$ , we say that two elements  $x, y \in X$  are comparable when  $x \mathcal{R} y$  or  $y \mathcal{R} x$ . Comparability is structurally symmetric. The *total* property of the relation  $\mathcal{R}$  can thus be rephrased as the completeness of the associated comparability relation, i.e. when any two elements are comparable. We relax this global property by requiring comparability to be solely *transitive*, so as to spread by contagion. Indeed, when any two elements are in relation with a third one, they are considered part of the relation. When applied to a poset  $(X, \lesssim)$ , this notion leads to a partition of  $X$  into the equivalence classes of the comparability relation associated to  $\lesssim$ , where two elements of different classes are incomparable, while the binary relation induced by  $\lesssim$  on each class is a total preorder. When applied to a finite set  $X$  ordered by  $\leq$ , the partition is finite and the induced relations on each class are total orders.

**Definition 3** (Partitioned linear order). *The binary relation on  $X$ , denoted  $\trianglelefteq$ , is a partitioned linear order iff  $\exists m \in \mathbb{N}^*$  such that  $X_1, \dots, X_m$  is a partition of  $X$  (i.e.  $\bigcup_{i=1..m} X_i = X$  and*



(a) PLO  $\leq$  over  $X$ .

(b) Corresponding  $\mathcal{R}^>$  and  $\mathcal{R}^<$ .

Fig. 2: PLO from Example 1

$\forall i, j \in \{1, \dots, m\}$  such that  $i \neq j$ ,  $X_i \cap X_j = \emptyset$  and  $\exists <_i$ , a linear order over  $X_i$ ,  $\forall i \in \{1, \dots, m\}$  such that  $(X, \leq) = \bigcup_{i=1..m} (X_i, <_i)$ .

**Proposition 1.** A poset  $(X, \leq)$ , with  $\approx$  its induced incomparability relation, is a PLOed set iff  $\forall x, y, z \in X$  if  $x \approx z$  does not hold and  $z \approx y$  does not hold, then  $x \approx y$  does not hold.

In other words, if  $x$  and  $z$  are comparable and  $z$  and  $y$  are also comparable, then  $x$  has to be comparable with  $y$ .

**Example 1.** Let us take the following set  $X = \{a, b, c, d, e, f, g, h\}$  and the partitioned linear order  $\leq = \{(c, c), (c, b), (c, a), (b, b), (b, a), (a, a), (f, f), (f, e), (f, d), (e, e), (e, d), (d, d), (h, h), (h, g), (g, g)\}$ . The following partition of  $X$ ,  $X_1 = \{a, b, c\}$ ,  $X_2 = \{d, e, f\}$  and  $X_3 = \{g, h\}$  respectively associated to each of the following linear orders  $\leq_1 = \{(c, c), (c, b), (c, a), (b, b), (b, a), (a, a)\}$ ,  $\leq_2 = \{(f, f), (f, e), (f, d), (e, e), (e, d), (d, d)\}$ , and  $\leq_3 = \{(h, h), (h, g), (g, g)\}$  leads to the 3 linearly ordered sets  $(X_1, \leq_1)$ ,  $(X_2, \leq_2)$ , and  $(X_3, \leq_3)$  such that  $(X, \leq) = \bigcup_{i \in \{1, 2, 3\}} (X_i, \leq_i)$ . Figure 2a represents these orders.

## B. Representations

It is customary to represent partial orders as the intersection of linear orders, or with the help of numerical functions. For instance, any total preorder  $\lesssim$  over a finite set  $X$  can be represented with a score function  $f : X \rightarrow \mathbb{N}$  such that  $x \lesssim y$  iff  $f(x) \leq f(y)$ . The relation  $\lesssim$  is a linear order iff  $f$  is injective. We propose both types of representation for partitioned linear orders.

a) *Bilinear orders:* Bilinear orders [7] correspond to the case  $n = 2$  of  $n$ -linear orders, defined as the category of product orders of  $n$  linear orders. It lies almost at the bottom of this hierarchy, just above the category of linear orders. Let  $\leq$  be a PLO over a finite set  $X$  leading to the partition  $X = \bigcup_{i=1}^n X_i$

into comparability classes. Let  $\mathcal{R}^>$  and  $\mathcal{R}^<$  the binary relations defined over  $X$  as follows:

$$\forall x, y \in X, x \mathcal{R}^> y \text{ iff } x \leq y \text{ or } x \in X_i, y \in X_j \text{ and } i > j \quad (1)$$

$$\forall x, y \in X, x \mathcal{R}^< y \text{ iff } x \leq y \text{ or } x \in X_i, y \in X_j \text{ and } i < j \quad (2)$$

In brief, it consists in linearizing the partition over  $X$  in increasing (resp. decreasing) order for  $\mathcal{R}^>$  (reps.  $\mathcal{R}^<$ ). Figure 2b represents the linear orders  $\mathcal{R}^>$  and  $\mathcal{R}^<$  associated to the PLO depicted in Figure 2a. In this specific case,  $\mathcal{R}^>$  is the concatenation of  $(X_1, \leq_1)$  followed by  $(X_2, \leq_2)$  then  $(X_3, \leq_3)$ , while  $\mathcal{R}^<$  is the concatenation of these posets in the reverse order.

**Proposition 2.**  $\mathcal{R}^>$  and  $\mathcal{R}^<$  are linear orders over  $X$  that refine a PLO  $\leq$  and such that  $\leq$  is the product order of  $\mathcal{R}^>$  and  $\mathcal{R}^<$  (i.e.  $x \leq y$  iff  $x \mathcal{R}^> y$  and  $x \mathcal{R}^< y$ ).

**Sketch of proof.** ( $\Rightarrow$ ) The proof is straightforward by applying the definition.

( $\Leftarrow$ ) Towards a contradiction, let us assume that  $\exists x, y \in X$  such that  $x \mathcal{R}^> y$  and  $x \mathcal{R}^< y$  and  $x \leq y$  does not hold. Thus, by definition we have two cases, either  $x \leq y$  or  $\exists i, j \in \{1..n\}$  two indexes of the partition such that  $x \in X_i, y \in X_j$  and  $i > j$  and  $\exists i', j' \in \{1..n\}$  such that  $x \in X_{i'}, y \in X_{j'}$  and  $i' < j'$ . The first case is obviously contradictory. Thus, in the second case, it follows from the fact that  $X_i, X_{i'}, X_j$  and  $X_{j'}$  are elements of a partition of  $X$  and that  $x \in X_i$  and  $x \in X_{i'}$ , and  $y \in X_j$  and  $y \in X_{j'}$ , that  $i = i'$  and  $j = j'$ . Thus, we have that  $i > j$  and  $i < j$  which is contradictory.

Note that both relations  $\mathcal{R}^>$  and  $\mathcal{R}^<$  depend on the arbitrary ordering of the comparability classes of  $(X, \leq)$ , but their product order does not. As we shall see in Section IV-C, Proposition 2 positions partitioned linear orders as a subcategory of bilinear orders. Moreover, it can be leveraged to produce three numerical representations for partitioned linear orders.

As a corollary of Proposition 2, PLO, being the intersection of two linear orders, are bilinear orders [7].

b) *Product representation:* A PLO over a finite set  $X$  can be represented with two score functions  $f, g : X \rightarrow \mathbb{N}$  such that  $x \leq y$  iff  $f(x) < f(y)$  and  $g(x) < g(y)$  as illustrated by Figure 3. This provides a lever for efficient computations, e.g. given a subset  $A \in X$  and two score functions  $f, g$  as above, determining the minimal elements of  $A$  can be performed in  $O(|A| \log |A|)$  time by sorting  $A$  in place according to  $f$ , initiating the returned list  $L$  with  $L := [\min_A f]$ , then enumerating  $A$  and piling on  $L$  each element of  $A$  that has a smaller image by  $g$  than the top of the pile.

c) *Interval representation:* It can be shown that the condition  $\forall x \in X, f(x) > g(x)$  can be imposed w.l.o.g. to the product representation presented above. Thus, each element of the set  $X$  can be represented by a non-empty interval  $[f(x), g(x)]$  [17], [18]. Incomparability of two elements is represented here by the inclusion/containment of their respective intervals, as shown on Figure 4. This implies that strict comparability, i.e.  $x < y$ , is encoded as  $f(x) < f(y)$  and

TABLE II:  $f, g, f', g' : X \mapsto \mathbb{R}$ , two pairs of score functions offering a product representation  $(f, g)$  or an interval representation  $(f', g')$  of  $\triangleleft$ .

	$a$	$b$	$c$	$d$	$e$	$f$	$g$	$h$
$f(x)$	1	2	3	4	5	6	7	8
$g(x)$	6	7	8	3	4	5	1	2
$f'(x)$	1	2	3	4	5	6	7	8
$g'(x)$	14	15	16	11	12	13	9	10

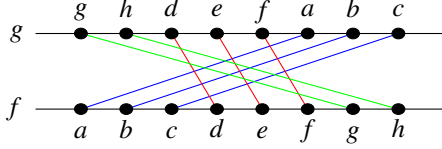


Fig. 3: Graphical representation of the functions  $f$  and  $g$ .

$g(x) < g(y)$  as presented above which diverges from interval orders' definition of strict comparability, that is  $g(x) < f(y)$ .

d) *Split linear order*: A PLO over a finite set with  $n$  comparability classes can be represented with a single injective score function  $f : X \rightarrow \mathbb{N}$  and a set of  $n - 1$  split points  $\tau_1, \dots, \tau_{n-1}$  in the following manner:  $x \triangleleft y$  iff  $f(x) \leq f(y)$  and  $\nexists i \in \{1, \dots, n - 1\} f(x) < \tau_i < f(y)$ .

**Example 2.** Table II displays a numerical representation of  $(X, \triangleleft)$  as the product order of two scores functions  $f$  and  $g$ . The graphical representation of  $f$  and  $g$  is shown in Figure 3. Elements of  $X$  are represented by segments such that the segments representing two objects cross iff these objects are incomparable. The graphical representation of  $f'$  and  $g'$  is shown in Figure 4. Elements of  $X$  are represented by intervals, such that if one interval contains the other, the two corresponding objects are incomparable.

### C. Forbidden Patterns

We are interested in positioning the category of partitioned linear orders inside the cartography of subcategories of partial orders that have been proposed in the literature. Proposition 2 shows that PLO form a subcategory of bilinear orders.

Another way of defining subcategories of partial orders is by forbidding some patterns to appear in the Hasse diagram, i.e. what these categories are unable to represent.

We recall the definitions of *interval orders*, *semiorders*, and *weak orders* in terms of the forbidden patterns presented Figure 1 as defined in [7]:

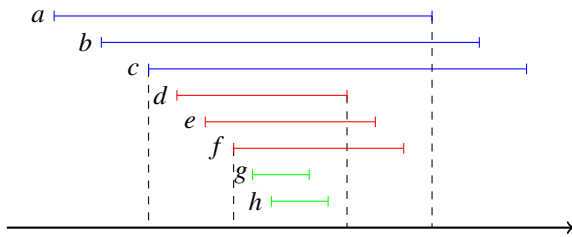


Fig. 4: Graphical representations of  $[f', g']$ .

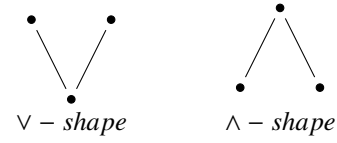


Fig. 5: PLO's Forbidden patterns.

- interval order : no 2+2
- semiorder : no 3+1, no 2+2
- weak order : no 2+1

The category of partitioned linear orders can be defined in terms of two forbidden patterns. Let  $(X, \triangleleft)$  be a poset and  $x, y, z \in X$  be three elements of  $X$ .

- $x, y, z$  form a  $\vee$ -pattern when  $z \triangleleft x, z \triangleleft y$  and neither  $x \triangleleft y$  nor  $y \triangleleft x$ ;
- $x, y, z$  form a  $\wedge$ -pattern when  $x \triangleleft z, y \triangleleft z$  and neither  $x \triangleleft y$  nor  $y \triangleleft x$ .

These two patterns are represented by Figure 5.

**Proposition 3.** A partial order  $\leq$  over a finite set  $X$  is a partitioned linear order iff there is no  $x, y, z \in X$  forming a  $\vee$ -pattern or a  $\wedge$ -pattern.

**Sketch of proof.** ( $\Rightarrow$ ) If a poset contains a  $\wedge$ -pattern (resp. a  $\vee$ -pattern)  $a, b, c$  where  $a$  and  $b$  are incomparable but both are comparable to  $c$ , the triplet cannot be partitioned into comparability classes. Hence the poset is not a PLO.

( $\Leftarrow$ ) Let  $(X, \triangleleft)$  be a poset where  $\vee$ -pattern and  $\wedge$ -pattern are forbidden. We prove  $X$  is partitioned into comparability classes. Let  $x, y$  such that  $x \asymp y$  and assume there is  $z \in X$  which is comparable to both  $x$  and  $y$ . There are 4 configurations to consider. The case where  $x \triangleleft z$  and  $z \triangleleft y$ , or  $z \triangleleft x$  and  $y \triangleleft z$  are excluded, because transitivity of  $\triangleleft$  yields to compare  $x$  and  $y$ : a contradiction. The case where  $z \triangleleft x$  and  $z \triangleleft y$  is a  $\vee$ -shape, and the case where  $x \triangleleft z$  and  $y \triangleleft z$  is a  $\wedge$ -shape. Thus, the comparability classes are pairwise disjoint, i.e. comparability is an equivalence relation and  $\triangleleft$  is a PLO.

Also, by definition, partitioned linear orders capture any pattern of the  $n + m$  kind. This follows directly from Definition 3 where PLOs are depicted as a collection of linear orders.

Figure 6 inserts partitioned linear orders into the proper inclusion diagram of the asymmetric part of certain posets proposed by Fishburn in [7]. On this diagram, non-connected classes are not included into one another and within two connected classes, the lower one is strictly included in the upper one. In brief:

- $\text{PLO}^* \not\subseteq \text{IO}^*$ : PLO can represent 2+2 patterns, while IO cannot.
- $\text{WO}^* \not\subseteq \text{PLO}^*$ :  $\wedge$ -shape and  $\vee$ -shape patterns are allowed in weak orders.
- $\text{PLO}^* \subseteq \text{BO}^*$ : derives from Proposition 2.
- $\text{LO}^* \subseteq \text{PLO}^*$ : A linearly ordered set  $(X, \leq)$  is a PLOed set  $(X, \triangleleft)$  where  $X$  is a partition of  $X$  and  $\triangleleft = \leq$ .

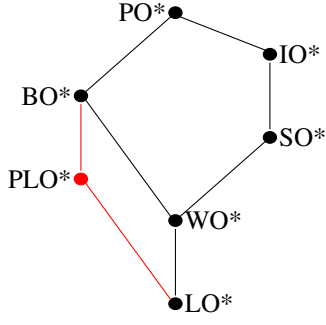


Fig. 6: Proper inclusion diagram of the asymmetric part of certain posets based on the one from [7]. PO = partial order; IO = interval order; SO = semiorder; BO = bilinear order; PLO = partitioned linear order; WO = weak order; and LO = linear order.  $R^*$  being the asymmetric part of the ordered structure  $R$ .

- All other relations derive from these and from the ones provided in [7].

In a nutshell, in this section, we study in detail *partitioned linear orders*, a fragment of bilinear orders that is expressive enough to capture abundant incomparability, while being finitely characterized with forbidden patterns, in contrast to the category of bilinear orders, which cannot [19].

## V. REVISION AND PLOs

As PLOs are useful to represent a huge family of incomparable beliefs, we propose a set of postulates capturing the notion of faithful assignment to a PLO.

**Example 3.** Let us consider the expert beliefs of a doctor on flu ( $f$ ) and coughing patients ( $c$ ). In general, it can order the plausibility of each possible situation like this:  $\{\neg c \wedge \neg f\} \leq \{c \wedge f\} \leq \{c \wedge \neg f\} \leq \{\neg c \wedge f\}$ . Indeed, it seems to be more plausible that people are in good health but, if a patient is coughing, it should be affected by the flu. The least plausible situation is when a patient has the flu but is not coughing.

If we consider now a new variable  $w$ , which represents "it's winter", it could be much easier to him to consider two independent local linear orders, depending on the context  $w$  and  $\neg w$ :  $\{\neg w, \neg c, \neg f\} \leq_1 \{\neg w, c, \neg f\} \leq_1 \{\neg w, c, f\} \leq_1 \{\neg w, \neg c, f\}$  and  $\{w, \neg c, \neg f\} \leq_2 \{w, c, f\} \leq_2 \{w, c, \neg f\} \leq_2 \{w, \neg c, f\}$ . The first linear order expresses the fact that, when it is not winter, it is more plausible that a coughing patient does not have the flu (for instance because of an allergy) whereas, when it is winter, it is more plausible that a coughing patient is affected by the flu. The union of these two orders leads to a PLO.

PLO structures can also be found in belief merging [20], [21], in particular when the plausibility of the beliefs provided by the agents are incommensurable [5]. This problem is linked to the one of interpersonal comparison of utility [22], [23]. PLOs give a locality property to revision. Such approach studying locality can also be found in [24] where the locality of the inconsistencies is considered.

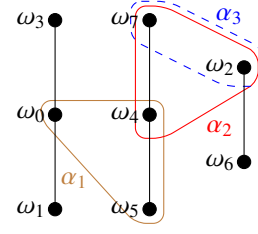


Fig. 7: PLO counter-examples for  $(R_2)$  and  $(R_6)$ .

### A. A New Rationality Postulate

First, we can remark that KM postulates do not allow capturing the structure of PLO and cannot be used for totally informative revision operators. For instance, consider the current beliefs  $\varphi$  of an agent such that  $\llbracket \varphi \rrbracket = \{\omega_1, \omega_5, \omega_6\}$  and a revision operator  $\circ_{PLO}$  leading to the PLO represented by Figure 7. Considering now a new information  $\alpha_1$  such that  $\llbracket \alpha_1 \rrbracket = \{\omega_0, \omega_4, \omega_5\}$ . If  $\circ_{PLO}$  satisfied  $(R_2)$ , as  $\llbracket \varphi \wedge \alpha_1 \rrbracket = \{\omega_5\}$ , the result of the revision should be such that  $\llbracket \varphi \circ_{PLO} \alpha_1 \rrbracket = \{\omega_5\}$ , while there is no reason to reject  $\omega_0$  that is no longer dominated. Similarly, a PLO revision operator should not satisfy  $(R_6)$ , by considering for example  $\alpha_2$  such that  $\llbracket \alpha_2 \rrbracket = \{\omega_2, \omega_4, \omega_7\}$  and  $\alpha_3$  such that  $\llbracket \alpha_3 \rrbracket = \{\omega_2, \omega_7\}$ . Then,  $\llbracket \varphi \circ_{PLO} (\alpha_2 \wedge \alpha_3) \rrbracket = \{\omega_2, \omega_7\}$ , whereas  $\llbracket \varphi \circ_{PLO} \alpha_2 \rrbracket = \{\omega_2, \omega_4\}$  and  $\llbracket (\varphi \circ_{PLO} \alpha_2) \wedge \alpha_3 \rrbracket = \{\omega_2\}$ .

This incompatibility with  $(R_2)$  and  $(R_6)$  is no surprise, since PLOs are partial orders that induce incomparabilities that KM operators cannot capture. We therefore place ourselves here in the framework of [6] to which we propose to add the following postulate  $(R_{PLO})$ .

$(R_{PLO})$  if  $\alpha \wedge \beta \vdash \perp$ , and  $(\varphi \circ \alpha) \vee (\varphi \circ \beta) \vdash \varphi \circ (\alpha \vee \beta)$ , then for every complete formula  $\gamma$ ,  $(\varphi \circ \alpha) \vee (\varphi \circ \gamma) \vdash \varphi \circ (\alpha \vee \gamma)$  or  $(\varphi \circ \beta) \vee (\varphi \circ \gamma) \vdash \varphi \circ (\beta \vee \gamma)$

The keystone of postulate  $(R_{PLO})$  is the entailment  $(\varphi \circ \alpha) \vee (\varphi \circ \beta) \vdash \varphi \circ (\alpha \vee \beta)$ . Once brought to ordered structures it leads to the incomparability between two items  $\alpha$  and  $\beta$  when  $\alpha$  and  $\beta$  are complete. Hence,  $(R_{PLO})$  states that for any trio of items where a pair of items is incomparable, there is at least another pair of incomparable items. Imposing  $(R_{PLO})$  to a transitive structure is equivalent to excluding the  $\vee$ -shape and  $\wedge$ -shape patterns.

This new postulate is compatible with the framework of [6], i.e. postulates  $(R_1)$ ,  $(R'_2)$ ,  $(R_3)$ – $(R_5)$ ,  $(R_7)$ ,  $(R_8)$ ,  $(R_{PLO})$  are consistent. We defer the proof to the next section – indeed, this is a corollary of Proposition 4.

Moreover, as expected, the following theorem shows that by adding  $(R_{PLO})$  to postulates of partial ordered revision  $((R_1)$ ,  $(R'_2)$ ,  $(R_3)$ ,  $(R_4)$ ,  $(R_5)$ ,  $(R_7)$ , and  $(R_8)$ ), it is possible to build a PLO-faithful assignment.

**Theorem 3.** A PLO-revision operator  $\circ$  satisfies postulates  $(R_1)$ ,  $(R'_2)$ ,  $(R_3)$ – $(R_5)$ ,  $(R_7)$ ,  $(R_8)$ ,  $(R_{PLO})$  iff there exists a  $p$ -faithful assignment that maps  $\varphi$  to a PLO  $\leq_\varphi$  such that:

$$\llbracket \varphi \circ \mu \rrbracket = \text{Min}(\llbracket \mu \rrbracket, \leq_\varphi).$$

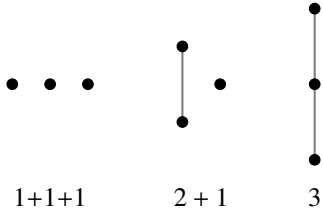


Fig. 8: Permitted patterns.

**Sketch of proof.** The property of partial order for  $\leq_\varphi$  follows directly from Theorem 2. The postulate  $(\mathbf{R}_{PLO})$ , for its part, constrained this partial order to have neither  $\vee$ -shape, nor  $\wedge$ -shape. By application of Proposition 3, this ensures that  $\leq_\varphi$  is a PLO.

More precisely, in a semantic point of view,  $(\mathbf{R}_{PLO})$  states that if two elements are incomparable, then every third element is incomparable with at least one of them. This leads to the three permitted patterns represented in Figure 8, which are the counterparts of the forbidden  $\vee$ -pattern and  $\wedge$ -pattern.

The next section presents an example of PLO-revision operator based on Hamming distance and resolute Voronoi diagrams.

### B. A Concrete PLO-revision Operator

We propose here a concrete PLO-revision operator. To this end, we use a resolute Voronoi operator coupled with two tie-breaking functions, and illustrate it in Example 4.

Tie-breaking functions are routinely used in collective decision-making and in (computational) social choice; see for instance [11], [12]. Given a set of alternatives, a tie-breaking function is able to select exactly a preferred element. For the purpose of this paper, we only consider deterministic tie-breaking functions  $\mathcal{T}$  on  $\Omega$  satisfying the following independence property (denoted  $\alpha$  in [25]).

$$\forall E, E' \in 2^\Omega \text{ s.t. } \mathcal{T}(E) \in E' \subseteq E, \text{ we have } \mathcal{T}(E') = \mathcal{T}(E).$$

**Definition 4** (Resolute Voronoi diagram). Given  $\varphi$  such that  $\llbracket \varphi \rrbracket = \{\omega_1, \dots, \omega_m\}$  and a tie-breaking function  $\mathcal{T}$ , then  $\Delta_\varphi^\mathcal{T} = \{V_{\omega_1}, \dots, V_{\omega_m}\}$  is the partition of  $\Omega$  such that:

$$\omega \in V_{\omega^*} \text{ iff } \mathcal{T}(\text{Argmin}_{\omega' \in \llbracket \varphi \rrbracket} d(\omega', \omega)) = \omega^*.$$

We can remark that this partition can equivalently be built using dilation operators [26].

**Definition 5** (Resolute Voronoi revision operator.). Given a formula  $\varphi$ , new information  $\alpha$ , and two tie-breaking functions  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , we define a binary relation  $\leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2}$  over  $\Omega$  by  $\omega \leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2} \omega'$  when  $\{\omega, \omega'\} \subseteq V_{\omega^*}$  and either  $d(\omega^*, \omega) < d(\omega^*, \omega')$  or  $d(\omega^*, \omega) = d(\omega^*, \omega')$  and  $\mathcal{T}_2(\{\omega, \omega'\}) = \omega$ , where  $V_{\omega^*} \in \Delta_\varphi^{\mathcal{T}_1}$  is a cell of the resolute Voronoi diagram of  $\varphi$ . The associated revision operator  $\circ_V$  is defined by:

$$\llbracket \varphi \circ_V \alpha \rrbracket = \text{Min}(\llbracket \alpha \rrbracket, \leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2})$$

Where  $\leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2}$  is a partitioned linear order whose comparability classes are the Voronoi cells  $\Delta_\varphi^{\mathcal{T}_1} = \{V_{\omega_1}, \dots, V_{\omega_m}\}$ .

$c \backslash ab$	$\{-a, \neg b\}$	$\{\neg a, b\}$	$\{a, b\}$	$\{a, \neg b\}$
$\{\neg c\}$	$\omega_0$	$\omega_2$	$\omega_6$	$\omega_4$
$\{c\}$	$\omega_1$	$\omega_3$	$\omega_7$	$\omega_5$

Fig. 9: Karnaugh map representing the cells induced by the dilation operation of  $\varphi$  from Example 4.

In other words, we use  $\mathcal{T}_1, \mathcal{T}_2$  as the two tie-breaking functions. In order to construct the PLO  $\leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2}$ , we will get the resolute Voronoi cells using Hamming distance around the models of  $\varphi$ . We require that each interpretation is associated to a unique closest model, and we break ties with  $\mathcal{T}_1$ . The partition of interpretations between cells centered around models of the initial beliefs can also be obtained by progressive dilation around the models.

Inside each cell, interpretations are partially ordered according to their distance to the center model, and we break ties using  $\mathcal{T}_2$ . Interpretations from different cells are incomparable. We thus end up with a PLO where the cells are exactly the comparability classes of the PLO and whose minimal elements are the models of the initial beliefs.

**Example 4.** As an example, let us take as the current belief of our agent  $\varphi$  such that  $\llbracket \varphi \rrbracket = \{\omega_1, \omega_5, \omega_6\}$ . Both for  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , we use the lexicographic order over the interpretations induces by the following order over the falsified variables :  $a < b < c$ .

Figure 9 shows the cells of the resolute Voronoi diagram of  $\varphi$  on a Karnaugh map<sup>1</sup> where  $V_{\omega_1}$ , the Voronoi cell containing  $\omega_1$ , is in blue,  $V_{\omega_5}$  is in red, and  $V_{\omega_6}$  is in green. On this figure, we can see that even though both  $\omega_4$  and  $\omega_7$  were at a Hamming distance of 1 to both  $\omega_5$  and  $\omega_6$ , they were allocated to  $\omega_5$  thanks to  $\prec^{\mathcal{T}_1}$ . This figure can be translated into the partial pre-order  $\prec_\varphi^{\mathcal{T}_1}$  shown Figure 10a. The usage of  $\prec^{\mathcal{T}_2}$  to sort out the indifferences leads us to the PLO  $\leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2}$  Figure 10b. Now, let us suppose that our agent is confronted to the new information  $\alpha = \neg c$ , we can compute  $\varphi \circ \alpha$  using the obtained  $\leq_\varphi^{\mathcal{T}_1, \mathcal{T}_2}$ . We get  $\llbracket \varphi \circ \alpha \rrbracket = \text{Min}(\llbracket \alpha \rrbracket, \leq_\varphi) = \{\omega_0, \omega_6, \omega_4\}$  as Figure 10b shows.

As expected, the following proposition states that any dilation using as a metric any KM compatible distance coupled with two tie-breaking functions is a PLO-revision operator.

**Proposition 4.** The revision operator  $\circ_V$  satisfies  $(\mathbf{R}_1)$ ,  $(\mathbf{R}'_2)$ ,  $(\mathbf{R}_3)$ – $(\mathbf{R}_5)$ ,  $(\mathbf{R}_7)$ ,  $(\mathbf{R}_8)$ , and  $(\mathbf{R}_{PLO})$ .

At first glance our operator can be confused with an update operator [27], [28], like the well-known PMA operator [29].

<sup>1</sup>The benefit of using a Karnaugh map representation is to have a Hamming distance of 1 between each cell and its direct neighbours.

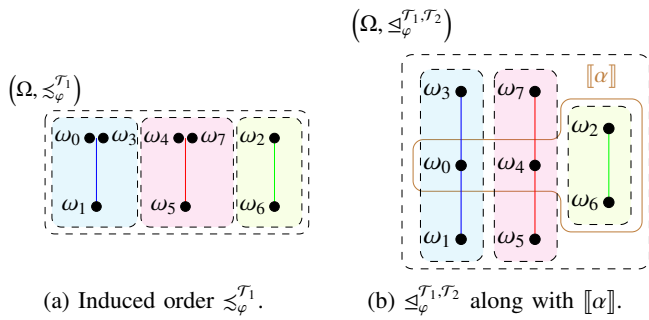


Fig. 10: Induced order structures from Example 4

In both cases, a family of orders is defined, each one of them corresponding to a model of the initial beliefs  $\varphi$ . Nevertheless, we can point out two significant differences. The first one is that the orders are not linear, but most of all, each orders we produce are not on the whole set of interpretation  $\Omega$  but only on the elements of a partition. Nevertheless, a thorough study on the links between update and PLO-operators remains to be conducted.

## VI. CONCLUSION AND PERSPECTIVES

We study in depth in this paper an order structure, generalizing linear orders, the partitioned linear orders PLOs, and study their properties in terms of expressiveness, graphical and numerical representations, and forbidden patterns. We place PLOs in a constellation of ordered structures. It cannot be captured either by interval orders, or by weak orders. It is also a rare fragment of bilinear orders which can be characterized in terms of forbidden patterns. After that, we propose a new postulate ( $\mathbf{R}_{PLO}$ ), which, coupled with the postulates of partially ordered revision, leads to a representation theorem capturing PLOs. We finally provide an example of an operator satisfying all these postulates.

These works open some interesting perspectives. The first that comes to mind is applying them to iterated belief revision [14]. In particular, an interesting question could be how to stay in the same ordered structures throughout the iteration process. It also seems to be natural to study partitioned weak orders, a direct generalization of PLO. Another question is to find some operators useful in practice. Those proposed in this article have been made for illustration purpose and to prove the consistency of our sets of postulates. As already mentioned, our works are linked to belief merging and share some similarity with update. The theoretical complexity of our revision operators remains an open question. Lastly, an in-depth computational study of PLO needs to be conducted in order to respond efficiently to certain complex requests, even independently of the belief revision context.

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