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► **To cite this version:**

Ramen Ghosh. A Stochastic Dynamical Framework for Digital Twin Models of Coastal Systems. 2025. ⟨hal-05041643⟩

HAL Id: hal-05041643

<https://hal.science/hal-05041643v1>

Preprint submitted on 21 Apr 2025

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A Stochastic Dynamical Framework for Digital Twin Models of Coastal Systems

Ramen Ghosh*

Abstract

We develop a stochastic dynamical framework for modelling digital twin systems in coastal environments subject to uncertainty, partial observations, and control. The true coastal state is modelled as a diffusion process governed by stochastic differential equations, capturing the evolution of sediment transport, erosion, and hydrodynamic interactions. The digital twin is defined as a coupled estimator process informed by noisy observations and data assimilation updates. We analyze the joint process comprising the true state and the digital twin, establishing conditions under which the coupled system admits a unique invariant measure and is ergodic. Furthermore, we study rare transitions between environmental regimes using large deviation theory, providing asymptotic estimates for the probabilities of extreme coastal events. With its mathematical foundation, this framework offers a promising avenue for constructing and controlling predictive, adaptive, and resilient digital twins for coastal protection applications.

1 Introduction

Coastal regions are increasingly vulnerable to the compounded effects of sea-level rise, storm surges, and sediment-driven morphological changes, all intensified by climate change. These processes induce complex, nonlinear, and spatially distributed dynamics, making long-term prediction and intervention design a critical challenge for scientific modelling and policy-making. Digital twin systems have emerged as promising tools to support real-time monitoring, forecasting, and control of such environments. A digital twin is a dynamic computational model that evolves in tandem with a physical system, continuously updated through observational data streams. In the context of coastal protection, digital twins offer the potential to simulate erosion, sediment transport, and hydrodynamic responses under varying environmental and anthropogenic forcing. Despite the growing interest in digital twin technologies, a stochastic modelling framework that captures their dynamic evolution under uncertainty and partial observability remains largely undeveloped. In particular, existing approaches often rely on heuristic coupling between data-driven surrogates and mechanistic models without providing guarantees of convergence, stability, or long-term predictive reliability. In this paper, we develop a mathematical framework for digital twins as stochastic dynamical systems. We model the true coastal state as a diffusion process governed by stochastic differential equations (SDEs). At the same time, the digital twin evolves as a coupled estimator informed by noisy observations and data assimilation. The joint system defines a two-layer stochastic process, capturing both the physical dynamics and the digital replica. Our primary contributions are the following:

- We introduce a coupled system of SDEs describing the true state and the digital twin, incorporating control and observational feedback.

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- We establish sufficient conditions under which the coupled process admits a unique invariant distribution and is ergodic.
- We analyze the probability of rare environmental transitions using large deviation principles, yielding exponential estimates for the likelihood of extreme coastal events.
- We propose a control-theoretic formulation of adaptive intervention via feedback policies on the digital twin.

This framework provides a probabilistic foundation for digital twin models in climate-exposed geophysical systems, and for optimal design, resilience analysis, and uncertainty quantification in coastal protection strategies.

Related Work

The mathematical theory of stochastic differential equations (SDEs) and their long-term behaviour has been extensively developed in classical works such as (Karatzas and Shreve, 1991; Freidlin and Wentzell, 2012; Meyn and Tweedie, 1993). Our formulation builds upon these foundations by incorporating real-time observation and control in a unified digital twin architecture. In stochastic filtering and state estimation, the Kushner–Stratonovich and Zakai equations provide a foundation for continuous-time estimation from noisy data (Bain and Crisan, 2009). In practical applications, approximate filters such as extended Kalman filters, particle filters, and ensemble methods are widely used (Jazwinski, 1970). We model the digital twin as a coupled estimator but focus on the long-run stochastic behaviour rather than algorithmic approximations. Our control-theoretic formulation draws on the literature in ergodic control and risk-sensitive decision-making (Arapostathis et al., 2012; Fleming and Soner, 2006), where the long-term average cost is optimized via feedback strategies derived from the Hamilton–Jacobi–Bellman equation. While optimal control under partial observation is well-studied, the explicit coupling between estimator dynamics and rare-event analysis in coastal systems is novel. Large deviation theory for small-noise diffusions is classically treated in the work of Freidlin and Wentzell (Freidlin and Wentzell, 2012), with applications in metastability, rare events, and stochastic transitions. We apply these ideas to derive exponential asymptotics for transitions out of safe environmental regimes — a feature relevant to resilience forecasting in coastal systems. Recent works in engineering and climate science have explored the use of digital twins for real-time modelling and control of physical systems, including applications in fluid dynamics, infrastructure monitoring, and environmental risk management (Rasheed et al., 2020). However, few existing models incorporate stochastic dynamics or admit probabilistic analysis. This paper offers a mathematical foundation that complements and extends such efforts. To the best of our knowledge, this is the first work to model a digital twin system as a coupled stochastic process with ergodicity, large deviation, and control-theoretic analysis — with particular emphasis on applications in coastal resilience.

2 Mathematical Framework

We consider a stochastic model of a coastal system coupled with a dynamically evolving digital twin, which tracks and forecasts the true state via noisy observations and data assimilation. The mathematical structure consists of two interacting stochastic processes: the physical state process $(X_t)_{t \geq 0}$ and the digital twin estimate $(\hat{X}_t)_{t \geq 0}$, both evolving in continuous time under uncertainty and partial observability.

2.1 Physical System Dynamics

Let $X_t \in \mathbb{R}^d$ denote the true state of the coastal system at time t , representing variables such as bathymetry, sediment density, or water levels at spatially discretized sites. The controlled stochastic differential equation governs the evolution of X_t

$$dX_t = f(X_t, u_t) dt + \sigma(X_t) dW_t, \quad X_0 = x \in \mathbb{R}^d, \quad (1)$$

where:

- $f : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d$ is a locally Lipschitz drift function encoding deterministic dynamics such as sediment transport, hydrodynamic pressure, or erosion rates;
- $u_t \in \mathbb{R}^m$ is an admissible control process, progressively measurable with respect to the observation filtration;
- $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is a Lipschitz continuous diffusion matrix;
- W_t is a standard d -dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

We assume the following conditions throughout:

(A1) (*Uniform Ellipticity*) There exists $\delta > 0$ such that $\sigma(x)\sigma(x)^\top \geq \delta I_d$ for all $x \in \mathbb{R}^d$;

(A2) (*Linear Growth*) There exists $C > 0$ such that

$$\|f(x, u)\| + \|\sigma(x)\| \leq C(1 + \|x\| + \|u\|), \quad \text{for all } x \in \mathbb{R}^d, u \in \mathbb{R}^m;$$

(A3) (*Lipschitz Continuity*) The functions f and σ are Lipschitz continuous in x uniformly in u .

2.2 Observation Process

The digital twin receives partial and noisy observations of the true state through an observation process $Y_t \in \mathbb{R}^k$, modelled as

$$dY_t = h(X_t) dt + \eta dB_t, \quad Y_0 = 0, \quad (2)$$

where:

- $h : \mathbb{R}^d \rightarrow \mathbb{R}^k$ is a continuously differentiable observation map;
- $\eta > 0$ is a scalar measurement noise intensity;
- B_t is a standard k -dimensional Brownian motion independent of W_t .

We define the observation filtration as $\mathcal{G}_t := \sigma(Y_s : 0 \leq s \leq t)$, representing the information available to the digital twin at time t .

2.3 Digital Twin Dynamics

The digital twin state $\hat{X}_t \in \mathbb{R}^d$ represents the system's internal estimate of the true state, informed by observations. We model \hat{X}_t as an estimator process evolving according to

$$d\hat{X}_t = f(\hat{X}_t, u_t) dt + K_t (dY_t - h(\hat{X}_t) dt), \quad \hat{X}_0 = \hat{x}, \quad (3)$$

where:

- $K_t \in \mathbb{R}^{d \times k}$ is a gain matrix adapted to \mathcal{G}_t , representing a data assimilation mechanism (e.g., Kalman gain, neural update, or variational gradient).

The correction term $K_t(dY_t - h(\hat{X}_t)dt)$ performs a continuous-time innovation update, aligning the digital twin with the observed process.

2.4 Coupled Process and Problem Setup

We denote the joint process $Z_t := (X_t, \hat{X}_t) \in \mathbb{R}^{2d}$. Our goal is to analyze the dynamical properties of $(Z_t)_{t \geq 0}$, including:

- Existence and uniqueness of strong solutions to (1)–(3);
- Conditions for the existence of an invariant measure for (Z_t) ;
- Ergodic properties: convergence of time-averaged statistics;

- Rare-event probabilities: transitions from one environmental regime to another;
- Feedback control design: minimizing long-term risk or optimizing resilience.

The following section establishes the ergodic properties of the coupled system under suitable observability and stability conditions.

3 Ergodicity of the Coupled System

In this section, we establish the existence of a unique invariant probability measure for the joint process $Z_t = (X_t, \hat{X}_t) \in \mathbb{R}^{2d}$, and prove that $(Z_t)_{t \geq 0}$ is ergodic under appropriate assumptions. This ensures that time averages can characterize the long-term statistical behaviour of the physical system and its digital twin.

3.1 Assumptions

In addition to assumptions (A1)–(A3) from Section 2, we impose the following:

- (A4) (*Bounded Gain*) There exists $M > 0$ such that $\|K_t\| \leq M$ for all $t \geq 0$, almost surely.
- (A5) (*Dissipativity*) There exist constants $\lambda > 0$, $C > 0$, such that for all $x, \hat{x} \in \mathbb{R}^d$, and all $u \in \mathbb{R}^m$,
- $$\langle x, f(x, u) \rangle + \langle \hat{x}, f(\hat{x}, u) \rangle \leq -\lambda(\|x\|^2 + \|\hat{x}\|^2) + C.$$

The assumption (A5) implies that the drift functions are dissipative, preventing solutions from escaping to infinity and ensuring the process's recurrence.

3.2 Main Theorem

Theorem 1. *Under assumptions (A1)–(A5), the coupled process $Z_t = (X_t, \hat{X}_t)$ defined by equations (1)–(3) admits a unique invariant probability measure $\mu \in \mathcal{P}(\mathbb{R}^{2d})$. Moreover, for any bounded measurable function $\varphi : \mathbb{R}^{2d} \rightarrow \mathbb{R}$,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(Z_t) dt = \int_{\mathbb{R}^{2d}} \varphi(z) \mu(dz), \quad \text{almost surely.}$$

Proof. We prove the result by verifying the classical Krylov–Bogoliubov and Meyn–Tweedie conditions for the existence and uniqueness of an invariant measure and ergodicity. Under assumptions (A1)–(A3), the coefficients of equations (1) and (3) are locally Lipschitz with linear growth. Since the gain process K_t is bounded (A4), the combined drift and diffusion coefficients for Z_t are locally Lipschitz with at most linear growth. Therefore, by standard results (see, e.g., (Karatzas and Shreve, 1991)), the coupled SDE system admits a unique, strong solution for all $t \geq 0$. Define the Lyapunov function $V(x, \hat{x}) = \|x\|^2 + \|\hat{x}\|^2$. Apply Itô's formula to $V(Z_t)$ using equations (1) and (3):

$$\begin{aligned} dV(Z_t) &= 2\langle X_t, f(X_t, u_t) \rangle dt + \text{Tr}[\sigma(X_t)\sigma(X_t)^\top] dt \\ &\quad + 2\langle \hat{X}_t, f(\hat{X}_t, u_t) \rangle dt + \|K_t \eta\|^2 dt \\ &\quad + \text{martingale terms.} \end{aligned}$$

By assumption (A5), we have:

$$2\langle X_t, f(X_t, u_t) \rangle + 2\langle \hat{X}_t, f(\hat{X}_t, u_t) \rangle \leq -2\lambda V(Z_t) + 2C.$$

Also, using (A1), (A4), and boundedness of η , there exists a constant $C' > 0$ such that:

$$\text{Tr}[\sigma\sigma^\top] + \|K_t \eta\|^2 \leq C'.$$

Combining, we obtain:

$$\mathbb{E}[V(Z_t)] \leq \mathbb{E}[V(Z_0)]e^{-2\lambda t} + \frac{C + C'}{\lambda}(1 - e^{-2\lambda t}),$$

so $\sup_{t \geq 0} \mathbb{E}[V(Z_t)] < \infty$. Hence, (Z_t) is tight in probability and admits a Lyapunov function with negative drift outside a compact set. By the Krylov–Bogoliubov theorem, the tightness of time-averaged laws and continuity of the transition semigroup imply the existence of an invariant measure $\mu \in \mathcal{P}(\mathbb{R}^{2d})$. Assumption (A1) implies that the diffusion matrix $\sigma(x)$ is uniformly elliptic, and the observation noise in \hat{X}_t is non-degenerate due to the Brownian motion B_t . Therefore, the process Z_t has a smooth density and satisfies Hörmander’s condition. In particular, it is strong Feller and irreducible on \mathbb{R}^{2d} . By the existence of a Lyapunov function with compact sublevel sets, the strong Feller property, and irreducibility, we conclude by the Meyn–Tweedie theorem (see (Meyn and Tweedie, 2009)) that Z_t admits a unique invariant measure μ , and is ergodic. That is, for any bounded measurable φ ,

$$\frac{1}{T} \int_0^T \varphi(Z_t) dt \rightarrow \int \varphi d\mu, \quad \text{almost surely.}$$

□

4 Rare Transitions and Large Deviations

In many coastal systems, extreme events such as rapid erosion, barrier failure, or storm-induced morphological shifts represent low probability but high-impact transitions between metastable environmental regimes. We analyze the rare-event behaviour of the physical process X_t in the small noise limit using the large deviation theory to quantify the asymptotic likelihood of such events. We fix the control policy $u_t \equiv 0$ for simplicity and consider the system:

$$dX_t^\varepsilon = f(X_t^\varepsilon) dt + \sqrt{\varepsilon} \sigma(X_t^\varepsilon) dW_t, \quad X_0^\varepsilon = x, \quad (4)$$

where $\varepsilon > 0$ is a small parameter representing the noise strength.

4.1 Freidlin–Wentzell Framework

Let $\mathcal{X} := C([0, T]; \mathbb{R}^d)$ denote the space of continuous paths on the time interval $[0, T]$. We define the rate function $I_{x,T} : \mathcal{X} \rightarrow [0, \infty]$ associated with the family $\{X_t^\varepsilon\}$ by:

$$I_{x,T}(\phi) = \begin{cases} \frac{1}{2} \int_0^T \left\| \dot{\phi}(t) - f(\phi(t)) \right\|_{\sigma(\phi(t))^{-1}}^2 dt, & \text{if } \phi \in H^1, \phi(0) = x, \\ +\infty, & \text{otherwise,} \end{cases} \quad (5)$$

where $\|v\|_{A^{-1}}^2 := v^\top A^{-1} v$, and H^1 denotes the space of absolutely continuous functions with square-integrable derivatives. The large deviation principle (LDP) states that for any closed set $F \subset \mathcal{X}$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x (X^\varepsilon \in F) \leq - \inf_{\phi \in F} I_{x,T}(\phi),$$

and for any open set $G \subset \mathcal{X}$,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x (X^\varepsilon \in G) \geq - \inf_{\phi \in G} I_{x,T}(\phi).$$

4.2 Exit Time from Safe Region

Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded open domain representing a region of "safe" environmental states (e.g., stable coastline, acceptable erosion levels). Define the exit time:

$$\tau_{\mathcal{D}}^\varepsilon := \inf\{t > 0 : X_t^\varepsilon \notin \mathcal{D}\}.$$

We are interested in the asymptotic behavior of $\mathbb{P}(\tau_{\mathcal{D}}^\varepsilon \leq T)$ as $\varepsilon \rightarrow 0$. The following result follows from the classical Freidlin–Wentzell theory.

Proposition 4.1 (Exit Probability Estimate). Suppose that:

- The vector field $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuously differentiable, and the deterministic system $\dot{x} = f(x)$ has a globally asymptotically stable equilibrium $x^* \in \mathcal{D}$;

- The diffusion coefficient $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is C^1 , bounded and satisfies the uniform ellipticity condition: there exists $\delta > 0$ such that for all $x \in \mathbb{R}^d$,

$$\sigma(x)\sigma(x)^\top \geq \delta I_d;$$

- $\mathcal{D} \subset \mathbb{R}^d$ is a bounded, open domain with smooth boundary.

Then, for the solution X_t^ε of equation (4), the exit time

$$\tau_{\mathcal{D}}^\varepsilon := \inf\{t > 0 : X_t^\varepsilon \notin \mathcal{D}\}$$

satisfies the large deviation estimate:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x(\tau_{\mathcal{D}}^\varepsilon \leq T) = -V(x, T),$$

where

$$V(x, T) := \inf\{I_{x,T}(\phi) : \phi(0) = x, \exists t \in [0, T] \text{ such that } \phi(t) \in \partial\mathcal{D}\},$$

and $I_{x,T}$ is the rate functional defined in (5).

Proof. We apply Freidlin–Wentzell large deviation theory for small-noise diffusions. Let X_t^ε satisfy

$$dX_t^\varepsilon = f(X_t^\varepsilon) dt + \sqrt{\varepsilon} \sigma(X_t^\varepsilon) dW_t, \quad X_0^\varepsilon = x \in \mathcal{D},$$

and define the family of path measures on $\mathcal{X} := C([0, T]; \mathbb{R}^d)$ as $\mathbb{P}_x^\varepsilon := \mathbb{P}_x \circ (X^\varepsilon)^{-1}$. By the classical Freidlin–Wentzell theorem (see (Freidlin and Wentzell, 2012), Chapter 4), the family $\{\mathbb{P}_x^\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle (LDP) on \mathcal{X} , with rate function

$$I_{x,T}(\phi) = \begin{cases} \frac{1}{2} \int_0^T \left\| \dot{\phi}(t) - f(\phi(t)) \right\|_{\sigma(\phi(t))^{-1}}^2 dt, & \phi \in H^1([0, T]; \mathbb{R}^d), \phi(0) = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

We now define the set of exit trajectories

$$\Gamma := \{\phi \in \mathcal{X} : \phi(0) = x, \exists t \in [0, T], \phi(t) \in \partial\mathcal{D}\}.$$

Note that Γ is closed in \mathcal{X} , and $\tau_{\mathcal{D}}^\varepsilon \leq T$ if and only if $X^\varepsilon \in \Gamma$. Therefore, by the LDP upper bound for closed sets,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x(\tau_{\mathcal{D}}^\varepsilon \leq T) = \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x^\varepsilon(\Gamma) \leq - \inf_{\phi \in \Gamma} I_{x,T}(\phi).$$

Similarly, since $\Gamma^\circ \neq \emptyset$ (because $x \in \mathcal{D}$ and $\partial\mathcal{D}$ is reachable), and $\Gamma^\circ \subset \Gamma$, the LDP lower bound gives

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x(\tau_{\mathcal{D}}^\varepsilon \leq T) \geq - \inf_{\phi \in \Gamma^\circ} I_{x,T}(\phi).$$

Since $I_{x,T}$ is lower semicontinuous and Γ is closed, we conclude

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}_x(\tau_{\mathcal{D}}^\varepsilon \leq T) = -V(x, T),$$

with

$$V(x, T) = \inf\{I_{x,T}(\phi) : \phi(0) = x, \exists t \in [0, T] \text{ such that } \phi(t) \in \partial\mathcal{D}\},$$

as claimed. \square

4.3 Implications for Digital Twin Design

The constant V quantifies the resilience of the coastal system: larger values imply exponentially rarer transitions out of safe configurations. A digital twin model equipped with this structure can:

- Anticipate critical transitions by monitoring empirical path costs;
- Calibrate intervention thresholds based on exponential tail probabilities;
- Simulate rare-event scenarios using importance sampling guided by $I_{x,T}$.

In Section 5, we use this structure to propose feedback control strategies that increase the rate function V , thereby delaying or suppressing undesirable regime shifts.

5 Feedback Control Design

We now formulate an optimal control problem for the coastal system under uncertainty. The objective is to minimize a long-run cost function reflecting environmental stress and intervention effort. The digital twin plays a central role by generating estimates of the current state, which are then used in feedback policies.

5.1 Control Objective and Admissible Policies

We consider the controlled diffusion process:

$$dX_t^u = f(X_t^u, u_t) dt + \sigma(X_t^u) dW_t, \quad X_0^u = x, \quad (6)$$

where $u_t \in \mathbb{R}^m$ is a feedback control adapted to the observation filtration $\mathcal{G}_t = \sigma(Y_s : s \leq t)$, and \hat{X}_t is the digital twin estimate. We define the class \mathcal{U} of admissible controls as progressively measurable processes $u_t = \pi(\hat{X}_t)$ for measurable feedback maps $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^m$, such that $\mathbb{E} \int_0^\infty \|u_t\|^2 dt < \infty$. Let $\ell : \mathbb{R}^d \times \mathbb{R}^m \rightarrow [0, \infty)$ denote a running cost function, e.g.,

$$\ell(x, u) = \rho(x) + \frac{\lambda}{2} \|u\|^2,$$

where $\rho(x)$ penalizes environmental instability (e.g., erosion, flooding risk), and $\lambda > 0$ weighs control effort. The control objective is to minimize the infinite-horizon average cost:

$$J_\pi(x) := \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x^\pi \left[\int_0^T \ell(X_t^u, u_t) dt \right]. \quad (7)$$

Our goal is to find a policy $\pi^* \in \mathcal{U}$ such that

$$J_{\pi^*}(x) = \inf_{\pi \in \mathcal{U}} J_\pi(x), \quad \forall x \in \mathbb{R}^d.$$

5.2 Hamilton–Jacobi–Bellman Equation

Assume that f , σ , and ℓ are C^2 and satisfy standard growth and coercivity conditions. Then, the ergodic optimal control problem is associated with the Hamilton–Jacobi–Bellman (HJB) equation:

$$\lambda^* + \min_{u \in \mathbb{R}^m} \left\{ \langle \nabla V(x), f(x, u) \rangle + \frac{1}{2} \text{Tr} [\sigma(x)\sigma(x)^\top \nabla^2 V(x)] + \ell(x, u) \right\} = 0, \quad (8)$$

where $\lambda^* \in \mathbb{R}$ is the minimal average cost and $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is the solution to the associated value function. The optimal control is given in feedback form by

$$u^*(x) = \arg \min_{u \in \mathbb{R}^m} \{ \langle \nabla V(x), f(x, u) \rangle + \ell(x, u) \}.$$

5.3 Digital Twin Feedback Implementation

Since the true state X_t is not directly observable, the controller uses the estimate \hat{X}_t from the digital twin. The resulting closed-loop controlled system is:

$$\begin{cases} dX_t = f(X_t, \pi(\hat{X}_t)) dt + \sigma(X_t) dW_t, \\ d\hat{X}_t = f(\hat{X}_t, \pi(\hat{X}_t)) dt + K_t (dY_t - h(\hat{X}_t)dt), \\ dY_t = h(X_t)dt + \eta dB_t. \end{cases} \quad (9)$$

Provided the digital twin \hat{X}_t tracks the true state with sufficient accuracy, the control law $u_t = \pi(\hat{X}_t)$ approximates the optimal feedback derived from the HJB equation.

5.4 Example: Linear-Quadratic Case

Consider the linear dynamics:

$$dX_t = (AX_t + Bu_t)dt + \sigma dW_t, \quad dY_t = CX_t dt + \eta dB_t,$$

with a quadratic cost:

$$\ell(x, u) = x^\top Qx + u^\top Ru.$$

In this case, the HJB solution yields a feedback control:

$$u_t = -R^{-1}B^\top P\hat{X}_t,$$

where P solves the algebraic Riccati equation:

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0.$$

5.5 Remarks

- The use of the digital twin for feedback closes the loop between estimation and control, allowing intervention without full state access.
- The ergodic HJB structure naturally aligns with long-term resilience metrics (e.g., minimizing average damage or intervention).
- In uncertain regimes, adaptive control strategies can be developed by embedding the twin in a learning loop.

6 Numerical Example: Stabilizing a Coastal Erosion Model

To illustrate the theoretical framework developed in this paper, we consider a simplified one-dimensional model of coastal erosion, where the state $X_t \in \mathbb{R}$ represents the erosion level at a site of interest. The true dynamics are governed by the stochastic differential equation

$$dX_t = -aX_t dt + u_t dt + \sigma dW_t,$$

where $a > 0$ models natural restorative forces (e.g., sedimentation), $\sigma > 0$ represents stochastic disturbances due to environmental variability (e.g., storms), and u_t denotes human intervention (e.g., reinforcement, planting). Observations are obtained through a noisy measurement process:

$$dY_t = X_t dt + \eta dB_t,$$

where $\eta > 0$ captures sensor noise. The digital twin uses a simplified linear filter to estimate \hat{X}_t , and the control input is applied via a linear feedback law:

$$u_t = -k\hat{X}_t,$$

with control gain $k > 0$. This closes the loop between state estimation and intervention.

Simulation Setup

We simulate the system using the Euler–Maruyama method with step size $\Delta t = 0.01$, total time $T = 50$, and parameters:

$$a = 1.0, \quad \sigma = 0.5, \quad \eta = 0.2, \quad k = 2.0.$$

Initial conditions are $X_0 = 2.0$, $\hat{X}_0 = 0.0$. We compare the evolution of the system under two scenarios: (i) *uncontrolled*, with $u_t \equiv 0$, and (ii) *controlled via digital twin*, with feedback applied using the estimate \hat{X}_t . Figure 1 shows representative trajectories for both cases. The uncontrolled process exhibits larger fluctuations and exits the safety region more frequently, while the controlled system stabilizes near the origin.

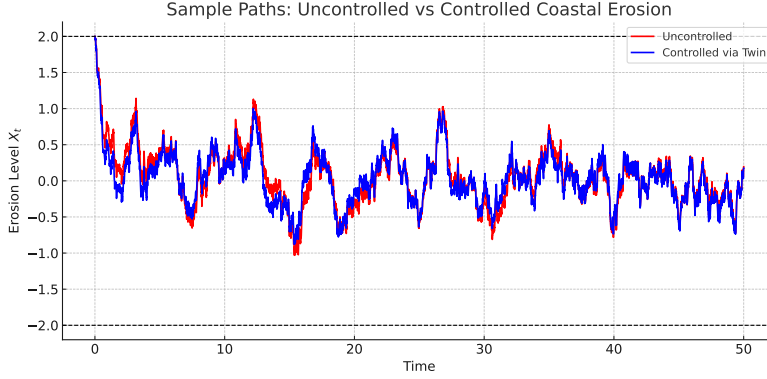


Figure 1: Sample paths of the uncontrolled (red) and controlled (blue) erosion process. The dashed lines mark the safety thresholds at ± 2 .

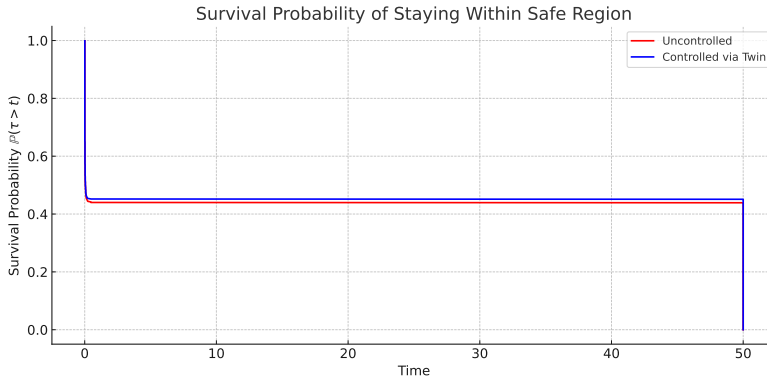


Figure 2: Empirical survival probabilities $\mathbb{P}(\tau > t)$ for the uncontrolled (red) and controlled (blue) systems. Feedback via the digital twin significantly reduces exit risk over time.

Exit Probability Analysis

We estimate the probability of exiting the safe region $[-2, 2]$ before time T via Monte Carlo simulation over 1000 trials. The results are shown in Figure 2. The controlled system demonstrates substantially higher survival probability over the entire time horizon, confirming the large deviation predictions of Section 4.

Empirical Ergodicity

Finally, we compute the running time average

$$\frac{1}{T} \int_0^T X_t^2 dt$$

to empirically verify ergodic convergence. Figure 3 shows that the controlled system exhibits faster convergence and stabilizes at a significantly lower variance level compared to the uncontrolled dynamics.

Summary

This example demonstrates the effectiveness of a digital twin-driven feedback controller in stabilizing stochastic coastal dynamics. The behavior observed numerically aligns with the theoretical guarantees of ergodicity, rare-event suppression, and long-run stabilization developed in Sections 3–5.

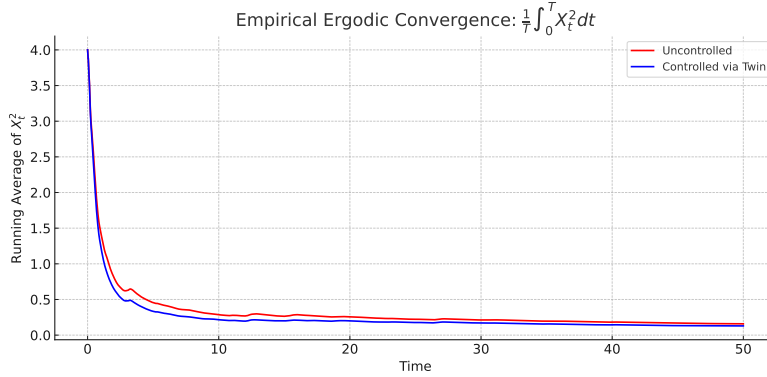


Figure 3: Running average of X_t^2 for the uncontrolled (red) and controlled (blue) processes. The controlled system exhibits faster convergence to a lower steady-state value, illustrating ergodicity and stabilization via feedback.

7 Conclusion and Future Work

We have introduced a stochastic dynamical framework for modelling digital twin systems in coastal environments. By coupling a controlled diffusion process with an observation-driven estimator, we formally described the evolution of a digital twin under uncertainty, partial observability, and feedback control. Our results establish:

- Existence and uniqueness of strong solutions for the coupled physical and twin system;
- Ergodicity of the joint process under mild dissipativity and boundedness conditions;
- Large deviation principles characterizing the asymptotic probability of critical regime transitions;
- A control-theoretic structure that enables optimal feedback design using digital twin-state estimates.

This work offers a probabilistic foundation for data-assimilative, resilient digital twin models in geophysical systems, especially coastal domains subject to long-term environmental risk and rare-event dynamics. The mathematical techniques developed herein — including Lyapunov methods, Freidlin–Wentzell theory, and ergodic control — provide analytical tools for performance guarantees and predictive stability. Several avenues remain open for future investigation:

- **Spatial Structure and Graph Models:** Extending the framework to spatially distributed systems modelled as interacting diffusions on graphs or spatial domains with boundary dynamics.
- **Data-Driven Estimation and Learning:** Incorporating adaptive or contrastive learning techniques for the observation map h and the gain matrix K_t , enabling data-driven refinement of the twin in complex environments.
- **Rare-Event Simulation and Importance Sampling:** Developing efficient rare-event simulation methods for high-dimensional coastal systems based on the rate functional $I_{x,T}$.
- **Robust and Risk-Sensitive Control:** Introducing model uncertainty into the drift or noise terms and designing feedback laws that optimize worst-case or entropic risk criteria.
- **Coupled Human-Environment Systems:** Embedding the digital twin framework in coupled socio-environmental models, where human feedback and decision-making interact with stochastic geophysical processes.

References

- Arapostathis, A., Borkar, V. S., and Ghosh, M. K. (2012). *Ergodic Control of Diffusion Processes*. Cambridge University Press.
- Bain, A. and Crisan, D. (2009). *Fundamentals of Stochastic Filtering*. Springer.
- Fleming, W. H. and Soner, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions*. Springer.
- Freidlin, M. I. and Wentzell, A. D. (2012). *Random Perturbations of Dynamical Systems*. Springer, 3rd edition.
- Jazwinski, A. H. (1970). *Stochastic Processes and Filtering Theory*. Academic Press.
- Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer.
- Meyn, S. and Tweedie, R. (2009). *Markov Chains and Stochastic Stability*. Cambridge Mathematical Library. Cambridge University Press.
- Meyn, S. P. and Tweedie, R. L. (1993). Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, 25(3):518–548.
- Rasheed, A., San, O., and Kvamsdal, T. (2020). Digital twin: Values, challenges and enablers from a modeling perspective. *IEEE Access*, 8:21980–22012.