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# Converse Lyapunov results for uniform stability properties

Ihab Haidar <sup>\*</sup>and Paolo Mason <sup>†</sup>

April 17, 2025

## Abstract

Converse Lyapunov results express the fact that, for a given dynamical model, the existence of a Lyapunov function is not only a sufficient condition for its stability, but also a necessary criterion. These results provide a solid justification for the Lyapunov approach as a tool to investigate stability, and also make it possible to prove auxiliary robustness properties of dynamical systems. This survey presents converse Lyapunov results concerned with uniform asymptotic stability of dynamical systems in the presence of parametric perturbations, giving a particular emphasis to the infinite-dimensional case.

## 1 Introduction

Lyapunov functions are extremely effective tools for assessing stability properties of dynamical systems without explicitly computing their trajectories. They are often natural objects when dealing with models coming from applications, the classical example being that of the energy of a dissipative mechanical system. The use of a Lyapunov function in order to determine a stability property is usually referred to as the second or direct Lyapunov method. On the other hand, a converse Lyapunov theorem is a result establishing that any system with certain stability properties admits a Lyapunov function.

The usefulness of a converse Lyapunov result is twofold. First of all, it justifies the use of numerical algorithms based on the search of a Lyapunov function as an effective test for stability (although such algorithms can only deal with a restricted family of candidate Lyapunov functions). Secondly, in several cases converse Lyapunov results can be used to provide some extra information

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on the dynamics. For example, converse Lyapunov results can be used to show that a certain stability property is robust with respect to perturbations (see, e.g., [8, 17, 25, 69] for the stability of interconnected systems, [20, 29, 68] for input-to-state stability) or to analyze the stability of a singularly perturbed system, i.e., a system whose components evolve on two different time scales, on the basis of the stability of reduced subsystems (see, e.g., [9, 23, 73]).

The present survey focuses on continuous-time systems admitting multiple solutions for each initial condition, including differential equations in the presence of parametric perturbations and differential inclusions. For this kind of systems, we review the main converse Lyapunov results associated with uniform asymptotic stability properties. Uniformity is intended here with respect to all possible solutions of the system and all possible initial times. We consider both finite-dimensional and infinite-dimensional models. For finite-dimensional systems the subject of converse Lyapunov theorems is nowadays well understood; converse Lyapunov results have been obtained for very general models [1, 36] and the limitations of the approach are rather clear [63]. We will briefly recall these results, as the survey [27] already provides a well-documented description of most converse Lyapunov results for finite-dimensional systems. In addition to these results, we will discuss, for the particular case of switched linear systems, the notion of universal class of Lyapunov functions. A universal class is, roughly speaking, a family of candidate Lyapunov functions large enough to identify all uniformly asymptotically stable switched systems, and sufficiently simple to handle for numerical algorithms.

The case of infinite-dimensional systems is more delicate and still the subject of ongoing research. In this paper, we specifically consider systems that evolve on an infinite-dimensional space and satisfy the axioms defining a forward complete dynamical system [52]. In this framework, one of the main differences, compared to the usual results for finite-dimensional systems, is that Lyapunov functions do not necessarily satisfy norm-dependent bounds. In particular, direct Lyapunov results exist which involve *non-coercive* Lyapunov functions, i.e., functions which are positive outside the origin but do not necessarily admit a norm-dependent lower bound. However, such results require additional assumptions on the dynamics in order to guarantee the stability of the system [15, 22, 51]. Concerning converse Lyapunov theorems for uniform stability properties of infinite-dimensional systems, several general results have been obtained in the last two decades [22, 26, 51]. Note that, unlike the finite-dimensional case, these results do not guarantee the smoothness of the associated Lyapunov function, as the known constructions lead to Lyapunov functions which can only be proved to be continuous or Lipschitz continuous on bounded sets.

The paper is organized as follows. In Section 2, we introduce the main stability notions used throughout the paper. In Section 3, after introducing the classical Lyapunov approach for the stability analysis in the finite-dimensional case, we recall some relevant converse Lyapunov results concerning uniform asymptotic stability. The most general among such results are obtained in the framework of differential inclusions. We also provide a Lyapunov characteriza-

tion of the notion of stability with respect to two measures and discuss some results concerning universal classes of Lyapunov functions for switched linear systems. In Section 4 we discuss direct and converse Lyapunov results in the infinite-dimensional framework of forward complete dynamical systems. The particular case of retarded functional differential equations is also discussed. Finally, in Section 5 we provide some examples of applications of converse Lyapunov results for finite and infinite-dimensional systems.

## 1.1 Notations

By  $\mathbb{R}$  we denote the set of real numbers, by  $\mathbb{R}_+$  the set of non-negative real numbers, and by  $\overline{\mathbb{R}}$  the extended real line.  $(\mathbb{R}^n, |\cdot|)$  denotes the  $n$ -dimensional Euclidean space, where  $n$  is a positive integer and  $|\cdot|$  is the Euclidean norm. By  $\mathbb{C}$  we denote the set of complex numbers and  $\Re(z)$  denotes the real part of  $z \in \mathbb{C}$ .

Given a Banach space  $(X, \|\cdot\|)$  and a bounded linear operator  $T$  on  $X$ , we write  $\|T\| := \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|}{\|x\|}$  to denote the usual operator norm of  $T$ . Given  $r > 0$ , and  $x \in X$ ,  $B(x, r)$  denotes the closed ball of center  $x$  and radius  $r$ . More generally, given a set  $C \subset X$ , we denote by  $B(C, r)$  the set of points  $x$  whose distance from  $C$ , defined as  $\|x\|_C := \inf_{y \in C} \|x - y\|$ , is less than or equal to  $r$ . In the special case  $X = \mathbb{R}^n$ , we denote as  $|x|_C$  the distance from  $x$  to  $C$  and, whenever we find it convenient, we also write  $C + B(0, r)$  to denote  $B(C, r)$ .

A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be of class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\alpha(0) = 0$ ; it is said to be of class  $\mathcal{K}_\infty$  if it is of class  $\mathcal{K}$  and unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be of class  $\mathcal{KL}$  if  $\beta(\cdot, t)$  is of class  $\mathcal{K}$  for each  $t \in \mathbb{R}_+$  and, for each  $s \in \mathbb{R}_+$ ,  $\beta(s, \cdot)$  is nonincreasing and converges to zero as  $t$  tends to infinity. A function  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be positive definite if it is continuous,  $\eta(0) = 0$  and  $\eta(s) > 0$  for every  $s > 0$ .

Given  $F \subset \mathbb{R}^n$ ,  $\overline{\text{co}} F$  denotes the smallest closed convex subset of  $\mathbb{R}^n$  containing  $F$ .

For a  $\mathcal{C}^1$  function (i.e., a continuously differentiable function)  $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote the derivative with respect to the first variable (time variable) as  $\frac{\partial}{\partial t} V$  and the derivative with respect to the second one (space variable) as  $\nabla_x V$ . If  $V$  does not depend on time we simply denote  $\nabla_x V$  by  $\nabla V$ .

A set-valued map  $F$  that associates with every element of a space  $X$  a subset of another space  $Y$  is denoted  $F : X \rightsquigarrow Y$ .

## 2 Stability Notions

In the following, we present the main stability notions used throughout the paper. In order to encompass all considered settings, here we adopt an abstract point of view by identifying a system  $\Sigma$  with a collection of time-dependent trajectories  $x : \mathbb{R}_+ \rightarrow X$ ,  $X$  being a Banach space endowed with a norm  $\|\cdot\|$ .

Let  $C$  be a closed subset of  $X$ , positively invariant with respect to the dynamics  $\Sigma$ , that is, if  $x(\cdot) \in \Sigma$  is such that  $x(t_0) \in C$  for some  $t_0 \in \mathbb{R}_+$ , then

$x(t) \in C$  for every  $t \geq t_0$ . The notion of stability of  $\Sigma$  with respect to  $C$  is given below.

**Definition 2.1.** We say that  $\Sigma$  is (Lyapunov) stable with respect to  $C$  if for every  $r > 0$  there exists  $r' > 0$  such that if  $x(\cdot) \in \Sigma$  satisfies  $x(t_0) \in B(C, r')$  for some  $t_0 \in \mathbb{R}_+$  then  $x(t) \in B(C, r)$  for every  $t \geq t_0$ .

Next, we introduce the notion of asymptotic stability of  $\Sigma$  with respect to  $C$ .

**Definition 2.2.**  $\Sigma$  is said to be asymptotically stable with respect to  $C$  (*C-AS*, for short), if it is stable and if it is locally attractive, that is, there exists  $r > 0$  such that, if  $x(\cdot) \in \Sigma$  satisfies  $x(t_0) \in B(C, r)$  for some  $t_0 \in \mathbb{R}_+$  then  $\lim_{t \rightarrow \infty} \|x(t)\|_C = 0$ .  $\Sigma$  is said to be globally asymptotically stable with respect to  $C$  (*C-GAS*, for short) if the condition  $\lim_{t \rightarrow \infty} \|x(t)\|_C = 0$  holds for every trajectory  $x(\cdot) \in \Sigma$ .

In the previous definition, different trajectories may tend to  $C$  with different convergence rates. It is reasonable, for instance for non-autonomous or parameter-dependent differential equations, to ask whether convergence of all trajectories occurs with a uniform rate. The notion of uniform asymptotic stability with respect to  $C$ , which fulfills this property, is introduced next.

**Definition 2.3.**  $\Sigma$  is said to be uniformly asymptotically stable with respect to  $C$  (*C-UAS*, for short), if there exist  $r > 0$  and a function  $\beta \in \mathcal{KL}$  such that the following inequality holds for every trajectory  $x(\cdot)$  of  $\Sigma$  and every  $t_0 \in \mathbb{R}_+$  such that  $x(t_0) \in B(C, r)$

$$\|x(t)\|_C \leq \beta(\|x(t_0)\|_C, t - t_0), \quad \forall t \geq t_0. \quad (1)$$

We say that system  $\Sigma$  is uniformly globally asymptotically stable with respect to  $C$  (*C-UGAS*, for short) if (1) holds for every  $x(t_0) \in X$ .

When  $C = \{0\}$ , the previous definitions reduce to the classical definitions of stability and asymptotic stability *at the origin*. In the sequel, such notions of asymptotic stability will be simply denoted as *AS*, *GAS*, *UAS* and *UGAS*.

We stress that several alternative notions of stability and asymptotic stability have been proposed in the history of Lyapunov-type results. For instance, non-uniform versions of the previous stability notions may be natural when considering trajectories of a non-autonomous differential equation (see, e.g. [43]). Stability notions that hold for every initial condition in a specified set  $D \subset X$  (e.g., the domain of attraction of a stable equilibrium) are often said to hold *in the large* [2, 3].

Whenever  $C = \{0\}$  and the convergence to the equilibrium occurs with an exponential rate, we have the following definition.

**Definition 2.4.** Given  $r > 0$ , we say that  $\Sigma$  is uniformly exponentially stable in  $B(0, r)$  (*UES in  $B(0, r)$* , for short) if (1) holds (with  $C = \{0\}$ ) whenever  $x(t_0) \in B(0, r)$ ,  $t_0 \in \mathbb{R}_+$  with  $\beta(s, t) = Me^{-\lambda t}$ , for some  $M \geq 1$  and  $\lambda > 0$ .

If the latter holds for some  $r > 0$ , then we say that  $\Sigma$  is uniformly locally exponentially stable at the origin (ULES, for short), while if it holds for every  $r > 0$  we say that  $\Sigma$  is uniformly globally exponentially stable at the origin (UGES, for short).  $\Sigma$  is said to be uniformly semi-globally exponentially stable at the origin (USGES, for short) if for every  $r > 0$  there exist  $M(r) \geq 1$  and  $\lambda(r) > 0$  such that (1) holds (with  $C = \{0\}$ ) in  $B(0, r)$  with  $\beta(s, t) = M(r)e^{-\lambda(r)t_s}$ .

### 3 Finite-Dimensional Systems

The Lyapunov direct method was originally developed for non-autonomous ordinary differential equations of the form

$$\dot{x} = f(t, x), \quad (2)$$

where  $x$  belongs to a finite-dimensional space and  $t \in \mathbb{R}_+$ . These early results (see, e.g., [28, 37, 39, 43]) were mainly concerned with local stability properties of the system at isolated equilibrium points. To fix ideas, we present below a Lyapunov-type result concerning the UGAS property of (2); see, e.g., [27, Theorem 2.3] for a more accurate picture, taking into account several other stability properties.

**Theorem 3.1.** *The existence of a continuously differentiable function  $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \bar{\alpha}(|x|), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

for some  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  and such that

$$\dot{V}(t, x) := \frac{\partial}{\partial t} V(t, x) + \nabla_x V(t, x) f(t, x) \leq -\eta(|x|), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

for some positive definite function  $\eta$  guarantees that (2) is UGAS.

A function  $V$  satisfying the conditions of Theorem 3.1 is called a *Lyapunov function* for (2). In the rest of the paper suitable variations of the previous notion of Lyapunov function will be adopted depending on the context.

The Lyapunov approach is particularly significant when the stability property under consideration is proved to be equivalent to the existence of a Lyapunov function, that is, when a *converse Lyapunov theorem* can be established. In fact, characterizing the stability by the existence of a Lyapunov function allows one to investigate additional properties associated with stable systems, as already mentioned in the introduction.

A first example of converse Lyapunov theorem has been given by A. M. Lyapunov [37] for the following autonomous linear system

$$\dot{x} = Ax, \quad (3)$$

where the square matrix  $A$  is Hurwitz. In fact, for a given positive definite (symmetric) matrix  $Q$ , the matrix

$$P = \int_0^\infty e^{tA^T} Q e^{tA} dt \quad (4)$$

is well-defined (since the norm of the integrand decays exponentially with  $t$ ), positive definite, and satisfies

$$A^T P + PA = -Q. \quad (5)$$

Therefore, the quadratic function  $V(x) = x^T P x$  satisfies  $\dot{V}(x) = -x^T Q x < 0$  for every  $x \in \mathbb{R}^n \setminus \{0\}$  and it is then a Lyapunov function for the dynamics  $\dot{x} = Ax$ . This can be formally written as follows.

**Theorem 3.2** ([37]). *Given any positive definite matrix  $Q$ , there exists a positive definite matrix  $P$  satisfying (5) if and only if (3) is UGAS.*

A somewhat similar construction as in (4) was proposed by J. L. Massera [43] for nonautonomous systems

$$\dot{x} = f(t, x), \quad (6)$$

whenever the dependence on time is periodic. Massera proved that, if the system (6) is UAS, then there exists a  $\mathcal{C}^1$  function  $V$  satisfying, in a neighborhood of the origin, the conditions of Theorem 3.1 of the form

$$V(t, z) = \int_t^\infty \varphi(|x(s)|) ds, \quad (7)$$

where  $x(\cdot)$  is the solution of the differential equation with boundary condition  $x(t) = z$  and  $\varphi$  is an appropriate  $\mathcal{K}$  function. Let us notice that a construction of a Lyapunov function, alternative to (7), for an asymptotically stable system has been given, for instance, in [74]. Such a Lyapunov function takes the form

$$V(t, z) = \sup_{s \geq t} \varphi(|x(s)|) e^{cs}, \quad (8)$$

where  $x(\cdot)$  is the solution of the differential equation with boundary condition  $x(t) = z$ ,  $c > 0$ , and  $\varphi$  is an appropriate  $\mathcal{K}_\infty$  function.

Several other early converse Lyapunov results could be mentioned, such as [38, 61]. A fairly complete picture of such early results has been given in the brief paper [45].

The first results establishing the existence of a *smooth* time-dependent Lyapunov function in the presence of an asymptotically stable equilibrium have been obtained by Massera and Kurzweil in [31, 44] under weak regularity assumptions on the vector field  $f$ .

In the next section, we provide a brief overview of existing converse Lyapunov theorems for uncertain dynamics.

### 3.1 Converse Lyapunov results for uniform stability properties

Lyapunov characterizations of uniform stability notions for systems with multiple solutions or with uncertain dynamics were developed as early as the 1960s. For example, Roxin [66] (see also [64, 65]) established the equivalence between a local asymptotic stability property and the existence of a suitable Lyapunov function for systems described via their time-dependent attainable sets. However, note that the results obtained in this context appear to be difficult to apply since the attainable sets at fixed times can rarely be characterized precisely. Furthermore, the corresponding Lyapunov functions are not guaranteed to be continuous and the derivatives along the possible dynamics, essentially defined as Dini derivatives, appear to be hard to compute explicitly.

For stable linear systems with time-dependent parametric perturbations, that is, systems of the form

$$\dot{x}(t) = A(\sigma(t))x(t), \quad x \in \mathbb{R}^n, \quad (9)$$

where  $A(\cdot)$  is a function defined from a set of parameters to a set of real  $n \times n$  matrices, a derivation of a continuously differentiable Lyapunov function has been obtained by A. Meilakhs [46]. In modern terminology this can be seen as a converse Lyapunov result for switched linear systems<sup>1</sup>, and the Lyapunov function is usually called *common Lyapunov function*. It is interesting to note that:

- the asymptotic stability of all systems corresponding to constant switching laws  $\sigma$ , i.e., the fact that each matrix  $A(\sigma)$  is Hurwitz, does not necessarily imply that (9) is UGAS [35, page 19];
- unlike linear time-invariant systems (3), asymptotically stable switched linear systems of the form (9) do not necessarily admit common quadratic Lyapunov functions, as shown for instance in [12] by means of an explicit example. Characterizations of linear systems of the form (9) admitting quadratic Lyapunov functions can be found e.g. in [67].

A generalization of the smooth converse Lyapunov theorems of J. L. Massera and J. Kurzweil [31, 44] to systems of the form

$$\dot{x}(t) = f(x(t), \sigma(t)), \quad (10)$$

where  $x(\cdot)$  takes values in  $\mathbb{R}^n$  and  $\sigma : \mathbb{R}_+ \rightarrow S$  is measurable,  $S$  being a compact subset of  $\mathbb{R}^m$ , is given in [36]. The right-hand side  $f$  is assumed to be locally Lipschitz with respect to the state variable independently of  $\sigma$  and continuous with respect to the parameter  $\sigma$ . The result in [36], given below, concerns the global asymptotic stability with respect to a closed subset  $C$  invariant with respect to the dynamics (10).

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<sup>1</sup>Recall that a switched system in  $\mathbb{R}^n$  is the collection of all dynamics  $\dot{x} = f_{\sigma(t)}(x(t))$ , where  $\{f_{\sigma} \mid \sigma \in S\}$  is a family of vector fields in  $\mathbb{R}^n$ ,  $S$  being a finite or infinite index set, and  $\sigma(\cdot)$  belongs to a family of *switching laws*  $\mathcal{S}$ , a set of functions from  $\mathbb{R}_+$  to  $S$ .

**Theorem 3.3** ([36]). *If  $C$  is compact then the following conditions are equivalent:*

- (i) *system (10) is C-GAS;*
- (ii) *system (10) is C-UGAS;*
- (iii) *there exists a smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that*

$$\underline{\alpha}(|x|_C) \leq V(x) \leq \bar{\alpha}(|x|_C)$$

*for some  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  and such that*

$$\nabla V(x)f(x, s) \leq -\eta(|x|_C), \quad \forall s \in S,$$

*for some positive definite function  $\eta$ .*

*The result is still true if  $C$  is not compact (i.e., if it is unbounded) under the additional assumption that all trajectories of the system are defined for every  $t \in \mathbb{R}$  (i.e., the system is complete).*

For switched linear systems of the form (9) the Lyapunov function  $V$  can be chosen as a homogeneous polynomial of even degree, as shown, e.g. in [6, 41].

It is interesting to observe that, for UGAS nonlinear systems of the form (10), nonsmooth Lyapunov functions can be formally derived by taking the supremum of expressions of the form (7) and (8) among all possible trajectories  $x(\cdot)$  satisfying  $x(t_0) = z$ .

### 3.2 Converse Lyapunov results for differential inclusions

It is well-known that solutions of a system of the form (10) can be equivalently represented as solutions of a differential inclusion  $\dot{x} \in F(x)$ , where  $F(x) := \{f(t, s) \mid s \in S\}$ . The differential inclusion representation is actually more handy, as it allows one to deal in a more natural way with systems with discontinuous right-hand side. A smooth converse Lyapunov result applying to differential inclusions

$$\dot{x}(t) \in F(x), \tag{11}$$

in the case in which  $F(\cdot)$  is upper-semicontinuous has been obtained in [10]. This result is actually the time-invariant version of a more general result given below concerning time-dependent differential inclusions of the form

$$\dot{x}(t) \in F(t, x), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \tag{12}$$

where  $F : \mathbb{R}_+ \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ . We need the following assumption:

- (H) There exists a set  $\mathcal{N} \subset \mathbb{R}_+$  of measure zero such that  $F$  satisfies the following conditions on  $D := (\mathbb{R}_+ \setminus \mathcal{N}) \times \mathbb{R}^n$ :
  - (i)  $F(t, x)$  is nonempty, convex and compact, for every  $(t, x) \in D$ ;

(ii)  $F$  is locally bounded: for every  $R > 0$  there exists  $M > 0$  such that

$$(t, x) \in D, |(t, x)| \leq R \quad \Rightarrow \quad F(t, x) \subset B(0, M);$$

(iii)  $F$  is upper semicontinuous: for every  $(t_0, x_0) \in D$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$(t, x) \in D, |(t - t_0, x - x_0)| \leq \delta \quad \Rightarrow \quad F(t, x) \subset B(F(t_0, x_0), \varepsilon).$$

**Theorem 3.4** ([1]). *Assume that (H) holds true. Then the differential inclusion (12) is UGAS if and only if there exists a smooth function  $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that*

$$\underline{\alpha}(|x|) \leq V(t, x) \leq \bar{\alpha}(|x|), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

for some  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  and such that

$$\frac{\partial}{\partial t} V(t, x) + \nabla_x V(t, x) f \leq -V(t, x), \quad \forall (t, x) \in D, \forall f \in F(t, x),$$

for some positive definite function  $\eta$ . Furthermore,  $V$  can be assumed to be time-independent (or periodic) if  $F$  is time-independent (resp. periodic).

**Remark 3.1.** *Theorem 3.4 requires the function  $V$  to decrease exponentially along every trajectory of the system. This implies, for instance, that in the case  $C = \{0\}$  the dissipation condition in Theorem 3.3 (iii) could be equivalently replaced by the assumption of an exponentially decaying Lyapunov function. An earlier Lyapunov result remarking the equivalence between the two formulations of the dissipation condition of  $V$  can be found e.g. in [32, Theorem 3.6.10]. We stress that the exponential convergence of  $V$  does not imply the exponential convergence of the state variable since the functions  $\underline{\alpha}, \bar{\alpha}$  do not necessarily depend polynomially on  $|x|$ . For instance, the function  $V(x) = e^{-1/|x|}$  is a Lyapunov function satisfying  $\dot{V} = -V$  along the solutions of the equation  $\dot{x} = -x|x|$ , which is asymptotically but not exponentially stable at the origin.*

Theorem 3.4 is an adaptation of analogous smooth converse Lyapunov results for (12) characterizing Lyapunov and Lagrange stability [63]. Comparing the results in [63] to Theorem 3.4, we remark the relevant difference that a Lagrange or Lyapunov stable time-independent system of the form (11) does not necessarily admit a time-independent smooth Lyapunov function.

We conclude this section by providing a Lyapunov characterization of a stability notion that is more general compared to the notion of asymptotic stability with respect to a closed set. The following definition has been introduced in [58] and considered e.g. in [33, 34, 70].

**Definition 3.1** ( $\mathcal{KL}$ -stability with respect to two measures). *Let  $\omega_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , for  $i = 1, 2$ , be continuous functions. We say that (11) is  $\mathcal{KL}$ -stable with respect to  $(\omega_1, \omega_2)$  if there exists  $\beta \in \mathcal{KL}$  such that*

$$\omega_1(x(t)) \leq \beta(\omega_2(x(0)), t)$$

for every trajectory of (11).

Note that although the above stability notion implies that all trajectories of the system converge towards the (closed) level set  $\omega_1^{-1}(0)$ , the latter is not necessarily positively invariant with respect to the dynamics. In that sense,  $\mathcal{KL}$ -stability with respect to two measures is more general compared to global uniform asymptotic stability with respect to a closed invariant set.

The next smooth converse Lyapunov result actually characterizes a slightly stronger stability notion, defined below. Given  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , consider the differential inclusion

$$\dot{x} \in F_\delta(x) := \overline{\text{co}} F(B(x, \delta(x))) + B(0, \delta(x)). \quad (13)$$

Note that  $F(x) \subset F_\delta(x)$  for every  $x \in \mathbb{R}^n$  so that, in general, the solution set of (13) includes that of (11).

**Definition 3.2** (Robust  $\mathcal{KL}$ -stability with respect to two measures). *We say that (11) is robustly  $\mathcal{KL}$ -stable with respect to  $(\omega_1, \omega_2)$  if there exists a continuous function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that:*

- (i) *the support of  $\delta$  coincides with  $\mathbb{R}^n \setminus \Omega$ ,  $\Omega$  being the largest subset of the level set  $\omega_1^{-1}(0)$  invariant with respect to (11);*
- (ii) *the differential inclusion (13) is  $\mathcal{KL}$ -stable with respect to  $(\omega_1, \omega_2)$ .*

We then have the following result.

**Theorem 3.5** ([70, Theorem 1]). *Assume that  $F(x)$  is nonempty, convex and compact for every  $x \in \mathbb{R}^n$  and that  $F(\cdot)$  is upper semicontinuous (see Assumption (H) above). Then (11) is robustly  $\mathcal{KL}$ -stable with respect to  $(\omega_1, \omega_2)$  if and only if there exists a smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that*

$$\underline{\alpha}(\omega_1(x)) \leq V(x) \leq \bar{\alpha}(\omega_2(x)), \quad \forall x \in \mathbb{R}^n,$$

for some  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  and such that

$$\nabla V(x)f \leq -V(x) \quad \forall x \in \mathbb{R}^n, \forall f \in F(x).$$

### 3.3 Universal classes for switched linear systems

Given a class  $\mathcal{S}$  of switched systems (e.g., the set of all switched linear systems of the form (9) under the assumption that the matrix  $A(\cdot)$  can only take a finite number of values) we say that a family  $\mathcal{F}$  of real-valued functions defined on  $\mathbb{R}^n$  is *universal* if every UGAS switched system in  $\mathcal{S}$  admits an element of  $\mathcal{F}$  as a common Lyapunov function. The concept of universal class of Lyapunov functions for switched linear systems was introduced informally in the nineties (see, e.g. [4, 5]) in the context of stabilization of linear control systems with uncertain dynamics. In the absence of inputs, such stabilization problem reduces to the problem of uniform asymptotic stability of switched linear systems.

The motivation for considering a universal class  $\mathcal{F}$  is that it allows us to reduce the problem of checking the stability of a system to the problem of numerically finding a Lyapunov function within  $\mathcal{F}$ .

Given a switched linear system, the family of smooth functions is, by the converse Lyapunov results recalled above, a universal class of Lyapunov functions. However, this class is too broad to admit a tractable parameterization, suitable for investigating numerically the existence of a Lyapunov function. A “good” universal class is instead a parametrized family of functions within which the properties of being positive definite and strictly decreasing along all admissible dynamics can be translated into numerically verifiable algebraic relations or inequalities (e.g., tractable via linear matrix inequalities or linear programming). Specifically, this is the case for the class of candidate Lyapunov functions considered in [6], which is that of homogeneous polynomial Lyapunov functions. Converse Lyapunov results involving the use of polyhedrals or piecewise quadratic functions had been obtained earlier by Molchanov and Pyatnitskiy [54, 55, 56, 57]. As stated precisely below, a general sufficient condition for a class  $\mathcal{F}$  to be universal is its density in the class of convex absolutely homogeneous functions for the topology of uniform convergence on compact sets. Before formally stating such a result, let us clarify the setting and provide some useful remarks. First of all, a switched linear system corresponds here to the collection of all possible dynamics

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}_+,$$

where  $x(\cdot)$  evolves in  $\mathbb{R}^n$  and the switching law  $A(\cdot)$  is a measurable function taking values in a given bounded set  $\mathcal{A}$  of  $n \times n$  real matrices. Stability properties of the systems are then completely determined by the set  $\mathcal{A}$ . We remark that switched linear systems are UGAS if and only if they are UGES (see, e.g., [12]). Then, for a UGAS system, setting

$$V(z) = \sup_{x(\cdot) : x(0)=z} \int_0^\infty |x(t)| dt, \quad \forall z \in \mathbb{R}^n,$$

where the supremum is computed among all trajectories of the system starting at  $z$ , we obtain a well-defined real-valued function which strictly decreases along all trajectories of the system. More precisely, one can easily show that  $V$  is a convex and absolutely homogeneous function (i.e.,  $V(\alpha z) = |\alpha|V(z)$ , for every  $\alpha \in \mathbb{R}$ ), and it satisfies  $\nabla V(x)Ax \leq -c|x|$  for some  $c > 0$ , for almost every  $x \in \mathbb{R}^n$  and  $A \in \mathcal{A}$ . On the other hand, the existence of a function satisfying such conditions guarantees that the system is UGES, hence it is a valid Lyapunov function. It is then natural to look for universal classes formed by convex absolutely homogeneous functions, as done in the next result.

**Proposition 3.1** ([42, Proposition 3.1]). *Let  $\mathcal{F}$  be a subset of the family of convex absolutely homogeneous functions from  $\mathbb{R}^n$  to  $\mathbb{R}_+$ . Assume that for every convex absolutely homogeneous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and every  $\delta > 0$  there exists a function  $W$  in  $\mathcal{F}$  such that  $|W(x) - V(x)| \leq \delta$  for every  $x$  in the unit sphere  $S^{n-1}$  of  $\mathbb{R}^n$ . Then  $\mathcal{F}$  is a universal class of Lyapunov functions.*

The well-established fact that polyhedral functions and homogeneous sums of squares form universal classes of Lyapunov functions for switched linear systems [6, 54, 56, 55, 57] follows easily from the previous result. Note that the

density property is clearly lost if one imposes a uniform bound on the degree of the candidate polynomial Lyapunov functions or on the number of linear maps defining polyhedral functions. Indeed, the next results show that if one imposes such bounds the universality property is no more valid.

**Theorem 3.6** ([42, Theorem 4.2 and Corollary 4.5]). *Let  $n \geq 2$ . If either*

- (i)  $\mathcal{F}$  is a compact subset of the space of continuous functions from  $\mathbb{R}^n$  to  $\mathbb{R}$  that are analytic on  $\mathbb{R}^n \setminus \{0\}$ , endowed with the topology of uniform convergence on bounded sets. Moreover  $\mathcal{F}$  does not contain the zero function

or

- (ii)  $\mathcal{F}$  is a finite-dimensional vector subspace of the space of continuous functions from  $\mathbb{R}^n$  to  $\mathbb{R}$  that are analytic on  $\mathbb{R}^n \setminus \{0\}$

then  $\mathcal{F}$  cannot be a universal class of Lyapunov functions. In particular, for every positive integer  $m$ , the set of polynomial functions of degree at most  $m$  from  $\mathbb{R}^n$  to  $\mathbb{R}$  is not a universal class of Lyapunov functions.

Note that the result above applies to candidate Lyapunov functions other than polynomials. For example, consider the class of absolutely homogeneous functions of degree two

$$\mathcal{F} = \left\{ x \mapsto e^{\frac{x^\top Q_1 x}{|x|^2}} x^\top Q_2 x \mid \|Q_1\| \leq 1, \|Q_2\| = 1 \right\}.$$

The level sets of each element of  $\mathcal{F}$  are obtained by deforming those of the quadratic functions  $x \mapsto x^\top Q_2 x$  by a positive 0-homogeneous term. As each function in  $\mathcal{F}$  is analytic outside the origin, by Theorem 3.6 the class  $\mathcal{F}$  is not universal, despite being richer than that of quadratic functions.

We next state a result which applies to functions involving maxima and minima within a finite family of polynomials such as the class of polyhedral functions  $V$  of the form

$$\begin{aligned} V(x) &= \max\{|l_1^\top x|, \dots, |l_N^\top x|\} \\ &= \max\{l_1^\top x, \dots, l_N^\top x, -l_1^\top x, \dots, -l_N^\top x\}, \end{aligned}$$

with  $l_1, \dots, l_N \in \mathbb{R}^n$ , where  $N$  is fixed.

**Theorem 3.7** ([42, Theorem 4.9]). *Let  $n \geq 2$  and  $\mathcal{F}_d^n$  be the family of polynomial functions in  $\mathbb{R}^n$  of degree at most  $d$  and  $l$  be a positive integer. Consider the family*

$$\begin{aligned} \mathcal{F}_{d,l}^n &= \{V \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}) \mid \exists V_1, \dots, V_l \in \mathcal{F}_d^n \\ &\text{s.t. } V(x) \in \{V_1(x), \dots, V_l(x)\}, \forall x \in \mathbb{R}^n\}. \end{aligned}$$

Then,  $\mathcal{F}_{d,l}^n$  is not universal.

## 4 Infinite-Dimensional Systems

It was noted already in early works, see, e.g., [14, 30, 44], that some direct Lyapunov results developed in the context of finite-dimensional systems remain valid even whenever the state space is an infinite-dimensional Banach space and if the solutions satisfy conditions of existence, uniqueness and continuous dependence with respect to initial data.

For what concerns converse Lyapunov results, a first result has been given by R. Datko in [11] for strongly continuous semigroups of linear operators (the semigroup properties are recalled below in Definition 4.2 in a more general nonlinear setting) on a complex Hilbert space  $\mathcal{H}$  endowed with a scalar product  $\langle \cdot, \cdot \rangle$  and the corresponding norm  $\| \cdot \|$ .

**Theorem 4.1** ([11]). *A strongly continuous semigroup  $(T(t))_{t \in \mathbb{R}_+}$  on  $\mathcal{H}$  satisfies the condition  $\|T(t)\| \leq Me^{-\lambda t}$ , where  $M \geq 1$  and  $\lambda > 0$ , if and only if there exists a positive definite Hermitian endomorphism  $P$  on  $\mathcal{H}$  such that the relation  $2\Re\langle PAx, x \rangle = -\|x\|^2$  holds for every  $x$  in the domain of  $A$ ,  $A$  being the infinitesimal generator of the semigroup  $(T(t))_{t \in \mathbb{R}_+}$ .*

Theorem 4.1 is the generalization of Theorem 3.2 to strongly continuous semigroup operators in Hilbert spaces. In particular, the function  $V(x) = \langle Px, x \rangle$  can be seen as a Lyapunov function associated with the strongly continuous semigroup  $(T(t))_{t \in \mathbb{R}_+}$ .

As underlined in Section 3, an important byproduct of converse Lyapunov results is that they make it possible to establish additional properties for systems which are asymptotically stable, for example, for what concerns the effects of perturbations. These results are usually obtained thanks to norm-dependent lower and upper bounds of the Lyapunov functions. For infinite-dimensional systems, however, the existence of norm-dependent bounds is not a natural requirement. For example, as reported in [59], given a UGES strongly continuous semigroup  $(T(t))_{t \in \mathbb{R}_+}$ , the functional  $V : \mathcal{H} \rightarrow \mathbb{R}_+$ ,  $\mathcal{H}$  being a Hilbert space, defined by

$$V(x) = \int_0^{+\infty} \|T(t)x\|^2 dt, \quad \forall x \in \mathcal{H}, \quad (14)$$

can be seen as a Lyapunov function of the form  $V(x) = \langle Px, x \rangle$ ,  $P$  being as in Theorem 4.1. The function  $V$  satisfies a condition of the type

$$0 < V(x) \leq \bar{\alpha}(\|x\|), \quad \forall x \in \mathcal{H} \setminus \{0\} \quad (15)$$

for some  $\bar{\alpha} \in \mathcal{K}_\infty$ , but does not necessarily admit a norm-dependent lower bound (Lyapunov functions satisfying (15) are called *non-coercive*, see Section 4.2 below for more details). For this reason, in [59], A. Pazy gives a characterization of all strongly continuous semigroups for which the Hermitian form  $P$  defines an equivalent norm on  $\mathcal{H}$ . More generally, characterizations of strongly continuous and analytic semigroups such that there exists a function  $V(x) = \langle Bx, x \rangle$ , for

some bounded and self-adjoint operator  $B$  on  $\mathcal{H}$ , satisfying

$$\begin{aligned} \underline{\alpha}\|x\|^2 \leq V(x) \leq \bar{\alpha}\|x\|^2, \\ \dot{V}(x) \leq -\eta\|x\|^2, \end{aligned} \quad \forall x \in \mathcal{H}$$

where  $\underline{\alpha}, \bar{\alpha}, \eta$  are positive scalars and  $\dot{V}$  is a suitable Dini derivative, have recently been given in [49].

In order to encompass various types of infinite-dimensional dynamics, we introduce below the general setting of forward complete dynamical systems.

#### 4.1 Forward complete dynamical systems

Let  $Q$  be a nonempty set. We denote by  $\mathcal{S}$  a set of functions, that we call *signals*, from  $\mathbb{R}_+$  to  $Q$ , and we assume that

- $\mathcal{S}$  is closed by time-shift, i.e., for all  $\sigma \in \mathcal{S}$  and all  $\tau \in \mathbb{R}_+$ , the  $\tau$ -shifted signal  $\sigma(\tau + \cdot)$  belongs to  $\mathcal{S}$ ;
- $\mathcal{S}$  is closed by concatenation, i.e., for all  $\sigma_1, \sigma_2 \in \mathcal{S}$  and all  $\tau > 0$  the signal  $\sigma$  defined by  $\sigma \equiv \sigma_1$  over  $[0, \tau]$  and by  $\sigma(\tau + t) = \sigma_2(t)$  for all  $t > 0$ , belongs to  $\mathcal{S}$ .

The following definition introduces the concept of forward complete dynamical system evolving in a Banach space  $X$ .

**Definition 4.1** ([52]). *The triplet  $\Sigma = (X, \mathcal{S}, \phi)$ , where  $\phi$  is a function from  $\mathbb{R}_+ \times X \times \mathcal{S}$  to  $X$  called transition map, is said to be a forward complete dynamical system if the following properties hold:*

- (i) *For every  $(x, \sigma) \in X \times \mathcal{S}$ , it holds that  $\phi(0, x, \sigma) = x$ ;*
- (ii) *For every  $(x, \sigma) \in X \times \mathcal{S}$ ,  $t \in \mathbb{R}_+$ , and  $\tilde{\sigma} \in \mathcal{S}$  such that  $\tilde{\sigma} = \sigma$  over  $[0, t]$ , it holds that  $\phi(t, x, \tilde{\sigma}) = \phi(t, x, \sigma)$ ;*
- (iii) *For every  $(x, \sigma) \in X \times \mathcal{S}$ , the map  $t \mapsto \phi(t, x, \sigma)$  is continuous;*
- (iv) *For every  $t, \tau \in \mathbb{R}_+$  and  $(x, \sigma) \in X \times \mathcal{S}$ , it holds that*

$$\phi(t, \phi(\tau, x, \sigma), \sigma(\tau + \cdot)) = \phi(t + \tau, x, \sigma).$$

Observe that if  $\Sigma$  is a forward complete dynamical system and  $\mathcal{S}$  contains a constant function  $\sigma$  then  $(\phi(t, \cdot, \sigma))_{t \in \mathbb{R}_+}$  is a strongly continuous nonlinear semigroup, whose definition is recalled below.

**Definition 4.2** ([53]). *Let  $T(t) : X \rightarrow X$ ,  $t \in \mathbb{R}_+$ , be a family of nonlinear maps. We say that  $(T(t))_{t \in \mathbb{R}_+}$  is a strongly continuous nonlinear semigroup if the following properties hold:*

- (i) *For every  $x \in X$ , it holds that  $T(0)x = x$ ;*

(ii) For every  $t_1, t_2 \in \mathbb{R}_+$ , it holds that  $T(t_1)T(t_2)x = T(t_1 + t_2)x$ ;

(iii) For every  $x \in X$ , the map  $t \mapsto T(t)x$  is continuous.

Next, three examples of forward complete dynamical systems are given, which will be exploited and developed more in detail in the remainder of the paper. We denote by  $\mathcal{S}_{\text{pc}}$  the set of piecewise-constant functions from  $\mathbb{R}_+$  to  $Q$  that are right-continuous and admit a finite number of discontinuities in each bounded interval.

**Example 4.1** (switched linear systems). Consider a family of strongly continuous nonlinear semigroups  $(T_q(t))_{t \in \mathbb{R}_+}$  in a Banach space  $X$ , parametrized by  $q \in Q$ . Let  $\sigma \in \mathcal{S}_{\text{pc}}$  be constantly equal to  $\sigma_k$  over  $[t_k, t_{k+1})$ , with  $t_0 = 0$  and  $t_k < t_{k+1}$  for every nonnegative integer  $k$ . By concatenating the flows  $(T_{\sigma_k}(t))_{t \in \mathbb{R}_+}$ , one can associate with  $\sigma$  the family of nonlinear evolution operators

$$T_\sigma(t) := T_{\sigma_k}(t - t_k)T_{\sigma_{k-1}}(t_k - t_{k-1}) \cdots T_{\sigma_1}(t_1), \quad t \in [t_k, t_{k+1}). \quad (16)$$

Then, the triple  $\Sigma = (X, \mathcal{S}_{\text{pc}}, \phi)$ , with the transition map  $\phi$  defined as

$$\phi(t, x_0, \sigma) = T_\sigma(t)x_0, \quad x_0 \in X, \sigma \in \mathcal{S}_{\text{pc}}, \quad (17)$$

is a forward complete dynamical system.

**Example 4.2** (switched semilinear control systems). Let  $X, U$  be two Banach spaces. Consider the switched semilinear control system

$$\begin{cases} \dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t), u(t)), & t \in \mathbb{R}_+, \\ x(0) = x_0 \in X, \end{cases} \quad (18)$$

where  $A$  is the infinitesimal generator of a strongly continuous semigroup  $(T(t))_{t \in \mathbb{R}_+}$  of bounded linear operators on  $X$ ,  $\sigma \in \mathcal{S}_{\text{pc}}$ ,  $f_{\bar{\sigma}} : X \times U \rightarrow X$  is a Lipschitz continuous nonlinear operator for every  $\bar{\sigma} \in Q$ , and  $u \in \mathcal{C}(\mathbb{R}_+, U)$  is the control input, which is assumed here to be given. For every  $\bar{\sigma} \in Q$  and  $x_0 \in X$  there exists a unique mild solution of (18) over  $\mathbb{R}_+$ , i.e., a continuous function  $x(\cdot)$  satisfying

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f_{\bar{\sigma}}(x(s), u(s)) ds,$$

for every  $t \in \mathbb{R}_+$ . This defines a family  $(T_{\bar{\sigma}}(t))_{t \in \mathbb{R}_+}$  of nonlinear maps by setting  $T_{\bar{\sigma}}(t)x_0 = x(t)$ , for  $t \in \mathbb{R}_+$ . As in Example 4.1, assuming  $\sigma \equiv \sigma_k$  on  $[t_k, t_{k+1})$ , by concatenating the flows  $(T_{\sigma_k}(t))_{t \in \mathbb{R}_+}$ , one can associate with  $\sigma$  the family of nonlinear evolution operators given by (16). Then the triple  $\Sigma = (X, \mathcal{S}_{\text{pc}}, \phi_u)$ , with the transition map  $\phi_u$  defined as in (17), is a forward complete dynamical system.

**Example 4.3** (Retarded functional differential equations). Let  $\Delta \in \mathbb{R}_+$  and set  $\mathcal{C} := (\mathcal{C}([-\Delta, 0], \mathbb{R}^n), \|\cdot\|_{\mathcal{C}})$ , the Banach space of continuous functions from  $[-\Delta, 0]$  to  $\mathbb{R}^n$ . Here  $\|\cdot\|_{\mathcal{C}}$  is the norm of uniform convergence defined, for  $\psi \in \mathcal{C}$ , as  $\|\psi\|_{\mathcal{C}} = \max_{\theta \in [-\Delta, 0]} |\psi(\theta)|$ . Consider the switched nonlinear retarded functional differential equation

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x_t), & t \in \mathbb{R}_+, \\ x(\theta) &= \varphi(\theta), & \theta \in [-\Delta, 0], \end{aligned} \quad (19)$$

where  $x(t) \in \mathbb{R}^n$ ,  $\varphi \in \mathcal{C}$ ,  $x_t : [-\Delta, 0] \rightarrow \mathbb{R}^n$  is the standard notation for the history function defined by

$$x_t(\theta) = x(t + \theta), \quad -\Delta \leq \theta \leq 0,$$

$\sigma \in \mathcal{S}_{\text{pc}}$ , and for every  $\bar{\sigma} \in Q$  the functional  $f_{\bar{\sigma}} : \mathcal{C} \rightarrow \mathbb{R}^n$  is assumed to be Lipschitz on bounded sets. For every  $\varphi \in \mathcal{C}$  and  $k \geq 0$ , there exists  $b > 0$  and a unique solution  $x$  over  $[-\Delta, b]$  of (19) with  $\sigma \equiv \bar{\sigma}$  (see, e.g., [21]). Assume that  $b = +\infty$  for every  $\varphi \in \mathcal{C}$  and  $\bar{\sigma} \in Q$ . This defines a family  $(T_{\bar{\sigma}}(t))_{t \geq 0}$  of nonlinear maps from  $\mathcal{C}$  into itself by setting

$$T_{\bar{\sigma}}(t)\varphi = x_t, \quad \forall t \geq 0.$$

As in (17), for each  $\varphi \in \mathcal{C}$  and  $\sigma \in \mathcal{S}_{\text{pc}}$  such that  $\sigma \equiv \sigma_k$  over intervals  $[t_k, t_{k+1})$ , we define the transition map  $\phi^\Delta$  corresponding to (19), by concatenating the flows  $(T_{\sigma_k}(t))_{t \geq 0}$ . We denote by  $\Sigma = (\mathcal{C}, \mathcal{S}_{\text{pc}}, \phi^\Delta)$  the corresponding forward complete dynamical system.

In the next section we focus on Lyapunov characterizations of uniform stability properties for forward complete dynamical systems. Notice that, as in the finite-dimensional case, the asymptotic stability of individual systems corresponding to constant signals is a necessary, but not a sufficient, condition for the uniform asymptotic stability of the corresponding switched system.

## 4.2 Direct Lyapunov theorems

Let  $(X, \|\cdot\|)$  be a Banach space. We introduce the following definition of Dini derivative of a functional  $V : X \rightarrow \mathbb{R}_+$  in the context of forward complete dynamical systems.

**Definition 4.3.** Consider a forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi)$ . Given  $\sigma \in \mathcal{S}$ , the upper- and lower-right Dini derivatives  $\bar{D}_\sigma V : X \rightarrow \mathbb{R}$  and  $\underline{D}_\sigma V : X \rightarrow \mathbb{R}$  of a functional  $V : X \rightarrow \mathbb{R}_+$  are defined, respectively, as

$$\bar{D}_\sigma V(x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} (V(\phi(h, x, \sigma)) - V(x)), \quad \forall x \in X,$$

and

$$\underline{D}_\sigma V(x) = \liminf_{h \rightarrow 0^+} \frac{1}{h} (V(\phi(h, x, \sigma)) - V(x)), \quad \forall x \in X.$$

The notions of coercive and non-coercive Lyapunov functions are given in the following definition.

**Definition 4.4.** A function  $V : X \rightarrow \mathbb{R}_+$  is called a coercive Lyapunov function for system  $\Sigma = (X, \mathcal{S}, \phi)$ , if there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  and  $\eta \in \mathcal{K}$  such that

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|) \quad \forall x \in X, \quad (20)$$

and the Dini derivative of  $V$  along the trajectories of  $\Sigma$  satisfies

$$\overline{D}_\sigma V(x) \leq -\eta(\|x\|) \quad \forall x \in X, \forall \sigma \in \mathcal{S}. \quad (21)$$

We call  $V$  a non-coercive Lyapunov function if instead of (20) we have  $V(0) = 0$  and

$$0 < V(x) \leq \bar{\alpha}(\|x\|) \quad \forall x \in X \setminus \{0\}, \quad (22)$$

and  $\overline{D}_\sigma V$  is replaced by  $\underline{D}_\sigma V$  in (21).<sup>2</sup>

Coercive and non-coercive Lyapunov functions should be interpreted here as “strong” and “weak” forms of Lyapunov functions, hence the different choices of Dini derivative. The fact that the weaker conditions characterizing non-coercive Lyapunov functions are, in principle, easier to check compared to coercive Lyapunov functions, justifies the usefulness of Lyapunov results involving such type of functions. On the other hand, the existence of a coercive Lyapunov function may be helpful in order to provide additional information on the system, e.g., robustness properties with respect to perturbations.

Note that if  $X$  is an Euclidean space, every radially unbounded function  $V$  that satisfies (22) also fulfills (20). However, this property does not hold in infinite-dimensional spaces  $X$ . More importantly, for infinite-dimensional systems, the existence of a non-coercive Lyapunov functional is not a sufficient condition for the uniform asymptotic stability. More precisely, even if there exists a functional  $V : X \rightarrow \mathbb{R}_+$  that satisfies inequality (22) and decreases uniformly along each trajectory of the system, additional technical assumptions are needed in order to guarantee the asymptotic stability of the system under consideration. Such a condition was first given in [72] in the case of a family of parameter-dependent strongly continuous (linear) semigroups  $(T_h(t))_{t \in \mathbb{R}_+}$  on  $X$ , with  $h \in \mathbb{R}$ , and later adapted in [22] to characterize the exponential stability of linear forward complete dynamical systems of type (17). Namely, in [22] it is shown that a system of this type is UGES if there exists a non-coercive Lyapunov function and if, in addition, the following condition is verified: there exist constants  $M \geq 1$  and  $\lambda > 0$  such that

$$\|T_\sigma(t)\| \leq Me^{\lambda t}, \quad \forall t \in \mathbb{R}_+, \forall \sigma \in \mathcal{S}. \quad (23)$$

**Remark 4.1.** Observe that inequality (23) is verified if one considers a single strongly continuous (linear) semigroup (see, e.g., [60]). On the other hand,

<sup>2</sup>The term “non-coercive” is a slight abuse of terminology and should be read as “not necessarily coercive”.

notice that this inequality is a nontrivial requirement for a general forward complete dynamical system  $\Sigma$ . Even in the linear case, with (23) satisfied for each constant  $\sigma \equiv q$ , uniformly with respect to  $q \in Q$ , it does not follow that a similar exponential bound holds for the corresponding system  $\Sigma$  (see [22, Example 1]).

For general forward complete dynamical systems, when dealing with non-coercive Lyapunov functionals, the assumption (23) must be replaced by suitable uniform growth conditions. This is done in [52] for the UGAS property and in [15] for exponential stability properties.

The following theorem characterizes the UGAS property of a forward complete dynamical system in terms of the existence of a non-coercive Lyapunov functional.

**Theorem 4.2** ([52, Theorem 3.5]). *Consider a forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi)$ . Assume that  $\Sigma$  is robustly forward complete in the sense that for any  $C > 0$  and any  $\tau > 0$  it holds that*

$$\sup_{\|x\| \leq C, t \in [0, \tau], \sigma \in \mathcal{S}} \|\phi(t, x, \sigma)\| < \infty. \quad (24)$$

Assume also that 0 is a robust equilibrium point of  $\Sigma$ , i.e., that for every  $\varepsilon, h > 0$ , there exists  $\delta = \delta(\varepsilon, h) > 0$ , so that

$$\|x\| \leq \delta \implies \|\phi(t, x, \sigma)\| \leq \varepsilon, \quad \forall t \in [0, h], \forall \sigma \in \mathcal{S}. \quad (25)$$

Then,  $\Sigma$  is UGAS if it admits a continuous non-coercive Lyapunov function.

Note that the conditions (24) and (25) hold true if and only if there exists a function  $\beta : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  which is of class  $\mathcal{K}$  in both variables and such that  $\|\phi(t, x, \sigma)\| \leq \beta(\|x\|, t)$  for every  $t \in \mathbb{R}_+$ ,  $x \in X$  and  $\sigma \in \mathcal{S}$ .

The following Lyapunov-type result is concerned with exponential stability properties of forward complete dynamical systems.

**Theorem 4.3** ([15, Theorem 4]). *Consider a forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi)$ . Let  $t_1 > 0$  and  $\gamma$  be a function of class  $\mathcal{K}_\infty$  such that  $\limsup_{r \rightarrow 0^+} \frac{\gamma(r)}{r}$  is finite and*

$$\|\phi(t, x, \sigma)\| \leq \gamma(\|x\|), \quad \forall t \in [0, t_1], \forall x \in X, \forall \sigma \in \mathcal{S}. \quad (26)$$

i) *Suppose that there exist  $R > 0$ , a non-coercive function  $V : B(0, R) \rightarrow \mathbb{R}_+$  and  $p = p_R, c = c_R > 0$  such that inequality (22) holds in  $B(0, R) \setminus \{0\}$  with  $\bar{\alpha}(s) = cs^p$  and inequality (21) holds in  $B(0, R)$  with  $\eta(s) = s^p$ . Suppose, in addition, that  $V(\phi(\cdot, x, \sigma))$  is continuous from the left at every  $t > 0$  such that  $\phi(t, x, \sigma) \in B(0, R)$ . Then system  $\Sigma$  is ULES;*

ii) *if i) holds true for every  $R > 0$  and, moreover,*

$$\limsup_{R \rightarrow +\infty} \gamma^{-1}(R) \min \left\{ 1, \left( \frac{t_1}{c_R} \right)^{\frac{1}{p_R}} \right\} = +\infty, \quad (27)$$

*then system  $\Sigma$  is USGES;*

iii) if  $\gamma(r) = G_0 r$  with  $G_0 \geq 1$  in (26) and there exist  $p, c > 0$  and a non-coercive functional  $V : X \rightarrow \mathbb{R}_+$  such that, for every  $x \in X$  and  $\sigma \in \mathcal{S}$ , the map  $t \mapsto V(\phi(t, x, \sigma))$  is continuous from the left, then system  $\Sigma$  is UGES.

It is easy to show that the existence of  $\gamma \in \mathcal{K}_\infty$  satisfying condition (26) implies both the robust forward completeness property (24) and the robust equilibrium point property (25). Compared to Theorem 4.2 the extra assumption  $\limsup_{r \rightarrow 0^+} \frac{\gamma(r)}{r}$ , together with the particular choices of  $\bar{\alpha}(\cdot)$  and  $\eta(\cdot)$  in Theorem 4.3, are crucial in order to obtain an exponentially (rather than asymptotically) stable behaviour at the origin.

### 4.3 Converse Lyapunov theorems

The notion of  $\mathcal{S}$ -uniform Lipschitz continuity of the transition map is given by the following definition.

**Definition 4.5.** *We say that the transition map  $\phi$  of  $\Sigma = (X, \mathcal{S}, \phi)$  is  $\mathcal{S}$ -uniformly Lipschitz continuous on bounded sets if, for every  $\bar{t} > 0$  and  $R > 0$ , there exists  $l(\bar{t}, R) > 0$  such that*

$$\|\phi(t, x, \sigma) - \phi(t, y, \sigma)\| \leq l(\bar{t}, R)\|x - y\|, \quad \forall t \in [0, \bar{t}], \forall x, y \in B(0, R), \forall \sigma \in \mathcal{S}.$$

If  $l(\bar{t}, R)$  can be taken independent of  $R$ , then we say that the transition map is  $\mathcal{S}$ -uniformly Lipschitz continuous.

A construction of Lipschitz continuous coercive Lyapunov functions for forward complete dynamical systems which are UGAS is given in [26].

**Theorem 4.4** ([26, Theorem 3.4]). *Consider a forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi)$  and assume that the transition map  $\phi$  is  $\mathcal{S}$ -uniformly Lipschitz continuous on bounded sets. If  $\Sigma$  is UGAS, then there exists a sequence of positive real numbers  $\{a_k\}_{k \geq 1}$  so that  $W : X \rightarrow \mathbb{R}_+$ , defined by*

$$W(x) := \sum_{k=1}^{\infty} a_k V_k(x) \quad \forall x \in X, \quad (28)$$

with

$$V_k(x) := \sup_{\sigma \in \mathcal{S}, t \in \mathbb{R}_+} e^t \max\{0, \rho(\|\phi(t, x, \sigma)\|) - 1/k\}, \quad k \geq 1, \quad (29)$$

for some  $\rho \in \mathcal{K}_\infty$ , is a coercive Lyapunov function for  $\Sigma$  which is Lipschitz continuous on bounded sets.

A similar construction is given in [51] in order to derive non-coercive Lyapunov functions for forward complete dynamical systems which are UGAS.

The following theorem states that the existence of a coercive Lyapunov functional is necessary for the uniform semi-global exponential stability of a forward complete dynamical system.

**Theorem 4.5** ([15, Theorem 5]). *Consider a forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi)$  and assume that the transition map  $\phi$  is  $\mathcal{S}$ -uniformly Lipschitz continuous on bounded sets. If  $\Sigma$  is USGES then, for every  $r > 0$  there exist  $\underline{c}_r, \bar{c}_r > 0$  and a functional  $V_r : X \rightarrow \mathbb{R}_+$  Lipschitz continuous on bounded sets such that*

$$\begin{aligned} \underline{c}_r \|x\| &\leq V_r(x) \leq \bar{c}_r \|x\|, \quad \forall x \in B(0, r), \\ \overline{D}_\sigma V_r(x) &\leq -\|x\|, \quad \forall x \in B(0, r), \forall \sigma \in \mathcal{S}. \end{aligned}$$

The following theorem characterizes the uniform global exponential stability of a forward complete dynamical system.

**Theorem 4.6** ([15, Corollary 1]). *Consider a forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi)$ . Assume that the transition map  $\phi$  is  $\mathcal{S}$ -uniformly Lipschitz continuous. If there exist  $t_1 > 0$  and  $G_0 \geq 1$  such that condition (26) holds with  $\gamma(r) = G_0 r$ , then the following statements are equivalent:*

- (i) *System  $\Sigma$  is UGES;*
- (ii) *there exists a continuous coercive function  $V$  and positive reals  $p, \underline{c}, \bar{c}$  such that inequality (20) holds with  $\underline{\alpha}(s) = \underline{c}s^p$ ,  $\bar{\alpha}(s) = \bar{c}s^p$  and inequality (21) holds with  $\eta(s) = s^p$ .*
- (iii) *there exist a non-coercive function  $V$ , positive reals  $p$  and  $\bar{c}$  such that inequality (22) holds with  $\bar{\alpha}(s) = \bar{c}s^p$ , inequality (21) holds with  $\eta(s) = s^p$  and such that  $t \mapsto V(\phi(t, x, \sigma))$  is continuous from the left.*

Theorem 4.6 can be seen as a generalization of the converse Lyapunov results [22, Theorem 3 and Theorem 6] obtained earlier for linear forward complete dynamical systems of type (17).

#### 4.4 Time delay systems as a particular class of infinite-dimensional systems

As observed in Example 4.3, the evolution of the history function  $x_t \in \mathcal{C}$  associated with a nonlinear retarded functional differential equation in presence of switching parameters can be described in terms of a forward complete dynamical system. Note that the state  $x(t)$  and the history function  $x_t$  share the same stability properties. In particular, the direct and converse Lyapunov results established in Sections 4.2 and 4.3 can be directly applied for nonlinear retarded functional differential equations. Actually, by exploiting peculiar characteristics of such systems, the equivalence between their uniform stability properties and the existence of suitable Lyapunov functions can be established under weakened assumptions.

Let us introduce the following two basic assumptions.

**Assumption 4.1.** *For each  $q \in Q$ ,  $f_q(0) = 0$ . Moreover,  $f_q(\cdot)$  is Lipschitz on bounded subsets of  $\mathcal{C}$ , uniformly with respect to  $q \in Q$ .*

**Assumption 4.2.** For each  $\phi \in \mathcal{C}$  and  $\sigma \in \mathcal{S}$ , the function  $t \mapsto f_{\sigma(t)}(\phi)$ ,  $t \in \mathbb{R}_+$ , is Lebesgue measurable.

Under Assumption 4.1 and Assumption 4.2, the local existence and uniqueness of a solution for system (19), as well as its continuous dependence with respect to the initial state, are guaranteed by classical results concerning retarded functional differential equations (see, e.g., [21]).

It is interesting to note that, under the previous assumptions, the forward completeness of system (19) does not imply the robust forward completeness property (24). Indeed, in [40] the authors exhibit an example of a time-delay system (with a single delay) evolving in the space  $\mathcal{C}$  which is forward complete but not robustly forward complete. Another example in the context of time-delay systems with uncertain delay showing that forward completeness does not imply robust forward completeness is given in [16] for a simple linear uncertain delay system evolving in  $L^2([-\Delta, 0], \mathbb{R}) \times \mathbb{R}$ , which can also be recast in the setting of forward complete dynamical systems introduced in Definition 4.1.

The converse Lyapunov results that we will state below concern the case of switched retarded systems evolving in  $\mathcal{C}$  and are taken from the recent papers [18, 19]. They make use of the notion of Driver's form derivative [13, 18] of a continuous functional  $V : \mathcal{C} \rightarrow \mathbb{R}_+$ , which is recalled below.

**Definition 4.6** ([19]). Consider system (19) and let  $q \in Q$ . For a continuous functional  $V : \mathcal{C} \rightarrow \mathbb{R}_+$ , its Driver's form derivative,

$$D_q^+ V : \mathcal{C} \rightarrow \overline{\mathbb{R}},$$

is defined as

$$D_q^+ V(\varphi) = \limsup_{h \rightarrow 0^+} \frac{V\left(\varphi_h^{\Sigma, q}\right) - V(\varphi)}{h}, \quad \forall \varphi \in \mathcal{C}, \quad (30)$$

where  $\varphi_h^{\Sigma, q} \in \mathcal{C}$  is defined, for  $h \in [0, \Delta]$  and  $\theta \in [-\Delta, 0]$ , as follows

$$\varphi_h^{\Sigma, q}(\theta) = \begin{cases} \varphi(\theta + h), & \theta \in [-\Delta, -h) \\ \varphi(0) + (\theta + h)f_q(\varphi), & \theta \in [-h, 0]. \end{cases}$$

In contrast to Dini's derivatives, Driver's type derivative do not involve the explicit expression of the solution (similarly to Lie derivatives of functions defined on finite-dimensional spaces), hence it can, in principle, be handled more easily.

**Remark 4.2.** Fix  $\varphi \in \mathcal{C}$ ,  $\sigma \in \mathcal{S}_{\text{pc}}$  and let  $x(\cdot)$  be the associated solution of (19) on a maximal interval  $[0, b)$ . If  $V$  is locally Lipschitz it holds that (see [19])

$$D_{\sigma(t)}^+ V(x_t) = \overline{D}_{\sigma(t)} V(x_t), \quad \forall t \in [0, b). \quad (31)$$

In general, when  $\sigma \in \mathcal{S}$ , the equality in (31) is satisfied almost everywhere in  $[0, b)$ .

The following theorem gathers some results from [18] (see also [19]).

**Theorem 4.7.** *Consider system (19) under Assumptions 4.1-4.2 and let  $\phi^\Delta$  be the transition map defined in Example 4.3. The following statements hold:*

- (i)  $\Sigma = (\mathcal{C}, \mathcal{S}, \phi^\Delta)$  is a UGAS forward complete dynamical system if and only if there exist a functional  $V : \mathcal{C} \rightarrow \mathbb{R}_+$ , Lipschitz on bounded sets, functions  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ , and  $\eta \in \mathcal{K}$  such that

$$\underline{\alpha}(|\varphi(0)|) \leq V(\varphi) \leq \bar{\alpha}(\|\varphi\|_{\mathcal{C}}), \quad \forall \varphi \in \mathcal{C}, \quad (32)$$

$$D_q^+ V(\varphi) \leq -\eta(|\varphi(0)|), \quad \forall \varphi \in \mathcal{C}, \forall q \in Q; \quad (33)$$

- (ii)  $\Sigma = (\mathcal{C}, \mathcal{S}, \phi^\Delta)$  is a UGES forward complete dynamical system if and only if there exist a functional  $V : \mathcal{C} \rightarrow \mathbb{R}_+$ , Lipschitz on bounded sets, and positive reals  $\underline{c}, \bar{c}$  and  $p$ , such that

$$\underline{c}\|\phi\|_{\mathcal{C}}^p \leq V(\phi) \leq \bar{c}\|\phi\|_{\mathcal{C}}^p, \quad \forall \phi \in \mathcal{C}, \quad (34)$$

$$D_q^+ V(\phi) \leq -\|\phi\|_{\mathcal{C}}^p, \quad \forall \phi \in \mathcal{C}, \forall q \in Q; \quad (35)$$

- (iii)  $\Sigma = (\mathcal{C}, \mathcal{S}, \phi^\Delta)$  is a ULES forward complete dynamical system if and only if there exist positive reals  $r, p, \underline{c}, \bar{c}$  and Lipschitz functional  $V : \mathcal{C}_r \rightarrow \mathbb{R}_+$  such that inequalities (34)-(35) hold in  $\mathcal{C}_r$ , where  $\mathcal{C}_r$  denotes the subset of continuous functions  $\phi$  with  $\|\phi\|_{\mathcal{C}} \leq r$ .

Functionals satisfying the conditions (32)-(33), i.e. admitting a “pointwise dependent” lower bound and a “pointwise dependent” bound on the decay rate, are peculiar to retarded functional differential equations and are called Lyapunov-Krasovskii functionals (see, e.g., [21]). Note that they are not necessarily coercive or non-coercive Lyapunov functions, in the sense of Definition 4.4.

In the case of switched linear retarded systems, i.e., when (19) is given by

$$\dot{x}(t) = \Gamma_{\sigma(t)} x_t, \quad t \in \mathbb{R}_+, \quad (36)$$

where  $\Gamma_q : \mathcal{C} \rightarrow \mathbb{R}^n$ ,  $q \in Q$ , is a family of linear operators, more precise characterizations of the Lyapunov functionals, compared to Theorem 4.6 and Theorem 4.7, are obtained in [16]. We note that for switched linear retarded systems, Assumption 4.1 is equivalent to the fact that the norm of the linear maps  $\Gamma_q$  is uniformly bounded with respect to  $q \in Q$ . This implies the forward completeness of (36). Before stating the converse Lyapunov results, let us first recall the definition of directional Fréchet derivative of a nonlinear functional  $V$  from a Banach space  $X$  to  $\mathbb{R}_+$ .

**Definition 4.7.**  *$V$  is said to be directionally differentiable in the sense of Fréchet at  $\psi \in X$  if there exists a functional  $V'(\psi, \cdot) : X \rightarrow \mathbb{R}$  satisfying  $V'(\psi, \lambda\xi) = \lambda V'(\psi, \xi)$  for every  $\lambda \in \mathbb{R}_+, \xi \in X$  and such that*

$$\frac{V(\psi + \xi) - V(\psi) - V'(\psi, \xi)}{\|\xi\|} \rightarrow 0 \quad \text{as } \xi \rightarrow 0.$$

*In particular, for every  $\xi \in X$ , the directional derivative of  $V$  at  $\psi$  in the direction  $\xi$  is well defined and equal to  $V'(\psi, \xi)$ .*

Recall that  $\phi^\Delta$  denotes the transition map associated with (36). Remark that  $\phi^\Delta$  is well-defined also in the Sobolev space  $H^1 := H^1([-\Delta, 0], \mathbb{R}^n)$  (see [16, Lemma 1]), which is a subspace of  $\mathcal{C}$  endowed with the norm

$$\|\psi\|_{H^1} = \left( \int_{-r}^0 \left( |\psi(\theta)|^2 + \left| \frac{d}{d\theta} \psi(\theta) \right|^2 \right) d\theta \right)^{\frac{1}{2}}.$$

**Theorem 4.8** ([16, Theorem 2]). *Let Assumptions 4.1-4.2 hold and consider the forward complete dynamical system  $\Sigma = (X, \mathcal{S}, \phi^\Delta)$  with  $X = \mathcal{C}$  or  $X = H^1$ . The following statements are equivalent:*

- (i) System  $\Sigma$  with  $X = \mathcal{C}$  is UGES;
- (ii) System  $\Sigma$  with  $X = H^1$  is UGES;
- (iii) There exists a functional  $V : \mathcal{C} \rightarrow \mathbb{R}_+$  such that  $\sqrt{V(\cdot)}$  is a norm on  $\mathcal{C}$  and positive reals  $\underline{c}, \bar{c}$  such that

$$\begin{aligned} \underline{c}\|\psi\|_{\mathcal{C}}^2 &\leq V(\psi) \leq \bar{c}\|\psi\|_{\mathcal{C}}^2, & \forall \psi \in \mathcal{C}, \\ \bar{D}_q V(\psi) &\leq -\|\psi\|_{\mathcal{C}}^2, & \forall \psi \in \mathcal{C}, \forall q \in Q. \end{aligned}$$

- (iv) There exists a directionally Fréchet differentiable functional  $V : H^1 \rightarrow \mathbb{R}_+$  such that  $\sqrt{V(\cdot)}$  is a norm on  $H^1$ ,

$$\begin{aligned} \underline{c}\|\psi\|_{H^1}^2 &\leq V(\psi) \leq \bar{c}\|\psi\|_{H^1}^2, & \forall \psi \in H^1 \\ |V'(\psi, \xi)| &\leq c\|\psi\|_{H^1}\|\xi\|_{H^1}, & \forall \psi, \xi \in H^1 \\ V'(\psi, \xi_1 + \xi_2) &\leq V'(\psi, \xi_1) + V'(\psi, \xi_2), & \forall \psi, \xi_1, \xi_2 \in H^1 \end{aligned}$$

for some constants  $c, \underline{c}, \bar{c} > 0$  and

$$\bar{D}_q V(\psi) \leq -\|\psi\|_{H^1}^2, \quad \forall \psi \in H^1, \forall q \in Q.$$

If  $\psi \in H^2([-\Delta, 0], \mathbb{R}^n)$  and  $\psi'(0) = \Gamma_q \psi$  then  $\bar{D}_q V(\psi) = V'(\psi, \psi')$ .

- (v) There exists a continuous functional  $V : \mathcal{C} \rightarrow \mathbb{R}_+$  and  $c > 0$  such that

$$\begin{aligned} V(\psi) &\leq c\|\psi\|_{\mathcal{C}}^2, & \forall \psi \in \mathcal{C}, \\ \underline{D}_q V(\psi) &\leq -|\psi(0)|^2, & \forall \psi \in \mathcal{C}, \forall q \in Q. \end{aligned}$$

- (vi) There exists a continuous functional  $V : H^1 \rightarrow \mathbb{R}_+$  and  $c > 0$  such that

$$\begin{aligned} V(\psi) &\leq c\|\psi\|_{H^1}^2, & \forall \psi \in H^1, \\ \underline{D}_q V(\psi) &\leq -|\psi(0)|^2, & \forall \psi \in H^1, \forall q \in Q. \end{aligned}$$

## 5 Applications of Converse Lyapunov Theorems

### 5.1 Stability of singularly perturbed systems

Consider the following differential inclusion

$$\begin{pmatrix} \dot{x}_\varepsilon(t) \\ \varepsilon \dot{y}_\varepsilon(t) \end{pmatrix} \in F(x_\varepsilon(t), y_\varepsilon(t)) = \begin{pmatrix} F_x(x_\varepsilon(t), y_\varepsilon(t)) \\ F_y(x_\varepsilon(t), y_\varepsilon(t)) \end{pmatrix}, \quad (37)$$

where  $t \in \mathbb{R}_+$ ,  $\varepsilon$  is a small positive parameter,  $(x_\varepsilon(t), y_\varepsilon(t)) \in \mathbb{R}^n \times \mathbb{R}^m$  and  $F$  is a Lipschitz set-valued map with linear growth and having nonempty convex compact values. Equation (37) is used to model a system with a slow variable  $x_\varepsilon$  and a fast variable  $y_\varepsilon$ . A typical question when dealing with (37) concerns its stability properties for small values of  $\varepsilon$ . Specifically, we ask: is there an  $\varepsilon_0 > 0$  such that (37) is UGAS for every  $\varepsilon \in (0, \varepsilon_0)$ ? The singular perturbation theory, originally introduced for ordinary differential equations [71], provides an interesting approach in this context. This consists in the study of two limit dynamics which approximately describe separately the evolution of the slow and fast variables. This approach has been extensively studied in the context of differential inclusions (see, e.g., [62, 73]). More precisely, consider a Lipschitz set-valued map  $K : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$  associating with every  $x \in \mathbb{R}^n$  a compact convex subset  $K(x)$  of  $\mathbb{R}^m$ . We define the differential inclusion

$$\dot{x}(t) \in \overline{\text{co}} F_x(x(t), K(x(t))). \quad (38)$$

We also associate with each fixed  $x \in \mathbb{R}^n$ , the dynamics

$$\frac{d}{d\tau} y(\tau) \in F_y(x, y(\tau)), \quad (39)$$

which is obtained via the time rescaling  $\tau = t/\varepsilon$  and by “freezing” the slow variable. Note that  $x$  plays the role of a parameter in (39). We will need the following assumptions.

**Assumption 5.1.** *There exists  $A \subset \mathbb{R}^n$  such that the differential inclusion (38) is UGAS with respect to  $A$ .*

**Assumption 5.2.** *For every  $x \in \mathbb{R}^n$  the differential inclusion (39) is UGAS with respect to the set  $K(x)$ .*

Under Assumptions 5.1 and 5.2, and by a slight generalization of Theorems 3.3 and 3.4, in [73] the author proves the existence of smooth Lyapunov functions for both systems (38) and (39). In particular, the Lyapunov function for system (39) smoothly depends on the parameter  $x$  (this also generalizes a converse Lyapunov result for parametric systems obtained in [23]). Based on this Lyapunov characterization, the main result of [73], stated below, essentially allows one to study the behaviour of the solutions (37) in terms of the auxiliary differential inclusions (38) and (39).

**Theorem 5.1** ([73]). *For any  $x \in \mathbb{R}^n$  there exists  $r > 0$  such that for every collection of solutions  $(x_\varepsilon, y_\varepsilon)_{\varepsilon>0}$  of (37) starting from  $(x, y)$ , with  $y \in B(K(x), r)$ , there exists a collection  $(\bar{x}_\varepsilon)_{\varepsilon>0}$  of solutions of (38) which start from  $x$  such that*

$$\begin{aligned} \forall t_1 > 0, \quad \sup_{t \geq t_1} |y_\varepsilon(t)|_{K(x_\varepsilon(t))} &\rightarrow_{\varepsilon \rightarrow 0} 0 \\ \forall T > 0, \quad \sup_{t \in [0, T]} |x_\varepsilon(t) - \bar{x}_\varepsilon(t)| &\rightarrow_{\varepsilon \rightarrow 0} 0. \end{aligned}$$

Moreover, for each  $\eta > 0$  there exists  $T_\eta > 0$  such that for all  $T \geq T_\eta$  there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \in (0, \varepsilon_0)$ , we have

$$\sup_{t \in [0, T]} |x_\varepsilon(t) - \bar{x}_\varepsilon(t)| + \sup_{t \geq T} |x_\varepsilon(t)|_A + \sup_{t \geq T} |\bar{x}_\varepsilon(t)|_A \leq \eta.$$

In [9], the authors construct an explicit Lyapunov function based on hypotheses analogous to Assumptions 5.1-5.2 in order to study the stability property of singularly perturbed switched systems.

## 5.2 Robustness of perturbed systems

Consider system (36). Let  $\mathcal{P}$  be a bounded subset of  $\mathcal{L}(\mathcal{C}, \mathbb{R}^n)$ . In the case where system (36) is UGES, the stability of the perturbed system

$$\dot{x}(t) = (\Gamma_{\sigma(t)} + \Lambda_{\sigma(t)})x_t, \quad \forall t \in \mathbb{R}_+ \quad (40)$$

where  $\Lambda_{\sigma(t)} \in \mathcal{P}$ , can be studied using the following proposition taken from [17], which can be easily obtained as a consequence of Theorem 4.8.

**Proposition 5.1.** *Suppose that the unperturbed system (36) is UGES and let  $V$  be the associated Lyapunov functional given by item (iii) of Theorem 4.8. Let  $\ell = \sup_{\Lambda \in \mathcal{P}} \|\Lambda\|_{\mathcal{L}(\mathcal{C}, \mathbb{R}^n)}$ . Then,*

$$\overline{D}_{\Gamma+\Lambda}V(\psi) \leq \overline{D}_\Gamma V(\psi) + 2\ell\bar{c}\|\psi\|_{\mathcal{C}}^2 \quad \forall \psi \in \mathcal{C},$$

where  $\overline{D}_\Gamma V$  and  $\overline{D}_{\Gamma+\Lambda}V$  are the upper right-hand Dini derivatives of  $V$  along systems (36) and (40), respectively.

According to Proposition 5.1 and Item (iii) of Theorem 4.8, if the unperturbed system is UGES and the perturbation is sufficiently small, in the sense that  $-1 + 2\ell\bar{c} < -\bar{\ell}$  for some  $\bar{\ell} > 0$ , then the perturbed system remains UGES. An explicit example showing the applicability of Proposition 5.1 can be found in [17].

## 5.3 Stability of sampled semilinear control systems

We consider here the problem of stabilizing the system (18) assuming that only discrete-time state measurements are available. We suppose that

- $A$  is the infinitesimal generator of a strongly continuous (linear) group  $(T(t))_{t \in \mathbb{R}}$ ,
- system (18) in closed-loop with

$$u(t) = K(x(t)), \quad \forall t \in \mathbb{R}_+, \quad (41)$$

where  $K : X \rightarrow U$  is a Lipschitz continuous function satisfying  $K(0) = 0$ , is UGES.

Let us give the following definition.

**Definition 5.1.** *Let  $0 < s_1 < \dots < s_k < \dots$  be an increasing sequence of times such that  $\lim_{k \rightarrow +\infty} s_k = +\infty$ . The instants  $s_k$  are called sampling instants and the quantity*

$$\delta = \sup_{k \in \mathbb{R}_+} (s_{k+1} - s_k)$$

*is called the maximal sampling time. By predictor-based sampled data controller we mean a feedback  $u(\cdot)$  of the type*

$$u(t) = K(T(t - s_k)x(s_k)), \quad \forall t \in [s_k, s_{k+1}), \quad \forall k \in \mathbb{R}_+. \quad (42)$$

We denote by  $\Sigma_0 = (X, \mathcal{S}_{\text{pc}}, \phi^{\Sigma_0})$  and  $\Sigma = (X, \mathcal{S}_{\text{pc}}, \phi^\Sigma)$  the forward complete dynamical systems corresponding to (18)–(41) and (18)–(42), respectively. The following theorem, which can be proved thanks to Theorem 4.5, gives sufficient conditions for the exponential stability of system  $\Sigma$  provided that  $\Sigma_0$  is uniformly semi-globally exponentially stable.

**Theorem 5.2** ([15, Theorem 8]). *If system  $\Sigma_0$  is USGES and*

$$\lim_{r \rightarrow \infty} \frac{r}{M(r)} = +\infty, \quad (43)$$

*where  $M(r)$  is as in Definition 2.4, then for every  $r > 0$  there exists  $\delta^*(r) > 0$  such that  $\Sigma$  is UES in  $B(0, r)$ , provided that the maximal sampling time of  $(s_k)_{k \in \mathbb{N}}$  is smaller than  $\delta^*(r)$ . In addition, if  $\inf_{r \in \mathbb{R}_+} \delta^*(r) > 0$  then  $\Sigma$  is USGES.*

Note that, due to the fact that  $A$  is in general an unbounded operator, taking  $K(x(s_k))$  as feedback control in each interval  $[s_k, s_{k+1})$  does not necessarily preserve the stability of the system.

When  $\Sigma_0$  is uniformly globally exponentially stable, we have the following result.

**Corollary 5.1.** *Suppose that system  $\Sigma_0$  is UGES. Then there exists  $\delta^* > 0$  such that system  $\Sigma$  is UGES provided that the maximal sampling time of  $(s_k)_{k \in \mathbb{N}}$  is smaller than  $\delta^*$ .*

The applicability of Corollary 5.1 to the exponential stabilization of a switched damped wave equation with sampled data feedback is shown in [15].

## 5.4 Input-to-state stability of perturbed stable systems

Let  $X$  and  $U$  be two Banach spaces and consider the control system (18). Assume that the set of admissible controls is  $L^p(U) := L^p(\mathbb{R}_+, U)$  with  $1 \leq p \leq +\infty$ . As in Example 4.2, we can define for all  $t \in \mathbb{R}_+$ ,  $x_0 \in X$ ,  $\sigma \in \mathcal{S}_{\text{pc}}$ , and  $u \in L^p(U)$ , the corresponding transition map  $\phi_u(t, x_0, \sigma)$ , which is continuous with respect to  $(t, x_0, u) \in \mathbb{R}_+ \times X \times L^p(U)$ .

A direct application of the converse theorems presented in Section 4.3 yields the following result, which demonstrates that the input-to-state map  $u \mapsto \phi_u(\cdot, 0, \sigma)$  has finite gain.

**Theorem 5.3** ([15, Theorem 9]). *Assume that the forward complete dynamical system  $(X, \mathcal{S}_{\text{pc}}, \phi_0)$  is UGES. Then for every  $1 \leq p \leq +\infty$  and  $\sigma \in \mathcal{S}_{\text{pc}}$ , the input-to-state map  $u \mapsto \phi_u(\cdot, 0, \sigma)$  is well defined as a map from  $L^p(U)$  to  $L^p(X)$  and has a finite  $L^p$ -gain independent of  $\sigma$ , i.e., there exists  $c_p > 0$  such that*

$$\|\phi_u(\cdot, 0, \sigma)\|_{L^p(X)} \leq c_p \|u\|_{L^p(U)}, \quad \forall u \in L^p(U), \forall \sigma \in \mathcal{S}_{\text{pc}}. \quad (44)$$

The input-to-state stability (ISS) properties of semilinear control systems have been widely studied in the literature. For a detailed overview of the literature in this field, see the recent works [7, 47, 48]. Note that, in the infinite-dimensional setting, ISS has been characterized through coercive [50] and non-coercive [24] Lyapunov functionals.

## 6 Conclusions

Converse Lyapunov theorems are particularly important in control theory, as they facilitate the study of various system properties, including robustness to external perturbations, parametric robustness, stability of interconnected systems, and many other interesting properties, as discussed in this paper.

We review the existing bibliography for different classes of dynamical systems in both finite and infinite dimension. For finite-dimensional systems, we summarize key results related to autonomous and non-autonomous differential systems, differential inclusions, and switched systems. In the infinite-dimensional setting, we revisit results within the general framework of forward-complete dynamical systems, which encompasses interesting classes of infinite-dimensional systems, including time-delay systems. The distinction between coercive and non-coercive Lyapunov functions, a characteristic feature of infinite-dimensional systems, is carefully highlighted. Finally, the paper provides examples of applications of converse Lyapunov theorems for both finite- and infinite-dimensional systems, illustrating their utility in practical settings.

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