



**HAL**  
open science

# Limiting absorption principle for long-range perturbation in a graphene setting

Nassim Athmouni, Marwa Ennaceur, Sylvain Golenia, Amel Jadlaoui

## ► To cite this version:

Nassim Athmouni, Marwa Ennaceur, Sylvain Golenia, Amel Jadlaoui. Limiting absorption principle for long-range perturbation in a graphene setting. 2025. <hal-05038643v2>

**HAL Id: hal-05038643**

**<https://hal.science/hal-05038643v2>**

Preprint submitted on 7 Jul 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# LIMITING ABSORPTION PRINCIPLE FOR LONG-RANGE PERTURBATION IN A GRAPHENE SETTING

NASSIM ATHMOUNI<sup>1</sup>, MARWA ENNACEUR<sup>2</sup>, SYLVAIN GOLÉNIA<sup>3</sup>, AND AMEL JADLAOUI<sup>4</sup>

ABSTRACT. In this paper, we examine the discrete Laplacian acting on a hexagonal lattice by introducing long-range modifications in both the metric and the potential. Our objective is to establish a Limiting Absorption Principle, excluding possible embedded eigenvalues. To this end, we employ the positive commutator technique as our method.

## CONTENTS

1. Introduction and main result	1
2. The Mourre theory	5
3. Proof of main result	6
3.1. Laplacian on the hexagonal lattice	6
3.2. The perturbed model	15
3.3. Proof of the main result	29
4. Appendix	30
References	41

## 1. INTRODUCTION AND MAIN RESULT

Over the last years, spectral graph theory has garnered research interest, particularly with regards to the study of various types of discrete Laplacians [AtDa, AtEnGo, AtEnGoJa, BaGoJe, Sa, GeGo, GüKe, AnTo, KoMø, KoSa] and their magnetic counterparts [GoTr, BoKeGoLiMü, GoMo, AtBaDaEn, PaRi1, HiSh]. One method for exploring the essential spectrum of these operators is using a positive commutator technique. With respect to previous research, [Sa, BoSa, Ta1] investigate the case of  $\mathbb{Z}^d$ , while [AlFr] and [GeGo] explore binary trees. On the other hand, [MäRiTt] analysed a general class of graphs. Moreover, [AtEnGo] examines a discrete version of cusps and funnels. Last, [AtEnGoJa] focuses on a triangular lattice graph.

In [AtEnGoJa], our research is centered on the discrete Laplacian acting on the triangular lattice. We aim to obtain spectral results for a wider class of perturbations. Specifically, we introduce long-range perturbations in both the potential and the metric. To this end, we employ a Mourre estimate technique. In the present work, we analyze the hexagonal lattice (see Figure 1), which is more complex due to its close resemblance to the analysis of a Dirac operator acting on a triangular lattice. It is highly similar to the structure of materials such as graphene, which exhibits unique electronic properties due to its hexagonal lattice. The mathematical framework developed here provides a foundation for understanding quantum phenomena behaviors, including particle wave interferences, which emerge in lattice systems. This structure allows for the observation of phenomena like Dirac fermions, which play a key role in quantum mechanics. In this paper, we seek spectral results, such as resolvent estimates, which in turn yield propagation estimates. The model was already studied in [PaRi, RiTs] through Floquet decomposition and analytic deformation. However, those approaches fail to treat the class of long-range perturbations. With this in mind, the aim of our work is precisely to cover this long-range regime. As in [AtEnGoJa], we implement long-range modifications of both the potential and the metric, relying on the Mourre estimate method. Moreover, thanks to the specific structure of the model, we can explicitly determine the set of thresholds (Theorem 1.1) by means of the Fourier transform, see also [AnIsMo]. This distinguishes our approach from that of [PaRi], where the authors adopt a general Floquet-Bloch method. By means an abstract reasoning, they assert the existence of these points in the spectrum

2010 *Mathematics Subject Classification.* 81Q10, 47B25, 47A10, 05C63, 08B15.

*Key words and phrases.* Commutator, Mourre estimate, Limiting Absorption Principle, Discrete Laplacian, Hexagonal lattice.

merely via a direct integral decomposition. We note that [Ta2] analyses the short-range and long-range potentials with another choice of conjugate operator, but does not accommodate metric perturbations, see also Remark 3.7. Concerning the perturbation of the metric, this is considered in [PaRi] but due to the abstract context, they do not succeed in dealing with the long range perturbation of the metric. Here, our choice of conjugate operator is different from the two other approaches it helps us to attain this goal. However, our approach is heavier in term of computation.

We begin by recalling definitions from some graph theory, which will form the foundation for our analysis. An infinite, connected *graph*  $\mathcal{G}$  is a triple  $(\mathcal{V}, \mathcal{E}, m)$ , where  $\mathcal{V}$  is a countable set of *vertices*,  $m : \mathcal{V} \rightarrow (0, \infty)$  is a *weight*, and  $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow [0, +\infty)$  (the edges) is symmetric. For two vertices  $n$  and  $l$ , we say they are *neighbors* if  $\mathcal{E}(n, l) > 0$ . We denote this relationship by  $n \sim l$  and define the set of neighbors of  $n$  as  $\mathcal{N}_{\mathcal{G}}(n)$ . We let  $\mathcal{C}(\mathcal{V}) := \{f : \mathcal{V} \rightarrow \mathbb{C}^d\}$  denote the space of complex-valued functions defined on the set of vertices  $\mathcal{V}$ . We define the Hilbert space

$$\ell^2(\mathcal{V}, m; \mathbb{C}^d) := \left\{ f \in \mathcal{C}(\mathcal{V}), \sum_{n \in \mathcal{V}} m(n) |f(n)|_{\mathbb{C}^d}^2 < \infty \right\},$$

equipped with the scalar product,  $\langle f, g \rangle := \sum_{n \in \mathcal{V}} \sum_{k=1}^d m(n) \overline{f_k(n)} g_k(n)$ . In general, we define the Laplacian  $\Delta_H$  on  $\mathcal{G}$  by:

$$(1) \quad \Delta_H f(x) := \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) f(y).$$

Here, we investigate a hexagonal lattice where:

$$\mathcal{V}_0 := \left\{ \sum_{j=1}^2 k_j v_j; k := (k_1, k_2) \in \mathbb{Z}^2 \right\}, v_1 := \left( \frac{3}{2}, \frac{\sqrt{3}}{2} \right), v_2 := (0, \sqrt{3}).$$

Let

$$p_1 := \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right), p_2 := (1, 0),$$

and define the vertex set  $\tilde{\mathcal{V}}$  by:

$$(2) \quad \tilde{\mathcal{V}} := \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{V}_i := p_i + \mathcal{V}_0.$$

Note that  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ . Let  $a_1 \in \mathcal{V}_1$  and  $a_2 \in \mathcal{V}_2$  such that  $a_1 = p_1 + (n_1, n_2)$  and  $a_2 = p_2 + (n_1, n_2)$  we have:

$$(3) \quad \begin{aligned} \mathcal{N}_{a_1} &= \{l \in \mathbb{R}^2; |l - a_1|_{\mathbb{R}^2} = 1, a_1 \in \mathcal{V}_1\} \cap \mathcal{V}_2 \\ &= \left\{ a_1 + \frac{v_1 + v_2}{3}, a_1 + \frac{v_1 - 2v_2}{3}, a_1 - \frac{2v_1 - v_2}{3} \right\}, \end{aligned}$$

$$(4) \quad \begin{aligned} \mathcal{N}_{a_2} &= \{l \in \mathbb{R}^2; |l - a_2|_{\mathbb{R}^2} = 1, a_2 \in \mathcal{V}_2\} \cap \mathcal{V}_1 \\ &= \left\{ a_2 + \frac{2v_1 - v_2}{3}, a_2 - \frac{v_1 - 2v_2}{3}, a_2 - \frac{v_1 + v_2}{3} \right\}. \end{aligned}$$

Given  $f \in \ell^2(\tilde{\mathcal{V}}, 1; \mathbb{C})$ ,  $f_i$  is its projection onto  $\ell^2(\mathcal{V}_i, 1; \mathbb{C})$ . Namely,

$$(5) \quad f_i(n_1, n_2) := f(p_i + (n_1, n_2)) =: f(n_1, n_2, p_i).$$

Using (2) and the following isomorphisms:

$$(6) \quad \begin{aligned} \ell^2(\tilde{\mathcal{V}}, 1; \mathbb{C}) &\rightarrow \ell^2(\mathcal{V}_1, 1; \mathbb{C}) \oplus \ell^2(\mathcal{V}_2, 1; \mathbb{C}) \rightarrow \ell^2(\mathcal{V}_0, 1; \mathbb{C}^2) \\ f &\mapsto (f_1, f_2) \mapsto {}^t(f_1, f_2), \end{aligned}$$

we conclude that:

$$(7) \quad \ell^2(\tilde{\mathcal{V}}, 1; \mathbb{C}) \simeq \ell^2(\mathcal{V}_1, 1; \mathbb{C}) \oplus \ell^2(\mathcal{V}_2, 1; \mathbb{C}) \simeq \ell^2(\mathcal{V}_0, 1; \mathbb{C}^2).$$

In this sequel, we often identify the vertices of  $\mathcal{V}_i$  with  $\mathbb{Z}^2$ , using the canonical map:

$$(8) \quad \begin{aligned} G_i &: \mathbb{Z}^2 \rightarrow \mathcal{V}_i \\ (k_1, k_2) &\mapsto k_1 v_1 + k_2 v_2 + p_i, \end{aligned}$$

where  $i \in \{1, 2\}$ . In other words, we have the following identifications:

$$(9) \quad \ell^2(\mathbb{Z}^2 \times \{p_i\}; \mathbb{C}) \simeq \ell^2(\mathcal{V}_i; \mathbb{C}),$$

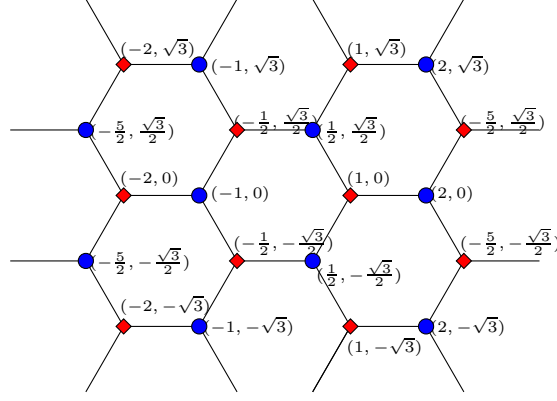


FIGURE 1. Hexagonal lattice  
 •: element of  $\mathcal{V}_1$ ; ♦: element of  $\mathcal{V}_2$ .

where  $i \in \{1, 2\}$ . From (6) and (9), we have the following isomorphism.

$$(10) \quad \ell^2(\mathbb{Z}^2 \times \{p_1, p_2\}; \mathbb{C}) \simeq \ell^2(\mathbb{Z}^2; \mathbb{C}^2) \simeq \ell^2(\tilde{\mathcal{V}}, 1; \mathbb{C}).$$

For all  $f \in \ell^2(\mathbb{Z}^2, 1)$ , we have:

$$\begin{aligned} U_1 f(n_1, n_2) &:= f(n_1 - 1, n_2), & U_1^* f(n_1, n_2) &= f(n_1 + 1, n_2), \\ U_2 f(n_1, n_2) &:= f(n_1, n_2 - 1), & U_2^* f(n_1, n_2) &= f(n_1, n_2 + 1). \end{aligned}$$

Recall (1). When  $m = 1$  and

$$\begin{aligned} \mathcal{E} : \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} &\rightarrow \{0, 1\} \\ (l, n) &\mapsto \mathcal{E}(l, n) := \begin{cases} 1, & \text{if } |l - n|_{\mathbb{R}^2} = 1; \\ 0, & \end{cases} \end{aligned}$$

where  $l = (l_1, l_2)$ ,  $n = (n_1, n_2)$  and  $|n|_{\mathbb{R}^2} := \sqrt{n_1^2 + n_2^2}$ . Under the identification (10) we rewrite  $\Delta_H$  on the hexagonal lattice as follows:

$$\Delta_H := \frac{1}{3} \begin{pmatrix} 0 & \Delta_{2,H} \\ \Delta_{1,H} & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & \text{id} + U_1 + U_2 \\ \text{id} + U_1^* + U_2^* & 0 \end{pmatrix}.$$

Now, we perturb the metric of the previous case, by an amount that remains small at infinity. Using (1) and (10) the Laplacian  $\Delta_{m,\mathcal{E}}$  is given by

$$(11) \quad \Delta_{m,\mathcal{E}} : \ell^2(\tilde{\mathcal{V}}, m) \rightarrow \ell^2(\tilde{\mathcal{V}}, m) \\ f \mapsto \Delta_{m,\mathcal{E}} f := \frac{1}{3} \begin{pmatrix} 0 & \Delta_{2,m,\mathcal{E}} \\ \Delta_{1,m,\mathcal{E}} & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

with

$$\begin{aligned} \Delta_{1,m,\mathcal{E}} f_1(n_1, n_2) &:= \frac{1}{m(n_1, n_2, p_2)} \left( \left( \mathcal{E}((Q_1, Q_2, p_1), (Q_1, Q_2, p_2)) + \mathcal{E}((Q_1, Q_2, p_1), (Q_1 + 1, Q_2, p_2)) U_1^* \right. \right. \\ &\quad \left. \left. + \mathcal{E}((Q_1, Q_2, p_1), (Q_1, Q_2 + 1, p_2)) U_2^* \right) f_1 \right)(n_1, n_2), \end{aligned}$$

$$\begin{aligned} \Delta_{2,m,\mathcal{E}} f_2(n_1, n_2) &:= \frac{1}{m(n_1, n_2, p_1)} \left( \left( \mathcal{E}((Q_1, Q_2, p_2), (Q_1, Q_2, p_1)) + \mathcal{E}((Q_1, Q_2, p_2), (Q_1 - 1, Q_2, p_1)) U_1 \right. \right. \\ &\quad \left. \left. + \mathcal{E}((Q_1, Q_2, p_2), (Q_1, Q_2 - 1, p_1)) U_2 \right) f_2 \right)(n_1, n_2), \end{aligned}$$

where  $m$  and  $\mathcal{E}$  respectively satisfy the following hypotheses:

$$(H_0) \quad m(n_1, n_2, p_i) := (1 + \eta(n_1, n_2, p_i)), \quad \text{where } \inf_{(n_1, n_2) \in \mathbb{Z}^2} \eta(n_1, n_2, p_i) > -1,$$

$$\eta(n_1, n_2, p_i) \rightarrow 0 \text{ if } |n| \rightarrow \infty,$$

$$(H_1) \quad \mathcal{E}((n_1, n_2, p_i), (n, p_j)) := (1 + \varepsilon((n_1, n_2, p_i), (n_1, n_2, p_j))), \text{ where}$$

$$\inf_{(n_1, n_2), (k_1, k_2) \in \mathbb{Z}^2} \varepsilon((n_1, n_2, p_i), (k_1, k_2, p_j)) > -1,$$

$$\varepsilon((n_1, n_2, p_i), (k_1, k_2, p_j)) \rightarrow 0 \text{ if } |(n_1, n_2), (k_1, k_2)| \rightarrow \infty, \text{ where } i, j \in \{1, 2\}.$$

Note that  $\mathcal{E}$  has the following properties:

- By symmetry of  $\mathcal{E}$ , we have  $\mathcal{E}((n_1, n_2, p_i), (n'_1, n'_2, p_j)) = \mathcal{E}((n'_1, n'_2, p_j), (n_1, n_2, p_i))$ .
- Note that  $\mathcal{E}((n_1, n_2, p_i), (n'_1, n'_2, p_j)) \neq 0$ , then  $p_i \neq p_j$ .
  - If  $i = 1$ , then  $(n'_1, n'_2) = (n_1, n_2)$  or  $(n_1 + 1, n_2)$  or  $(n_1, n_2 + 1)$ .
  - If  $i = 2$ , then  $(n'_1, n'_2) = (n_1, n_2)$  or  $(n_1 - 1, n_2)$  or  $(n_1, n_2 - 1)$ .

The operator  $\Delta_{m, \varepsilon}$  is self-adjoint and bounded on  $\ell^2(\tilde{\mathcal{V}}, m)$ . Moreover, in the case where  $\eta = 0$  and  $\varepsilon = 0$ , we have  $\Delta_{m, \varepsilon} = \Delta_H$  and  $\sigma(\Delta_H) = [-1, 1] = \sigma_{\text{ac}}(\Delta_H)$ , where  $\sigma_{\text{ac}}(\cdot)$  is the absolutely continuous spectrum, see [AnIsMo] and Lemma 3.6. Given a function  $W : \tilde{\mathcal{V}} \rightarrow \mathbb{C}$ , we denote by  $W(Q_1, Q_2)$  the operator of multiplication by  $W$ . In particular  $(W(Q_1, Q_2)f)(n_1, n_2, p_i) := W(n_1, n_2, p_i)f(n_1, n_2, p_i)$ , for all  $f \in \mathcal{D}(W(Q_1, Q_2))$ . By (6), (7) and Proposition 3.21, we have  $W = W_1 \oplus W_2$ , with  $W_i := (\cdot, \cdot, p_i)$ . It acts as follows:

$$W : \ell^2(\tilde{\mathcal{V}}, m; \mathbb{C}) \rightarrow \ell^2(\mathcal{V}_1, m; \mathbb{C}) \oplus \ell^2(\mathcal{V}_2, m; \mathbb{C})$$

$$(f_1, f_2) \mapsto (W_1 f_1, W_2 f_2).$$

Hence,  $\mathcal{D}(W) = \mathcal{D}(W_1) \oplus \mathcal{D}(W_2)$  and

$$\mathcal{D}(W_i(Q_1, Q_2)) := \{f \in \ell^2(\mathcal{V}_i, m); (n_1, n_2) \mapsto W_i(n_1, n_2)f(n_1, n_2, p_i) \in \ell^2(\mathcal{V}_i, m)\}.$$

Let  $V : \tilde{\mathcal{V}} \rightarrow \mathbb{R}^2$  and  $H := \Delta_{m, \varepsilon} + V(Q)$  such that:

$$(H_2) \quad V(n_1, n_2, p_i) \rightarrow 0 \text{ if } |(n_1, n_2)| \rightarrow \infty,$$

where  $i \in \{1, 2\}$ . Note  $V(Q_1, Q_2)$  is a compact operator, as uniform limit of finite rank operators given by  $1_{\|\cdot\|_{\mathbb{R}^2} \leq R} V$ , with  $R \in \mathbb{N}$ . The operator  $H_{m, \varepsilon}$  is self-adjoint on  $\ell^2(\tilde{\mathcal{V}}, m)$ . In fact, it can be regarded as a compact perturbation of  $\Delta_H$ , see Proposition 3.24 for a precise statement. Moreover, we have  $\sigma_{\text{ess}}(H_{m, \varepsilon}) = \sigma_{\text{ess}}(\Delta_H)$ , where  $\sigma_{\text{ess}}(\cdot)$  is the essential spectrum.

We now seek to refine the spectral property and ask for more decay. We set:

$$(12) \quad \Lambda_H(\cdot) := \Lambda(\cdot) \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix},$$

where  $\Lambda(n_1, n_2) = \langle n_1 \rangle + \langle n_2 \rangle$  and  $\langle \cdot \rangle := \sqrt{\frac{1}{2} + |\cdot|^2}$ . Note that  $\Lambda(Q_1, Q_2)$  is an unbounded self-adjoint operator. We fix  $\gamma > 0$ , we assume a long-range decay for  $V$ ,  $\eta$  and  $\varepsilon$ : Let  $l_1, l_2 \in \{0, 1\}$ . Let  $k \in \mathbb{N}^*$ ,  $h \in \{0, 1, 2\}$ ,  $b \in \mathbb{N}$ . We say that  $G : (\mathbb{Z}^2 \times \{p_1, p_2\})^k \rightarrow \mathbb{C}$  satisfies the hypothesis  $(H_{3,k})$  if the following condition hold

$$(H_{3,k}) \quad \max_{h \in \{1, 2\}} \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |G(J_{k, h, 1}(n_1, n_2, p_1)) - G(J_{k, h, 0}(n_1, n_2, p_2))| < \infty,$$

which is equivalent to

$$\max_{h \in \{1, 2\}} \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |G(J_{k, h, 1}(n_1, n_2, p_2)) - G(J_{k, h, 0}(n_1, n_2, p_1))| < \infty,$$

where

$$J_{k, h, b} : (\mathbb{Z}^2 \times \{p_1, p_2\}) \rightarrow (\mathbb{Z}^2 \times \{p_1, p_2\})^k$$

$$(n_1, n_2, p_i) \mapsto \left( (n_1 - (-1)^i b \delta_{h, 1}, n_2 - (-1)^i b \delta_{h, 2}, p_j), \underbrace{(n_1, n_2, p_i)}_{k-1 \text{ times}} \right)$$

and  $\delta_{h, r}$  is the Kronecker's delta symbol.

We denote by  $\sigma_p(\cdot)$  the set of pure point spectrum. We now state our main theorem:

**Theorem 1.1.** *Suppose that  $(H_0)$ ,  $(H_1)$  and  $(H_2)$  are true. We assume that  $\eta$  and  $V$  satisfy  $(H_{3,1})$  and  $\varepsilon$  satisfy  $(H_{3,2})$  and we take  $s > \frac{1}{2}$ . We obtain the following assertions:*

1.  $\sigma_{\text{ess}}(H_{m, \varepsilon}) = \sigma_{\text{ess}}(\Delta_H)$ .

2. The eigenvalues of  $H_{m,\varepsilon}$  distinct from  $\kappa$  are of finite multiplicity and can accumulate only toward  $\kappa$ , where  $\kappa := \{-1, -\frac{1}{3}, 0, \frac{1}{3}, 1\}$ , the set of thresholds.
3. The singular continuous spectrum of  $H_{m,\varepsilon}$  is empty.
4. The following limit exists and is finite:

$$\lim_{\rho \rightarrow 0^+} \sup_{\lambda \in [a,b]} \|\Lambda^{-s}(Q)(H_{m,\varepsilon} - \lambda - i\rho)^{-1}\Lambda^{-s}(Q)\| < \infty,$$

where  $[a, b]$  is included in  $\mathbb{R} \setminus (\kappa \cup \sigma_p(H_{m,\varepsilon}))$ . Moreover, in the norm topology of bounded operators, the boundary values of the resolvent:

$$[a, b] \ni \lambda \mapsto \lim_{\rho \rightarrow 0^\pm} \Lambda^{-s}(Q)(H_{m,\varepsilon} - \lambda - i\rho)^{-1}\Lambda^{-s}(Q) \text{ exists and is continuous.}$$

5. There exists  $c > 0$  such that for all  $f \in \ell^2(\tilde{\mathcal{V}})$ , we have:

$$\int_{\mathbb{R}} \|\Lambda^{-s}(Q)e^{-itH_{m,\varepsilon}} E_{[a,b]}(H_{m,\varepsilon})f\|^2 dt \leq c\|f\|^2,$$

where  $E_{[a,b]}(H_{m,\varepsilon})$  is the spectral projection of  $H_{m,\varepsilon}$  above  $[a, b]$ .

We now explain the hypothesis  $(H_{3,k})$  in more detail in our setting. The hypotheses on  $V$  correspond to the following cases: If  $k = 1$ , we take  $G \equiv V$ . For  $h = 1$ , the hypothesis  $(H_{3,1})$  gives

$$\begin{aligned} & \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |V(J_{1,1,1}(n_1, n_2, p_1)) - V(J_{1,1,0}(n_1, n_2, p_2))|, \\ & = \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |V(n_1 + 1, n_2, p_1) - V(n_1, n_2, p_2)| < \infty, \end{aligned}$$

For  $h = 2$ , the hypothesis  $(H_{3,1})$  gives

$$\begin{aligned} & \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |V(J_{1,2,1}(n_1, n_2, p_1)) - V(J_{1,2,0}(n_1, n_2, p_2))|, \\ & = \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |V(n_1, n_2 - 1, p_1) - V(n_1, n_2, p_2)| < \infty, \end{aligned}$$

Similarly for  $G \equiv \eta$ . The hypotheses on  $\varepsilon$  correspond to the following cases: If  $k = 2$ , we take  $G \equiv \varepsilon$ . For  $h = 1$ , the hypothesis  $(H_{3,2})$  gives

$$\begin{aligned} & \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |\varepsilon(J_{2,1,1}(n_1, n_2, p_1)) - \varepsilon(J_{2,1,0}(n_1, n_2, p_2))|, \\ & = \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |\varepsilon((n_1 + 1, n_2, p_1), (n_1, n_2, p_2)) \\ & \quad - \varepsilon((n_1, n_2, p_2), (n_1, n_2, p_1))| < \infty, \end{aligned}$$

For  $h = 2$ , the hypothesis  $(H_{3,2})$  gives

$$\begin{aligned} & \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |\varepsilon(J_{2,2,1}(n_1, n_2, p_1)) - \varepsilon(J_{2,2,0}(n_1, n_2, p_2))|, \\ & = \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |\varepsilon((n_1, n_2 - 1, p_1), (n_1, n_2, p_2)) \\ & \quad - \varepsilon((n_1, n_2, p_1), (n_1, n_2, p_2))| < \infty, \end{aligned}$$

In point 1. we only need hypotheses  $(H_0)$ ,  $(H_1)$  and  $(H_2)$ . Points 2.–5. are typical results derived from Mourre's theory. Specifically, we establish a Mourre estimate and verify the regularity assumptions. This is the most technical part of our analysis. For historical context and an introduction to the topic, refer to Section 2. Point 4. is called a *Limiting Absorption Principle*, which implies that the spectrum is purely absolutely continuous above  $\mathbb{R} \setminus (\kappa \cup \sigma_p(H_{m,\varepsilon}))$ . Particularly, Riemann-Lebesgue Theorem ensures that the solution of the Schrödinger equation dissipates at infinity. That is, for  $f$  belonging to the absolutely continuous subspace of  $H_{m,\varepsilon}$  and  $n \in \tilde{\mathcal{V}}$ ,

$$(13) \quad \lim_{|t| \rightarrow \infty} (e^{itH_{m,\varepsilon}} f)(n) = 0.$$

This can be interpreted as the particle moving off to infinity, while point 5. suggests that the particle focuses where  $\Lambda_H^\gamma$  is large. Point 5. corresponds to the observation that  $\Lambda_H^\gamma$  is locally  $H$ -regular over  $[a, b]$ , for example, [ReSi, Section VIII.C].

Thanks to the concrete framework of this work, we can explicitly define the set of the critical points  $\kappa$ . It corresponds to the energy where, after Fourier transform, the symbol of  $\Delta_H$  is zero, at this energies see Lemma 3.3. Intuitively, there is no propagation, see Lemma 3.16. In [PaRi], the authors use a general

and abstract Floquet Bloch approach. This ensures the existence of critical points by using the direct integral decomposition, see also [GéNi] for a general theory. They do not give this set explicitly.

We now outline the structure of our paper. Section 2 provides a brief overview of Mourre theory, and Subsection 3.1 analyzes the model and establishes the Mourre estimate for the Laplacian operating on a hexagonal lattice. Moreover, 3.2 introduces a perturbation to the metric and incorporates the potential. Lastly, Subsection 3.3, we prove Theorem 1.1.

## 2. THE MOURRE THEORY

In 1956, C.R. Putnam introduced a criterion ensuring that the spectrum of a self-adjoint operator  $H$  is purely absolutely continuous, based on the existence of a bounded, self-adjoint operator  $B$  satisfying  $[H, iB] > 0$ . However, the boundedness of  $B$  presents a significant limitation for practical applications. The Mourre Theory has attracted significant interest since its introduction in 1980 (cf., [Mo1, Mo2]). Numerous works have proved the importance of Mourre's commutator theory for understanding the point and continuous spectra of a sufficiently large class of self-adjoint operators. Notable contributions include [GeGéMø, GeGo, JeMoPe, Sa], as well as the book [AmBoGe], the master courses [Go], and more recent results such as [GoJe, Gé, GoMa].

Now, we recall Mourre's commutator theory. Let  $H$  and  $A$  be two self-adjoint operators acting on a complex Hilbert space  $\mathcal{H}$ . Suppose also  $H \in \mathcal{B}(\mathcal{H})$ . We denote by  $\|\cdot\|$  the norm of bounded operators on  $\mathcal{H}$ . Thanks to the operator  $A$ , we study several spectral properties of  $H$ . Given  $k \in \mathbb{N}$ , we say that  $H \in \mathcal{C}^k(A)$  if for all  $f \in \mathcal{H}$  the map  $\mathbb{R} \ni t \mapsto e^{itA} H e^{-itA} f$  has the usual  $\mathcal{C}^k(\mathbb{R})$  regularity, i.e.  $\mathbb{R} \ni t \mapsto e^{itA} H e^{-itA} \in \mathcal{B}(\mathcal{H})$  has the usual  $\mathcal{C}^k(\mathbb{R})$  regularity with  $\mathcal{B}(\mathcal{H})$  endowed with the strong operator topology. We say that  $H \in \mathcal{C}^{k,u}(A)$  if the map  $\mathbb{R} \ni t \mapsto e^{itA} H e^{-itA} \in \mathcal{B}(\mathcal{H})$  has the usual  $\mathcal{C}^k(\mathbb{R})$  regularity, with  $\mathcal{B}(\mathcal{H})$  endowed with the norm operator topology. The form  $[H, iA]$  is defined on  $\mathcal{D}(A) \times \mathcal{D}(A)$  by  $\langle f, [H, iA]g \rangle := i(\langle H^* f, Ag \rangle + \langle Af, Hg \rangle)$ . By [AmBoGe, Lemma 6.2.9]  $H \in \mathcal{C}^1(A)$  if and only if the form  $[H, iA]$  extends to a bounded operator in which case we denote by  $[H, iA]_o$ . We say that  $H \in \mathcal{C}^{0,1}(A)$  if

$$\int_0^1 \|[H, e^{itA}]\| \frac{dt}{t} < \infty$$

and that  $H \in \mathcal{C}^{1,1}(A)$  if

$$\int_0^1 \|[ [H, e^{itA}], e^{itA} ]\| \frac{dt}{t^2} < \infty.$$

Thanks to [AmBoGe, p. 205], we have the following of vector spaces inclusions:

$$(14) \quad \mathcal{C}^2(A) \subset \mathcal{C}^{1,1}(A) \subset \mathcal{C}^{1,u}(A) \subset \mathcal{C}^1(A) \subset \mathcal{C}^{0,1}(A).$$

The *Mourre estimate* for  $H$  on an open interval  $\mathcal{I}$  of  $\mathbb{R}$  holds if there exist  $c > 0$  and a compact operator  $K$  such that:

$$(15) \quad E_{\mathcal{I}}(H)[H, iA]_o E_{\mathcal{I}}(H) \geq E_{\mathcal{I}}(H)(c + K) E_{\mathcal{I}}(H),$$

where  $E_{\mathcal{I}}(H)$  the spectral measure of  $H$  above  $\mathcal{I}$ . The aim of Mourre's commutator theory is to prove a *Limiting Absorption Principle* (LAP), see [AmBoGe, Theorem 7.6.8].

**Theorem 2.1.** *Let  $H$  be a self-adjoint operator, with  $\sigma(H) \neq \mathbb{R}$ . Assume that  $H \in \mathcal{C}^1(A)$  and the Mourre estimate (15) true for  $H$  in  $\mathcal{I}$ . Then:*

1. *If  $K = 0$ , then  $H$  has no eigenvalues in  $\mathcal{I}$ .*
2. *The number of eigenvalues of  $H$  in  $\mathcal{I}$  counted with multiplicity is finite.*
3. *If  $H \in \mathcal{C}^{1,1}(A)$ ,  $\gamma > \frac{1}{2}$  and  $\mathcal{I}'$  a compact sub-interval of  $\mathcal{I}$  containing no eigenvalue, then*

$$\sup_{\Re(z) \in \mathcal{I}', \Im(z) \neq 0} \|\langle A \rangle^{-\gamma} (H - z)^{-1} \langle A \rangle^{-\gamma}\| \text{ is finite.}$$

4. *In the norm topology of bounded operators, the boundary values of the resolvent:*

$$\mathcal{I}' \ni \lambda \mapsto \lim_{\rho \rightarrow 0^\pm} \langle A \rangle^{-\gamma} (H - \lambda - i\rho)^{-1} \langle A \rangle^{-\gamma} \text{ exists and is continuous.}$$

For further details, we refer to [AmBoGe, Proposition 7.2.10, Corollary 7.2.11, Theorem 7.5.2].

## 3. PROOF OF MAIN RESULT

We now prove Theorem 1.1. Subsection 3.1 studies the Laplacian on a hexagonal lattice and proves its Mourre estimate. In Subsection 3.2 the metrics are perturbed and the potential is added. Finally in Subsection 3.3, we prove Theorem 1.1.

**3.1. Laplacian on the hexagonal lattice.** Let,

$$\mathcal{S} := \left\{ f : \mathbb{Z}^2 \rightarrow \mathbb{C}^2 \text{ such that, for all } N \in \mathbb{N} \sup_{(n_1, n_2) \in \mathbb{Z}^2} |(1 + n_1^2 + n_2^2)^N f(n_1, n_2)|_{\mathbb{C}^2} < \infty \right\},$$

it is the discrete Schwartz space. We denote by  $\mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$  the set of functions defined on  $[-\pi, \pi]^2$  of class  $\mathcal{C}^\infty$  and  $2\pi$ -periodic. Now, we define the Fourier transform  $\mathcal{F} : \ell^2(\mathbb{Z}^2 \times \{p_1, p_2\}; \mathbb{C}) \rightarrow L^2([-\pi, \pi]^2 \times \{p_1, p_2\}; \mathbb{C})$  through

$$\mathcal{F}f(x, p_i) := \frac{1}{2\pi} \sum_n f(n, p_i) e^{-i\langle n, x \rangle}, \quad \forall f \in \ell^2(\mathbb{Z}^2 \times \{p_1, p_2\}; \mathbb{C}).$$

The inverse Fourier transform  $\mathcal{F}^{-1} : L^2([-\pi, \pi]^2 \times \{p_1, p_2\}; \mathbb{C}) \rightarrow \ell^2(\mathbb{Z}^2 \times \{p_1, p_2\}; \mathbb{C})$  is then given by

$$\mathcal{F}^{-1}f(n, p_i) = \frac{1}{2\pi} \int_{[-\pi, \pi]^2} f(x, p_i) e^{i\langle n, x \rangle} dx, \quad \forall f \in L^2([-\pi, \pi]^2 \times \{p_1, p_2\}; \mathbb{C}),$$

where  $i \in \{1, 2\}$ . By (10), we rewrite the Fourier transform  $\mathcal{F} : \ell^2(\mathbb{Z}^2; \mathbb{C}^2) \rightarrow L^2([-\pi, \pi]^2; \mathbb{C}^2)$  through

$$\mathcal{F}f(x) = \frac{1}{2\pi} \sum_n f(n) e^{-i\langle n, x \rangle}, \quad \forall f \in \ell^2(\mathbb{Z}^2; \mathbb{C}^2).$$

The inverse Fourier transform  $\mathcal{F}^{-1} : L^2([-\pi, \pi]^2; \mathbb{C}^2) \rightarrow \ell^2(\mathbb{Z}^2; \mathbb{C}^2)$  is then given by

$$\mathcal{F}^{-1}f(n) = \frac{1}{2\pi} \int_{[-\pi, \pi]^2} f(x) e^{i\langle n, x \rangle} dx, \quad \forall f \in L^2([-\pi, \pi]^2; \mathbb{C}^2).$$

**Remark 3.1.** Note that  $\mathcal{F}(\mathcal{S}) = \mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$ .

First, we rewrite the Laplacian on a hexagonal lattice.

**Lemma 3.2.** With the (10), we have:

$$\Delta_H = \frac{1}{3} \begin{pmatrix} 0 & \Delta_{2,H} \\ \Delta_{1,H} & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & \text{id} + U_1 + U_2 \\ \text{id} + U_1^* + U_2^* & 0 \end{pmatrix}.$$

*Proof.* Recalling (3), (4) and (8). Let  $f \in \ell^2(\mathbb{Z}^2; \mathbb{C}^2)$ , we infer

$$\begin{aligned} (\Delta_H f)(n) &= \frac{1}{3} \begin{pmatrix} f_2(n - V_1) + f_2(n - V_2) + f_2(n - V_3) \\ f_1(n - V_1) + f_1(n - V_2) + f_1(n - V_3) \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} f_2(n_1, n_2) + f_2(n_1 - 1, n_2) + f_2(n_1, n_2 - 1) \\ f_1(n_1, n_2) + f_1(n_1 + 1, n_2) + f_1(n_1, n_2 + 1) \end{pmatrix}, \end{aligned}$$

where  $V_1 := \frac{v_1 + v_2}{3}$ ,  $V_2 := \frac{v_1 - 2v_2}{3}$  and  $V_3 := \frac{2v_1 - v_2}{3}$ . This gives the result.  $\square$

We turn to the Fourier transform.

**Lemma 3.3.** For all  $f \in L^2([-\pi, \pi]^2; \mathbb{C}^2)$ , we have:

$$((\mathcal{F}\Delta_H\mathcal{F}^{-1})f)(x_1, x_2) := (F(Q_1, Q_2)f)(x_1, x_2) = F(x_1, x_2)f(x_1, x_2),$$

with

$$F(x_1, x_2) := \frac{1}{3} \begin{pmatrix} 0 & 1 + e^{-ix_1} + e^{-ix_2} \\ 1 + e^{ix_1} + e^{ix_2} & 0 \end{pmatrix}.$$

*Proof.* Let  $f \in L^2([-\pi, \pi]^2; \mathbb{C}^2)$ , we have:

$$\begin{aligned} \mathcal{F}(U_1 \otimes 1 \mathcal{F}^{-1}f)(x) &= \frac{1}{2\pi} \sum_n (U_1 \mathcal{F}^{-1}f)(n) e^{-i\langle x, n \rangle} \\ &= \frac{1}{2\pi} \sum_n (\mathcal{F}^{-1}f)(n_1 - 1, n_2) e^{-i\langle x, n \rangle} \\ &= \frac{1}{2\pi} \sum_n (\mathcal{F}^{-1}f)(n_1, n_2) e^{-i\langle x_1, n_1 + 1 \rangle + \langle x_2, n_2 \rangle} \end{aligned}$$

$$= \frac{1}{2\pi} \sum_n (\mathcal{F}^{-1}f)(n_1, n_2) e^{-ix_1} e^{-i\langle x, n \rangle}.$$

Then,  $\mathcal{F}(U_1 \mathcal{F}^{-1}f)(x) = e^{-ix_1} f(x)$ . The other terms are treated in the same way. We obtain the result.  $\square$

Now, we turn to the diagonalization of the matrix  $F$ .

**Lemma 3.4.** *For all  $(x_1, x_2) \in \Xi := [-\pi, \pi]^2 \setminus \{(-\frac{2\pi}{3}, \frac{2\pi}{3}), (\frac{2\pi}{3}, -\frac{2\pi}{3})\}$ , we have:  $F = PDP^{-1}$ , with*

$$D(x_1, x_2) := \begin{pmatrix} \frac{\sqrt{\beta(x_1, x_2)}}{3} & 0 \\ 0 & -\frac{\sqrt{\beta(x_1, x_2)}}{3} \end{pmatrix},$$

$$P(x_1, x_2) := \begin{pmatrix} 1 & 1 \\ \frac{\sqrt{\beta(x_1, x_2)}}{1+e^{-ix_1}+e^{-ix_2}} & -\frac{1+e^{ix_1}+e^{ix_2}}{\sqrt{\beta(x_1, x_2)}} \end{pmatrix},$$

$$P^{-1}(x_1, x_2) := \begin{pmatrix} \frac{1}{2} & \frac{1+e^{-ix_1}+e^{-ix_2}}{2\sqrt{\beta(x_1, x_2)}} \\ \frac{1}{2} & -\frac{1+e^{-ix_1}+e^{-ix_2}}{2\sqrt{\beta(x_1, x_2)}} \end{pmatrix}$$

and  $\beta(x) := |1 + e^{ix_1} + e^{ix_2}|^2 = 3 + 2(\cos(x_1) + \cos(x_2) + \cos(x_1 - x_2))$ .  
If  $(x_1, x_2) \in \{(\frac{2\pi}{3}, \frac{2\pi}{3}), (-\frac{2\pi}{3}, -\frac{2\pi}{3})\}$ , then

$$D(x_1, x_2) = F(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

*Proof.* It suffices to verify that  $\det F(x_1, x_2) \neq 0$ , thus  $P$  is invertible and  $\det(F - \lambda \text{id}) = \lambda^2 - \frac{\beta(x)}{9}$ .  $\square$

**Lemma 3.5.** *On  $\Xi$  the critical points of  $\sqrt{\beta}$  is the set*

$$(16) \quad \Gamma := \{(-\pi, -\pi); (-\pi, \pi); (\pi, -\pi); (\pi, \pi); (-\pi, 0); (\pi, 0); (0, -\pi); (0, \pi); (0, 0)\}$$

and  $\sqrt{\beta}(\Gamma) = \{1, 3\}$ .

*Proof.* We have  $\sqrt{\beta} \in \mathcal{C}^1$  on  $\Xi$ . We obtain:

$$(17) \quad \nabla \left( \sqrt{\beta} \right) (x_1, x_2) = \frac{\nabla \beta(x_1, x_2)}{2\sqrt{\beta}(x_1, x_2)}.$$

We introduce the functions:

$$X := (x_1, x_2) \mapsto \cos\left(\frac{x_1 + x_2}{2}\right), \quad Y := (x_1, x_2) \mapsto \cos\left(\frac{x_1 - x_2}{2}\right).$$

Using trigonometric formulas, we obtain:

$$\beta = 1 + 2(2XY + 2Y^2).$$

Recall  $\beta$  is a smooth on  $\Xi$ . We have:

$$\nabla \beta = 4(Y \nabla X + (X + 2Y) \nabla Y),$$

where:

$$\nabla X(x_1, x_2) = -\frac{1}{2} \sin\left(\frac{x_1 + x_2}{2}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \nabla Y(x_1, x_2) = \frac{1}{2} \sin\left(\frac{x_1 - x_2}{2}\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

By (17),  $\nabla(\sqrt{\beta})(x_1, x_2) = 0$ , then  $\nabla \beta(x_1, x_2) = 0$ . In particular,  $\nabla X$  and  $\nabla Y$  are orthogonal. Thus, they are independent vectors unless one of them is zero. Therefore:

$$\begin{aligned} \nabla \beta(x_1, x_2) = 0 &\iff (\nabla X(x_1, x_2) = 0 = \nabla Y(x_1, x_2)) \\ &\text{or } (\nabla X(x_1, x_2) = 0, \nabla Y(x_1, x_2) \neq 0, \text{ and } (X + 2Y)(x_1, x_2) = 0) \\ &\text{or } (\nabla Y(x_1, x_2) = 0, \nabla X(x_1, x_2) \neq 0, \text{ and } Y(x_1, x_2) = 0) \\ &\text{or } (\nabla X, \nabla Y(x_1, x_2) \text{ are independent, } Y(x_1, x_2) = 0, (X + 2Y)(x_1, x_2) = 0). \end{aligned}$$

We observe that we cannot simultaneously obtain that:

$$Y(x_1, x_2) = 0 \text{ and } \nabla Y(x_1, x_2) = 0.$$

In the case of independent  $\nabla X(x_1, x_2)$  and  $\nabla Y(x_1, x_2)$ , we have  $X(x_1, x_2) = Y(x_1, x_2) = 0$ , hence  $\sqrt{\beta(x_1, x_2)} = 1$ . This actually occurs for  $(x_1, x_2) \in \{(-\pi, 0); (\pi, 0); (0, -\pi); (0, \pi)\}$ .

In the case  $\nabla X(x_1, x_2) = 0 = \nabla Y(x_1, x_2)$ , we must have  $|X(x_1, x_2)| = |Y(x_1, x_2)| = 1$ . Then  $\sqrt{\beta(x_1, x_2)} = 3$  if  $X(x_1, x_2)Y(x_1, x_2) > 0$  and  $\sqrt{\beta(x_1, x_2)} = 1$  if  $X(x_1, x_2)Y(x_1, x_2) < 0$ . The first case occurs at  $(x_1, x_2) \in \{(0, 0); (-\pi, -\pi); (-\pi, \pi); (\pi, -\pi); (\pi, \pi)\}$ .

In the last case, we have  $\nabla X(x_1, x_2) = 0, \nabla Y(x_1, x_2) \neq 0$ , and  $(X+2Y)(x_1, x_2) = 0$ . Since  $\nabla Y(x_1, x_2) \neq 0, x_1 \neq x_2$  thus  $-\pi < \frac{(x_1+x_2)}{2} < \pi$ , yielding  $X(x_1, x_2) > -1$ . Since  $\nabla X(x_1, x_2) = 0, |X(x_1, x_2)| = 1$ . Thus,  $X(x_1, x_2) = 1$  and  $Y(x_1, x_2) = -\frac{1}{2}$ . Then  $\sqrt{\beta(x_1, x_2)} = 0$ . This case occurs for  $(x_1, x_2) = (-\frac{2\pi}{3}, \frac{2\pi}{3})$  and  $(x_1, x_2) = (\frac{2\pi}{3}, -\frac{2\pi}{3})$ , which does not belong  $\Xi$ . We have shown that  $\sqrt{\beta(\Gamma)} = \{1, 3\}$ .  $\square$

Next, we turn to the spectrum.

**Lemma 3.6.**  $\sigma(F(Q_1, Q_2)) = [-1, 1]$ .

*Proof.* By [ReSi, Vol 1, p. 229] and Lemma 3.4, we obtain  $\sigma(F(Q_1, Q_2)) := \left\{ \pm \frac{\sqrt{\beta(x)}}{3}, \forall x \in [-\pi, \pi]^2 \right\}$ .

In the other hand, we have  $\beta(x_1, x_2) = 3 + 6\phi(x_1, x_2)$ , where

$$\phi(x_1, x_2) := \frac{1}{3} (\cos(x_1) + \cos(x_2) + \cos(x_1 - x_2)).$$

By [AtEnGoJa, Lemma 3.3], we obtain that  $\sigma(\phi(Q_1, Q_2)) = [-\frac{1}{2}, 1]$ . We conclude that  $\sigma(F(Q_1, Q_2)) = [-1, 1]$ .  $\square$

Let  $g \in C_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C})$ , we set:

$$\begin{aligned} \widehat{A}g &:= \frac{i}{2} \left( \nabla \beta^{\frac{5}{2}} \cdot \nabla + \nabla \cdot \nabla \beta^{\frac{5}{2}} \right) g \\ &= \frac{i}{2} \left( \widehat{A}_1 + \widehat{A}_2 \right) g. \end{aligned}$$

**Remark 3.7.** The power  $\frac{1}{2}$  should be the natural try but it does not seem to lead to a self-adjoint operator. Here we take power  $\frac{5}{2}$  for two reasons: First it allows to obtain a self-adjoint operator and second, it can be explicitly set back to the original space which, in turn, permits to deal with the perturbation of the metric. Note that in [Ta2] he chose to localize in energy away from 0 to circumvent the difficulty. However, he does not deal with the metric perturbation.

**Lemma 3.8.** For  $g \in C_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C})$ , we have:

$$\begin{aligned} \widehat{A}g(x_1, x_2) &:= 5i\beta^{\frac{3}{2}}(x_1, x_2) \left( (-\sin(x_1) - \sin(x_1 - x_2)) \frac{\partial g}{\partial x_1}(x_1, x_2) \right. \\ &\quad \left. + (-\sin(x_2) + \sin(x_1 - x_2)) \frac{\partial g}{\partial x_2}(x_1, x_2) \right) \\ &\quad + 15i\sqrt{\beta(x_1, x_2)} \left( \sin^2(x_1) + \sin^2(x_2) + 2\sin^2(x_1 - x_2) \right. \\ &\quad \left. + 2\sin(x_1 - x_2) (\sin(x_1) - \sin(x_2)) \right) f(x_1, x_2) \\ &\quad - 5i\beta^{\frac{3}{2}}(x_1, x_2) \left( \cos(x_1) + 2\cos(x_1 - x_2) + \cos(x_2) \right) g(x_1, x_2). \end{aligned}$$

*Proof.* Let  $g \in C_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C})$ , we have:

$$\begin{pmatrix} \partial_{x_1} \beta^{\frac{5}{2}} \\ \partial_{x_2} \beta^{\frac{5}{2}} \end{pmatrix} (x_1, x_2) = 5 \begin{pmatrix} \beta^{\frac{3}{2}}(x_1, x_2) (-\sin(x_1) - \sin(x_1 - x_2)) \\ \beta^{\frac{3}{2}}(x_1, x_2) (-\sin(x_2) + \sin(x_1 - x_2)) \end{pmatrix}$$

and

$$\begin{aligned} \left\langle \begin{pmatrix} \partial_{x_1} \beta \\ \partial_{x_2} \beta \end{pmatrix}, \begin{pmatrix} \partial_{x_1} g \\ \partial_{x_2} g \end{pmatrix} \right\rangle (x_1, x_2) &= 5\beta^{\frac{3}{2}}(x_1, x_2) \left( (-\sin(x_1) - \sin(x_1 - x_2)) \frac{\partial g}{\partial x_1}(x_1, x_2) \right. \\ &\quad \left. + (-\sin(x_2) + \sin(x_1 - x_2)) \frac{\partial g}{\partial x_2}(x_1, x_2) \right). \end{aligned}$$

This concludes the Lemma.  $\square$

**Lemma 3.9.** *We next denote  $A_{\mathcal{F}} := PA_D P^{-1}$  the conjugate operator associated to  $F(Q)$ , where*

$$A_D := \begin{pmatrix} \widehat{A} & 0 \\ 0 & -\widehat{A} \end{pmatrix}.$$

Let  $f = {}^t(f_1, f_2) \in C_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$ , we have

1. If  $(x_1, x_2) \in \Xi$ , we obtain

$$A_{\mathcal{F}} f(x_1, x_2) = \begin{pmatrix} 0 & \widehat{A}_1 \frac{1+e^{-ix_1+e^{-ix_2}}}{\sqrt{\beta(x_1, x_2)}} + \widehat{A}_2 \frac{1+e^{-ix_1+e^{-ix_2}}}{\sqrt{\beta(x_1, x_2)}} \\ \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \widehat{A}_1 + \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \widehat{A}_2 & 0 \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} (x_1, x_2).$$

2. If  $(x_1, x_2) \in \{(-\frac{2\pi}{3}, \frac{2\pi}{3}), (\frac{2\pi}{3}, -\frac{2\pi}{3})\}$ , we have  $A_{\mathcal{F}} f$  is extendable by continuity in  $(x_1, x_2)$  and

$$A_{\mathcal{F}} f(x_1, x_2) := 0.$$

*Proof.*

1. After a straightforward calculation.
2. We have

$$(18) \quad \begin{aligned} \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \widehat{A}_1 f(x_1, x_2) &= \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \nabla \beta^{\frac{5}{2}} \cdot \nabla f(x_1, x_2) \\ &= 5(1+e^{ix_1+e^{ix_2}})^2 (1+e^{-ix_1+e^{-ix_2}}) \\ &\quad \times \left( -(\sin(x_1) + \sin(x_1 - x_2)) \frac{\partial f}{\partial x_1}(x_1, x_2) \right. \\ &\quad \left. + (-\sin(x_2) + \sin(x_1 - x_2)) \frac{\partial f}{\partial x_2}(x_1, x_2) \right), \end{aligned}$$

$$(19) \quad \begin{aligned} \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \widehat{A}_2 f(x_1, x_2) &= \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \nabla \cdot \nabla \beta^{\frac{5}{2}} f(x_1, x_2) \\ &= 15(1+e^{ix_1+e^{ix_2}}) \left( \sin^2(x_1) + \sin^2(x_2) + 2\sin^2(x_1 - x_2) \right. \\ &\quad \left. + 2\sin(x_1 - x_2)(\sin(x_1) - \sin(x_2)) \right) f(x_1, x_2) \\ &\quad - 5(1+e^{ix_1+e^{ix_2}})^2 (1+e^{-ix_1+e^{-ix_2}}) \\ &\quad \times \left( \cos(x_1) + 2\cos(x_1 - x_2) + \cos(x_2) \right) f(x_1, x_2). \end{aligned}$$

By (18) and (19), we have

$$\begin{aligned} &\lim_{(x_1, x_2) \rightarrow \pm(\frac{2\pi}{3}, -\frac{2\pi}{3})} \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \widehat{A}_1 f(x_1, x_2) \\ &= \lim_{(x_1, x_2) \rightarrow \pm(\frac{2\pi}{3}, -\frac{2\pi}{3})} \frac{1+e^{ix_1+e^{ix_2}}}{\sqrt{\beta(x_1, x_2)}} \widehat{A}_2 f(x_1, x_2) = 0, \end{aligned}$$

then  $A_{\mathcal{F}} f$  is extendable by continuity. □

We now define the conjugate operator  $A_H$  on  $\mathcal{S}$ . We denote by  $A_H := \mathcal{F} A_{\mathcal{F}} \mathcal{F}^{-1}$ , where

$$A_H := \begin{pmatrix} 0 & A_{2,H} \\ A_{1,H} & 0 \end{pmatrix}$$

is the conjugate operator associated to  $\Delta_H$ .

**Lemma 3.10.** *On  $\mathcal{S}$ , we have*

$$\begin{aligned} A_{1,H} = & \frac{5i}{2} Q_1 (1 + U_1^* + U_2^*)^2 (1 + U_1 + U_2) (U_1^* - U_1 + U_1^* U_2 - U_1 U_2^*) \\ & + \frac{5i}{2} Q_2 (1 + U_1^* + U_2^*)^2 (1 + U_1 + U_2) (U_2^* - U_2 - (U_1^* U_2 - U_1 U_2^*)) \\ & - \frac{5i}{8} \left( 3(1 + U_1^* + U_2^*) \left( (U_1^* - U_1)^2 + (U_2^* - U_2)^2 + 2(U_1^* U_2 - U_1 U_2^*)^2 \right. \right. \\ & \left. \left. + 2(U_1^* U_2 - U_1 U_2^*) (U_1^* - U_1 - (U_2^* - U_2)) \right) \right. \\ & \left. + 2(1 + U_1^* + U_2^*)^2 (1 + U_1 + U_2) (U_1^* + U_1 + U_2^* + U_2 + 2(U_1^* U_2 + U_1 U_2^*)) \right). \end{aligned}$$

Similarly, we obtain  $A_{2,H}$  as the adjoint operator of  $A_{1,H}$ .

On  $\mathcal{S}$ , we can rewrite  $A_{1,H}$  as follows:

$$\begin{aligned} A_{1,H} = & R_{0,0}(U_1, U_2) + \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1 l_2}(U_1, U_2) \\ = & \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{0,0} U_1^i U_2^j + \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) \\ & \times \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{l_1, l_2} U_1^i U_2^j. \end{aligned}$$

where

$$\begin{aligned} R_{0,0}(U_1, U_2) := & -\frac{5i}{8} \left( 3(1 + U_1^* + U_2^*) \left( (U_1^* - U_1)^2 + (U_2^* - U_2)^2 + 2(U_1^* U_2 - U_1 U_2^*)^2 \right. \right. \\ & \left. \left. + 2(U_1^* U_2 - U_1 U_2^*) (U_1^* - U_1 - (U_2^* - U_2)) \right) + 2(1 + U_1^* + U_2^*)^2 \right. \\ & \left. \times (1 + U_1 + U_2) (U_1^* + U_1 + U_2^* + U_2 + 2(U_1^* U_2 + U_1 U_2^*)) \right) \\ = & -\frac{5i}{8} \left( 19(U_1^*)^2 + 5U_1^2 + 19(U_2^*)^2 + 5U_2^2 + 5U_1^* + 5U_2^* + 5(U_1^*)^3 \right. \\ & + 5(U_2^*)^3 + 16U_1 + 16U_2 + 15(U_1^*)^2 U_2^2 + 15U_1^2 (U_2^*)^2 + 4U_1 U_2 \\ & + 12U_1^2 U_2^* + 25U_1 (U_2^*)^2 + 9U_1^* (U_2^*)^2 + 18U_1^* U_2^* + 8(U_1^*)^3 U_2^2 \\ & + 7(U_1^*)^3 U_2 + 9(U_1^*)^2 U_2^* + 8(U_1)^2 (U_2^*)^3 + 7U_1 (U_2^*)^3 + 25(U_1^*)^2 U_2 \\ & \left. + 24U_1 U_2^* + 24U_1^* U_2 + 2 \right) \\ = & \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{0,0} U_1^i U_2^j, \end{aligned}$$

where

$$\begin{aligned} (20) \quad & \alpha_{0,0}^{0,0} = 2, \alpha_{-2,0}^{0,0} = \alpha_{0,-2}^{0,0} = 19, \alpha_{2,0}^{0,0} = \alpha_{0,2}^{0,0} = 5, \alpha_{-3,0}^{0,0} = \alpha_{0,-3}^{0,0} = 5, \\ & \alpha_{0,1}^{0,0} = \alpha_{1,0}^{0,0} = 16, \alpha_{-2,-1}^{0,0} = \alpha_{-1,-2}^{0,0} = 9, \alpha_{1,-2}^{0,0} = \alpha_{-2,1}^{0,0} = 25, \\ & \alpha_{1,-1}^{0,0} = \alpha_{-1,1}^{0,0} = 24, \alpha_{-1,2}^{0,0} = \alpha_{2,-1}^{0,0} = 12, \alpha_{-3,1}^{0,0} = \alpha_{1,-3}^{0,0} = 7, \\ & \alpha_{-2,2}^{0,0} = \alpha_{2,-2}^{0,0} = 15, \alpha_{2,-3}^{0,0} = \alpha_{-3,2}^{0,0} = 8, \alpha_{-1,-1}^{0,0} = 18, \alpha_{1,1}^{0,0} = 4 \\ & \alpha_{-1,0}^{0,0} = \alpha_{0,-1}^{0,0} = 5. \end{aligned}$$

$$\begin{aligned} R_{1,0}(U_1, U_2) := & \frac{5i}{2} \left( -4 + 4U_1^* - 5U_1 - U_1^2 - 2U_2 + (U_1^*)^3 + 5(U_1^*)^2 + (U_2^*)^2 \right. \\ & + (U_1^*)^3 U_2 + 2(U_1^*)^2 U_2^* + 2(U_1^*)^2 U_2 + 5U_1^* U_2^* + U_1^* (U_2^*)^2 \\ & \left. - 5U_1 U_2^* - U_1 U_2 - U_1 (U_2^*)^2 - 2U_1^2 U_2^* - U_1^2 (U_2^*)^2 \right), \end{aligned}$$

$$= \frac{5i}{2} \sum_{i,j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{1,0} U_1^i U_2^j,$$

with

$$(21) \quad \begin{aligned} \alpha_{0,0}^{1,0} &= -4, \quad \alpha_{-3,0}^{1,0} = \alpha_{0,-2}^{1,0} = \alpha_{-3,1}^{1,0} = \alpha_{-1,-2}^{1,0} = 1, \quad \alpha_{2,0}^{1,0} = \alpha_{1,1}^{1,0} = \alpha_{1,-2}^{1,0} = \alpha_{2,-2}^{1,0} = -1, \\ \alpha_{-2,-1}^{1,0} &= \alpha_{-2,1}^{1,0} = 2, \quad \alpha_{0,1}^{1,0} = \alpha_{2,-1}^{1,0} = -2, \quad \alpha_{-2,0}^{1,0} = \alpha_{-1,-1}^{1,0} = 5, \quad \alpha_{1,-1}^{1,0} = \alpha_{1,0}^{1,0} = -5, \\ \alpha_{-1,0}^{1,0} &= 4. \end{aligned}$$

$$\begin{aligned} R_{0,1}(U_1, U_2) &:= \frac{5i}{2} \left( -4 + 4U_2^* - 5U_2 - U_2^2 + U_2^3 + 5(U_2^*)^2 + (U_1^*)^2 + U_1(U_2^*)^3 \right. \\ &\quad + 2U_1^*(U_2^*)^2 + 2U_1(U_2^*)^2 + 5U_1^*U_2^* + (U_1^*)^2U_2^* - 5U_1^*U_2 \\ &\quad \left. - U_1U_2 - (U_1^*)^2U_2 - 2U_1^*U_2^2 - (U_1^*)^2U_2^2 \right) \\ &= \frac{5i}{2} \sum_{i,j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{0,1} U_1^i U_2^j, \end{aligned}$$

with

$$(22) \quad \begin{aligned} \alpha_{0,0}^{0,1} &= -4, \quad \alpha_{0,-3}^{0,1} = \alpha_{-2,0}^{0,1} = \alpha_{1,-3}^{0,1} = \alpha_{-2,-1}^{0,1} = 1, \quad \alpha_{0,2}^{0,1} = \alpha_{1,1}^{0,1} = \alpha_{-2,1}^{0,1} = \alpha_{-2,2}^{0,1} = -1, \\ \alpha_{-1,-2}^{0,1} &= \alpha_{1,-2}^{0,1} = 2, \quad \alpha_{-1,2}^{0,1} = \alpha_{1,0}^{0,1} = -2, \quad \alpha_{0,-2}^{0,1} = \alpha_{-1,-1}^{0,1} = 5, \quad \alpha_{-1,1}^{0,1} = \alpha_{0,1}^{0,1} = -5, \\ \alpha_{0,-1}^{0,1} &= 4. \end{aligned}$$

$$\begin{aligned} R_{1,1}(U_1, U_2) &:= \frac{5i}{2} \left( 4U_1^* + 6U_1^* + 6U_2^* + U_1 + U_2 + 2(U_1^*)^2 + 2(U_2^*)^2 + 2U_1^*U_2^* \right. \\ &\quad + 5U_1^*U_2 + 5U_1U_2^* + 5(U_1^*)^2U_2 + 5U_1(U_2^*)^2 + U_1^*U_2^2 \\ &\quad + U_1^2U_2^* + 2(U_1^*)^2U_2^2 + 2U_1^2(U_2^*)^2 + (U_1^*)^3U_2 \\ &\quad \left. + U_1(U_2^*)^3 + (U_1^*)^3U_2^2 + U_1^2(U_2^*)^3 \right) \\ &= \frac{5i}{2} \sum_{i,j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{1,1} U_1^i U_2^j, \end{aligned}$$

with

$$(23) \quad \begin{aligned} \alpha_{-1,0}^{1,1} &= 4, \quad \alpha_{0,1}^{1,1} = \alpha_{-1,2}^{1,1} = \alpha_{-3,1}^{1,1} = \alpha_{-3,2}^{1,1} = 1, \quad \alpha_{2,-2}^{1,1} = \alpha_{0,-2}^{1,1} = -2, \\ \alpha_{1,0}^{1,1} &= \alpha_{2,-3}^{1,1} = \alpha_{1,-3}^{1,1} = \alpha_{2,-1}^{1,1} = -1, \quad \alpha_{-2,0}^{1,1} = \alpha_{-2,2}^{1,1} = 2, \\ \alpha_{-1,1}^{1,1} &= \alpha_{-2,1}^{1,1} = 5, \quad \alpha_{1,-1}^{1,1} = \alpha_{1,-2}^{1,1} = -5, \quad \alpha_{0,-1}^{1,1} = -4. \end{aligned}$$

Similarly, we rewrite  $A_{2,H}$ .

**Lemma 3.11.** *There exists  $C > 0$  such that for all  $f \in \mathcal{S}$ , we have:*

$$\|A_H f\|^2 \leq C \|\Lambda_H(Q)f\|^2.$$

*Proof.* Let  $f \in \mathcal{S}$ . Here, all constants are denoted by  $C$  and are independent of  $f$ . We have:

$$\|A_H f\|^2 = \|A_{1,H} f_1\|^2 + \|A_{2,H} f_2\|^2.$$

So, we obtain:

$$\begin{aligned} \|A_{1,H} f_1\|^2 &= \sum_{(n_1, n_2) \in \mathbb{Z}^2} \left| \left( R_{0,0}(U_1, U_2) + \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) \right) f_1(n_1, n_2) \right|^2 \\ &\leq \sum_{(n_1, n_2) \in \mathbb{Z}^2} 2 |R_{0,0}(U_1, U_2) f_1(n_1, n_2)|^2 \\ &\quad + 2 \left| \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) f_1(n_1, n_2) \right|^2 \end{aligned}$$

$$\leq C \left( \|(Q_1^2 + \frac{1}{2})^{\frac{1}{2}} f_1\|^2 + \|(Q_2^2 + \frac{1}{2})^{\frac{1}{2}} f_1\|^2 + \|f_1\|^2 \right) \leq C \|\Lambda(Q) f_1\|^2.$$

Similarly, we show that:

$$\|A_{2,H} f_2\|^2 \leq C \|\Lambda(Q) f_2\|^2.$$

This concludes the result.  $\square$

**Remark 3.12.** By applying Lemma 3.11 and since  $\|\langle A_H \rangle^0 f\|^2 \leq \|\Lambda^0(Q) f\|^2$ , for all  $f \in \mathcal{S}$ , by real interpolation, e.g. [BeJoLoJo, Theorem 4.1.2, p.88], for all  $\gamma \in [0, 1]$ , we remark that there is  $C_\gamma$  such that

$$\|\langle A_H \rangle^\gamma f\|^2 \leq C_\gamma \|\Lambda_H^\gamma f\|^2, \text{ for all } f \in \mathcal{S}.$$

**Lemma 3.13.**  $A_H$  is essentially self-adjoint on  $\mathcal{S}$ . We write  $A_H$  for its closure in the sequel.

*Proof.* First, we recall that  $A_H$  is symmetric operator on  $\mathcal{S}$ . By Lemma 3.11, there exists  $C$  such that for all  $f \in \mathcal{S}$ , we have:

$$\|A_H f\|^2 \leq C \|\Lambda(Q) f\|^2.$$

By the Nelson's Lemma, e.g. [ReSi, Theorem X.37], it suffices to prove

$$\exists C > 0, \forall f \in \mathcal{S}, \left| \langle f, [\Lambda(Q), A_H] f \rangle \right| \leq C \|\Lambda^{\frac{1}{2}}(Q) f\|.$$

to ensure to that  $A_H$ , defined on  $\mathcal{S}$ , extends to a self-adjoint operator. Let  $f \in \mathcal{S}$ . We denote all constants by  $C$ , we have:

$$\begin{aligned} |\langle f, [\Lambda_H(Q), A_H] f \rangle| &\leq |\langle f_2, [\Lambda(Q), A_{1,H}] f_1 \rangle| + |\langle f_1, [\Lambda(Q), A_{2,H}] f_2 \rangle| \\ &= 2 |\langle f_2, [\Lambda(Q), A_{1,H}] f_1 \rangle|. \end{aligned}$$

Then,

$$\begin{aligned} &|\langle f_2, [\Lambda(Q), A_{1,H}] f_1 \rangle| \\ &= \left| \sum_{(n_1, n_2) \in \mathbb{Z}^2} \overline{f_2(n_1, n_2)} \left( \Lambda(n_1, n_2) \left( R_{0,0}(U_1, U_2) + \sum_{(l_1, l_2) \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (n_1 l_1 - n_2 l_2) R_{l_1, l_2}(U_1, U_2) \right) \right. \right. \\ &\quad \left. \left. - \left( R_{0,0}(U_1, U_2) + \sum_{(l_1, l_2) \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (n_1 l_1 - n_2 l_2) R_{l_1, l_2}(U_1, U_2) \right) \Lambda(n_1, n_2) \right) f_1(n_1, n_2) \right| \\ &= \left| \sum_{(n_1, n_2) \in \mathbb{Z}^2} \overline{f_2(n_1, n_2)} \left( \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{0,0} (\Lambda(n_1, n_2) - \Lambda(n_1 + i, n_2 + j)) \right. \right. \\ &\quad \left. \left. + \sum_{(l_1, l_2) \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (n_1 l_1 - n_2 l_2) \right. \right. \\ &\quad \left. \left. \times \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{l_1, l_2} (\Lambda(n_1, n_2) - \Lambda(n_1 + i, n_2 + j)) \right) f_1(n_1 + i, n_2 + j) \right| \\ &= \left| \sum_{(n_1, n_2) \in \mathbb{Z}^2} \overline{f_2(n_1, n_2)} \left( \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{0,0} \left( \frac{\Lambda^2(n_1, n_2) - \Lambda^2(n_1 + i, n_2 + j)}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \right) \right. \right. \\ &\quad \left. \left. + \sum_{(l_1, l_2) \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (n_1 l_1 - n_2 l_2) \right. \right. \\ &\quad \left. \left. \times \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{l_1, l_2} \left( \frac{\Lambda^2(n_1, n_2) - \Lambda^2(n_1 + i, n_2 + j)}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \right) \right) f_1(n_1 + i, n_2 + j) \right| \\ &= \left| \sum_{(n_1, n_2) \in \mathbb{Z}^2} \overline{f_2(n_1, n_2)} \left( \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{0,0} \left( \frac{-2in_1 - 2jn_2 + i^2 + j^2}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \right) \right. \right. \\ &\quad \left. \left. + \sum_{(l_1, l_2) \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (n_1 l_1 - n_2 l_2) \right. \right. \\ &\quad \left. \left. \times \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{l_1, l_2} \left( \frac{-2in_1 - 2jn_2 + i^2 + j^2}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \right) \right) f_1(n_1 + i, n_2 + j) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{(n_1, n_2) \in \mathbb{Z}^2} |f_2(n_1, n_2)| \left( \left| \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{0,0} \left( \frac{-2in_1 - 2jn_2 + i^2 + j^2}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \right) f_1(n_1 + i, n_2 + j) \right| \right. \\
&\quad \left. + \left| \sum_{(l_1, l_2) \in \{0, 1\}^2, (l_1, l_2) \neq (0, 0)} (n_1 l_1 - n_2 l_2) \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i, j}^{l_1, l_2} \left( \frac{-2in_1 - 2jn_2 + i^2 + j^2}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \right) \right. \right. \\
&\quad \left. \left. \times f_1(n_1 + i, n_2 + j) \right| \right) \\
&\leq C \left( \|f_2(\cdot)\| \|f_1(\cdot)\| + \|\Lambda^{\frac{1}{2}}(Q) f_2(\cdot)\| \|\Lambda^{\frac{1}{2}}(Q) f_1(\cdot)\| \right), \text{ due to } \frac{-2in_1 - 2jn_2 + i^2 + j^2}{\Lambda(n_1, n_2) + \Lambda(n_1 + i, n_2 + j)} \text{ is bounded.} \\
&\leq C \left( \|f_2(\cdot)\|^2 + \|f_1(\cdot)\|^2 + \|\Lambda^{\frac{1}{2}}(Q) f_2(\cdot)\|^2 + \|\Lambda^{\frac{1}{2}}(Q) f_1(\cdot)\|^2 \right) \\
&\leq C \|\Lambda_H^{\frac{1}{2}}(Q) f\|^2.
\end{aligned}$$

As  $\mathcal{S}$  is a core for  $\Lambda_H(Q)$ , applying [ReSi, Theorem X.37], we prove that:

$$|\langle f, [\Lambda_H(Q), A_H] f \rangle| \leq C \left\| \Lambda_H^{\frac{1}{2}}(Q) f \right\|^2.$$

□

**Lemma 3.14.** *On  $\mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$ , we have:*

$$(24) \quad A_{\mathcal{F}} := \mathcal{F} A_H \mathcal{F}^{-1}.$$

*Proof.* We have:

$$\begin{aligned}
(\mathcal{F} U_1^* \mathcal{F}^{-1} f)(x) &= e^{ix_1} f(x), \quad (\mathcal{F} U_2^* \mathcal{F}^{-1} f)(x) = e^{ix_2} f(x), \\
\frac{1}{2i} (\mathcal{F} U_1^* - U_1 \mathcal{F}^{-1} f)(x) &= \frac{1}{2i} (e^{ix_1} - e^{-ix_1}) f(x) = \sin(x_1) f(x), \\
\frac{1}{2i} (\mathcal{F} (U_1 U_2^* - U_1^* U_2) \mathcal{F}^{-1} f)(x) &= \sin(x_1 - x_2) f(x)
\end{aligned}$$

and

$$(\mathcal{F}(-iQ_i) \mathcal{F}^{-1} f)(x) = \frac{\partial f}{\partial x_i}(x),$$

where  $i \in \{1, 2\}$ . Similarly, we treat the other terms. So, we obtain the result.

**Remark 3.15.** *Since  $\mathcal{F}(\mathcal{S}) = \mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$ , we now recall Lemma 3.14 by density we have,  $A_{\mathcal{F}}$  is essentially self-adjoint on  $\mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$  and we denote by  $A_{\mathcal{F}}$  its closure. Note that (24) extends to the closure and  $\mathcal{D}(A_H) = \mathcal{F}^{-1} \mathcal{D}(A_{\mathcal{F}})$ .*

**Lemma 3.16.** *We have,  $\sqrt{\beta(Q)} \in \mathcal{C}^1(\widehat{A})$  and therefore  $\Delta_H \in \mathcal{C}^1(A_H)$ . Moreover,*

$$(25) \quad \left[ \sqrt{\beta(Q)}, i\widehat{A} \right]_{\circ} = \frac{5}{4} \left\| \sqrt{\beta} \nabla \beta \right\|^2(Q).$$

*Proof.* For  $f \in \mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C})$ , we have:

$$\begin{aligned}
\left[ \sqrt{\beta(Q)}, i\widehat{A} \right] f(x_1, x_2) &= - \left[ \sqrt{\beta(Q)}, \nabla \cdot \nabla \beta^{\frac{5}{2}}(Q) \right] f(x_1, x_2) \\
&= - \left[ \sqrt{\beta(Q)}, \sum_{j=1}^2 \partial_{x_j} \left( \partial_{x_j} \beta^{\frac{5}{2}} \right) (Q) \right] f(x_1, x_2) \\
&= - \frac{5}{2} \left[ \sqrt{\beta(Q)}, \sum_{j=1}^2 \partial_{x_j} \left( \beta^{\frac{3}{2}} \partial_{x_j} \beta \right) (Q) \right] f(x_1, x_2) \\
&= - \frac{5}{2} \sum_{j=1}^2 \sqrt{\beta(x_1, x_2)} \partial_{x_j} \left( \beta^{\frac{3}{2}} \partial_{x_j} \beta f \right) (x_1, x_2) \\
&\quad + \partial_{x_j} \left( \beta^2 \partial_{x_j} \beta f \right) (x_1, x_2) \\
&= \frac{5}{4} \beta(x_1, x_2) (\partial_{x_j} \beta)^2(x_1, x_2) f(x_1, x_2).
\end{aligned}$$

As  $\beta\nabla\beta \in L^\infty([-\pi, \pi]^2; \mathbb{C})$  then there exists  $c > 0$  such that  $\left\| \left[ \sqrt{\beta(Q)}, i\hat{A} \right] f \right\| \leq c\|f\|$ . By density and thanks to [AmBoGe, Lemma 6.2.9], we obtain  $\sqrt{\beta(Q)} \in \mathcal{C}^1(\hat{A})$ . Recalling  $\mathcal{F}(\mathcal{S}) = \mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2; \mathbb{C}^2)$ , as the Fourier transform is unitary, we obtain  $\|[\Delta_H, iA_H]g\| \leq c\|g\|$  for all  $g \in \mathcal{S}$  which ensures that  $\Delta_H \in \mathcal{C}^1(A_H)$ .  $\square$

We now focus on the Mourre estimate for the unperturbed Laplacian.

**Proposition 3.17.** *We have  $\sqrt{\beta(Q)} \in \mathcal{C}^1(\hat{A})$ . Moreover, let  $\mathcal{I}$  be an open interval such that its closure is included in  $[-1, 1] \setminus \kappa$ , where  $\kappa := \pm \frac{\sqrt{\beta(\Gamma)}}{3} \cup \{0\}$ , with  $\Gamma$  is given in (16), there exists  $c > 0$  such that:*

$$(26) \quad E_{\mathcal{I}}(\sqrt{\beta(Q)})[\sqrt{\beta(Q)}, i\hat{A}] \circ E_{\mathcal{I}}(\sqrt{\beta(Q)}) \geq cE_{\mathcal{I}}(\sqrt{\beta(Q)}).$$

Equivalently, we have:

$$(27) \quad E_{\mathcal{I}}(\Delta_H)[\Delta_H, iA_H] \circ E_{\mathcal{I}}(\Delta_H) \geq cE_{\mathcal{I}}(\Delta_H).$$

The set  $\kappa$  is called the set of thresholds.

*Proof.* We work in  $L^2([-\pi, \pi]^2; \mathbb{C})$ . The  $\mathcal{C}^1$  property is given in Lemma 3.16. Let  $\mathcal{I}$  be an open interval such that its closure is included in  $[-1, 1] \setminus \kappa$ . Since  $\mathcal{I}$  is bounded then by Bolzano-Weierstrass Theorem, its closure is compact. There exists  $c > 0$ , such that for all  $(x_1, x_2) \in \sqrt{\beta}^{-1}(\mathcal{I})$ , we have:  $\frac{5}{4} \left\| \sqrt{\beta(Q)}\nabla\beta \right\|^2(x_1, x_2) \geq c$ . Recalling  $\left[ \sqrt{\beta(Q)}, i\hat{A} \right] \circ = \frac{5}{4} \left\| \sqrt{\beta(Q)}\nabla\beta \right\|^2(Q)$ , by functional calculus, we have:

$$E_{\mathcal{I}}(\sqrt{\beta(Q)})[\sqrt{\beta(Q)}, i\hat{A}] \circ E_{\mathcal{I}}(\sqrt{\beta(Q)}) \geq cE_{\mathcal{I}}(\sqrt{\beta(Q)}).$$

Using (7), (10), from (26) and thanks to Lemma 3.9 and Lemma 3.10, we obtain (27) by going back to  $\ell^2(\tilde{\mathcal{V}}; \mathbb{C})$ .  $\square$

**Proposition 3.18.** *We have  $\sqrt{\beta(Q)} \in \mathcal{C}^2(\hat{A})$  and therefore  $\Delta_H \in \mathcal{C}^2(A_H)$ .*

*Proof.* We denote the constant by  $c$ . By Lemma 3.16 and for  $f \in \mathcal{C}_{2\pi}^\infty([-\pi, \pi]^2)$ , we obtain:

$$\left[ \left[ \sqrt{\beta(Q)}, i\hat{A} \right] \circ, i\hat{A} \right] f = \left[ \frac{5}{4} \left\| \sqrt{\beta}\nabla\beta \right\|^2(Q), i\hat{A} \right] f(x_1, x_2).$$

Hence,

$$\begin{aligned} & \left[ \frac{5}{4} \left\| \sqrt{\beta}\nabla\beta \right\|^2(Q), i5i\beta^{\frac{3}{2}}(Q) (-\sin(Q_1) - \sin(Q_1 - Q_2)) \frac{\partial}{\partial x_1} \right] f(x_1, x_2) \\ &= \frac{25}{4} \beta^{\frac{3}{2}}(Q) (-\sin(x_1) - \sin(x_1 - x_2)) \frac{\partial \left\| \sqrt{\beta}\nabla\beta \right\|^2}{\partial x_1}(x_1, x_2) f(x_1, x_2). \end{aligned}$$

As  $\frac{5\partial \left\| \sqrt{\beta}\nabla\beta \right\|^2}{4\partial x_1} \in L^\infty([-\pi, \pi]^2)$ , we deduce:

$$\left\| \left[ \frac{5}{4} \left\| \sqrt{\beta}\nabla\beta \right\|^2(Q), 5\beta^{\frac{3}{2}}(Q) (-\sin(Q_1) - \sin(Q_1 - Q_2)) \frac{\partial}{\partial x_1} \right] f \right\|^2 \leq c\|f\|^2.$$

Similarly, we treat the other terms. Hence, by density and [AmBoGe, Proposition 5.2.2]. As in Lemma 3.16, we also obtain that  $\Delta_H \in \mathcal{C}^2(A_H)$ , this proves the lemma.  $\square$

**3.2. The perturbed model.** In this subsection, we perturb the earlier case by modifying the metric and introducing a potential. We require both the modified metric and potential to be in some sense, small at infinity. We address some technical details and begin with the properties of  $\Lambda_H$ .

**Proposition 3.19.**  *$\Lambda_H$  satisfies the following assertions:*

1.  $\mathcal{D}(\Lambda_H(Q)) \subset \mathcal{D}(A_H)$ .
2. There is  $c > 0$  such that for all  $r > 0$ ,  $-ir$  belongs to the resolvent set of  $\Lambda_H(Q)$  and  $r\|(\Lambda_H(Q) + ir)^{-1}\|_{\mathcal{B}(\ell^2(\tilde{\mathcal{V}}, 1))} \leq c$ .
3.  $t \rightarrow e^{it\Lambda_H(Q)}$  has a polynomial growth in  $\ell^2(\tilde{\mathcal{V}}, 1)$ .
4. Given  $\xi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$  such that  $\xi(x) = 0$  near 0 and 1 near infinity and  $T \in \mathcal{B}(\ell^2(\tilde{\mathcal{V}}, 1))$  symmetric, if

$$(28) \quad \int_1^\infty \left\| \xi \left( \frac{\Lambda_H(Q)}{r} \right) T \right\|_{\mathcal{B}(\ell^2(\tilde{\mathcal{V}}, 1))} dr < \infty,$$

then  $T \in \mathcal{C}^{0,1}(A_H)$ .

*Proof.* See [AtEnGoJa, Proposition 3.14].  $\square$

**Corollary 3.20.** *With the notation of Proposition 3.19, let  $\gamma \in (0, 1)$  and  $T \in \mathcal{B}(\ell^2(\tilde{\mathcal{V}}, 1))$  symmetric. Assume that*

$$\langle \Lambda_H \rangle^\gamma T \in \mathcal{B}(\ell^2(\tilde{\mathcal{V}}, 1)),$$

*then  $T \in \mathcal{C}^{0,1}(A_H)$ . Moreover, if  $T \in \mathcal{C}^1(A_H)$ ,  $T \in \mathcal{C}^{1,1}(A_H)$ .*

**3.2.1. Unitary transformation.** The perturbation of the metric introduces a second Hilbert space. To address this difficulty, we apply the following transformation:

**Proposition 3.21.** *Set the following map*

$$T_{1 \rightarrow m} : \ell^2(\tilde{\mathcal{V}}, 1; \mathbb{C}) \rightarrow \ell^2(\tilde{\mathcal{V}}, m; \mathbb{C})$$

$$f \mapsto T_{1 \rightarrow m} f(n) := \frac{1}{\sqrt{m(n)}} f(n).$$

*Then, the transformation  $T_{1 \rightarrow m}$  is unitary.*

*Proof.* See [AtEnGoJa, Proposition 3.17].  $\square$

Recalling (11) and the hypothesis  $(H_0)$  and  $(H_1)$ . We can now send  $\Delta_{m,\varepsilon}$  in  $\ell^2(\tilde{\mathcal{V}}, 1)$  with the help of this unitary transformation. Namely, let  $\tilde{\Delta} := T_{1 \rightarrow m}^{-1} \Delta_{m,\varepsilon} T_{1 \rightarrow m}$ . We have the following result:

**Proposition 3.22.** *We have:*

$$\tilde{\Delta} = \begin{pmatrix} 0 & \tilde{\Delta}_2 \\ \tilde{\Delta}_1 & 0 \end{pmatrix},$$

where

$$(29) \quad \tilde{\Delta}_1 = \frac{1}{3\sqrt{m_2(Q_1, Q_2)}} \left( \frac{\mathcal{E}((Q_1, Q_2, p_1), (Q_1, Q_2, p_2))}{\sqrt{m_1(Q_1, Q_2)}} + \frac{\mathcal{E}((Q_1, Q_2, p_1), (Q_1 + 1, Q_2, p_2))}{\sqrt{m_1(Q_1 + 1, Q_2)}} U_1^* \right. \\ \left. + \frac{\mathcal{E}((Q_1, Q_2, p_1), (Q_1, Q_2 + 1, p_2))}{\sqrt{m_1(Q_1, Q_2 + 1)}} U_2^* \right).$$

and

$$(30) \quad \tilde{\Delta}_2 = \frac{1}{3\sqrt{m_1(Q_1, Q_2)}} \left( \frac{\mathcal{E}((Q_1, Q_2, p_2), (Q_1, Q_2, p_1))}{\sqrt{m_2(Q_1, Q_2)}} + \frac{\mathcal{E}((Q_1, Q_2, p_2), (Q_1 - 1, Q_2, p_1))}{\sqrt{m_2(Q_1 - 1, Q_2)}} U_1 \right. \\ \left. + \frac{\mathcal{E}((Q_1, Q_2, p_2), (Q_1, Q_2 - 1, p_1))}{\sqrt{m_2(Q_1, Q_2 - 1)}} U_2 \right)$$

We establish the following expression for the perturbation:

**Proposition 3.23.**

1. *For all  $i, j \in \{1, 2\}$ ,  $i \neq j$ , we have:*

$$(31) \quad D_i := \Delta_{i,H} - \tilde{\Delta}_i : \ell^2(\mathcal{V}_i, 1) \rightarrow \ell^2(\mathcal{V}_j, 1)$$

$$f_1 \mapsto \frac{1}{3} (L_{i,H} + T_{i,H} + S_{i,H}) f_1(n),$$

where

$$(L_{i,H} f_i)(n) := \left( 1 - \frac{\mathcal{E}((Q_1, Q_2, p_i), (Q_1, Q_2, p_j))}{\sqrt{m_i(Q_1, Q_2)} \sqrt{m_j(Q_1, Q_2)}} \right) f_i(n),$$

$$(T_{i,H} f_1)(n) := \left( 1 - \frac{\mathcal{E}((Q_1, Q_2, p_i), (Q_1 - i + j, Q_2, p_j))}{\sqrt{m_i(Q_1 - i + j, Q_2)} \sqrt{m_j(Q_1, Q_2)}} \right) U_1^{i-j} f_i(n)$$

and

$$(S_{i,H} f_i)(n) := \left( 1 - \frac{\mathcal{E}((Q_1, Q_2, p_i), (Q_1, Q_2 - i + j, p_j))}{\sqrt{m_i(Q_1, Q_2 - i + j)} \sqrt{m_j(Q_1, Q_2)}} \right) U_2^{i-j} f_i(n).$$

2. *We have  $D_i$  is a compact operator in  $\ell^2(\mathcal{V}_i, 1)$ .*

*Proof.*

1. This is by Proposition 3.22 and straightforward calculation.

2. We have

$$\begin{aligned}
& (\Delta_{i,H} - \widetilde{\Delta}_i) f_i(n_1, n_2) \\
& := \sum_{\sigma \in \{0, j-i\}} \sum_{(h,k) \in \{(0, j-1)\}^2, h+k=\sigma} \left( \left( 1 - \frac{\mathcal{E}((Q_1, Q_2, p_i), (Q_1+h, Q_2+k, p_j))}{\sqrt{m_i(Q_1+h, Q_2+k)}\sqrt{m_j(Q_1, Q_2)}} \right) U_1^{-h} U_2^{-k} f \right) (n_1, n_2) \\
& = \sum_{\sigma \in \{0, j-i\}} \sum_{(h,k) \in \{(0, j-1)\}^2} \left( \left( m_j(Q_1, Q_2) (\eta_i(Q_1+h, Q_2+k) - \varepsilon((Q_1, Q_2, p_i), (Q_1+h, Q_2+k, p_j))) \right. \right. \\
& + \mathcal{E}((Q_1, Q_2, p_i), (Q_1+h, Q_2+k, p_j)) (\eta_j(Q_1, Q_2) - \varepsilon((Q_1, Q_2, p_i), (Q_1+h, Q_2+k, p_j))) \\
& \times \frac{1}{\sqrt{m_i(Q_1+h, Q_2+k)}\sqrt{m_j(Q_1, Q_2)}} \\
& \left. \left. \times \frac{1}{\left( \sqrt{m_1(Q_1+h, Q_2+k)}\sqrt{m_2(Q_1, Q_2)} + \mathcal{E}((Q_1, Q_2, p_i), (Q_1+h, Q_2+k, p_j)) \right)} \right) U_1^{-h} U_2^{-k} f_i \right) (n_1, n_2).
\end{aligned}$$

By using the hypotheses  $(H_0)$  and  $(H_1)$ , we have  $\eta_1((n_1+h, n_2+k)) - \varepsilon((n_1, n_2, p_i), (n_1+h, n_2+k, p_j)) \rightarrow 0$  if  $(n_1, n_2) \rightarrow \infty$  and  $\eta_2(n_1, n_2) - \varepsilon((n_1, n_2, p_i), (n_1+h, n_2+k, p_j)) \rightarrow 0$  if  $(n_1, n_2) \rightarrow \infty$  and  $U_1^{-h} U_2^{-k}$  is bounded. So have that the operator  $\Delta_{i,H} - \widetilde{\Delta}_i$  is a finite sum of compact operators. This concludes the result.  $\square$

We have a first spectral and general result.

**Proposition 3.24.** *Let  $\varepsilon, \eta$  and  $V$  be three real-valued bounded functions satisfying respectively  $(H_0)$ ,  $(H_1)$  and  $(H_2)$ . We have:*

1.  $H_{m,\mathcal{E}}$  is self-adjoint and bounded.
2.  $\sigma_{\text{ess}}(H_{m,\mathcal{E}}) = \sigma_{\text{ess}}(\Delta_H)$ .

*Proof.*

1. By hypotheses  $(H_2)$ , we deduce that  $V(Q)$  is compact. By using the fact that  $\Delta_H$  is self-adjoint and thanks to [ReSi, Theorem XIII.14], we conclude that  $H_{m,\mathcal{E}}$  is self-adjoint.
2. Thanks to Proposition 3.23 and using the fact that  $V(Q)$  is compact, we infer that  $\sigma_{\text{ess}}(\widetilde{\Delta} + V(Q)) = \sigma_{\text{ess}}(\Delta_H)$ . The unitary transformation assures that  $\sigma_{\text{ess}}(H_{m,\mathcal{E}}) = \sigma_{\text{ess}}(\Delta_H)$ .  $\square$

3.2.2. *Perturbed potential.* We start focus on the potential  $V$ . The perturbation of the metric is more complicated. It will be treated in the following subsection. We start with same auxiliary and general results.

**Remark 3.25.** 1. For all  $k \in \mathbb{N}^*$ ,  $J_{k,0,0} \equiv J_{k,1,0} \equiv J_{k,2,0}$ .

2. If  $h = 0$  and  $\forall k \in \mathbb{N}$ ,  $\forall b \in \mathbb{N}$ , we have  $J_{k,0,b} \equiv J_{k,0,0}$ .

**Lemma 3.26.** *Let  $G : (\mathbb{Z}^2 \times \{p_1, p_2\})^k \rightarrow \mathbb{C}$  satisfying the hypothesis  $(H_{3,k})$ , then we have*

$$\begin{aligned}
(H_{4,k}) \quad & \max_{h \in \{1,2\}} \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |G(J_{k,h,0}(n_1, n_2, p_i)) \\
& - G(J_{k,h,0}(n_1, n_2, p_j))| < \infty,
\end{aligned}$$

where  $k \in \mathbb{N}^*$ ,  $\{i, j\} = \{1, 2\}$ .

*Proof.* Let  $h \in \{1, 2\}$ ,  $h' \in \{1, 2\} \setminus \{h\}$ . For all  $l_1, l_2 \in \{0, 1\}$ , we have:

$$\begin{aligned}
& \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1, n_2, p_i))| \\
& \leq \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle \left( |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1, n_2, p_j))| \right. \\
& \left. + |G(J_{k,h,1}(n_1, n_2, p_j)) - G(J_{k,h',1}(n_1, n_2, p_i))| + |G(J_{k,h',1}(n_1, n_2, p_i)) - G(J_{k,h',0}(n_1, n_2, p_j))| \right).
\end{aligned}$$

Thanks to the hypothesis  $(H_{3,k})$ , we deduce the result.  $\square$

**Lemma 3.27.** *Let  $G : (\mathbb{Z}^2 \times \{p_1, p_2\})^k \rightarrow \mathbb{C}$  satisfying the hypothesis  $(H_{3,k})$  and Lemma 3.26, then we have*

$$\begin{aligned}
(H_{5,k}) \quad & \max_{i \in \{1,2\}} \max_{j \in \{1,2\}, j \neq i} \max_{h \in \{1,2\}} \max_{h' \in \{1,2\}, h' \neq h} \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle \\
& \left( |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1 - (-1)^j \delta_{h',1}, n_2 - (-1)^j \delta_{h',2}, p_j))| \right) < \infty,
\end{aligned}$$

where  $k \in \mathbb{N}^*$  and  $\{i, j\} \in \{1, 2\}$ .

*Proof.* Let  $h \in \{1, 2\}$  and  $h' \in \{1, 2\} \setminus \{h\}$ . For all  $l_1, l_2 \in \{0, 1\}$ , we have:

$$\begin{aligned} & \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle \left( |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1 - (-1)^j \delta_{h',1}, n_2 - (-1)^j \delta_{h',2}, p_j))| \right) \\ \leq & \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle \left( |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1, n_2, p_j))| \right. \\ & + |G(J_{k,h,1}(n_1, n_2, p_j)) - G(J_{k,h,0}(n_1, n_2, p_i))| \\ & + |G(J_{k,h',0}(n_1, n_2, p_i)) - G(J_{k,h',1}(n_1, n_2, p_j))|, \quad \text{by Remark 3.25} \\ & + |G(J_{k,h,0}(n_1 + (-1)^j \delta_{h',1}, n_2 - (-1)^j \delta_{h',2}, p_j)) - G(J_{k,h,0}(n_1 + (-1)^i \delta_{h',1}, n_2 - (-1)^i \delta_{h',2}, p_i))| \\ & \left. + |G(J_{k,h,0}(n_1 + (-1)^i \delta_{h',1}, n_2 - (-1)^j \delta_{h',2}, p_i)) - G(J_{k,h,1}(n_1 + (-1)^j \delta_{h',1}, n_2 - (-1)^j \delta_{h',2}, p_j))| \right). \end{aligned}$$

Thanks to the hypothesis  $(H_{3,k})$  and Lemma 3.26, we deduce the result.  $\square$

We turn to the regularity of  $V$  with respect to  $A_H$ .

**Proposition 3.28.** *Let  $V : \tilde{\mathcal{V}} \rightarrow \mathbb{R}^2$  be a function. We assume that  $V$  satisfying  $(H_2)$  and  $(H_{3,1})$  holds true, then  $V(Q) \in \mathcal{C}^1(A_H)$  and  $[V(Q), A_H]_0 \in \mathcal{C}^{0,1}(A_H)$ . In particular, we obtain  $V(Q) \in \mathcal{C}^{1,1}(A_H)$ .*

*Proof.* We show this proposition in two steps. First, we prove that  $V(Q) \in \mathcal{C}^1(A_H)$ . We establish there is  $c > 0$  such that

$$\|[V(Q), iA_H]\|^2 \leq c \|f\|^2, \quad \forall f \in \mathcal{S}.$$

Second, we take  $\gamma \in [0, 1)$  such that  $\gamma' < \gamma$ . We prove that  $[V(Q), iA_H]_0 \in \mathcal{C}^{0,1}(A_H)$ . We show there exists  $c_{\gamma'} > 0$  such that

$$\|\Lambda^\gamma(Q) [V(Q), iA_H]\|^2 \leq c_{\gamma'} \|f\|^2, \quad \forall f \in \mathcal{S}.$$

We have

$$\begin{aligned} [V(Q), iA_H] &= \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} 0 & iA_{2,H} \\ iA_{1,H} & 0 \end{pmatrix} - \begin{pmatrix} 0 & iA_{2,H} \\ iA_{2,H} & 0 \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i(V_1 A_{2,H} - A_{2,H} V_2) \\ i(V_2 A_{1,H} - A_{1,H} V_1) & 0 \end{pmatrix}. \end{aligned}$$

We are going to treat the term in  $V_2 A_{1,H} - A_{1,H} V_1$ . We work on  $\mathcal{S}$ . We infer:

$$\begin{aligned} & (V_2 A_{1,H} - A_{1,H} V_1) f_1(n_1, n_2) \\ &= \left( V_2 \left( R_{0,0}(U_1, U_2) + \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) \right) \right. \\ & \quad \left. - \left( R_{0,0}(U_1, U_2) + \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) \right) V_1 \right) f_1(n_1, n_2) \\ &= \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{0,0} (V_2(n_1, n_2) - V_1(n_1 + i, n_2 + j)) f_1(n_1 + i, n_2 + j) \\ & \quad + \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (Q_1 l_1 - Q_2 l_2) \\ & \quad \times \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{l_1, l_2} (V_2(n_1, n_2) - V_1(n_1 + i, n_2 + j)) f_1(n_1 + i, n_2 + j) \\ &= \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{0,0} (V_2(n_1, n_2) - V_1(n_1, n_2)) f_1(n_1 + i, n_2 + j) \\ & \quad + \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{0,0} (V_1(n_1, n_2) - V_1(n_1 + i, n_2 + j)) f_1(n_1 + i, n_2 + j) \\ & \quad + \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (Q_1 l_1 - Q_2 l_2) \\ & \quad \times \sum_{i, j \in \{-3, -2, -1, 0, 1, 2, 3\}} \alpha_{i,j}^{l_1, l_2} (V_2(n_1, n_2) - V_1(n_1, n_2)) f_1(n_1 + i, n_2 + j) \\ & \quad + \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (Q_1 l_1 - Q_2 l_2) \end{aligned}$$

$$\begin{aligned}
& \times \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{l_1,l_2} (V_1(n_1, n_2) - V_1(n_1 + i, n_2 + j)) f_1(n_1 + i, n_2 + j) \\
= & \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{0,0} (V_2(n_1, n_2) - V_1(n_1, n_2)) f_1(n_1 + i, n_2 + j) \\
& + \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{0,0} \left( \sum_{h=0}^{|j|-1} (V_2(n_1, n_2 + \operatorname{sgn}(j)h) - V_1(n_1, n_2 + \operatorname{sgn}(j)(h+1))) \right) \\
& + \sum_{h=0}^{|i|-1} V_2(n_1 + \operatorname{sgn}(i)h, n_2 + j) - V_1(n_1 + \operatorname{sgn}(i)(h+1), n_2 + j) \Big) f_1(n_1 + i, n_2 + j) \\
& + \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} (Q_1 l_1 - Q_2 l_2) \\
& \times \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{l_1, l_2} (V_2(n_1, n_2) - V_1(n_1, n_2)) f_1(n_1 + i, n_2 + j) \\
& + \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} \sum_{i,j \in \{-3,-2,-1,0,1,2,3\}} \alpha_{i,j}^{l_1, l_2} (Q_1 l_1 - Q_2 l_2) \\
& \times \left( \sum_{h=0}^{|i|-1} V_2(n_1 + \operatorname{sgn}(i)h, n_2) - V_1(n_1 + \operatorname{sgn}(i)(h+1), n_2) \right) \\
& + \sum_{h=0}^{|j|-1} V_2(n_1 + i, n_2 + \operatorname{sgn}(j)h) - V_1(n_1 + i, n_2 + \operatorname{sgn}(j)(h+1)) \Big) f_1(n_1 + i, n_2 + j).
\end{aligned}$$

We assume that  $(H_2)$ ,  $(H_{3,1})$  and  $(H_{4,1})$  are true. Let  $f_1 \in \mathcal{S}$ , we obtain:

$$\begin{aligned}
& \left\| \Lambda^\gamma(Q_1, Q_2) (V_2 A_{1,H} - A_{1,H} V_1) f_1 \right\| \\
\leq & \left\| \Lambda^\gamma(Q_1, Q_2) \sum_{i,j} \alpha_{i,j}^{0,0} (V_2(Q_1, Q_2) - V_1(Q_1, Q_2)) f_1 \right\| \\
& + \left\| \Lambda^\gamma(Q_1, Q_2) \sum_{i,j} \alpha_{i,j}^{0,0} \sum_{h=0}^{|j|-1} (V_2(Q_1, Q_2 + \operatorname{sgn}(j)h) - V_1(Q_1, Q_2 + \operatorname{sgn}(j)(h+1))) f_1 \right\| \\
(32) \quad & + \left\| \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} \sum_{i,j} \alpha_{i,j}^{l_1, l_2} (Q_1 l_1 - Q_2 l_2) (V_2(Q_1, Q_2) - V_1(Q_1, Q_2)) f_1 \right\| \\
& + \left\| \sum_{l_1, l_2 \in \{0,1\}^2, (l_1, l_2) \neq (0,0)} \sum_{i,j} \alpha_{i,j}^{l_1, l_2} (Q_1 l_1 - Q_2 l_2) \right. \\
& \quad \times \left. \left( \sum_{h=0}^{|i|-1} V_2(Q_1 + \operatorname{sgn}(i)h, Q_2) - V_1(Q_1 + \operatorname{sgn}(i)(h+1), Q_2) \right) f_1 \right\| \\
\leq & c_{\gamma'} \|f_1\|,
\end{aligned}$$

where  $\operatorname{sgn}(i) := 1$ , if  $i \geq 0$ , and  $:= -1$ , otherwise. The other terms are treated similarly. Then, by [AmBoGe, Lemma 6.2.9], we obtain  $V(Q) \in \mathcal{C}^1(A_H)$ . Now, to prove that  $[V(Q), A_H]_0 \in \mathcal{C}^{0,1}(A_H)$  it is enough to use (32), there is  $c$  such that  $\left\| \Lambda^\gamma(Q) [V(Q), iA_H]_0 f \right\| \leq c_{\gamma'} \|f\|$ ,  $\forall f \in \mathcal{S}$ . Finally, thanks to Corollary 3.20, we apply [AmBoGe, Proposition 7.5.7] and the result follows.  $\square$

**3.2.3. Perturbed metric.** We now move to the most technical section. We start with technical lemmata.

**Lemma 3.29.** *Let  $G : (\mathbb{Z}^2 \times \{p_1, p_2\})^k \rightarrow \mathbb{C}$  satisfying the hypothesis  $(H_{3,k})$  and Lemma 3.26, then for  $k \in \mathbb{N}^*$ , we have*

$$(H_{6,k}) \quad \max_{h \in \{1,2\}} \max_{i \in \{1,2\}} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1, n_2, p_i))| < \infty.$$

*Proof.* Let  $\{i, j\} \in \{1, 2\}$ ,  $h \in \{1, 2\}$ . For all  $l_1, l_2 \in \{0, 1\}$ , we have:

$$\Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,1}(n_1, n_2, p_i))|$$

$$\begin{aligned} &\leq \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle \left( |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,0}(n_1, n_2, p_j))| \right. \\ &\quad \left. + |G(J_{k,h,0}(n_1, n_2, p_j)) - G(J_{k,h,1}(n_1, n_2, p_i))| \right). \end{aligned}$$

Thanks to  $(H_{3,k})$  and Lemma 3.26, we obtain the result.  $\square$

**Lemma 3.30.** *Let  $k \in \mathbb{N}^*$ . Let  $G : (\mathbb{Z}^2 \times \{p_1, p_2\})^k \rightarrow \mathbb{C}$  satisfying the hypothesis  $(H_{3,k})$  and*

$$(H_{7,k}) \quad \max_{h \in \{1,2\}} \max_{j \in \{1,2\}} G(J_{k,h,b}(n_1, n_2, p_j)) \rightarrow 0 \text{ if } b \rightarrow \infty,$$

for all  $(n_1, n_2) \in \mathbb{Z}^2$ . Then we have:

$$(H_{8,k}) \quad \max_{i \in \{1,2\}} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) |G(J_{k,h,0}(n_1, n_2, p_i))| < \infty.$$

*Proof.* We fix  $i \in \{1, 2\}$ . We suppose that  $(H_{7,k})$  hold true, then for all  $j \in \{1, 2\}$ , we have:

$$\begin{aligned} &\Lambda^\gamma(n_1, n_2) |G(J_{k,h,0}(n_1, n_2, p_i))| \\ &= \lim_{b \rightarrow +\infty} \Lambda^\gamma(n_1, n_2) |G(J_{k,h,0}(n_1, n_2, p_i)) - G(J_{k,h,b}(n_1, n_2, p_j))| \end{aligned}$$

First of all, by using  $(H_{3,k})$  and  $(H_{6,k})$ , for  $i, j \in \{1, 2\}$  and  $\sigma \in \{0, 1\}$ , we have

$$\begin{aligned} \mathcal{M}_{k,h,\sigma,i,j} &:= \max_{(l_1, l_2)} \sup_{(n_1, n_2) \in \mathbb{Z}^2} \Lambda^\gamma(n_1, n_2) \langle n_1 l_1 - n_2 l_2 \rangle \left| G(J_{k,h,0}(n_1, n_2, p_i)) \right. \\ &\quad \left. - G(J_{k,h,\sigma}(n_1, n_2, p_j)) \right| < \infty. \end{aligned}$$

To get  $(H_{8,k})$ , it suffices to find an upper bound for the function

$$(n_1, n_2) \mapsto \Lambda^\varepsilon(n_1, n_2) |G(J_{k,h,0}(n_1, n_2, p_i))|$$

on the set  $\mathbb{Z}^2 \setminus \{(0, 0)\}$ .

First case:  $|n_1| \geq 1$  and  $|n_1| \geq |n_2|$ . Let  $j \in \{1, 2\}$  such that  $(-1)^{j+1} n_1 = |n_1|$ . For  $b \in \mathbb{N}$ , we have

$$\begin{aligned} f_b(n_1, n_2, p_i) &:= \Lambda^\gamma(n_1, n_2) |G(J_{k,1,b+1}(n_1, n_2, p_j)) - G(J_{k,1,0}(n_1, n_2, p_i))| \\ &\leq \Lambda^\gamma(n_1, n_2) \left( |G(J_{k,1,b+1}(n_1, n_2, p_j)) - G(J_{k,1,0}(n_1, n_2, p_j))| \right. \\ &\quad \left. + |G(J_{k,1,0}(n_1, n_2, p_j)) - G(J_{k,1,0}(n_1, n_2, p_i))| \right) \\ &\leq \Lambda^\gamma(n_1, n_2) \left( \sum_{t=0}^b |G(J_{k,1,t+1}(n_1, n_2, p_j)) - G(J_{k,1,t}(n_1, n_2, p_j))| \right. \\ &\quad \left. + |G(J_{k,1,0}(n_1, n_2, p_j)) - G(J_{k,1,0}(n_1, n_2, p_i))| \right) \\ &= \Lambda^\gamma(n_1, n_2) \left( \sum_{t=0}^b |G(n_1 + (-1)^{j+1}(t+1), n_2, p_i) - G(n_1 + (-1)^{j+1}t, n_2, p_i)| \right. \\ &\quad \left. + |G(J_{k,1,0}(n_1, n_2, p_j)) - G(J_{k,1,0}(n_1, n_2, p_i))| \right) \\ &\leq \Lambda^\gamma(n_1, n_2) \left( \sum_{t=0}^b |G(J_{k,1,1}(n_1 + (-1)^{j+1}t, n_2, p_j)) - G(J_{k,1,0}(n_1 + (-1)^{j+1}t, n_2, p_j))| \right. \\ &\quad \left. + \mathcal{M}_{k,h,0,i,j} \Lambda^{-\gamma}(n_1, n_2) \langle n_1 \rangle^{-1} \right), \text{ by } (H_{4,k}), \text{ see Lemma 3.26.} \end{aligned}$$

Changing the index in the last sum and using  $|n_2| \leq |n_1|$ , by  $(H_{6,k})$ , we get:

$$\begin{aligned} f_b(n_1, n_2, p_i) &\leq \mathcal{M}_{k,h,0,i,i} \langle n_1 \rangle^{-1} + \mathcal{M}_{k,h,1,j,j} \Lambda^\gamma(n_1, n_2) \sum_{\ell=n_1}^{n_1+(b+1)(-1)^{j+1}} \Lambda^{-\gamma}(\ell, n_2) \langle \ell \rangle^{-1} \\ &\leq \mathcal{M} \left( 1 + 2^\gamma \langle n_1 \rangle^\gamma \sum_{\ell=n_1+(-1)^{j+1}}^{n_1+(b+1)(-1)^{j+1}} \langle \ell \rangle^{-1-\gamma} + \langle n_1 \rangle^{-1} \right) \\ &\leq \mathcal{M} \left( 3 + 2^\gamma \langle n_1 \rangle^\gamma \int_{|n_1|}^{\infty} \langle s \rangle^{-1-\gamma} ds \right) \leq \mathcal{M} \left( 3 + 2^\gamma \langle n_1 \rangle^\gamma \int_{|n_1|}^{\infty} s^{-1-\gamma} ds \right) \end{aligned}$$

$$\leq \mathcal{M} \left( 3 + \frac{2^\gamma}{\gamma} \cdot \frac{\langle n_1 \rangle^\gamma}{|n_1|^\gamma} \right) \leq \mathcal{M} \left( 3 + \frac{(2\sqrt{2})^\gamma}{\gamma} \right),$$

with  $\mathcal{M} := \max(\mathcal{M}_{k,h,0,i,j}, \mathcal{M}_{k,h,1,j,j})$ . Since  $G$  tends to 0 at infinity, we may take the limit  $b \rightarrow \infty$

$$\Lambda^\gamma(n_1, n_2) |G(J_{k,h,0}(n_1, n_2, p_i))| = \lim_{b \rightarrow \infty} f_b(n_1, n_2, p_i) \leq \mathcal{M} \left( 3 + \frac{(2\sqrt{2})^\gamma}{\gamma} \right).$$

Second case:  $|n_2| \geq 1$  and  $|n_2| \geq |n_1|$ . Take  $j \in \{1, 2\}$  such that  $(-1)^{j+1}n_2 = |n_2|$ . □

Since explicitly writing out the expression for the commutator is extremely laborious (requiring thousands of elementary operations), we adopt a more systematic approach to simplify the process. In particular, we introduce an important lemma that serves as a computational algorithm in the following proof. The proof itself is provided in the appendix.

**Lemma 3.31.** *We set:*

$$\begin{aligned} I_{l_1, l_2}^0 &:= \{(i, j) \in \mathbb{Z}^2, \text{ such that } \alpha_{i,j}^{l_1, l_2} \neq 0\}, \\ I_{l_1, l_2}^1 &:= \{(i, j) \in I_{l_1, l_2}^0, \text{ such that } (i+1, j) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^2 &:= \{(i, j) \in I_{l_1, l_2}^1, \text{ such that } (i, j+1) \notin I_{l_1, l_2}^1\}, \\ I_{l_1, l_2}^3 &:= \{(i, j) \in I_{l_1, l_2}^0, \text{ such that } (-i, -j) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^4 &:= \{(i, j) \in I_{l_1, l_2}^0, \text{ such that } (-i-1, -j) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^5 &:= \{(i, j) \in I_{l_1, l_2}^0, \text{ such that } (-i, -j-1) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^6 &:= \{(i, j) \in I_{l_1, l_2}^2, \text{ such that } (i-1, j+1) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^7 &:= \{(i, j) \in I_{l_1, l_2}^3, \text{ such that } (-i-1, -j) \notin I_{l_1, l_2}^0 \text{ and } (-i, -j-1) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^8 &:= \{(i, j) \in I_{l_1, l_2}^4, \text{ such that } (-i-2, -j) \notin I_{l_1, l_2}^0 \text{ and } (-i-1, -j-1) \notin I_{l_1, l_2}^0\}, \\ I_{l_1, l_2}^9 &:= \{(i, j) \in I_{l_1, l_2}^5, \text{ such that } (-i-1, -j-1) \notin I_{l_1, l_2}^0, (-i, -j-2) \notin I_{l_1, l_2}^0, \\ &\quad (i, j+1) \notin I_{l_1, l_2}^0 \text{ and } (i+1, j+1) \notin I_{l_1, l_2}^0\}. \end{aligned}$$

We denote

$$I_{l_1, l_2}^{3+\min(l_1, l_2)} := \{(i, j) \in I_{l_1, l_2}^0, \text{ such that } (-i - \min(l_1, l_2), -j) \notin I_{l_1, l_2}^0\},$$

where  $I_{l_1, l_2}^{3+\min(l_1, l_2)} = I_{l_1, l_2}^3$  if  $l_1, l_2 \in \{(0, 1), (1, 0), (0, 0)\}$  and  $I_{l_1, l_2}^{3+\min(l_1, l_2)} = I_{l_1, l_2}^4$  if  $l_1, l_2 \in \{(1, 1)\}$ .

1. Let  $(i, j) \in I_{l_1, l_2}^{4-\min(l_1, l_2)}$ , then at least one of the following conditions is satisfied :

$$(33) \quad \begin{aligned} &i. (-i-2 + \min(l_1, l_2), -j) \in I_{l_1, l_2}^1, \quad ii. (-i-1 + \min(l_1, l_2), -j-1) \in I_{l_1, l_2}^2, \\ &iii. (-i - \min(l_1, l_2) + 1, j) \in I_{l_1, l_2}^{3+\min(l_1, l_2)}. \end{aligned}$$

2. Let  $(i, j) \in I_{l_1, l_2}^5$ , then at least one of the following conditions is satisfied:

$$(34) \quad \begin{aligned} &i. (-i-1, -j-1) \in I_{l_1, l_2}^1 \quad ii. (-i, -j-2) \in I_{l_1, l_2}^2 \quad iii. (-i-1, -j+1) \in I_{l_1, l_2}^4. \end{aligned}$$

We denote

$$\begin{aligned} I_{l_1, l_2}^{7+\min(l_1, l_2)} &:= \{(i, j) \in I_{l_1, l_2}^{3+\min(l_1, l_2)}, \text{ such that } (i-1 - \min(l_1, l_2), -j) \notin I_{l_1, l_2}^1 \\ &\quad \text{and } (i - \min(l_1, l_2), -j-1) \notin I_{l_1, l_2}^2\}, \end{aligned}$$

where  $I_{l_1, l_2}^{7+\min(l_1, l_2)} = I_{l_1, l_2}^7$  if  $(l_1, l_2) \in \{(0, 1), (1, 0), (0, 0)\}$  and  $I_{l_1, l_2}^{7+\min(l_1, l_2)} = I_{l_1, l_2}^8$  if  $(l_1, l_2) \in \{(1, 1)\}$ .

3. We have  $I_{l_1, l_2}^9 = \emptyset$  and  $I_{l_1, l_2}^{7+\min(l_1, l_2)} \subset I_{l_1, l_2}^1 \cup I_{l_1, l_2}^2 \cup I_{l_1, l_2}^{8-\min(l_1, l_2)}$ .

4. For all  $(i, j) \in I_{l_1, l_2}^0$  we have:

$$(35) \quad \alpha_{i,j}^{l_1, l_2} + \alpha_{i-1,j}^{l_1, l_2} + \alpha_{i,j-1}^{l_1, l_2} = (-1)^{\max(l_1, l_2)} \left( \alpha_{-i,-j}^{l_1, l_2} + \alpha_{-i-1,-j}^{l_1, l_2} + \alpha_{-i,-j-1}^{l_1, l_2} \right).$$

5. For all  $(i, j) \in I_{l_1, l_2}^1$  we obtain:

$$(36) \quad \alpha_{i,j}^{l_1, l_2} + \alpha_{i+1,j}^{l_1, l_2} + \alpha_{i+1,j-1}^{l_1, l_2} = (-1)^{\max(l_1, l_2)} \left( \alpha_{-i-1,-j}^{l_1, l_2} + \alpha_{-i-2,-j}^{l_1, l_2} + \alpha_{-i-1,-j-1}^{l_1, l_2} \right).$$

6. For all  $(i, j) \in I_{l_1, l_2}^2$  we get:

$$(37) \quad \alpha_{i,j}^{l_1, l_2} + \alpha_{i, j+1}^{l_1, l_2} + \alpha_{i-1, j+1}^{l_1, l_2} = (-1)^{\max(l_1, l_2)} \left( \alpha_{-i, -j-1}^{l_1, l_2} + \alpha_{-i-1, -j-1}^{l_1, l_2} + \alpha_{-i-1, -j-2}^{l_1, l_2} \right).$$

7. For all  $(i, j) \in I_{l_1, l_2}^4$  we get:

$$(38) \quad \alpha_{i,j}^{l_1, l_2} + \alpha_{i+1, j}^{l_1, l_2} + \alpha_{i+1, j-1}^{l_1, l_2} = (-1)^{\max(l_1, l_2)} \left( \alpha_{-i-1, -j}^{l_1, l_2} + \alpha_{-i-2, -j}^{l_1, l_2} + \alpha_{-i-1, -j-1}^{l_1, l_2} \right).$$

We turn to the regularity of metric with respect of  $A_H$ .

**Proposition 3.32.** *We assume that  $(H_0)$  and  $(H_1)$  hold true,  $\eta$  satisfy  $(H_{3,1})$  and  $\varepsilon$  satisfy  $(H_{3,2})$ , we have  $\tilde{\Delta} \in \mathcal{C}^1(A_H)$ . Moreover  $[\tilde{\Delta}, iA_H]_\circ \in \mathcal{C}^{0,1}(A_H)$ . In particular,  $\tilde{\Delta} \in \mathcal{C}^{1,1}(A_H)$ .*

*Proof.* We prove this proposition in two steps. In the first step, we will establish that  $\tilde{\Delta} \in \mathcal{C}^1(A_H)$ . We show there is  $c$ , such that:

$$(39) \quad \left\| [\Delta_H - \tilde{\Delta}, iA_H] f \right\|^2 \leq c \|f\|^2, \quad \forall f \in \mathcal{S}.$$

Hence, by density and thanks to Proposition 3.18 and [AmBoGe, Proposition 6.2.9], we obtain that  $\tilde{\Delta} \in \mathcal{C}^1(A)$ . In the second step, we aim to show that  $[\tilde{\Delta}, iA_H]_\circ \in \mathcal{C}^{0,1}(A_H)$ . Using  $(H_0)$ ,  $(H_1)$ ,  $(H_{3,k})$ ,  $(H_{4,k})$ ,  $(H_{5,k})$ ,  $(H_{6,k})$  and thanks to Lemma 3.30, we derive, for all  $\gamma' < \gamma$  such that  $\gamma' \in ]0, 1[$ , there exists an integer  $c_{\gamma'}$ , such that:

$$(40) \quad \left\| \Lambda^{\gamma'}(Q_1, Q_2) [\Delta_H - \tilde{\Delta}, iA_H] f \right\|^2 \leq c_{\gamma'} \|f\|^2, \quad \forall f \in \mathcal{S}.$$

So, by density, we obtain  $[\Delta_H - \tilde{\Delta}, iA_H]_\circ \in \mathcal{C}^{0,1}(A_H)$ . Therefore, thanks to Corollary 3.20 and by Proposition 3.18, we deduce that  $[\tilde{\Delta}, iA_H]_\circ \in \mathcal{C}^{0,1}(A_H)$ . In particular, by Proposition 3.18, we get  $\tilde{\Delta} \in \mathcal{C}^{1,1}(A_H)$ .

Now, we proceed to prove (40), from which (39) follows as a consequence. We recall (31). We have:

$$[\Delta_H - \tilde{\Delta}, iA_H] = i \begin{pmatrix} D_2 A_{1,H} - A_{2,H} D_1 & 0 \\ 0 & D_1 A_{2,H} - A_{1,H} D_2 \end{pmatrix}.$$

We denote  $Diag(\mathbb{Z}^4) := \{(n, n), n \in \mathbb{Z}^2\}$ . In the sequel, to simplify matters, for  $((n_1, n_2), (n_1, n_2)) \in Diag(\mathbb{Z}^4)$ , we write

$$\mathcal{E}_{Diag(\mathbb{Z}^4) \times \{p_1, p_2\}}((n_1, n_2, p_1), (n_1, n_2, p_2)) = \mathcal{E}(n_1, n_2, p_1, p_2)$$

and we write

$$m(n_1, n_2, p_i) := m_i(n_1, n_2), \quad \text{where } i, j \in \{1, 2\}.$$

We next treat the term  $i(D_1 A_{2,H} - A_{1,H} D_2)$ . It gives three types of terms, the ones in:  $f_2(n_1 + i, n_2 + j)$ ,  $f_2(n_1 + i + 1, n_2 + j)$  and  $f_2(n_1 + i, n_2 + j + 1)$ . We have:

$$\begin{aligned} & i(D_1 A_{2,H} - A_{1,H} D_2) f_2(n) \\ &= i \left( (\Delta_{1,H} - \tilde{\Delta}_1) A_{2,H} - A_{1,H} (\Delta_{2,H} - \tilde{\Delta}_2) \right) f_2(n) \\ &= \frac{i}{3} \left( (L_{1,H} + T_{1,H} + S_{1,H}) A_{2,H} - A_{1,H} (L_{2,H} + T_{2,H} + S_{2,H}) \right) f_2(n) \\ &= \frac{1}{3} \left( (L_{1,H} + T_{1,H} + S_{1,H}) \left( R_{0,0}^*(U_1, U_2) + \sum_{l_1, l_2 \in \{0,1\}} R_{l_1, l_2}^*(U_1, U_2) (Q_1 l_1 - Q_2 l_2) \right) \right. \\ & \quad \left. + \left( R_{0,0}(U_1, U_2) + \sum_{l_1, l_2 \in \{0,1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) \right) (L_{2,H} + T_{2,H} + S_{2,H}) \right) f_2(n). \end{aligned}$$

As the first step, we analyse the terms in  $R_{0,0}$ , we have

$$\begin{aligned} & \left( (L_{1,H} + T_{1,H} + S_{1,H}) R_{0,0}^*(U_1, U_2) + R_{0,0}(U_1, U_2) (L_{2,H} + T_{2,H} + S_{2,H}) \right) f_2(n) \\ &= \left( 1 - \frac{\mathcal{E}(n_1, n_2, p_1, p_2)}{\sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \sum_{(i,j) \in I_{0,0}^0} \alpha_{i,j}^{0,0} f_2(n_1 + i, n_2 + j) \end{aligned}$$

$$\begin{aligned}
& + \sum_{(i,j) \in I_{0,0}^0} \left( 1 - \frac{\mathcal{E}(n_1 - i, n_2 - j, p_2, p_1)}{\sqrt{m_1(n_1 - i, n_2 - j)} \sqrt{m_2(n_1 - i, n_2 - j)}} \right) \alpha_{i,j}^{0,0} f_2(n_1 - i, n_2 - j) \\
& + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \sum_{(i,j) \in I_{0,0}^0} \alpha_{i,j}^{0,0} f_2(n_1 + i + 1, n_2 + j) \\
& + \sum_{(i,j) \in I_{0,0}^0} \left( 1 - \frac{\mathcal{E}((n_1 - i - 1, n_2 - j, p_2), (n_1 - i, n_2 - j, p_1))}{\sqrt{m_1(n_1 - i, n_2 - j)} \sqrt{m_2(n_1 - i - 1, n_2 - j)}} \right) \alpha_{i,j}^{0,0} f_2(n_1 - i - 1, n_2 - j) \\
& + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \sum_{(i,j) \in I_{0,0}^0} \alpha_{i,j}^{0,0} f_2(n_1 + i, n_2 + j + 1) \\
& + \sum_{(i,j) \in I_{0,0}^0} \left( 1 - \frac{\mathcal{E}((n_1 - i, n_2 - j - 1, p_2), (n_1 - i, n_2 - j, p_1))}{\sqrt{m_1(n_1 - i, n_2 - j)} \sqrt{m_2(n_1 - i, n_2 - j - 1)}} \right) \alpha_{i,j}^{0,0} f_2(n_1 - i, n_2 - j - 1) \\
& = \sum_{(i,j) \in I_{0,0}^0} \left( \left( 1 - \frac{\mathcal{E}(n_1, n_2, p_1, p_2)}{\sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i,j}^{0,0} + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i-1,j}^{0,0} \right. \\
& + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i,j-1}^{0,0} \\
& + \left( 1 - \frac{\mathcal{E}(n_1 + i, n_2 + j, p_2, p_1)}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} \right) \alpha_{-i,-j}^{0,0} \\
& + \left( 1 - \frac{\mathcal{E}((n_1 + i, n_2 + j, p_2), (n_1 + i + 1, n_2 + j, p_1))}{\sqrt{m_1(n_1 + i + 1, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} \right) \alpha_{-i-1,-j}^{0,0} \\
& \left. + \left( 1 - \frac{\mathcal{E}((n_1 + i, n_2 + j, p_2), (n_1 + i, n_2 + j + 1, p_1))}{\sqrt{m_1(n_1 + i, n_2 + j + 1)} \sqrt{m_2(n_1 + i, n_2 + j)}} \right) \alpha_{-i,-j-1}^{0,0} \right) f_2(n_1 + i, n_2 + j).
\end{aligned}$$

We turn to the norm, by Lemma 3.30, we infer

$$\begin{aligned}
& \left\| \Lambda^{\gamma'}(n_1, n_2) \left( (L_{1,H} + T_{1,H} + S_{1,H}) R_{0,0}^*(U_1, U_2) + R_{0,0}(U_1, U_2) (L_{2,H} + T_{2,H} + S_{2,H}) \right) f_2(n) \right\|^2 \\
& \leq \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(i,j) \in I_{l_2, l_2}^0} \left( m_1(n_1, n_2) (\eta_2(n_1, n_2) - \varepsilon(n_1, n_2, p_1, p_2)) \right. \right. \\
& \quad \left. \left. + \mathcal{E}(n_1, n_2, p_1, p_2) (\eta_1(n_1, n_2) - \varepsilon(n_1, n_2, p_1, p_2)) \right) \frac{1}{\sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \right. \\
& \quad \left. \times \frac{1}{\sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)} + \mathcal{E}(n_1, n_2, p_1, p_2)} \alpha_{i,j}^{0,0} f_2(n_1 + i, n_2 + j) \right|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(i,j) \in I_{l_2, l_2}^0} \left( m_1(n_1 + 1, n_2) (\eta_2(n_1, n_2) - \varepsilon((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))) \right. \right. \\
& \quad \left. \left. + \mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2)) (\eta_1(n_1 + 1, n_2) - \varepsilon((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))) \right) \right. \\
& \quad \left. \times \frac{1}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right. \\
& \quad \left. \times \frac{1}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)} + \mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))} \alpha_{i-1,j}^{0,0} f_2(n_1 + i, n_2 + j) \right|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(i,j) \in I_{l_2, l_2}^0} \left( m_1(n_1, n_2 + 1) (\eta_2(n_1, n_2) - \varepsilon((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2)) (\eta_1(n_1, n_2 + 1) - \varepsilon((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))) \Big) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1, n_2 + 1)}\sqrt{m_2(n_1, n_2)}} \\
& \quad \times \frac{1}{\sqrt{m_1(n_1, n_2 + 1)}\sqrt{m_2(n_1, n_2)} + \mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))} \alpha_{i,j-1}^{0,0} f_2(n_1 + i, n_2 + j) \Big|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(i,j) \in I_{l_2, l_2}^0} \left( m_1(n_1 + i, n_2 + j) (\eta_2(n_1 + i, n_2 + j) - \varepsilon(n_1 + i, n_2 + j, p_1, p_2)) \right. \right. \\
& \quad \left. \left. + \mathcal{E}(n_1 + i, n_2 + j, p_1, p_2) (\eta_1(n_1 + i, n_2 + j) - \varepsilon(n_1 + i, n_2 + j, p_1, p_2)) \right) \right. \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)}\sqrt{m_2(n_1 + i, n_2 + j)}} \left( \sqrt{m_1(n_1 + i, n_2 + j)}\sqrt{m_2(n_1 + i, n_2 + j)} \right. \\
& \quad \left. + \mathcal{E}(n_1 + i, n_2 + j, p_1, p_2) \right)^{(-1)} \alpha_{-i,-j}^{0,0} f_2(n_1 + i, n_2 + j) \Big|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(i,j) \in I_{l_2, l_2}^0} \left( m_1(n_1 + i + 1, n_2 + j) (\eta_2(n_1 + i, n_2 + j) - \varepsilon(n_1 + i, n_2 + j, p_1, p_2)) \right. \right. \\
& \quad \left. \left. + \mathcal{E}((n_1 + i + 1, n_2 + j, p_1), (n_1 + i, n_2 + j, p_2)) \right) \right. \\
& \quad \times (m_1(n_1 + i + 1, n_2 + j) - \mathcal{E}((n_1 + i + 1, n_2 + j, p_1), (n_1 + i, n_2 + j, p_2))) \Big) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i + 1, n_2 + j)}\sqrt{m_2(n_1 + i, n_2 + j)}} \left( \sqrt{m_1(n_1 + i + 1, n_2 + j)}\sqrt{m_2(n_1 + i, n_2 + j)} \right. \\
& \quad \left. + \mathcal{E}((n_1 + i + 1, n_2 + j, p_1), (n_1 + i, n_2 + j, p_2)) \right)^{(-1)} \alpha_{-i-1,-j}^{0,0} f_2(n_1 + i, n_2 + j) \Big|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(i,j) \in I_{l_2, l_2}^0} \left( m_1(n_1 + i, n_2 + j + 1) \right. \right. \\
& \quad \times (\eta_2(n_1 + i, n_2 + j) - \varepsilon((n_1 + i, n_2 + j, p_1), (n_1 + i, n_2 + j + 1, p_2))) \\
& \quad \left. \left. + \mathcal{E}((n_1 + i, n_2 + j, p_1), (n_1 + i, n_2 + j + 1, p_2)) \right) \right. \\
& \quad \times (\eta_1(n_1 + i, n_2 + j + 1) - \varepsilon((n_1 + i, n_2 + j, p_1), (n_1 + i, n_2 + j + 1, p_2))) \Big) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j + 1)}\sqrt{m_2(n_1 + i, n_2 + j)}} \left( \sqrt{m_1(n_1 + i, n_2 + j + 1)}\sqrt{m_2(n_1 + i, n_2 + j)} \right. \\
& \quad \left. + \mathcal{E}((n_1 + i, n_2 + j, p_1), (n_1 + i, n_2 + j + 1, p_2)) \right)^{(-1)} \alpha_{-i,-j-1}^{0,0} f_2(n_1 + i, n_2 + j) \Big|^2 \\
& \leq c \|f_2\|^2.
\end{aligned}$$

Let  $S := \{(1, 0), (0, 1), (1, 1)\}$ . Now, we treat the terms in  $R_{l_1, l_2}$ , we have:

$$\begin{aligned}
& \left( (L_{1,H} + T_{1,H} + S_{1,H}) \sum_{(l_1, l_2) \in S} R_{l_1, l_2}^*(U_1, U_2) (Q_1 l_1 - Q_2 l_2) \right. \\
& \quad \left. + \sum_{(l_1, l_2) \in S} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) (L_{2,H} + T_{2,H} + S_{2,H}) \right) f_2(n) \\
& = \left( 1 - \frac{\mathcal{E}(n_1, n_2, p_1, p_2)}{\sqrt{m_1(n_1, n_2)}\sqrt{m_2(n_1, n_2)}} \right) \sum_{(l_1, l_2) \in S} \sum_{(i,j) \in I_{l_2, l_2}^0} ((n_1 + i)l_1 - (n_2 + j)l_2) \alpha_{i,j}^{l_1, l_2} f_2(n_1 + i, n_2 + j)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_2, l_2}^0} (n_1 l_1 - n_2 l_2) \left( 1 - \frac{\mathcal{E}(n_1 - i, n_2 - j, p_2, p_1)}{\sqrt{m_1(n_1 - i, n_2 - j)} \sqrt{m_2(n_1 - i, n_2 - j)}} \right) \\
& \quad \times \alpha_{i, j}^{l_1, l_2} f_2(n_1 - i, n_2 - j) \\
& + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_2, l_2}^0} ((n_1 + i + 1)l_1 - (n_2 + j)l_2) \\
& \quad \times \alpha_{i, j}^{l_1, l_2} f_2(n_1 + i + 1, n_2 + j) \\
& + \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_2, l_2}^0} (n_1 l_1 - n_2 l_2) \left( 1 - \frac{\mathcal{E}((n_1 - i - 1, n_2 - j, p_2), (n_1 - i, n_2 - j, p_1))}{\sqrt{m_1(n_1 - i - 1, n_2 - j)} \sqrt{m_2(n_1 - i - 1, n_2 - j)}} \right) \\
& \quad \times \alpha_{i, j}^{l_1, l_2} f_2(n_1 - i - 1, n_2 - j) \\
& + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_2, l_2}^0} ((n_1 + i)l_1 - (n_2 + j + 1)l_2) \\
& \quad \times \alpha_{i, j}^{l_1, l_2} f_2(n_1 + i, n_2 + j + 1) \\
& + \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_2, l_2}^0} (n_1 l_1 - n_2 l_2) \left( 1 - \frac{\mathcal{E}((n_1 - i, n_2 - j - 1, p_2), (n_1 - i, n_2 - j, p_1))}{\sqrt{m_1(n_1 - i, n_2 - j)} \sqrt{m_2(n_1 - i, n_2 - j - 1)}} \right) \\
& \quad \times \alpha_{i, j}^{l_1, l_2} f_2(n_1 - i, n_2 - j - 1).
\end{aligned}$$

Using 1., 2. and 3. in Lemma 3.31, and a change of variable, we derive:

$$\begin{aligned}
& \left( (L_{1, H} + T_{1, H} + S_{1, H}) \sum_{l_1, l_2 \in \{0, 1\}} R_{l_1, l_2}^*(U_1, U_2) (Q_1 l_1 - Q_2 l_2) \right. \\
& \quad \left. + \sum_{l_1, l_2 \in \{0, 1\}} (Q_1 l_1 - Q_2 l_2) R_{l_1, l_2}(U_1, U_2) (L_{2, H} + T_{2, H} + S_{2, H}) \right) f_2(n) \\
& = ((M_{\{1, 0, 1, 0\}} + M_{\{1, 1, 1, 0\}} + M_{\{1, 0, 1, 1\}} + M_{\{-1, -\min(l_1, l_2), -1, 0\}}) f_2)(n_1, n_2),
\end{aligned}$$

where

$$\begin{aligned}
& (M_{\{1, 0, 1, 0\}} f_2)(n_1, n_2) \\
& := \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^0} \left( ((n_1 + i)l_1 - (n_2 + j)l_2) \left( \left( 1 - \frac{\mathcal{E}(n_1, n_2, p_1, p_2)}{\sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i, j}^{l_1, l_2} \right. \right. \\
& \quad \left. \left. + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i-1, j}^{l_1, l_2} + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i, j-1}^{l_1, l_2} \right) \right) \\
& \quad + (n_1 l_1 - n_2 l_2) \left( \left( 1 - \frac{\mathcal{E}(n_1 + i, n_2 + j, p_2, p_1)}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} \right) \alpha_{-i, -j}^{l_1, l_2} \right. \\
& \quad \left. + \left( 1 - \frac{\mathcal{E}((n_1 + i, n_2 + j, p_2), (n_1 + i + 1, n_2 + j, p_1))}{\sqrt{m_1(n_1 + i + 1, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} \right) \alpha_{-i-1, -j}^{l_1, l_2} \right. \\
& \quad \left. + \left( 1 - \frac{\mathcal{E}((n_1 + i, n_2 + j, p_2), (n_1 + i, n_2 + j + 1, p_1))}{\sqrt{m_1(n_1 + i, n_2 + j + 1)} \sqrt{m_2(n_1 + i, n_2 + j)}} \right) \alpha_{-i, -j-1}^{l_1, l_2} \right) \right) f_2(n_1 + i, n_2 + j), \\
& (M_{\{1, 1, 1, 0\}} f_2)(n_1, n_2) \\
& := \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^1} \left( ((n_1 + i + 1)l_1 - (n_2 + j)l_2) \left( \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i, j}^{l_1, l_2} \right. \right. \\
& \quad \left. \left. + \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i+1, j-1}^{l_1, l_2} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& + (n_1 l_1 - n_2 l_2) \left( \left( 1 - \frac{\mathcal{E}(n_1 + i + 1, n_2 + j, p_2, p_1)}{\sqrt{m_1(n_1 + i + 1, n_2 + j)} \sqrt{m_2(n_1 + i + 1, n_2 + j)}} \right) \alpha_{-i-1, -j}^{l_1, l_2} \right. \\
& + \left( 1 - \frac{\mathcal{E}((n_1 + i + 1, n_2 + j, p_2), (n_1 + i + 2, n_2 + j, p_1))}{\sqrt{m_1(n_1 + i - 2, n_2 + j)} \sqrt{m_2(n_1 + i + 1, n_2 + j)}} \right) \alpha_{-i-2, -j}^{l_1, l_2} \\
& \left. + \left( 1 - \frac{\mathcal{E}((n_1 + i + 1, n_2 + j, p_2), (n_1 + i + 1, n_2 + j + 1, p_1))}{\sqrt{m_1(n_1 + i + 1, n_2 + j + 1)} \sqrt{m_2(n_1 + i + 1, n_2 + j)}} \right) \alpha_{-i-1, -j-1}^{l_1, l_2} \right) f_2(n_1 + i + 1, n_2 + j), \\
& (M_{\{1, 0, 1, 1\}} f_2)(n_1, n_2) \\
& := \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^6} \left( ((n_1 + i)l_1 - (n_2 + j + 1)l_2) \left( \left( 1 - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i, j}^{l_1, l_2} \right) \right. \\
& + (n_1 l_1 - n_2 l_2) \left( \left( 1 - \frac{\mathcal{E}(n_1 + i, n_2 + j + 1, p_2, p_1)}{\sqrt{m_1(n_1 + i, n_2 + j + 1)} \sqrt{m_2(n_1 + i, n_2 + j + 1)}} \right) \alpha_{-i, -j-1}^{l_1, l_2} \right. \\
& + \left( 1 - \frac{\mathcal{E}((n_1 + i, n_2 + j + 1, p_2), (n_1 + i + 1, n_2 + j + 1, p_1))}{\sqrt{m_1(n_1 + i + 1, n_2 + j + 1)} \sqrt{m_2(n_1 + i, n_2 + j + 1)}} \right) \alpha_{-i-1, -j-1}^{l_1, l_2} \\
& \left. \left. + \left( 1 - \frac{\mathcal{E}((n_1 + i, n_2 + j + 1, p_2), (n_1 + i, n_2 + j + 2, p_1))}{\sqrt{m_1(n_1 + i, n_2 + j + 2)} \sqrt{m_2(n_1 + i, n_2 + j + 1)}} \right) \alpha_{-i, -j-2}^{l_1, l_2} \right) \right) f_2(n_1 + i, n_2 + j + 1), \\
& (M_{\{-1, -\min(l_1, l_2), -1, 0\}} f_2)(n_1, n_2) \\
& := \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^{7+\min(l_1, l_2)}} (n_1 l_1 - n_2 l_2) \\
& \left( \left( 1 - \frac{\mathcal{E}((n_1 - i, n_2 - j - \min(l_1, l_2), p_2), (n_1 - i, n_2 - j, p_1))}{\sqrt{m_1(n_1 - i, n_2 - j)} \sqrt{m_2(n_1 - i, n_2 - j - \min(l_1, l_2))}} \right) \alpha_{-i, -j - \min(l_1, l_2)}^{l_1, l_2} \right. \\
& + \left( 1 - \frac{\mathcal{E}((n_1 - i - \min(l_1, l_2), n_2 - j, p_2), (n_1 - i + (-1)^{\min(l_1, l_2)}, n_2 - j, p_1))}{\sqrt{m_1(n_1 - i + (-1)^{\min(l_1, l_2)}, n_2 - j)} \sqrt{m_2(n_1 - i - \min(l_1, l_2), n_2 - j)}} \right) \alpha_{i - (-1)^{\min(l_1, l_2)}, j}^{l_1, l_2} \\
& \left. + \left( 1 - \frac{\mathcal{E}((n_1 - i - \min(l_1, l_2), n_2 - j, p_2), (n_1 - i - \min(l_1, l_2), n_2 - j + 1, p_1))}{\sqrt{m_1(n_1 - i - \min(l_1, l_2), n_2 - j + 1)} \sqrt{m_2(n_1 - i - \min(l_1, l_2), n_2 - j)}} \right) \alpha_{i + \min(l_1, l_2), j-1}^{l_1, l_2} \right) \right) \\
& \times f_2(n_1 - i - \min(l_1, l_2), n_2 - j).
\end{aligned}$$

We estimate on the norms. Firstly, we treat the terms on  $f_2(n_1 + i, n_2 + j)$ , the sum is over  $I_{l_1, l_2}^0$ , we obtain:

$$\begin{aligned}
& \left\| \Lambda^\gamma(n_1, n_2) (M_{\{1, 0, 1, 0\}} f_2)(n_1, n_2) \right\|^2 \\
& \leq \sum_n \left| \Lambda^\gamma(n_1, n_2) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^0} (n_1 l_1 - n_2 l_2) \left( \sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \right. \right. \\
(41) \quad & \times (\varepsilon(n_1 + i, n_2 + j, p_2, p_1) - \varepsilon((n_1 + i, n_2 + j, p_2), (n_1 + i + 1, n_2 + j, p_1))) \\
& \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \sqrt{m_1(n_1 + i + 1, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} \\
(42) \quad & + \mathcal{E}(n_1 + i, n_2 + j, p_2, p_1) \left( m_1(n_1 + i + 1, n_2 + j) (\eta_2(n_1 + i, n_2 + j) - \eta_1(n_1 + i, n_2 + j)) \right. \\
(43) \quad & \left. + m_1(n_1 + i, n_2 + j) (\eta_1(n_1 + i + 1, n_2 + j) - \eta_2(n_1 + i, n_2 + j)) \right) \\
& \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \sqrt{m_1(n_1 + i + 1, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} \\
& \left. \times \frac{1}{m_1(n_1 + i, n_2 + j) m_2(n_1 + i, n_2 + j) + m_1(n_1 + i + 1, n_2 + j) m_2(n_1 + i, n_2 + j)} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \alpha_{-i-1,-j}^{l_1,l_2} f_2(n_1+i, n_2+j) \Big|^2 \\
(44) \quad & + \left| \Lambda^{\gamma'}(n_1, n_2) (n_1 l_1 - n_2 l_2) \left( \sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} \right. \right. \\
& \times (\varepsilon(n_1+i, n_2+j, p_2, p_1) - \varepsilon((n_1+i, n_2+j, p_2), (n_1+i, n_2+j+1, p_1))) \\
& \times \frac{1}{\sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} \sqrt{m_1(n_1+i, n_2+j+1)} \sqrt{m_2(n_1+i, n_2+j)}} \\
(45) \quad & + \mathcal{E}(n_1+i, n_2+j, p_2, p_1) \left( m_1(n_1+i, n_2+j+1) (\eta_2(n_1+i, n_2+j) - \eta_1(n_1+i, n_2+j)) \right. \\
(46) \quad & + m_1(n_1+i, n_2+j) (\eta_1(n_1+i, n_2+j+1) - \eta_2(n_1+i, n_2+j)) \Big) \\
& \times \frac{1}{\sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} \sqrt{m_1(n_1+i, n_2+j+1)} \sqrt{m_2(n_1+i, n_2+j)}} \\
& \times \frac{1}{m_1(n_1+i, n_2+j) m_2(n_1+i, n_2+j) + m_1(n_1+i, n_2+j+1) m_2(n_1+i, n_2+j)} \Big) \\
& \times \alpha_{-i,-j-1}^{l_1,l_2} f_2(n_1+i, n_2+j) \Big|^2 \\
(47) \quad & + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(l_1,l_2) \in \mathcal{S}} \sum_{(i,j) \in \mathcal{I}_{l_1,l_2}^0} (i l_1 - j l_2) \left( \left( \mathcal{E}(n_1+i, n_2+j, p_2, p_1) \right. \right. \right. \\
& \times (\eta_1(n_1+i, n_2+j) - \varepsilon(n_1+i, n_2+j, p_2, p_1)) \\
(48) \quad & + m_1(n_1+i, n_2+j) (\eta_2(n_1+i, n_2+j) - \varepsilon(n_1+i, n_2+j, p_2, p_1)) \Big) \\
& \times \frac{1}{\sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)}} \\
& \times \frac{1}{\sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} + \mathcal{E}((n_1+i, n_2+j, p_2), (n_1+i, n_2+j, p_1))} \Big) \\
& \left( \alpha_{i,j}^{l_1,l_2} + \alpha_{i-1,j}^{l_1,l_2} + \alpha_{i,j-1}^{l_1,l_2} \right) f_2(n_1+i, n_2+j) \Big|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(l_1,l_2) \in \mathcal{S}} \sum_{(i,j) \in \mathcal{I}_{l_1,l_2}^0} (i l_1 - j l_2) \left( \sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} \right. \right. \\
& \times (\varepsilon(n_1+i, n_2+j, p_1, p_2) - \varepsilon(n_1, n_2, p_1, p_2)) \\
& \times \frac{1}{\sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} \sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \\
(49) \quad & + \mathcal{E}(n_1+i, n_2+j, p_1, p_2) \left( m_1(n_1, n_2) (\eta_2(n_1, n_2) - \eta_1(n_1+i, n_2+j)) \right. \\
& + m_1(n_1+i, n_2+j) (\eta_1(n_1, n_2) - \eta_2(n_1+i, n_2+j)) \Big) \\
& \times \frac{1}{\sqrt{m_1(n_1+i, n_2+j)} \sqrt{m_2(n_1+i, n_2+j)} \sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \\
& \times \frac{1}{m_1(n_1, n_2) m_2(n_1, n_2) + m_1(n_1+i, n_2+j) m_2(n_1+i, n_2+j)} \Big) \\
& \times \alpha_{i,j}^{l_1,l_2} f_2(n_1+i, n_2+j) \Big|^2
\end{aligned}$$

$$\begin{aligned}
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^0} (il_1 - jl_2) \left( \sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \right. \right. \\
& \quad \times (\varepsilon(n_1 + i, n_2 + j, p_1, p_2) - \varepsilon((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \\
& \quad + \mathcal{E}(n_1 + i, n_2 + j, p_1, p_2) \left( m_1(n_1 + 1, n_2) (\eta_2(n_1, n_2) - \eta_1(n_1 + i, n_2 + j)) \right. \\
& \quad \left. + m_1(n_1 + i, n_2 + j) (\eta_1(n_1 + 1, n_2) - \eta_2(n_1 + i, n_2 + j)) \right) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \\
& \quad \left. \times \frac{1}{m_1(n_1 + 1, n_2) m_2(n_1, n_2) + m_1(n_1 + i, n_2 + j) m_2(n_1 + i, n_2 + j)} \right) \\
& \quad \times \alpha_{i-1, j}^{l_1, l_2} f_2(n_1 + i, n_2 + j) \Big|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^0} (il_1 - jl_2) \left( \sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \right. \right. \\
& \quad \times (\varepsilon(n_1 + i, n_2 + j, p_1, p_2) - \varepsilon((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \\
& \quad + \mathcal{E}(n_1 + i, n_2 + j, p_1, p_2) \left( m_1(n_1, n_2 + 1) (\eta_2(n_1, n_2) - \eta_1(n_1 + i, n_2 + j)) \right. \\
& \quad \left. + m_1(n_1 + i, n_2 + j) (\eta_1(n_1, n_2 + 1) - \eta_2(n_1 + i, n_2 + j)) \right) \\
& \quad \times \frac{1}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)} \sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \\
& \quad \left. \times \frac{1}{m_1(n_1, n_2 + 1) m_2(n_1, n_2) + m_1(n_1 + i, n_2 + j) m_2(n_1 + i, n_2 + j)} \right) \\
& \quad \times \alpha_{i, j-1}^{l_1, l_2} f_2(n_1 + i, n_2 + j) \Big|^2 \\
& + \sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^0} ((n_1 + i)l_1 - (n_2 + j)l_2) \right. \\
(50) \quad & \left( \left( \frac{\mathcal{E}(n_1 + i, n_2 + j, p_1, p_2)}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} - \frac{\mathcal{E}(n_1, n_2, p_1, p_2)}{\sqrt{m_1(n_1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i, j}^{l_1, l_2} \right. \\
& + \left( \frac{\mathcal{E}(n_1 + i, n_2 + j, p_1, p_2)}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1 + 1, n_2, p_2))}{\sqrt{m_1(n_1 + 1, n_2)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i-1, j}^{l_1, l_2} \\
& + \left. \left( \frac{\mathcal{E}(n_1 + i, n_2 + j, p_1, p_2)}{\sqrt{m_1(n_1 + i, n_2 + j)} \sqrt{m_2(n_1 + i, n_2 + j)}} - \frac{\mathcal{E}((n_1, n_2, p_1), (n_1, n_2 + 1, p_2))}{\sqrt{m_1(n_1, n_2 + 1)} \sqrt{m_2(n_1, n_2)}} \right) \alpha_{i, j-1}^{l_1, l_2} \right) \\
& \quad \times f_2(n_1 + i, n_2 + j) \Big|^2 \\
& \leq c_{\gamma'} \|f_2\|^2
\end{aligned}$$

We treat (50) as follows:

$$\sum_n \left| \Lambda^{\gamma'}(n_1, n_2) \sum_{(l_1, l_2) \in S} \sum_{(i, j) \in I_{l_1, l_2}^0} \sum_{h=0}^{|i|-1} (n_1 l_1 - n_2 l_2) \right.$$

$$\begin{aligned}
& \times \left( \sqrt{m_1(n_1 + \operatorname{sgn}(i)(h+1), n_2)} \sqrt{m_2(n_1 + \operatorname{sgn}(i)(h+1), n_2)} \right. \\
(51) \quad & \times \left. \left( \varepsilon(n_1 + \operatorname{sgn}(i)(h+1), n_2, p_1, p_2) - \varepsilon(n_1 + \operatorname{sgn}(i)h, n_2, p_1, p_2) \right) \right) \\
& \times \frac{1}{\sqrt{m_1(n_1 + \operatorname{sgn}(i)(h+1), n_2)} \sqrt{m_2(n_1 + \operatorname{sgn}(i)(h+1), n_2)}} \\
& \times \frac{1}{\sqrt{m_1(n_1 + \operatorname{sgn}(i)h, n_2)} \sqrt{m_2(n_1 + \operatorname{sgn}(i)h, n_2)}} \\
(52) \quad & + \mathcal{E}(n_1 + \operatorname{sgn}(i)(h+1), n_2, p_1, p_2) \left( m_1(n_1 + \operatorname{sgn}(i)h, n_2) \left( \eta_2(n_1 + \operatorname{sgn}(i)h, n_2) \right. \right. \\
& \quad \left. \left. \times -\eta_1(n_1 + \operatorname{sgn}(i)(h+1), n_2) \right) \right) \\
(53) \quad & + m_1(n_1 + \operatorname{sgn}(i)(h+1), n_2) \left( \eta_1(n_1 + \operatorname{sgn}(i)h, n_2) - \eta_2(n_1 + \operatorname{sgn}(i)(h+1), n_2) \right) \\
& \times \frac{1}{\sqrt{m_1(n_1 + \operatorname{sgn}(i)(h+1), n_2)} \sqrt{m_2(n_1 + \operatorname{sgn}(i)(h+1), n_2)}} \\
& \times \frac{1}{\sqrt{m_1(n_1 + \operatorname{sgn}(i)h, n_2)} \sqrt{m_2(n_1 + \operatorname{sgn}(i)h, n_2)}} \\
& \times \left( m_1(n_1 + \operatorname{sgn}(i)h, n_2) m_2(n_1 + \operatorname{sgn}(i)h, n_2) \right. \\
& \quad \left. + m_1(n_1 + \operatorname{sgn}(i)(h+1), n_2) m_2(n_1 + \operatorname{sgn}(i)(h+1), n_2) \right)^{(-1)} \\
& \times \alpha_{i,j}^{l_1, l_2} f_2(n_1 + i, n_2 + j) \Big|^2
\end{aligned}$$

and we estimate (49) like in the proof of Proposition 3.28. Here, we have used  $(H_{3,2})$  for (41) and (44),  $(H_{3,1})$  for (43) and (46),  $(H_{4,1})$  for (42) and (45),  $(H_{8,1})$  and  $(H_{8,2})$  for (47),  $(H_{5,2})$  for (48),  $(H_{3,2})$  for (51) and  $(H_{3,1})$  for (52) and (53). The other terms have the same treatment.

Then, there exists a constant  $c_{\gamma'} > 0$  such that

$$\left\| \Lambda^{\gamma'}(n_1, n_2) \left( M_{\{1,0,1,0\}} + M_{\{1,1,1,0\}} + M_{\{1,0,1,1\}} + M_{\{-1, -\min(l_1, l_2), -1, 0\}} \right) f_2(n_1, n_2) \right\|^2 \leq c_{\gamma'} \|f_2\|^2.$$

To conclude, we obtain (39) and (40).  $\square$

**3.3. Proof of the main result.** The main result of this section is Theorem 1.1. We begin by establishing the Mourre estimate in the case of perturbation. Since the ambient space is now  $\ell^2(\tilde{\mathcal{V}}, m; \mathbb{C})$ , we extend the operators acting on  $\ell^2(\tilde{\mathcal{V}}, m; \mathbb{C})$  into this space. First, let us begin with a remark.

**Remark 3.33.** Let  $A_{m,\varepsilon} := T_{1 \rightarrow m} A_H T_{1 \rightarrow m}^{-1}, \forall z \in \mathbb{C} \setminus \mathbb{R}, T_{1 \rightarrow m} (A_H - z)^{-1} T_{1 \rightarrow m}^{-1} = (A_{m,\varepsilon} - z)^{-1}$ . By functional calculus, this implies  $T_{1 \rightarrow m} e^{itA_H} T_{1 \rightarrow m}^{-1} = e^{itA_{m,\varepsilon}}$ . Moreover if  $S$  bounded in  $\ell^2(\tilde{\mathcal{V}}, 1; \mathbb{C})$ , then  $S \in \mathcal{C}^\alpha(A_H) \Leftrightarrow T_{1 \rightarrow m} S T_{1 \rightarrow m}^{-1} \in \mathcal{C}^\alpha(A_{m,\varepsilon})$ , where  $\alpha \in \{1; 2; 0, 1; 1, 1\}$  and for  $\alpha = 1$ , we also have  $[T_{1 \rightarrow m} S T_{1 \rightarrow m}^{-1}, iA_{m,\varepsilon}]_\circ = T_{1 \rightarrow m} [S, iA_H]_\circ T_{1 \rightarrow m}^{-1}$ .

Next, since  $V(Q)$  is an operator of multiplication, so  $V(Q) := T_{1 \rightarrow m} V(Q) T_{1 \rightarrow m}^{-1}$ . Consequently, we have

$$T_{1 \rightarrow m} (\tilde{\Delta} + V(Q)) T_{1 \rightarrow m}^{-1} = T_{1 \rightarrow m} \tilde{\Delta} T_{1 \rightarrow m}^{-1} + T_{1 \rightarrow m} V(Q) T_{1 \rightarrow m}^{-1} = \Delta_{m,\varepsilon} + V(Q) := H_{m,\varepsilon}.$$

**Theorem 3.34.** Let  $V : \tilde{\mathcal{V}} \rightarrow \mathbb{R}^2$ . We assume that  $(H_0), (H_1), (H_2), (H_3)$  and  $(H_4)$  hold true. Then  $\tilde{\Delta} + V(Q) \in \mathcal{C}^{1,1}(A_H)$ . Furthermore, for all compact interval  $\mathcal{I} \subset [-1, 1] \setminus \kappa(H_{m,\varepsilon})$ , there are  $c > 0$  and a compact operator  $\tilde{K}$  such that:

$$\begin{aligned}
(54) \quad & E_{\mathcal{I}}(\tilde{\Delta} + V(Q)) \left[ \tilde{\Delta} + V(Q), iA_H \right]_\circ E_{\mathcal{I}}(\tilde{\Delta} + V(Q)) \\
& \geq c E_{\mathcal{I}}(\tilde{\Delta} + V(Q)) + \tilde{K}.
\end{aligned}$$

Equivalently, we obtain:

$$(55) \quad E_{\mathcal{I}}(H_{m,\varepsilon}) [H_{m,\varepsilon}, iA_{m,\varepsilon}]_\circ E_{\mathcal{I}}(H_{m,\varepsilon}) \geq c E_{\mathcal{I}}(H_{m,\varepsilon}) + \widetilde{K_{m,\varepsilon}},$$

where  $\widetilde{K_{m,\varepsilon}} := T_{1 \rightarrow m} \tilde{K} T_{1 \rightarrow m}^{-1}$ .

*Proof.* By Proposition 3.32, Proposition 3.28 and Proposition 3.18, we have  $\tilde{\Delta} + V(Q) \in \mathcal{C}^{1,1}(A_H)$ . By hypothesis  $V(Q)$  is compact and by Proposition 3.23, we have  $(\Delta_H - \tilde{\Delta})$  is a compact operator. Then, by using Proposition 3.17 and by [AmBoGe, Theorem 7.2.9], we obtain (54). Using the transformation unitary  $T_{1 \rightarrow m}$ , Remark 3.33 proves (55).  $\square$

**Proof of Theorem 1.1:** Proposition 3.24 provides point 1. and Theorem 3.34 gives the points 2. To prove point 4., it suffices to consider  $s > \frac{1}{2}$ . Applying [AmBoGe, Proposition 7.5.6], we obtain:

$$\lim_{\rho \rightarrow 0^+} \sup_{\lambda \in [a,b]} \|\langle A_{m,\varepsilon} \rangle^{-s} (H_{m,\varepsilon} - \lambda - i\rho)^{-1} \langle A_{m,\varepsilon} \rangle^{-s}\| \text{ is finite.}$$

Furthermore, in the norm topology of bounded operators, the boundary values of the resolvent:

$$[a, b] \ni \lambda \mapsto \lim_{\rho \rightarrow 0^\pm} \langle A_{m,\varepsilon} \rangle^{-s} (H_{m,\varepsilon} - \lambda - i\rho)^{-1} \langle A_{m,\varepsilon} \rangle^{-s} \text{ exists and is continuous,}$$

where  $[a, b]$  is included in  $\mathbb{R} \setminus (\kappa \cup \sigma_p(H_{m,\varepsilon}))$ . In particular, this proves Point 3. By Remark 3.12, there exists  $c > 0$  such that:

$$\|\langle H_{m,\varepsilon} \rangle^s f\| \leq c \|\Lambda^s(Q) f\|,$$

for all  $f \in \mathcal{D}(\Lambda^s(Q))$ . We conclude that:

$$\lim_{\rho \rightarrow 0^+} \sup_{\lambda \in [a,b]} \|\Lambda^{-s}(Q) (H_{m,\varepsilon} - \lambda - i\rho)^{-1} \Lambda^{-s}(Q)\| \text{ is finite.}$$

The point 5. follows directly as a consequence of 4.  $\square$

#### 4. APPENDIX

The appendix presents the proof of Lemma 3.31, using structured tables. First we recall (17), (21), (22) and (23). The highlighted cells correspond to the pairs  $(i, j)$  such that  $\alpha_{i,j}^{l_1, l_2} = 0$ .

1. The following tables provide points 1., 2. and 3.. The cells highlighted in same color other than pink, contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{l_1, l_2} = 0$  and satisfying one of the following properties:

$$(56) \quad \text{If } (-h, -k-1) \in I_{l_1, l_2}^5, \text{ then } (h-1, k) \in I_{l_1, l_2}^1 \text{ or } (h, k+1) \in I_{l_1, l_2}^2 \\ \text{or } (-h - \min(l_1, l_2), -k) \in I_{l_1, l_2}^{4-\min(l_1, l_2)}.$$

$$(57) \quad \text{If } (-h-1 + \min(l_1, l_2), -k) \in I_{l_1, l_2}^{4-\min(l_1, l_2)}, \text{ then } (h-1, k) \in I_{l_1, l_2}^1 \text{ or } (h, k-1) \in I_{l_1, l_2}^2 \\ \text{or } (-h - \min(l_1, l_2), -k) \in I_{l_1, l_2}^{3+\min(l_1, l_2)}.$$

To sum up, (56) establishes Point 1., (57) proves Point 2. and Point 3. follows directly from (56) and (57).

Note that the colors of one table are independent of the colors an other table.

TABLE 1.  $R_{0,1}$

In this case, we have  $\min(l_1, l_2) = \min(0, 1) = 0$ , then  $I^{7+\min(l_1, l_2)} = I^7$ . For example, here if we consider the lime cells, we have  $(h, k) = (-1, 0)$  therefore  $(-h-1 + \min(0, 1), -k) = (0, 0) \in I_{0,1}^4$ ,  $(h-1, k) = (-2, 0) \in I_{0,1}^1$ ,  $(h, k-1) = (-1, -1) \in I_{0,1}^2$  and  $(-h - \min(0, 1), -k) = (1, 0) \in I_{0,1}^3$ . For the yellow cells  $(h, k) = (2, -1)$ , we have  $(-h, -k-1) = (-2, 0) \in I_{0,1}^5$  and  $(-h - \min(0, 1), -k) = (-2, 1) \in I_{0,1}^3$ . For the blue cells  $(h, k) = (2-3)$ , we have  $(-h, -k-1) = (-2, 2) \in I_{0,1}^5$  and  $(h-1, k) \in I_{0,1}^1$ .

From this table we can easily see that:

$$\begin{aligned} I_{0,1}^1 &= \{(-2, 0), (1, -3), (0, 2), (1, 1), (1, -2), (0, -1), (1, 0)\}, \\ I_{0,1}^2 &= \{(0, 2), (1, 1), (-2, 2), (1, -2), (-1, 2), (-1, -1)\}, \\ I_{0,1}^3 &= \{(0, -3), (-2, 0), (1, -3), (-2, -1), (-2, 1), (-1, -2), (-1, 1), (2, 0)\}, \\ I_{0,1}^4 &= \{(0, 0), (0, -3), (1, -3), (-2, 1)\}, \\ I_{0,1}^5 &= \{(-2, 0), (-2, -1), (-2, 1), (-2, 2)\}, \\ I_{0,1}^6 &= \{(0, 2), (-2, 2), (-1, 2)\}, \\ I_{0,1}^7 &= \{(-2, 1)\}, \\ I_{0,1}^8 &= I_{0,1}^9 = \emptyset. \end{aligned}$$

$(i,j)$	$(i+1,j)$	$(i,j+1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$
(0,0)	(1,0)	(0,1)	(0,0)	(-1,0)	(0,-1)
(0,-3)	(1,-3)	(0,-2)	(0,3)	(-1,3)	(0,2)
(-2,0)	(-1,0)	(-2,1)	(2,0)	(1,0)	(2,-1)
(1,-3)	(2,-3)	(1,-2)	(-1,3)	(-2,3)	(-1,2)
(-2,-1)	(-1,-1)	(-2,0)	(2,1)	(1,1)	(2,0)
(0,2)	(1,2)	(0,3)	(0,-2)	(-1,-2)	(0,-3)
(1,1)	(2,1)	(1,2)	(-1,-1)	(-2,-1)	(-1,-2)
(-2,1)	(-1,1)	(-2,2)	(2,-1)	(1,-1)	(2,-2)
(-2,2)	(-1,2)	(-2,3)	(2,-2)	(1,-2)	(2,-3)
(-1,-2)	(0,-2)	(-1,-1)	(1,2)	(0,2)	(1,1)
(1,-2)	(2,-2)	(1,-1)	(-1,2)	(-2,2)	(-1,1)
(-1,2)	(0,2)	(-1,3)	(1,-2)	(0,-2)	(1,-3)
(0,-2)	(1,-2)	(0,-1)	(0,2)	(-1,2)	(0,1)
(-1,-1)	(0,-1)	(-1,0)	(1,1)	(0,1)	(1,0)
(-1,1)	(0,1)	(-1,2)	(1,-1)	(0,-1)	(1,-2)
(0,1)	(1,1)	(0,2)	(0,-1)	(-1,-1)	(0,-2)
(0,-1)	(1,-1)	(0,0)	(0,1)	(-1,1)	(0,0)
(1,0)	(2,0)	(1,1)	(-1,0)	(-2,0)	(-1,-1)

TABLE 2.  $R_{1,0}$ 

Here, we have  $\min(l_1, l_2) = \min(1, 0) = 0$ , hence  $I^{7+\min(l_1, l_2)} = I^7$ . From this table we can easy see that:

$$\begin{aligned}
I_{1,0}^1 &= \{(2, 0), (1, 1), (2, -2), (-2, 1), (2, -1), (-1, -1)\}, \\
I_{1,0}^2 &= \{(0, -2), (-3, 1), (2, 0), (1, 1), (-2, 1), (0, 1), (-1, 0)\}, \\
I_{1,0}^3 &= \{(-3, 0), (0, -2), (-3, 1), (-1, -2), (1, -2), (2, -2), (-2, 1), (0, 1), (1, -1)\}, \\
I_{1,0}^4 &= \{(0, -2), (-1, -2), (1, -2), (2, -2)\}, \\
I_{1,0}^5 &= \{(0, 0), (-3, 0), (-3, 1), (1, -2)\}, \\
I_{1,0}^6 &= \{(0, 1), (-3, 1), (1, 1), (-2, 1)\}, \\
I_{1,0}^7 &= \{(1, -2)\}, \\
I_{1,0}^8 &= \{(0, -2)\}, \\
I_{1,0}^9 &= \emptyset.
\end{aligned}$$

$(i,j)$	$(i+1,j)$	$(i,j+1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$
(0,0)	(1,0)	(0,1)	(0,0)	(-1,0)	(0,-1)
(-3,0)	(-2,0)	(-3,1)	(3,0)	(2,0)	(3,-1)
(0,-2)	(1,-2)	(0,-1)	(0,2)	(-1,2)	(0,1)
(-3,1)	(-2,1)	(-3,2)	(3,-1)	(2,-1)	(3,-2)
(-1,-2)	(0,-2)	(-1,-1)	(1,2)	(0,2)	(1,1)
(2,0)	(3,0)	(2,1)	(-2,0)	(-3,0)	(-2,-1)
(1,1)	(2,1)	(1,2)	(-1,-1)	(-2,-1)	(-1,-2)
(1,-2)	(2,-2)	(1,-1)	(-1,2)	(-2,2)	(-1,1)
(2,-2)	(3,-2)	(2,-1)	(-2,2)	(-3,2)	(-2,1)
(-2,-1)	(-1,-1)	(-2,0)	(2,1)	(1,1)	(2,0)
(-2,1)	(-1,1)	(-2,2)	(2,-1)	(1,-1)	(2,-2)
(0,1)	(1,1)	(0,2)	(0,-1)	(-1,-1)	(0,-2)
(2,-1)	(3,-1)	(2,0)	(-2,1)	(-3,1)	(-2,0)
(-2,0)	(-1,0)	(-2,1)	(2,0)	(1,0)	(2,-1)
(-1,-1)	(0,-1)	(-1,0)	(1,1)	(0,1)	(1,0)
(1,-1)	(2,-1)	(1,0)	(-1,1)	(-2,1)	(-1,0)
(1,0)	(2,0)	(1,1)	(-1,0)	(-2,0)	(-1,-1)
(-1,0)	(0,0)	(-1,1)	(1,0)	(0,0)	(1,-1)

TABLE 3.  $R_{0,0}$ 

Here, we have  $\min(l_1, l_2) = \min(0, 0) = 0$ , hence  $I^{7+\min(l_1, l_2)} = I^7$ . From this table we can easy see that:

$$\begin{aligned}
I_{0,0}^1 &= \{(2,0), (1,1), (2,-2), (2,-1), (0,2), (2,-3)\}, \\
I_{0,0}^2 &= \{(2,0), (1,1), (0,2), (-1,2), (-2,2), (-3,2)\}, \\
I_{0,0}^3 &= \{(-3,0), (-3,1), (-1,-2), (-2,-1), (0,-3), (1,-3), (2,-3), (-3,2)\}, \\
I_{0,0}^4 &= \{(0,-3), (1,-3), (2,-3)\}, \\
I_{0,0}^5 &= \{(-3,0), (-3,1), (-3,2)\}, \\
I_{0,0}^6 &= \{(0,2), (-1,2), (-2,2), (3,-2)\}, \\
I_{0,0}^7 &= I_{0,0}^8 = I_{0,0}^9 = \emptyset.
\end{aligned}$$

(i,j)	(i+1,j)	(i,j+1)	(-i,-j)	(-i-1,-j)	(-i,-j-1)
(0,0)	(1,0)	(0,1)	(0,0)	(-1,0)	(0,-1)
(-3,0)	(-2,0)	(-3,1)	(3,0)	(2,0)	(3,-1)
(0,-2)	(1,-2)	(0,-2)	(0,2)	(-1,2)	(0,1)
(-3,1)	(-2,1)	(-3,2)	(3,-1)	(2,-1)	(3,-2)
(-1,-2)	(0,-2)	(-1,-1)	(1,2)	(0,2)	(1,1)
(2,0)	(3,0)	(2,1)	(-2,0)	(-3,0)	(-2,-1)
(1,1)	(2,1)	(1,2)	(-1,-1)	(-2,-1)	(-1,-2)
(1,-2)	(2,-2)	(1,-1)	(-1,2)	(-2,2)	(-1,1)
(2,-2)	(3,-2)	(2,-1)	(-2,2)	(-3,2)	(-2,1)
(-2,-1)	(-1,-1)	(-2,0)	(2,1)	(1,1)	(2,0)
(-2,1)	(-1,1)	(-2,2)	(2,-1)	(1,-1)	(2,-2)
(0,1)	(1,1)	(0,2)	(0,-1)	(-1,-1)	(0,-2)
(2,-1)	(3,-1)	(2,0)	(-2,1)	(-3,1)	(-2,0)
(-2,0)	(-1,0)	(-2,1)	(2,0)	(1,0)	(2,-1)
(-1,-1)	(0,-1)	(-1,0)	(1,1)	(0,1)	(1,0)
(1,-1)	(2,-1)	(1,0)	(-1,1)	(-2,1)	(-1,0)
(1,0)	(2,0)	(1,1)	(-1,0)	(-2,0)	(-1,-1)
(0,2)	(1,2)	(0,3)	(0,-2)	(-1,-2)	(0,-3)
(0,-3)	(1,-3)	(0,-2)	(0,3)	(-1,3)	(0,2)
(-1,1)	(0,1)	(-1,2)	(1,-1)	(0,-1)	(1,-2)
(-1,2)	(0,2)	(-1,3)	(1,-2)	(0,-2)	(1,-3)
(1,-3)	(2,-3)	(1,-2)	(-1,3)	(-2,3)	(-1,2)
(-2,2)	(-1,2)	(-2,3)	(2,-2)	(1,-2)	(2,-3)
(2,-3)	(3,-3)	(2,-2)	(-2,3)	(-3,3)	(-2,2)
(-3,2)	(-2,2)	(-3,3)	(3,-2)	(2,-2)	(3,-3)
(0,-1)	(1,-1)	(0,0)	(0,1)	(-1,1)	(0,0)
(-1,0)	(0,0)	(-1,1)	(1,0)	(0,0)	(1,-1)

TABLE 4.  $R_{1,1}$ 

In this table, we have  $\min(l_1, l_2) = \min(1, 1) = 1$ , so  $I^{7+\min(l_1, l_2)} = I^7$ . From this table we can easy see that:

$$I_{1,1}^1 = \{(-1, 0), (0, 1), (-1, 2), (2, -2), (1, 0), (2, -3), (2, -1)\},$$

$$I_{1,1}^2 = \{(0, 1), (-1, 2), (-3, 2), (1, 0), (2, -1), (-2, 2), (0, -1)\},$$

$$I_{1,1}^3 = \{(-3, 1), (-3, 2), (0, -2), (2, -3), (1, -3), (2, 0)\},$$

$$I_{1,1}^4 = \{(-1, 0), (0, 1), (2, -3), (1, -3)\},$$

$$I_{1,1}^5 = \{(-3, 1), (-3, 2), (1, 0), (0, -1)\},$$

$$I_{1,1}^6 = \{(-1, 2), (-3, 2), (-2, 2)\},$$

$$I_{1,1}^8 = \{(0, 1)\},$$

$$I_{1,1}^7 = I_{1,1}^9 = \emptyset.$$

$(i,j)$	$(i+1,j)$	$(i,j+1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$
$(-1,0)$	$(0,0)$	$(-1,1)$	$(1,0)$	$(0,0)$	$(1,-1)$
$(0,1)$	$(1,1)$	$(0,2)$	$(0,-1)$	$(-1,-1)$	$(0,-2)$
$(-1,2)$	$(0,2)$	$(-1,3)$	$(1,-2)$	$(0,-2)$	$(1,-3)$
$(-3,1)$	$(-2,1)$	$(-3,2)$	$(3,-1)$	$(2,-1)$	$(3,-2)$
$(-3,2)$	$(-2,2)$	$(-3,3)$	$(3,-2)$	$(2,-2)$	$(3,-3)$
$(2,-2)$	$(3,-2)$	$(2,-1)$	$(-2,2)$	$(-3,2)$	$(-2,1)$
$(0,-2)$	$(1,-2)$	$(0,-1)$	$(0,2)$	$(-1,2)$	$(0,1)$
$(1,0)$	$(2,0)$	$(1,1)$	$(-1,0)$	$(-2,0)$	$(-1,-1)$
$(2,-3)$	$(3,-3)$	$(2,-2)$	$(-2,3)$	$(-3,3)$	$(-2,2)$
$(1,-3)$	$(2,-3)$	$(1,-2)$	$(-1,3)$	$(-2,3)$	$(-1,2)$
$(2,-1)$	$(3,-1)$	$(2,0)$	$(-2,1)$	$(-3,1)$	$(-2,0)$
$(2,0)$	$(-1,0)$	$(-2,1)$	$(2,0)$	$(1,0)$	$(2,-1)$
$(-2,2)$	$(-1,2)$	$(-2,3)$	$(2,-2)$	$(1,-2)$	$(2,-3)$
$(-1,1)$	$(0,1)$	$(-1,2)$	$(1,-1)$	$(0,-1)$	$(1,-2)$
$(-2,1)$	$(-1,1)$	$(-2,2)$	$(2,-1)$	$(1,-1)$	$(2,-2)$
$(1,-1)$	$(2,-1)$	$(1,0)$	$(-1,1)$	$(-2,1)$	$(-1,0)$
$(1,-2)$	$(2,-2)$	$(1,-1)$	$(-1,2)$	$(-2,2)$	$(-1,1)$
$(0,-1)$	$(1,-1)$	$(0,0)$	$(0,1)$	$(-1,1)$	$(0,0)$

2. The following four tables provide point 1. of the lemma. For all  $(l_1, l_2) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ , we define

$$S_{1,0}^{l_1, l_2} := \alpha_{i,j}^{l_1, l_2} + \alpha_{i-1,j}^{l_1, l_2} + \alpha_{i,j-1}^{l_1, l_2}$$

and

$$S_{2,0}^{l_1, l_2} := \alpha_{-i,-j}^{l_1, l_2} + \alpha_{-i-1,-j}^{l_1, l_2} + \alpha_{-i,-j-1}^{l_1, l_2}$$

TABLE 5. Property of  $R_{0,0}$  in  $I_{0,0}^0$  - computing  $S_{1,0}^{0,0}$  and  $S_{2,0}^{0,0}$ 

The highlighted cells are not included in  $I_{0,0}^0$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,0} = 0$ . Thanks to Table 3, we have:

$(i,j)$	$(i-1,j)$	$(i,j-1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$	$S_{1,0}^{0,0}$	$S_{2,0}^{0,0}$
(0, 0)	(-1, 0)	(0, -1)	(0, 0)	(-1, 0)	(0, -1)	12	12
(-3, 0)	(-4, 0)	(-3, -1)	(3, 0)	(2, 0)	(3, -1)	5	5
(0, -2)	(-1, -2)	(0, -3)	(0, 2)	(-1, 2)	(0, 1)	33	33
(-3, 1)	(-4, 1)	(-3, 0)	(3, -1)	(2, -1)	(3, -2)	12	12
(-1, -2)	(-2, -2)	(-1, -3)	(1, 2)	(0, 2)	(1, 1)	9	9
(2, 0)	(1, 0)	(2, -1)	(-2, 0)	(-3, 0)	(-2, -1)	33	33
(1, 1)	(0, 1)	(1, 0)	(-1, -1)	(-2, -1)	(-1, -2)	36	36
(1, -2)	(0, -2)	(1, -3)	(-1, 2)	(-2, 2)	(-1, 1)	51	51
(2, -2)	(1, -2)	(2, -3)	(-2, 2)	(-3, 2)	(-2, 1)	48	48
(-2, -1)	(-3, -1)	(-2, -2)	(2, 1)	(1, 1)	(2, 0)	9	9
(-2, 1)	(-3, 1)	(-2, 0)	(2, -1)	(1, -1)	(2, -2)	51	51
(0, 1)	(-1, 1)	(0, 0)	(0, -1)	(-1, -1)	(0, -2)	42	42
(2, -1)	(1, -1)	(2, -2)	(-2, 1)	(-3, 1)	(-2, 0)	51	51
(-2, 0)	(-3, 0)	(-2, -1)	(2, 0)	(1, 0)	(2, -1)	33	33
(-1, -1)	(-2, -1)	(-1, -2)	(1, 1)	(0, 1)	(1, 0)	36	36
(1, -1)	(0, -1)	(1, -2)	(-1, 1)	(-2, 1)	(-1, 0)	54	54
(1, 0)	(0, 0)	(1, -1)	(-1, 0)	(-2, 0)	(-1, -1)	42	42
(0, 2)	(-1, 2)	(0, 1)	(0, -2)	(-1, -2)	(0, -3)	33	33
(0, -3)	(-1, -3)	(0, -4)	(0, 3)	(-1, 3)	(0, 2)	5	5
(-1, 1)	(-2, 1)	(-1, 0)	(1, -1)	(0, -1)	(1, -2)	54	54
(-1, 2)	(-2, 2)	(-1, 1)	(1, -2)	(0, -2)	(1, -3)	51	51
(1, -3)	(0, -3)	(1, -4)	(-1, 3)	(-2, 3)	(-1, 2)	12	12
(-2, 2)	(-3, 2)	(-2, 1)	(2, -2)	(1, -2)	(2, -3)	48	48
(2, -3)	(1, -3)	(2, -4)	(-2, 3)	(-3, 3)	(-2, 2)	15	15
(-3, 2)	(-4, 2)	(-3, 1)	(3, -2)	(2, -2)	(3, -3)	15	15
(0, -1)	(-1, -1)	(0, -2)	(0, 1)	(-1, 1)	(0, 0)	42	42
(-1, 0)	(-2, 0)	(-1, -1)	(1, 0)	(0, 0)	(1, -1)	42	42

TABLE 6. Property of  $R_{0,1}$  in  $I_{0,1}^0$  - computing  $S_{1,0}^{0,1}$  and  $S_{2,0}^{0,1}$ 

The highlighted cells are not included in  $I_{0,1}^0$ , because they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,1} = 0$ .  
By Table 1, we obtain:

$(i,j)$	$(i-1,j)$	$(i,j-1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$	$S_{1,0}^{0,1}$	$S_{2,0}^{0,1}$
(0, 0)	(-1, 0)	(0, -1)	(0, 0)	(-1, 0)	(0, -1)	0	0
(0, -3)	(-1, -3)	(0, -4)	(0, 3)	(-1, 3)	(0, 2)	1	-1
(-2, 0)	(-3, 0)	(-2, -1)	(2, 0)	(1, 0)	(2, -1)	2	-2
(1, -3)	(0, -3)	(1, -4)	(-1, 3)	(-2, 3)	(-1, 2)	2	-2
(-2, -1)	(-3, -1)	(-2, -2)	(2, 1)	(1, 1)	(2, 0)	1	-1
(0, 2)	(-1, 2)	(0, 1)	(0, -2)	(-1, -2)	(0, -3)	-8	8
(1, 1)	(0, 1)	(1, 0)	(-1, -1)	(-2, -1)	(-1, -2)	-8	8
(-2, 1)	(-3, 1)	(-2, 0)	(2, -1)	(1, -1)	(2, -2)	0	0
(-2, 2)	(-3, 2)	(-2, 1)	(2, -2)	(1, -2)	(2, -3)	-2	2
(-1, -2)	(-2, -2)	(-1, -3)	(1, 2)	(0, 2)	(1, 1)	2	-2
(1, -2)	(0, -2)	(1, -3)	(-1, 2)	(-2, 2)	(-1, 1)	8	-8
(-1, 2)	(-2, 2)	(-1, 1)	(1, -2)	(0, -2)	(1, -3)	-8	8
(1, 0)	(0, 0)	(1, -1)	(-1, 0)	(-2, 0)	(-1, -1)	-6	6
(0, -2)	(-1, -2)	(0, -3)	(0, 2)	(-1, 2)	(0, 1)	8	-8
(-1, -1)	(-2, -1)	(-1, -2)	(1, 1)	(0, 1)	(1, 0)	8	-8
(-1, 1)	(-2, 1)	(-1, 0)	(1, -1)	(0, -1)	(1, -2)	-6	6
(0, 1)	(-1, 1)	(0, 0)	(0, -1)	(-1, -1)	(0, -2)	-14	14
(0, -1)	(-1, -1)	(0, -2)	(0, 1)	(-1, 1)	(0, 0)	14	-14

TABLE 7. Property of  $R_{1,0}$  in  $I_{1,0}^0$  - computing  $S_{1,0}^{1,0}$  and  $S_{2,0}^{1,0}$ 

The highlighted cells are not included in  $I_{1,0}^0$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,0} = 0$ . Using Table 2, we get:

$(i,j)$	$(i-1,j)$	$(i,j-1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$	$S_{1,0}^{1,0}$	$S_{2,0}^{1,0}$
(0,0)	(-1,0)	(0,-1)	(0,0)	(-1,0)	(0,-1)	0	0
(-3,0)	(-4,0)	(-3,-1)	(3,0)	(2,0)	(3,-1)	1	-1
(0,-2)	(-1,-2)	(0,-3)	(0,2)	(-1,2)	(0,1)	2	-2
(-3,1)	(-4,1)	(-3,0)	(3,-1)	(2,-1)	(3,-2)	2	-2
(-1,-2)	(-2,-2)	(-1,-3)	(1,2)	(0,2)	(1,1)	1	-1
(2,0)	(1,0)	(2,-1)	(-2,0)	(-3,0)	(-2,-1)	-8	8
(1,1)	(0,1)	(1,0)	(-1,-1)	(-2,-1)	(-1,-2)	-8	8
(1,-2)	(0,-2)	(1,-3)	(-1,2)	(-2,2)	(-1,1)	0	0
(2,-2)	(1,-2)	(2,-3)	(-2,2)	(-3,2)	(-2,1)	-2	2
(-2,-1)	(-3,-1)	(-2,-2)	(2,1)	(1,1)	(2,0)	2	-2
(-2,1)	(-3,1)	(-2,0)	(2,-1)	(1,-1)	(2,-2)	8	-8
(0,1)	(-1,1)	(0,0)	(0,-1)	(-1,-1)	(0,-2)	-6	6
(2,-1)	(1,-1)	(2,-2)	(-2,1)	(-3,1)	(-2,0)	-8	8
(-2,0)	(-3,0)	(-2,-1)	(2,0)	(1,0)	(2,-1)	8	-8
(-1,-1)	(-2,-1)	(-1,-2)	(1,1)	(0,1)	(1,0)	8	-8
(1,-1)	(0,-1)	(1,-2)	(-1,1)	(-2,1)	(-1,0)	-6	6
(1,0)	(0,0)	(1,-1)	(-1,0)	(-2,0)	(-1,-1)	-14	14
(-1,0)	(-2,0)	(-1,-1)	(1,0)	(0,0)	(1,-1)	14	-14

TABLE 8. Property of  $R_{1,1}$  in  $I_{1,1}^0$  - computing  $S_{1,0}^{1,1}$  and  $S_{2,0}^{1,1}$ 

The highlighted cells are not included in  $I_{1,1}^0$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,1} = 0$ . From Table 4, we have:

$(i,j)$	$(i-1,j)$	$(i,j-1)$	$(-i,-j)$	$(-i-1,-j)$	$(-i,-j-1)$	$S_{1,0}^{1,1}$	$S_{2,0}^{1,1}$
(-1,0)	(-2,0)	(-1,-1)	(1,0)	(0,0)	(1,-1)	6	-6
(0,1)	(-1,1)	(0,0)	(0,-1)	(-1,-1)	(0,-2)	6	-6
(-1,2)	(-2,2)	(-1,1)	(1,-2)	(0,-2)	(1,-3)	8	-8
(-3,1)	(-4,1)	(-3,0)	(3,-1)	(2,-1)	(3,-2)	1	-1
(-3,2)	(-4,2)	(-3,1)	(3,-2)	(2,-2)	(3,-3)	2	-2
(2,-2)	(1,-2)	(2,-3)	(-2,2)	(-3,2)	(-2,1)	-8	8
(0,-2)	(-1,-2)	(0,-3)	(0,2)	(-1,2)	(0,1)	-2	2
(1,0)	(0,0)	(1,-1)	(-1,0)	(-2,0)	(-1,-1)	-6	6
(2,-3)	(1,-3)	(2,-4)	(-2,3)	(-3,3)	(-2,2)	-2	2
(1,-3)	(0,-3)	(1,-4)	(-1,3)	(-2,3)	(-1,2)	-1	1
(2,-1)	(1,-1)	(2,-2)	(-2,1)	(-3,1)	(-2,0)	-8	8
(-2,0)	(-3,0)	(-2,-1)	(2,0)	(1,0)	(2,-1)	2	-2
(-2,2)	(-3,2)	(-2,1)	(2,-2)	(1,-2)	(2,-3)	8	-8
(-1,1)	(-2,1)	(-1,0)	(1,-1)	(0,-1)	(1,-2)	14	-14
(-2,1)	(-3,1)	(-2,0)	(2,-1)	(1,-1)	(2,-2)	8	-8
(1,-1)	(0,-1)	(1,-2)	(-1,1)	(-2,1)	(-1,0)	-14	14
(1,-2)	(0,-2)	(1,-3)	(-1,2)	(-2,2)	(-1,1)	-8	8
(0,-1)	(-1,-1)	(0,-2)	(0,1)	(-1,1)	(0,0)	-6	6

3. Point 2. of the Lemma is shown in the following four tables, where

$$S_{1,1}^{l_1,l_2} := \alpha_{i,j}^{l_1,l_2} + \alpha_{i+1,j}^{l_1,l_2} + \alpha_{i+1,j-1}^{l_1,l_2}$$

and

$$S_{2,1}^{l_1,l_2} := \alpha_{-i-1,-j}^{l_1,l_2} + \alpha_{-i-2,-j}^{l_1,l_2} + \alpha_{-i-1,-j-1}^{l_1,l_2},$$

for all  $(l_1, l_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$ .

TABLE 9. Property of  $R_{0,0}$  in  $I_{0,0}^1$  - computing  $S_{1,1}^{0,0}$  and  $S_{2,1}^{0,0}$ 

The highlighted cells are not included in  $I_{0,0}^1$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,0} = 0$ . By Table 3, we own:

$(i,j)$	$(i+1,j)$	$(i+1,j-1)$	$(-i-1,-j)$	$(-i-2,-j)$	$(-i-1,-j-1)$	$S_{1,1}^{0,0}$	$S_{2,1}^{0,0}$
(2,0)	(3,0)	(3,-1)	(-3,0)	(-4,0)	(-3,-1)	5	5
(1,1)	(2,1)	(2,0)	(-2,-1)	(-3,-1)	(-2,-2)	9	9
(2,-2)	(3,-2)	(3,-3)	(-3,2)	(-4,2)	(-3,1)	15	15
(2,-1)	(3,-1)	(3,-2)	(-3,1)	(-4,1)	(-3,0)	12	12
(0,2)	(1,2)	(1,1)	(-1,-2)	(-2,-2)	(-1,-3)	9	9
(2,-3)	(3,-3)	(3,-4)	(-3,3)	(-4,3)	(-3,2)	8	8

TABLE 10. Property of  $R_{1,0}$  in  $I_{1,0}^1$  - computing  $S_{1,1}^{0,1}$  and  $S_{2,1}^{0,1}$ 

The highlighted cells are not included in  $I_{1,0}^1$ , because they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,0} = 0$ . From Table 2, we obtain:

$(i,j)$	$(i+1,j)$	$(i+1,j-1)$	$(-i-1,-j)$	$(-i-2,-j)$	$(-i-1,-j-1)$	$S_{1,1}^{0,1}$	$S_{2,1}^{0,1}$
$(-2,0)$	$(-1,0)$	$(-1,-1)$	$(1,0)$	$(0,0)$	$(1,-1)$	6	-6
$(1,-3)$	$(2,-3)$	$(2,-4)$	$(-2,3)$	$(-3,3)$	$(-2,2)$	1	-1
$(0,2)$	$(1,2)$	$(1,1)$	$(-1,-2)$	$(-2,-2)$	$(-1,-3)$	-2	2
$(1,1)$	$(2,1)$	$(2,0)$	$(-2,-1)$	$(-3,-1)$	$(-2,-2)$	-1	1
$(1,-2)$	$(2,-2)$	$(2,-3)$	$(-2,2)$	$(-3,2)$	$(-2,-3)$	2	-2
$(0,-1)$	$(1,-1)$	$(1,-2)$	$(-1,1)$	$(-2,1)$	$(-1,0)$	6	-6
$(1,0)$	$(2,0)$	$(2,-1)$	$(-2,0)$	$(-3,0)$	$(-2,-1)$	-2	2

TABLE 11. Property of  $R_{1,1}$  in  $I_{0,1}^1$  - computing  $S_{1,1}^{1,0}$  and  $S_{2,1}^{1,0}$ 

The highlighted are not included in  $I_{0,1}^1$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,1} = 0$ . Thanks to Table 4, we obtain:

$(i,j)$	$(i+1,j)$	$(i+1,j-1)$	$(-i-1,-j)$	$(-i-2,-j)$	$(-i-1,-j-1)$	$S_{1,1}^{1,0}$	$S_{2,1}^{1,0}$
$(2,-3)$	$(3,-3)$	$(3,-4)$	$(-3,3)$	$(-4,3)$	$(-3,2)$	-1	1
$(-1,0)$	$(0,0)$	$(0,-1)$	$(0,0)$	$(-1,0)$	$(0,-1)$	0	0
$(0,1)$	$(1,1)$	$(1,0)$	$(-1,-1)$	$(-2,-1)$	$(1,-2)$	0	0
$(-1,2)$	$(0,2)$	$(0,1)$	$(0,-2)$	$(-1,-2)$	$(0,-3)$	2	-2
$(2,-2)$	$(3,-2)$	$(3,-3)$	$(-3,2)$	$(-4,2)$	$(-3,1)$	-2	2
$(1,0)$	$(2,0)$	$(2,-1)$	$(-2,0)$	$(-3,0)$	$(-2,-1)$	-2	2
$(2,-3)$	$(3,-3)$	$(3,-4)$	$(-3,3)$	$(-4,3)$	$(-3,2)$	-1	1

TABLE 12. Property of  $R_{0,1}$  in  $I_{1,1}^1$  - computing  $S_{1,1}^{1,1}$  and  $S_{2,1}^{1,1}$ 

The highlighted cells are not included in  $I_{1,1}^1$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,1} = 0$ . Using Table 1, we get:

$(i,j)$	$(i+1,j)$	$(i+1,j-1)$	$(-i-1,-j)$	$(-i-2,-j)$	$(-i-1,-j-1)$	$S_{1,1}^{1,1}$	$S_{2,1}^{1,1}$
$(0,-2)$	$(0,-1)$	$(-1,-1)$	$(0,1)$	$(0,0)$	$(-1,1)$	6	-6
$(-3,1)$	$(-3,2)$	$(-4,2)$	$(3,-2)$	$(3,-3)$	$(2,-2)$	1	-1
$(2,0)$	$(2,1)$	$(1,1)$	$(-2,-1)$	$(-2,-2)$	$(-3,-1)$	-2	2
$(1,1)$	$(1,2)$	$(0,2)$	$(-1,-2)$	$(-1,-3)$	$(-2,-2)$	-1	1
$(-2,1)$	$(-2,2)$	$(-3,2)$	$(2,-2)$	$(2,-3)$	$(1,-2)$	2	-2
$(0,1)$	$(0,2)$	$(-1,2)$	$(0,-2)$	$(0,-3)$	$(-1,-2)$	-2	2
$(-1,0)$	$(-1,1)$	$(-2,1)$	$(1,1)$	$(1,-2)$	$(0,-1)$	6	-6

4. The following tables provide the proof of point 3.. Let

$$S_{1,2}^{l_1,l_2} := \alpha_{i,j}^{l_1,l_2} + \alpha_{i,j+1}^{l_1,l_2} + \alpha_{i-1,j+1}^{l_1,l_2}$$

and

$$S_{2,2}^{l_1,l_2} := \alpha_{-i,-j-1}^{l_1,l_2} + \alpha_{-i-1,-j-1}^{l_1,l_2} + \alpha_{-i,-j-2}^{l_1,l_2}$$

for all  $(l_1, l_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$ .

TABLE 13. Property of  $R_{0,0}$  in  $I_{0,0}^2$  - computing  $S_{1,2}^{0,0}$  and  $S_{2,2}^{0,0}$ 

The highlighted cells are not included in  $I_{0,0}^2$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,0} = 0$ . By Table 3, we have:

$(i,j)$	$(i,j+1)$	$(i-1,j+1)$	$(-i,-j-1)$	$(-i,-j-2)$	$(-i-1,-j-1)$	$S_{1,2}^{0,0}$	$S_{2,2}^{0,0}$
(0,-2)	(0,-1)	(-1,-1)	(0,1)	(0,0)	(-1,1)	42	42
(2,0)	(2,1)	(1,1)	(-2,-1)	(-2,-2)	(-3,-1)	9	9
(1,1)	(1,2)	(0,2)	(-1,-2)	(-1,-3)	(-2,-2)	9	9
(-1,-1)	(-1,0)	(-2,0)	(1,0)	(1,-1)	(0,0)	42	42
(0,2)	(0,3)	(-1,3)	(0,-3)	(0,-4)	(-1,-3)	5	5
(-1,2)	(-1,3)	(-2,3)	(1,-3)	(1,-4)	(0,-3)	12	12
(-2,2)	(-2,3)	(-3,3)	(2,-3)	(2,-4)	(1,-3)	15	15
(-3,2)	(-3,3)	(-4,3)	(3,-3)	(3,-4)	(2,-3)	8	8

TABLE 14. Property of  $R_{1,0}$  in  $I_{1,0}^2$  - computing  $S_{1,2}^{1,0}$  and  $S_{2,2}^{1,0}$ 

The highlighted cells are not included in  $I_{1,0}^2$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,0} = 0$ . From Table 2, we get:

$(i,j)$	$(i,j+1)$	$(i-1,j+1)$	$(-i,-j-1)$	$(-i,-j-2)$	$(-i-1,-j-1)$	$S_{1,2}^{1,0}$	$S_{2,2}^{1,0}$
(0,-2)	(0,-1)	(-1,-1)	(0,1)	(0,0)	(-1,1)	6	-6
(-3,1)	(-3,2)	(-4,2)	(3,-2)	(3,-3)	(2,-2)	1	-1
(2,0)	(2,1)	(1,1)	(-2,-1)	(-2,-2)	(-3,-1)	-2	2
(1,1)	(1,2)	(0,2)	(-1,-2)	(-1,-3)	(-2,-2)	-1	1
(-2,1)	(-2,2)	(-3,2)	(2,-2)	(2,-3)	(1,-2)	2	-2
(0,1)	(0,2)	(-1,2)	(0,-2)	(0,-3)	(-1,-2)	-2	2
(-1,0)	(-1,1)	(-2,1)	(1,1)	(1,-2)	(0,-1)	6	-6

TABLE 15. Property of  $R_{0,1}$  in  $I_{0,1}^2$  - computing  $S_{1,2}^{0,1}$  and  $S_{2,2}^{0,1}$ 

The highlighted cells are not included in  $I_{0,1}^2$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,1} = 0$ . Thanks to Table 1, we obtain:

$(i,j)$	$(i,j+1)$	$(i-1,j+1)$	$(-i,-j-1)$	$(-i,-j-2)$	$(-i-1,-j-1)$	$S_{1,2}^{0,1}$	$S_{2,2}^{0,1}$
(-2,2)	(-2,3)	(-3,3)	(2,-3)	(2,-4)	(1,-3)	-1	1
(-1,2)	(-1,3)	(-2,3)	(1,-3)	(1,-4)	(0,-3)	-2	2
(-1,-1)	(-1,0)	(-2,0)	(1,0)	(1,-1)	(0,0)	1	-1
(0,2)	(0,3)	(-1,3)	(0,-3)	(0,-4)	(-1,-3)	6	-6
(1,1)	(1,2)	(0,2)	(-1,-2)	(-1,-3)	(-2,-2)	-2	2
(1,-2)	(1,-1)	(0,-1)	(-1,1)	(-1,0)	(-2,1)	6	-6

TABLE 16. Property of  $R_{1,1}$  in  $I_{1,1}^2$  - computing  $S_{1,2}^{1,1}$  and  $S_{2,2}^{1,1}$ 

The highlighted cells are not included in  $I_{1,1}^2$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,1} = 0$ . From Table 4, we have:

(i,j)	(i,j+1)	(i-1,j+1)	(-i,-j-1)	(-i,-j-2)	(-i-1,-j-1)	$S_{1,2}^{1,1}$	$S_{2,2}^{1,1}$
(0,1)	(0,2)	(-1,2)	(0,-2)	(0,-3)	(-1,-2)	2	-2
(-1,2)	(-1,3)	(-2,3)	(1,-3)	(1,-4)	(0,-3)	1	-1
(-3,2)	(-3,3)	(-4,3)	(3,-3)	(3,-4)	(2,-3)	1	-1
(1,0)	(1,1)	(0,1)	(-1,-1)	(-1,-2)	(-2,-1)	0	0
(2,-1)	(2,0)	(1,0)	(-2,0)	(-2,-1)	(-3,0)	-2	2
(-2,2)	(-2,3)	(-3,3)	(2,-3)	(2,-4)	(1,-3)	2	-2
(0,-1)	(0,0)	(-1,0)	(0,0)	(0,-1)	(-1,0)	0	0

5. Point 4. is demonstrated in the following tables. Let

$$S_{1,4}^{l_1,l_2} := \alpha_{i,j}^{l_1,l_2} + \alpha_{i+1,j}^{l_1,l_2} + \alpha_{i-1,j+1}^{l_1,l_2}$$

and

$$S_{2,4}^{l_1,l_2} := \alpha_{-i-1,-j}^{l_1,l_2} + \alpha_{-i-2,-j}^{l_1,l_2} + \alpha_{-i-1,-j-1}^{l_1,l_2},$$

where  $(l_1, l_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$ .

TABLE 17. Property of  $R_{0,0}$  in  $I_{0,0}^4$  - computing  $S_{1,4}^{0,0}$  and  $S_{2,4}^{0,0}$

The highlighted cells are not included in  $I_{0,0}^4$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,0} = 0$ . By Table 3, we own:

(i,j)	(i+1,j)	(i+1,j-1)	(-i-1,-j)	(-i-2,-j)	(-i-1,-j-1)	$S_{1,4}^{0,0}$	$S_{2,4}^{0,0}$
(0,-3)	(1,-3)	(1,-4)	(-1,3)	(-2,3)	(-1,2)	12	12
(1,-3)	(2,-3)	(2,-4)	(-2,-3)	(-3,3)	(-2,2)	-2	2
(2,-3)	(3,-3)	(3,-4)	(-3,3)	(-4,3)	(-3,2)	-1	1
(2,-1)	(3,-1)	(3,-2)	(-3,1)	(-4,1)	(-3,0)	-1	-1

TABLE 18. Property of  $R_{1,0}$  in  $I_{1,0}^4$  - computing  $S_{1,4}^{1,0}$  and  $S_{2,4}^{1,0}$

The highlighted cells are not included in  $I_{1,0}^4$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,0} = 0$ . From Table 2, we get:

(i,j)	(i+1,j)	(i+1,j-1)	(-i-1,-j)	(-i-2,-j)	(-i-1,-j-1)	$S_{1,4}^{1,0}$	$S_{2,4}^{1,0}$
(0,-2)	(1,-2)	(1,3)	(-1,2)	(-2,2)	(-1,1)	0	0
(-1,-2)	(0,-2)	(0,-3)	(0,2)	(-1,2)	(0,1)	2	-2
(1,-2)	(2,-2)	(2,-3)	(-2,2)	(-3,2)	(-2,1)	-2	2
(2,-2)	(3,-2)	(3,-3)	(-3,2)	(-4,2)	(-3,1)	-1	1

TABLE 19. Property of  $R_{0,1}$  in  $I_{0,1}^4$  - computing  $S_{1,4}^{0,1}$  and  $S_{2,4}^{0,1}$

The highlighted cells are not included in  $I_{0,1}^4$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{0,1} = 0$ . Using Table 1, we obtain:

(i,j)	(i+1,j)	(i+1,j-1)	(-i-1,-j)	(-i-2,-j)	(-i-1,-j-1)	$S_{1,4}^{0,1}$	$S_{2,4}^{0,1}$
(0,0)	(1,0)	(1,-1)	(-1,0)	(-2,0)	(-1,-1)	-14	14
(0,-3)	(1,-3)	(1,-4)	(-1,3)	(-2,3)	(-1,2)	2	-2
(1,-3)	(2,-3)	(2,-4)	(-2,3)	(-3,3)	(-2,2)	-1	1
(-2,1)	(-1,1)	(-1,0)	(1,-1)	(0,-1)	(1,-2)	6	-6

TABLE 20. Property of  $R_{1,1}$  in  $I_{1,1}^4$  - computing  $S_{1,4}^{1,1}$  and  $S_{2,4}^{1,1}$ 

The highlighted cells are not included in  $I_{1,1}^4$ , as they contain the pairs  $(h, k)$  such that  $\alpha_{h,k}^{1,1} = 0$ . From Table 4, we have:

$(i,j)$	$(i+1,j)$	$(i+1,j-1)$	$(-i-1,-j)$	$(-i-2,-j)$	$(-i-1,-j-1)$	$S_{1,4}^{1,1}$	$S_{2,4}^{1,1}$
$(-1,0)$	$(0,0)$	$(0,-1)$	$(0,0)$	$(-1,0)$	$(0,-1)$	0	0
$(0,1)$	$(1,1)$	$(1,0)$	$(-1,-1)$	$(-2,-1)$	$(-1,-2)$	0	0
$(2,-3)$	$(3,-3)$	$(3,-4)$	$(-3,3)$	$(-4,3)$	$(-3,2)$	-1	1
$(1,-3)$	$(2,-3)$	$(2,-4)$	$(-2,3)$	$(-3,3)$	$(-2,2)$	-2	2

## REFERENCES

- [AmBoGe] W.O. Amrein, A. Boutet de Monvel, and V. Georgescu,  *$C_0$ -groups, commutator methods and spectral theory of  $N$ -body hamiltonians*, Birkhäuser (1996), <https://doi.org/10.1007/978-3-0348-7762-6-9>.
- [AnIsMo] K. Ando, I. Isozaki, and M. Morioka, *Spectral properties of Schrödinger operators on perturbed Lattices*, *Annales Henri Poincaré* 17 (2016), 2103–2171, <https://doi.org/10.1007/s00023-015-0430-0>.
- [AnTo] C. Anné and N. Torki-Hamza: *The Gauss-Bonnet operator of an infinite graph*, *Anal. Math. Phys.*, **5**, No. 2, 137–159 (2015), <https://doi.org/10.1007/s13324-014-0090-0>.
- [AlFr] C. Allard and R. Froese: *A Mourre estimate for a Schrödinger operator on a binary tree*, *Rev. Math. Phys.* **12** (2000), no. 12, 1655–1667, <https://doi.org/10.1142/S0129055X00000575>.
- [AtBaDaEn] N. Athmouni, H. Baloudi, M. Damak and M. Ennaceur: *The magnetic discrete laplacian inferred from the Gauss-Bonnet operator and application*, *Annals of Functional Analysis* 12,33 (2021), <https://doi.org/10.1007/s43034-021-00119-8>.
- [AtDa] N. Athmouni and M. Damak, *On the non-propagation theorem and applications*, *Turkish Journal of Mathematics: Vol* (2013), <https://doi.org/10.3906/mat-1010-75>.
- [AtEnGo] N. Athmouni, M. Ennaceur, and S. Golénia, *Spectral analysis of the Laplacian acting on discrete cusps and funnels*, *Complex analysis and operator theory* (2021), <https://doi.org/10.1007/s11785-020-01053-8>.
- [AtEnGoJa] N. Athmouni, M. Ennaceur, S. Golénia, and A. Jadlaoui, *Limiting Absorption Principle for long-range perturbation in the discrete triangular lattice setting*, *arXiv: no. 2403.06578v2* (2024), <https://doi.org/10.48550/arXiv.2403.06578>.
- [BaGoJe] H. Baloudi, S. Golenia, and A. Jeribi: *The adjacency matrix and the discrete Laplacian acting on forms*, *Math Phys Anal Geom* (2019) 22: 9, <https://doi.org/10.1007/s11040-019-9301-0>.
- [BeJoLoJo] Bergh, Jöran; Löfström, Jörgen (1976), *Interpolation spaces. An introduction*, *Grundlehren der Mathematischen Wissenschaften*, vol. 223, Berlin-New York: Springer-Verlag, pp. x+207, ISBN 978-3-540-07875-3.
- [BoKeGoLiMü] M. Bonnefont, M. Keller, S. Golénia, S.Liu, and F. Münch: *Magnetic sparseness and Schrödinger operators on graphs*, *Annales Henri Poincaré* 21 (2020), 1489–1516, <https://doi.org/10.1007/s00023-020-00885-6>.
- [BoSa] A. Boutet de Monvel and J. Sahbani: *On the spectral properties of discrete Schrödinger operators: the multi-dimensional case*, *Rev. Math. Phys.* **11** (1999), no. 9, 1061–1078, [https://doi.org/10.1016/S0764-4442\(98\)80078-5](https://doi.org/10.1016/S0764-4442(98)80078-5).
- [BrKe] J. Breuer and M. Keller: *Spectral Analysis of Certain Spherically Homogeneous Graphs*, *Oper. Matrices* **7** (2013), no. 4, 825–847, [dx.doi.org/10.7153/oam-07-46](https://doi.org/10.7153/oam-07-46).
- [GeGéMø] V. Georgescu, C. Gérard, and J.S. Møller: *Commutators,  $C_0$ -semigroups and resolvent estimates*, *J. Funct. Anal.* **216** (2004), no. 2, 303–361, <https://doi.org/10.1016/j.jfa.2004.03.004>.
- [GeGo] V. Georgescu and S. Golénia: *Isometries, Fock spaces, and spectral analysis of Schrödinger operators on trees*, *J. Funct. Anal.* **227** (2005), no. 2, 389–429, <https://doi.org/10.1016/j.jfa.2005.01.001>.
- [Gé] C. Gérard: *A proof of the abstract limiting absorption principle by energy estimates*, *J. Funct. Anal.* **254** (2008), no. 11, 2707–2724, <https://doi.org/10.1016/j.jfa.2008.02.01>.
- [GéNi] C. Gérard and F. Nier: *The Mourre theory for analytically fibered operators*, *J. Funct. Anal.* **152** (1998), no. 1, 202–219, <https://doi.org/10.1006/JFAN.1997.3154>.
- [Go] S. Golénia: *On the absolute continuous spectrum of discrete operators* *Sémin. Congr.*, 32 Société Mathématique de France, Paris 2018, 87–145, <https://doi.org/10.1016/j.jmaa.2017.03.002>.
- [GoJe] S. Golénia and T. Jecko: *Weighted Mourre’s commutator theory, application to Schrödinger operators with oscillating potential*, *J. of Operator Theory.* **70** (2012), <https://doi.org/10.7900/jot.2011may06.1922>.
- [GoTr] S. Golénia and F. Truc: *The magnetic Laplacian acting on discrete cusp*, *Doc. Math.* **22** (2017), 1709–1727, <http://dx.doi.org/10.7900/jot.2011may06.1922>.
- [GoMa] S. Golénia and M. Mandich: *Propagation estimates for one commutator regularity*, *Integral Equations Operator Theory.* **90** (2018), no. 4, Paper No.47,40 pp, <https://doi.org/10.1007/s00020-018-2471-5>.
- [GoMo] S. Golénia and S. Moroianu: *Spectral analysis of magnetic Laplacians on conformally cusp manifolds*, *Ann. Henri Poincaré.* **9** (2008), no. 1, 131–179, <https://doi.org/10.1007/s00023-007-0353-5>.
- [GüKe] B. Güneysu and M. Keller: *Scattering the geometry of weighted graphs*, *Math. Phys. Anal. Geom.* **21** (3). 28 (2018), <https://doi.org/10.1007/s11040-018-9285-1>.
- [HiNo] Y. Higuchi and Y. Nomura: *Spectral structure of the Laplacian on a covering graph*, *European Journal of Combinatorics*, **30** (2009), 570–585, <https://doi.org/10.1016/j.ejc.2008.03.008>.

- [HiSh] Y. Higuchi and T. Shirai: *Weak Bloch property for discrete magnetic Schrödinger operators*, Nagoya Math. J. **161** (2001), 127–154, <https://doi.org/10.1017/S0027763000022157>.
- [KoMø] E. Korotyaev and J. Møller: *Weighted estimates for the Laplacian on the cube lattice*, Ark. Mat. **57**,(2019), 397–428, <https://doi.org/10.4310/ARKIV.2019.v57.n2.a8>.
- [KoSa] E. Korotyaev and N. Saburova: *Spectral estimates for Schrödinger operators on periodic discrete graphs* St. Petersburg Math. J. **30**, (2019), no. 4, 667–698, <https://doi.org/10.1090/spmj/1565>.
- [JeMoPe] A. Jensen, E. Mourre, and P. Perry: *Multiple commutator estimates and resolvent smoothness in quantum scattering theory*, Ann. Inst. H. Poincaré. **vol. 41** (1984), no. 2, 207–225, <https://doi.org/10.1090/S0002-9947-1985-0797050-9>.
- [MāRiTī] M. Măntoiu, S. Richard, and R. Tiedra de Aldecoa: *Spectral analysis for adjacency operators on graphs*, Ann. Henri Poincaré **8** (2007), no. 7, 1401–1423, <https://doi.org/10.1007/s00023-007-0339-3>.
- [Mo1] E. Mourre: *Absence of singular continuous spectrum for certain self-adjoint operators*. Comm. Math. Phys. **78** (1980), 391–408, <https://doi.org/10.1007/BF01942331>.
- [Mo2] E. Mourre: *Opérateurs conjugués et propriétés de propagation*, Comm. Math. Phys. **91** (1983), 279–300, <https://doi.org/10.1007/BF01211163>.
- [PaRi1] D. Parra and S. Richard: *Continuity of the spectra for families of magnetic operators on  $\mathbb{Z}^2$* , Anal. Math. phys. **6** (2016), 327–343, <https://doi.org/10.1007/s13324-015-0121-5>.
- [PaRi] D. Parra and S. Richard: *Spectral and scattering theory for Schrödinger operators on perturbed topological crystals*, Rev. Math. Phys. **30** (2018), no. 4, 1850009, 39 pp, <https://doi.org/10.1142/S0129055X18500095>.
- [ReSi] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, Tome I–IV: Analysis of operators* Academic Press, <https://doi.org/10.1088/0031-9112/23/12/045>.
- [RiTs] S. Richard and N. Tsuzu: *Spectral and scattering theory for topological crystals perturbed by infinitely many new edges*, Rev. Math. Phys. **34**, (2022), no. 4, 2250010, <https://doi.org/10.1142/S0129055X22500106>
- [Sa] J. Sahbani: *Mourre’s theory for some unbounded Jacobi matrices*, J. Approx. Theory. **135** (2005), no. 2, 233–244, <https://doi.org/10.1016/j.jat.2005.04.009>
- [Ta1] Y. Tadano: *Long-range scattering for discrete Schrödinger operators*, Ann. Henri Poincaré **20** (2019), 1439–1469, <https://doi.org/10.1007/s00023-019-00763>.
- [Ta2] Y. Tadano: *Long-range scattering theory for discrete Schrödinger operators on graphene*, J. Math. Phys. **60** (2019), 052–107, <http://doi.org/10.1063/1.5087013>.

<sup>1</sup>UNIVERSITÉ DE GAFSA, CAMPUS UNIVERSITAIRE, 2112, TUNISIE

*Email address:* [nassim.athmouni@fsgf.u-gafsa.tn](mailto:nassim.athmouni@fsgf.u-gafsa.tn)

<sup>2</sup>DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF HAIL, HA’IL 81451, SAUDI ARABIA

*Email address:* [mar.ennaceur@uoh.edu.sa](mailto:mar.ennaceur@uoh.edu.sa)

<sup>3</sup>UNIV. BORDEAUX, BORDEAUX INP, CNRS, IMB, UMR 5251, F-33400 TALENCE, FRANCE

*Email address:* [sylvain.golenia@math.u-bordeaux.fr](mailto:sylvain.golenia@math.u-bordeaux.fr)

<sup>4</sup>UNIVERSITÉ DE SFAX. ROUTE DE LA SOUKRA KM 4 - B.P. n° 802 - 3038 SFAX, TUNISIE

*Email address:* [amel.jadlaoui@yahoo.com](mailto:amel.jadlaoui@yahoo.com)