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Accurate semiclassical analysis of light propagation on tilted hyperplanes

Patrick GIOIA* VŨ NGỌC SAN†

June 4, 2025

Abstract

In the scalar light model given by Helmholtz' equation in $\mathbb{R}^{1+d}$, we consider the transformation of an initial scene (a hologram) in $\{0\} \times \mathbb{R}^d$ by an arbitrary affine transformation (which can be viewed as a propagation into a tilted hyperplane). In the high frequency regime, we use microlocal and semiclassical analysis to describe the propagator as a semiclassical Fourier integral operator, thus generalising the well-known Angular Spectrum formula from optics. We then prove new precise Egorov theorems, including subprincipal terms, which indicate how to take into account the propagation along rays of geometric optics.

1 Introduction

The initial motivation for this work has its roots in a technological challenge currently encountered by researchers in computational holography. The problem, described for instance in [12, 2], is to find a new phase space model for propagating holograms in virtual reality headsets. This turns out to involve sophisticated mathematical objects, including semiclassical Fourier integral operators. The goal of this paper is to present these objects and derive a new formula for accurately approximating propagated wave functions — such as holograms — in the high-frequency regime. In return, our mathematical analysis has direct applications to the numerical treatment of holograms: not only does it provide a new numerical scheme for coding and propagating holograms (see [14, 15]), but also it explains artefacts arising in the twisted angular spectrum method, which stem from the non-injectivity of the canonical transformation (see Theorem 5.7).

The mathematical setting is as follows. Let $\mathcal{U} \subset \mathbb{R}^{1+d}$, $d \geq 1$, be a domain contained in the half space $(-C, \infty) \times \mathbb{R}^d$ for some constant $C > 0$.

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For a given $k > 0$, let ψ be a non-trivial solution in $\mathcal{U}$ to the Helmholtz equation

$$\Delta\psi + k^2\psi = 0.$$

(Typically, ψ is the monochromatic light wave of frequency $kc/2\pi$ — where c is the speed of light in $\mathcal{U}$ — emitted from a source object located in the left half-space $\{(x_0, x_1, \dots, x_d); x_0 < -C\}$.) Our question is the following: let $\mathcal{G}$ be an affine transformation of $\mathbb{R}^{1+d}$; suppose we know only the initial hologram, *i.e.* the trace of ψ on the hyperplane $P_0 := \{x_0 = 0\}$ (assuming this trace is well-defined), then how to describe the trace of $\psi \circ \mathcal{G}$ on P_0 (the transformed hologram)? More precisely, how to describe the operator

$$U_{\mathcal{G}} : \psi|_{P_0} \mapsto \psi \circ \mathcal{G}|_{P_0} ?$$

(In the sequel we often call $U_{\mathcal{G}}$ the *propagator*.)

Two special cases are worth mentioning. First, if $\mathcal{G}$ just a translation along the x_0 axis: $\mathcal{G}(X) = X + \gamma$ for some $\gamma = (\gamma_0, 0, \dots, 0)$, then the above question translates into:

Given a hologram at position x_0 , how to compute the hologram at position $x_0 + \gamma_0$?

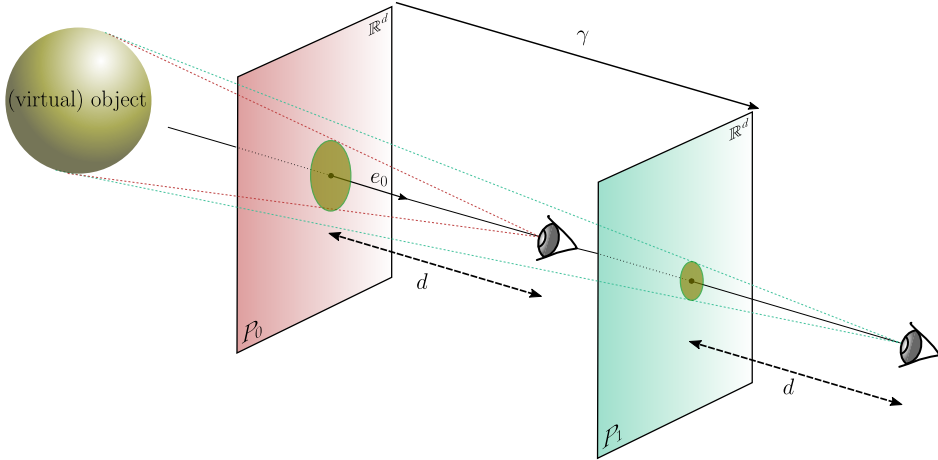


Figure 1 – Translating a hologram along the viewer axis. The initial hologram is the restriction of the light wave ψ emitted from the (virtual) object to the plane P_0 . The transformed hologram is the restriction of $\psi \circ \mathcal{G}$ to P_0 , which, when $\mathcal{G}(X) = X + \gamma$, is the same as the restriction of ψ to the translated plane $P_1 := \mathcal{G}(P_0)$. Each hologram is presented to the viewer, at a fixed distance d from the eye. When looking at the second hologram, the viewer sees the object at a distance increased by $\|\gamma\|$.

Thus, a viewer located on the right hand side, looking at the transformed hologram, will have the impression that the object has moved backward

by a distance γ_0 (Figure 1). This transformation (a translation along the viewer axis) is well known in optics, and can be efficiently computed using the *angular spectrum method* (see Section 2).

The second interesting situation is the case of a Euclidean rotation $\mathcal{G} \in \text{SO}(1 + d)$. Of course, an internal rotation (within the hyperplane P_0) is trivial to obtain: it suffices to rotate the initial hologram. However, when the hyperplane itself is rotated into a different hyperplane $\mathcal{G}P_0$, then the result is not obvious. This amounts to answering the question

Given an initial hologram on P_0 , how to compute the hologram on the tilted hyperplane $\mathcal{G}P_0$?

Thus, a viewer looking at the transformed hologram will have the impression that the object has *rotated*, or that the object is fixed while they actually *turn around the object*.

A general affine transformation is a composition of both translations and rotations, but also of dilations in various directions, which can emulate the effect of optical lenses. We introduce this “tilted plane generalisation” of the angular spectrum method in Section 3.

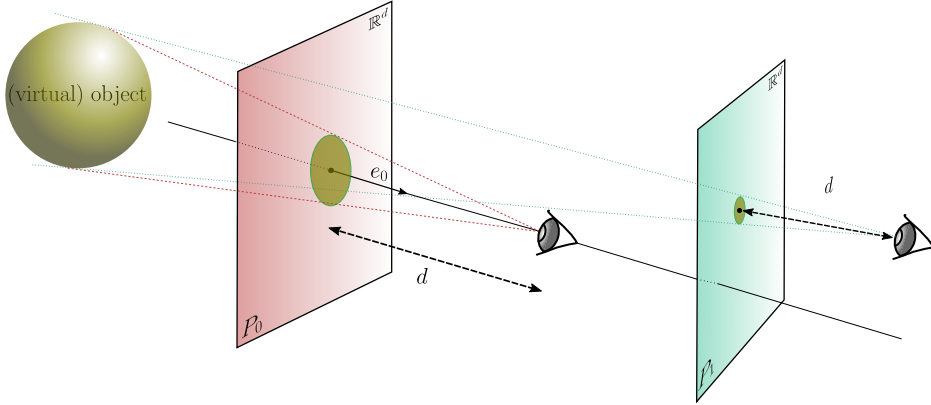


Figure 2 – Translating and rotating a hologram. On the transformed hologram P_1 , the viewer now starts to see the right-hand side of the object, which was hidden in the initial configuration on P_0 .

The goal of this paper is to cast the analysis of the operator $U_{\mathcal{G}}$ in the framework of *microlocal semiclassical analysis*. In the regime of small wavelength (compared to the details of the object), it is known that a natural and useful approach to wave optics is to use semiclassical analysis (which was initially devised for quantum equations with a small “Planck constant” $\hbar$), see Section 4. The semiclassical analysis of Helmholtz’ equation has been discussed by many authors, see for instance [4] and references therein.

Although we could not find a precise reference recovering geometric optics from our Helmholtz propagator, a similar and very nice discussion, based on the Fresnel formula (which is an approximation to the angular spectrum formula, in the paraxial regime) can be found in [19]. Thus, our first result is to find the conditions under which $U_{\mathcal{G}}$ is a good semiclassical Fourier integral operator, with a Hörmander non-degenerate phase function (Section 5), and to determine when it is associated with a canonical transformation (Proposition 5.5); we then compute explicitly its canonical transformation (Theorem 5.7), which in general is not injective. We explain the geometric meaning of these results, which allows us to recover the laws of geometric optics in Section 5.4.

The second contribution of this paper, motivated by the initial hologram analysis challenge, is to obtain precise information on the transformed hologram $U_{\mathcal{G}}(\psi_0)$ *without actually computing it*. This is achieved first by studying the action of pseudodifferential operators (viewed as space-frequency filters) on $U_{\mathcal{G}}$; in Propositions 6.2 and 6.3, we obtain semiclassical expansions of the Schwartz kernel of the product of $U_{\mathcal{G}}$ with arbitrary pseudo-differential operators. Then, as a consequence, we can predict the observation of the transformed hologram by means of precise Egorov theorems, keeping track of all *subprincipal* terms (*i.e.* terms of order $\mathcal{O}(\hbar)$, which correct the first order approximation given by geometric optics), see Section 6. More precisely, we prove two Egorov-type formulas with $\mathcal{O}(\hbar^2)$ remainders: one for $U_{\mathcal{G}}^{-1}PU_{\mathcal{G}}$ (Theorem 6.4), and one for $U_{\mathcal{G}}^*PU_{\mathcal{G}}$ (Theorem 7.3). Indeed, $U_{\mathcal{G}}$ is not (even microlocally) unitary — and the physics literature is often unclear on this issue. We compute the defect of unitarity in Section 7, which allows us to prove Theorem 7.3 from Theorem 6.4.

2 Angular spectrum

One of the most used methods for computing the propagation in free space of a scalar light wave emanating from a 3D scene is called the Angular Spectrum Method [17]. It is easy to implement, well studied, applicable to many situations, and reasonably fast thanks to the use of the Fast Fourier Transform, see for instance [33]. In this section, we recall the well-known Angular Spectrum formula from Fourier Optics, upon which this method is based. For the purpose of our work, we shall consider a Euclidean space of arbitrary dimension $1 + d$, as this presents no additional difficulty, although the physically relevant case for optics is naturally $1 + d = 3$.

Our initial data $\psi_0(x_1, \dots, x_d)$ will be a screen, or hologram, which is the trace of a light signal on the hyperplane $P_0 = \{0\} \times \mathbb{R}^d$, and our main direction of propagation is along x_0 . In the optical community, the coordinates x_0, x_1, x_2 of Euclidean 3-dimensional space are traditionally called z, x, y , and propagation occurs along (or close to) the z axis. We assume

that the light source is located in the left half-space $x_0 < 0$; as we shall see, it can be convenient to have a more precise assumption (for instance, that the light source is compactly supported inside that half-space).

We work in the scalar wave approximation, and restrict ourselves to monochromatic waves, with frequency $\omega/2\pi$, *i.e.* of the form

$$(t, x_0, \dots, x_d) \mapsto \psi(x_0, \dots, x_d) e^{-i\omega t}.$$

The total wave function $\psi = \psi(x_0, x_1, \dots, x_d)$ is described on source-free domains $\mathcal{U} \subset \mathbb{R}^{1+d}$ by the Helmholtz equation with wave number $k = \omega/c$ (c being the speed of light in vacuum, and the refractive index is assumed here to be constant equal to 1), through the incomplete Cauchy problem

$$\begin{cases} \Delta \psi + k^2 \psi = 0 \\ \psi|_{x_0=0} = \psi_0, \end{cases} \quad (1)$$

where $\Delta = \sum_{j=0}^d \frac{\partial^2}{\partial x_j^2}$ is the analysts' Laplacian. In order to perform Fourier analysis, we assume that $\psi_0 \in L^2(\mathbb{R}^d)$, or that ψ_0 is a tempered distribution. (Of course, since $-\Delta$ has continuous spectrum $[0, +\infty)$, we cannot expect ψ to be globally in $L^2(\mathbb{R}^{1+d})$, if $\psi_0 \neq 0$. For instance, it was proven in [3] that, if $\mathcal{U}$ is an open sector containing a half-space, then the only solution in $L^2(\mathcal{U})$ is the zero function.) Let $\mathcal{F}_d$ be the partial Fourier transform in the $(x_1, \dots, x_d)$ variables, *i.e.*

$$(\mathcal{F}_d \psi)(x_0, \zeta_1, \dots, \zeta_d) = \int_{\mathbb{R}^d} e^{-i\langle x, \zeta \rangle_d} \psi(X) dx_1 \cdots dx_d,$$

where

$$X := (x_0, x_1, \dots, x_d) \quad x := (x_1, \dots, x_d) \quad \zeta := (\zeta_1, \dots, \zeta_d)$$

and

$$\langle x, \zeta \rangle_d := \sum_{j=1}^d x_j \zeta_j.$$

In optical terminology, we may refer to $\mathcal{F}_d \psi$ as the “spectrum” of ψ on the parallel plane $P_{x_0} := \{x_0\} \times \mathbb{R}^d$. By the Fourier inversion formula applied on each P_{x_0} , one has

$$\psi(x_0, x_1, \dots, x_d) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \zeta \rangle_d} \mathcal{F}_d \psi(x_0, \zeta_1, \dots, \zeta_d) d\zeta_1 \cdots d\zeta_d. \quad (2)$$

Inserting (2) in Helmholtz' equation (1) and applying $\mathcal{F}_d$ we obtain

$$\partial_{x_0}^2 \mathcal{F}_d \psi + (k^2 - \|\zeta\|^2) \mathcal{F}_d \psi = 0.$$

In this differential equation, the variables $(\zeta_1, \dots, \zeta_d)$ can be seen as parameters; this shows that two qualitatively different regimes may occur depending on the Fourier domain we are interested in, based on the sign of $(k^2 - \|\zeta\|^2)$. If we assume that

$$\|\zeta\|^2 \leq k^2 \quad (3)$$

then we have an oscillatory solution of the form

$$\mathcal{F}_d \psi(x_0, \zeta) = A(\zeta) e^{ix_0 \sqrt{k^2 - \|\zeta\|^2}} + B(\zeta) e^{-ix_0 \sqrt{k^2 - \|\zeta\|^2}}, \quad (4)$$

with $A(\zeta) + B(\zeta) = \hat{\psi}_0(\zeta) := \mathcal{F}_d \psi_0(0, \zeta)$. In order to make an educated guess for A and B , we need to supplement the initial condition with a physically acceptable condition at infinity. Suppose we have chosen the square roots in (4) to lie in the upper half-plane for negative numbers. Then, in the propagation direction we are interested in, $x_0 > 0$, the first exponential $e^{ix_0 \sqrt{k^2 - \|\zeta\|^2}}$ gives an evanescent wave when $k^2 < \|\zeta\|^2$, which is physically acceptable (and mathematically amenable to the inverse Fourier transform), contrary to the term $e^{-ix_0 \sqrt{k^2 - \|\zeta\|^2}}$ which grows exponentially. This invites us to take $A = \hat{\psi}_0$ and $B = 0$, which yields the so-called Angular Spectrum representation for ψ , see for instance [17, 3.10.2]:

$$\psi(x_0, x_1, \dots, x_d) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\langle x, \zeta \rangle_d + x_0 \sqrt{k^2 - \|\zeta\|^2})} \hat{\psi}_0(\zeta) d\zeta, \quad (5)$$

which, as long as the involved quantities stay in a class where the Fourier transform $\mathcal{F}_d$ is applicable, solves the Helmholtz equation (1). In the high frequency regime, global estimates for Helmholtz' solutions, in more general settings (variable refraction, limit radiation term) have been studied by many authors, see [30] and references therein. In fact, as we shall see below (Remark 5.1), the choice made in the Angular Spectrum formula is essentially of microlocal nature, and will have important implications in our analysis.

3 Tilted planes

In this work, we are interested in the restriction of the light signal ψ from (5) to a (tilted) hyperplane, and more precisely in the map that sends ψ_0 to that restriction, see Figure 3.

Let $(e_0, e_1, \dots, e_d)$ be the canonical basis of $\mathbb{R}^{1+d}$, and denote the corresponding coordinates by $X = (x_0, x_1, \dots, x_d)$. A general hyperplane $P_{a,\beta} \subset \mathbb{R}^{1+d}$ is defined by the equation

$$\langle a, X \rangle = \beta, \quad (6)$$

where $a \in S^d$ is a unit vector in $\mathbb{R}^{1+d}$ and $\beta \geq 0$. Let us pick up an orthogonal transformation $G \in \text{SO}(d+1)$ such that $a = G^{\text{t}-1} e_0 = G e_0$. Recall that our

reference hyperplane is $P_0 = \{0\} \times \mathbb{R}^d$. Since $P_0 = P_{e_0,0}$, it follows from (6) that

$$P_{a,\beta} = G(P_0 + \beta e_0).$$

Let $\mathcal{G} := \tilde{X} \mapsto G(\tilde{X} + \beta e_0)$. The affine map $\mathcal{G} : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^{1+d}$ is a diffeomorphism that sends P_0 to $P_{a,\beta}$, so we may use it to parameterise $P_{a,\beta}$ by P_0 through the new coordinates $\tilde{X} = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_d)$ defined by

$$X = \mathcal{G}\tilde{X}.$$

Consider now Equation (5), which we may rewrite as

$$\psi(X) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle X, Z \rangle} \hat{\psi}_0(\zeta) d\zeta, \quad X \in \mathcal{H}_+,$$

where $\mathcal{H}_+$ is the right half-space: $\mathcal{H}_+ = \{X \in \mathbb{R}^{1+d}, x_0 \geq 0\}$, and

$$Z = Z(\zeta) = \left(\sqrt{k^2 - \|\zeta\|^2}, \zeta_1, \dots, \zeta_d \right). \quad (7)$$

Expressing ψ in the new coordinates (*via* $\tilde{\psi}(\tilde{X}) = \psi(X)$) we get $\tilde{\psi} = \psi \circ \mathcal{G}$. Hence the expression of the angular spectrum restricted to the new hyperplane $P_{a,\beta}$, in the new coordinates $\tilde{X} = (0, \tilde{x}_1, \dots, \tilde{x}_d)$, is simply

$$\tilde{\psi}(\tilde{X}) = \psi(\mathcal{G}(\tilde{X})) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \mathcal{G}(\tilde{X}), Z \rangle} \hat{\psi}_0(\zeta) d\zeta. \quad (8)$$

We shall call this formula the “rotated angular spectrum”. It holds in general only when $\mathcal{G}(\tilde{X}) \in \mathcal{H}_+$; if we assume $\text{supp } \hat{\psi}_0 \subset \{\|\zeta\|^2 \leq k^2\}$ (see Section 4 for a justification of this hypothesis), it can be extended to any $\tilde{X} \in \mathbb{R}^{1+d}$, however this is physically relevant only when $\mathcal{G}(\tilde{X})$ stays in a source-free region.

Remark 3.1 In the 3D case ($d = 2$), a similar analysis was carried out in [27] for tilted planes with no translation ($\beta = 0$). However, an incorrect formula (similar to (8), but involving a superfluous Jacobian) was presented, and this was partly clarified in a subsequent paper [26]: since $\beta = 0$, one can conveniently express $\tilde{\psi}$ as the inverse Fourier transform of some planar signal. This alternative formula, which we extend here to any dimension, requires an additional geometric assumption on $P_{a,\beta}$; indeed, since $\mathcal{G} = G$, we can write

$$\tilde{\psi}(\tilde{X}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \tilde{X}, G^t Z \rangle} \hat{\psi}_0(\zeta) d\zeta,$$

where G^t is the adjoint of G for the Euclidean structure. Since $\tilde{X} = (0, \tilde{x})$, we see that the scalar product in the phase involves only the last d components of $G^t Z$. Now, assume that $G^t Z$ induces a diffeomorphism on the hyperplane $\{0\} \times \mathbb{R}^d$, *i.e.* there is a diffeomorphism $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $G^t Z(\zeta) =$

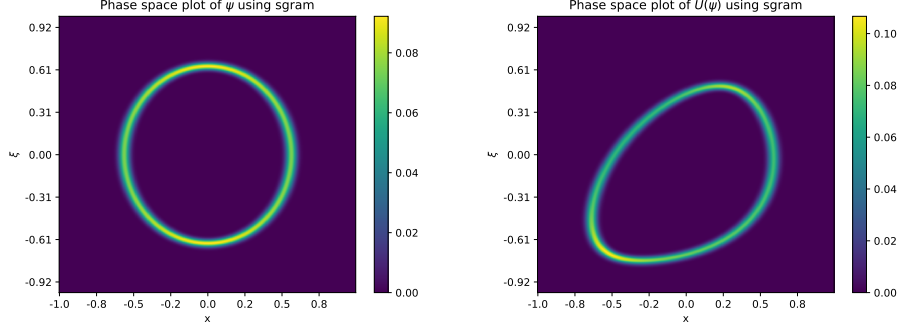


Figure 3 – Here ψ_0 is a highly excited 1D Hermite function, whose phase space representation is an ellipse (left figure). On the right figure, we display the phase space representation of its image $\tilde{\psi} = \psi \circ \mathcal{G}$, where $\mathcal{G}$ is a rotation of angle 10° followed by a translation of vector $\gamma = (0.3, 0) \in \mathbb{R}^2$, obtained by implementing Formula (8). The shape of the transformed signal will be explained by Theorem 5.7, see Figure 10. The phase space representations are obtained using the `sgram` command from the `ltfat` library.

$(\tilde{t}(\zeta), g(\zeta))$ for some smooth function $\tilde{t}$. Under this assumption, one could make the change of variables $\tilde{\zeta} = g(\zeta)$:

$$\begin{aligned} \tilde{\psi}(\tilde{X}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \tilde{X}, \tilde{Z} \rangle} \hat{\psi}_0(\zeta) d\zeta \quad \text{with } \tilde{Z} = G^t Z \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \tilde{x}, \tilde{\zeta} \rangle_d} \hat{\psi}_0(g^{-1}(\tilde{\zeta})) J(\tilde{\zeta}) d\tilde{\zeta} \\ &= \mathcal{F}_d^{-1}((\hat{\psi}_0 \circ g^{-1})J)(\tilde{x}), \end{aligned}$$

where $J(\tilde{\zeta})$ is the Jacobian of g^{-1} . In other words, viewing J as a multiplication operator, we would have

$$G^* \psi(0, \tilde{x}) = \mathcal{F}_d^{-1} \circ J \circ (g^{-1})^* \circ \mathcal{F}_d \psi_0(\tilde{x}). \quad (9)$$

(We use the superscript “ $*$ ” for the pull-back operator, for instance:

$$G^* \psi = \psi \circ G).$$

However, this formula is still not correct, simply because the assumption that g is a diffeomorphism can never be realised, due to the square root in (7), see Section 5.2, which brings several difficulties in: first, one has to assume that $\hat{\psi}_0$ should be supported in the unit disc, otherwise Z might take complex values; secondly, and most importantly, g is generally not injective, but rather “2 to 1”. Hence one has to carefully choose an integration sheet in the spectral variable. For these reasons, we see that a “microlocal” (*i.e.* local in phase space) approach will be more appropriate for a good understanding of this transform. On the numerical side, failure to take into account the

non-injectivity may result in parts of the viewed objects being hidden as one performs a rotation via the angular spectrum method.

Notice also that, if g were a linear transformation, the right-hand side of (9) would simply be equal to $g^*\psi_0(\tilde{x})$. But, of course, even when G is linear, g itself is generally non-linear, due to the square root in (7). In some sense, it is the goal of this paper, in addition to the introduction of the microlocal setting, to deal with this non-linearity. $\triangle$

Remark 3.2 Since, in this section, we chose G to be an orthogonal matrix, we could replace G^t by G^{-1} ; however, keeping the notation G^t will allow us to consider general linear transformations, which corresponds to additional scaling and shearing of the hologram; this notation also reminds us that, while G acts on position variables $(x_0, \dots, x_d)$, the transpose G^t should act on frequency (*i.e.* “Fourier”) variables $(\xi_0, \dots, \xi_d)$. Note however that we use the same canonical basis $(e_0, \dots, e_d)$ for both position and Fourier spaces $\mathbb{R}^{1+d}$. $\triangle$

Remark 3.3 The map $\mathcal{G}$ preserves P_0 if and only if $\beta = 0$ and the linear part G preserves P_0 (and hence $Ge_0 = \pm e_0$). This is exactly when the map g of Remark 3.1 is linear. Hence, in this case,

$$\tilde{\psi}(0, \tilde{x}) = \psi_0(g(\tilde{x}))$$

i.e. we simply perform a rotation within the initial state ψ_0 .

Similarly, the hyperplane $P_{a,\beta}$ is *parallel* to P_0 if and only if the linear part G preserves P_0 (and hence $Ge_0 = \pm e_0$). In this case again, G acts on P_0 as a linear map g , and Formula (8) becomes

$$\tilde{\psi}(0, \tilde{x}) = \mathcal{F}_d^{-1} \left(e^{i\beta\sqrt{k^2 - \|\zeta\|^2}} \mathcal{F}_d \psi_0 \right) (g(\tilde{x})),$$

which, modulo the internal rotation g , is the original Angular Spectrum formula (5). $\triangle$

Remark 3.4 It would be interesting to treat the case of a curved deformation of the initial hyperplane P_0 , *i.e.* when the transformation $\mathcal{G}$ is non-linear. $\triangle$

4 Semiclassical limit

Having in mind the natural regime when the spacial frequency k is very large, we write $k = \frac{1}{\hbar}$ for a small parameter $\hbar > 0$ (in physical terms, the “Planck constant” $\hbar$ used here is effectively the wave length λ). Helmholtz’ equation becomes

$$\hbar^2 \Delta \psi + \psi = 0. \tag{10}$$

The analogy between the semiclassical limit in quantum mechanics and the limit of geometric optics from Fourier optics has been known for a long time, see for instance [19, 31]. However, it seems that the geometric content of the (rotated) Angular Spectrum formulas that we develop here, based on the semiclassical intuition, was not investigated before.

To start with, since Equation (10), *i.e.* $-\hbar^2 \Delta \psi = \psi$, is nothing but a semiclassical Schrödinger equation at energy 1, this suggests that, in the limit $\hbar \rightarrow 0$, solutions have to be microlocalized on the unit sphere in the classical momentum variables $(\xi_0, \dots, \xi_d)$. To be precise, let us introduce the corresponding scaling, that is, $\xi = (\xi_1, \dots, \xi_d) = \hbar \zeta$. Since the semiclassical wave front of ψ is contained in the sphere $\sum_{j=0}^d \xi_j^2 = 1$, we must have $|\xi_0| < 1$ in the physical region. Then

$$Z_0 = \sqrt{k^2 - \|\zeta\|^2} = \hbar^{-1} \sqrt{1 - \|\xi\|^2},$$

and the unit sphere in the full semiclassical Fourier variables $(\hbar Z_0, \xi_1, \dots, \xi_d)$ corresponds to the unit disc $\|\xi\|^2 \leq 1$, which is the oscillatory region (3) of Helmholtz' equation, and the “physically accessible” region of phase space for the classical mechanical limit of Schrödinger's equation at energy 1. Up to an $\mathcal{O}(\hbar^\infty)$ error, we may truncate ψ in the neighbourhood of this unit co-sphere, which ensures that the trace ψ_0 on the vertical hyperplane $\mathcal{P}_0 := \{\xi_0 = 0\}$ is well-defined. The rotated Angular Spectrum formula (8) can be written as

$$\tilde{\psi}(\tilde{X}) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} \langle \mathcal{G}(\tilde{X}), \sigma(\xi) \rangle} \hat{\psi}_0(\xi/\hbar) d\xi,$$

where $\tilde{X} = (0, \tilde{x}_1, \dots, \tilde{x}_d) \in P_0$ and

$$\sigma(\xi) = \left(\sqrt{1 - \|\xi\|^2}, \xi \right) \in S^d \subset \mathbb{R}^{1+d}. \quad (11)$$

The map σ will play an important role in our analysis; it lifts a vector $\xi \in \mathbb{R}^d$ from $\mathcal{P}_0$ to the “right hemisphere” ($\xi_0 \geq 0$) of the unit sphere S^d in $\mathbb{R}^{1+d}$, see Figure 4.

Notice also that $\hat{\psi}_0(\xi/\hbar) = \mathcal{F}_\hbar \psi_0(\xi)$, where $\mathcal{F}_\hbar$ is the semiclassical Fourier transform given by

$$\mathcal{F}_\hbar = \int_{\mathbb{R}^d} e^{-\frac{i}{\hbar} \langle x, \xi \rangle} dx$$

(whose inverse is $\mathcal{F}_\hbar^{-1} = \frac{1}{(2\pi\hbar)^d} \mathcal{F}_\hbar^*$, where $\mathcal{F}_\hbar^*$ is the L^2 -adjoint). This yields the semiclassical expression for the value of the wave function on the tilted hyperplane:

$$\tilde{\psi}(0, \tilde{x}) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} \langle \mathcal{G}\tilde{X}, \sigma(\xi) \rangle} \mathcal{F}_\hbar \psi_0(\xi) d\xi \quad (12)$$

$$= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{\frac{i}{\hbar} (\langle \mathcal{G}\tilde{X}, \sigma(\xi) \rangle - \langle x, \xi \rangle_d)} \psi_0(x) dx d\xi. \quad (13)$$

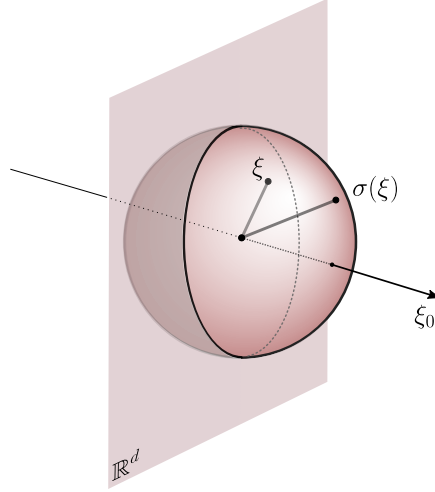


Figure 4 – The map σ

As we shall now investigate, from this formula, the operator that sends ψ_0 to the map $\tilde{x} \mapsto \tilde{\psi}(0, \tilde{x})$ has the structure of a semiclassical Fourier integral operator. As a byproduct, we will recover, without imposing any axiom, the usual structures of geometric optics, thanks to the symplectic geometry obtained in the limit $\hbar \rightarrow 0$.

5 Fourier integral operators

Fourier integral operators, or FIOs, are (microlocal) operators U acting on functions ψ , of the form

$$(U\psi)(\tilde{x}) = \iint e^{\frac{i}{\hbar}\phi(\tilde{x}, x, \theta)} a_{\hbar}(\tilde{x}, x, \theta) \psi(x) dx d\theta, \quad (14)$$

where $(\tilde{x}, x, \theta) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^N$ (the “auxiliary dimension” $N \leq d$ may depend on the operator), $a_{\hbar}$ is called the amplitude and ϕ , the phase. A relatively recent account of FIOs can be found for instance [20], but most of the theory was initially developed by Maslov [25] and Hörmander [24] (for the latter, without the small parameter $\hbar$); see also [10, 11], and the nice introductory paper [21]. Since the beginning of microlocal analysis, FIOs have been used to simplify many partial differential equations by ways of *normal forms*, see [32] and references therein. Interestingly, more recently, FIOs have also become part of the signal processing toolbox and can be analysed (theoretically and numerically) via Gabor frames, see for example [7], and specifically for the Helmholtz equation in [5].

In order to have a good FIO calculus, one asks that the phase be non-degenerate (in the sense of Hörmander), which essentially means that the

set

$$C_\phi := \{(\tilde{x}, x, \theta) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^N; \quad \partial_\theta \phi(\tilde{x}, x, \theta) = 0\}$$

must be a smooth manifold of dimension $2d$. (In view of the stationary phase lemma, we see that the Schwartz kernel of the integral operator (14) can be reduced, modulo a very small error of order $\mathcal{O}(\hbar^\infty)$, to a “usual” kernel depending only on the variables $(\tilde{x}, x)$. In other words, the Hörmander condition allows us to get rid of the auxiliary variable θ .) What is important for us is that with such an operator is associated a *canonical relation* in phase space, given by the image $\Lambda_\phi \subset \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ of the map

$$C_\phi \ni (\tilde{x}, x, \theta) \mapsto (\tilde{x}, \partial_{\tilde{x}} \phi, x, -\partial_x \phi).$$

The canonical relation Λ_ϕ is always Lagrangian in $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ (with the symplectic form $d\tilde{\xi} \wedge d\tilde{x} - d\xi \wedge dx$), but is not necessarily the graph of an honest transformation. For this to hold locally, it is sufficient that the matrix $\partial_{\tilde{x}} \partial_x \phi$ be everywhere invertible. Since in our case x and $\tilde{x}$ have the same dimension, it will then imply that the canonical transformation is locally invertible: it is a local symplectic diffeomorphism $\kappa_\phi : (x, -\partial_x \phi) \mapsto (\tilde{x}, \partial_{\tilde{x}} \phi)$.

The best known example of a non-trivial Fourier Integral Operator is the semiclassical Fourier transform $\mathcal{F}_\hbar$: there, $\phi(\tilde{x}, x) = -\langle \tilde{x}, x \rangle$ (there is no auxiliary variable θ , hence it is automatically Hörmander non-degenerate), and $\partial_{\tilde{x}} \partial_x \phi$ is minus the identity matrix. We have

$$\Lambda_\phi = (\tilde{x}, -x, x, \tilde{x})$$

so the associated symplectic transformation is $(x, \tilde{x}) \mapsto (\tilde{x}, -x)$.

In both formulas (12) and (13), we have obtained the target signal $\tilde{\psi}(\tilde{x}_1, \tilde{x}_2, 0)$ from the original one $\psi_0(x_1, x_2)$ by applying integral operators, which have the form of Fourier integral operators. Both formulas are actually interesting, and the goal of this section is to study them in details. While we believe that these results are new, we note that the use of FIOs for wave equations has a long tradition, and related formulas can be found in recent works like [22]. Moreover, in the paraxial regime (ψ_0 is nearly co-normal), the Helmholtz equation becomes a time-dependent Schrödinger equation (see [9, 8]), whose propagator has naturally the structure of a semiclassical Fourier integral operator, see for instance [29].

Remark 5.1 By definition, the Schwartz kernel K_U of U is called a Lagrangian distribution of $(\tilde{x}, x)$:

$$K_U(\tilde{x}, x) = \iint e^{\frac{i}{\hbar} \phi(\tilde{x}, x, \theta)} a_\hbar(\tilde{x}, x, \theta) d\theta,$$

associated with the Lagrangian submanifold Λ_ϕ . Let us compare to the choice we made for the Helmholtz solution (5), which writes

$$\psi(x_0, x_1, \dots, x_d) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} (\langle x, \xi \rangle_d + x_0 \sqrt{1 - \|\xi\|^2})} \hat{\psi}_0(\xi/\hbar) d\xi.$$

If we assume that $\hat{\psi}_0(\xi/\hbar) = \chi(\xi)$ for some smooth function χ , then we see that ψ is a Lagrangian distribution of x (also called Maslov-WKB state) whose Lagrangian submanifold is contained in the set $\{((x_0, x), (\xi_0, \xi)) \in T^*\mathbb{R}^{1+d}, \xi_0 = \sqrt{1 - \|\xi\|^2}\}$: the frequency variable (ξ_0, ξ) belongs to the *right* hemisphere $\{\|\xi\| = 1, \xi_0 > 0\}$. This particular choice of Lagrangian submanifold of the unit cosphere bundle underpins our entire analysis; it correspond to propagation in the $x_0 > 0$ direction (see Section 5.4), and induces a kind of *caustic* behaviour (in the frequency variable) at $\|\xi\| = 1$. $\triangle$

5.1 Decomposition of the Angular Spectrum propagation

We now consider a general affine transformation $\mathcal{G}$ of $\mathbb{R}^{1+d}$, of the form $\mathcal{G}(\tilde{X}) = G\tilde{X} + \gamma$, where $G \in \text{GL}_{1+d}(\mathbb{R})$ is not necessary orthogonal. This allows us to shrink or expand the light rays in specific directions, in addition to the rotation component.

Let us denote by $U_{\mathcal{G}}$ the operator given by (13), that is:

$$[U_{\mathcal{G}}\psi](\tilde{x}) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{\frac{i}{\hbar}(\langle \mathcal{G}\tilde{X}, \sigma(\xi) \rangle - \langle x, \xi \rangle_d)} \psi(x) dx d\xi.$$

It can be interesting to express $U_{\mathcal{G}}$ as a composition of simpler operators. For instance, it is clear from (12) that $U_{\mathcal{G}} = V_{\mathcal{G}} \circ \mathcal{F}_{\hbar}$, with

$$[V_{\mathcal{G}}\phi](\tilde{x}) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\langle \mathcal{G}\tilde{X}, \sigma(\xi) \rangle} \phi(\xi) d\xi \quad (15)$$

Remark that if $\mathcal{G} = \text{Id}$, then $V_{\mathcal{G}} = \mathcal{F}_{\hbar}^{-1}$, so $U_{\mathcal{G}} = \text{Id}$.

Lemma 5.2 *For $G \in \text{GL}_{1+d}(\mathbb{R})$, we identify G with the transformation $\tilde{X} \mapsto G\tilde{X}$, and for $\gamma \in \mathbb{R}^{1+d}$, we let $\tau_{\gamma} : \tilde{X} \mapsto \tilde{X} + \gamma$. To simplify notation, we write U_{γ} for $U_{\tau_{\gamma}}$. The following holds formally:*

1. For any $\gamma_1, \gamma_2 \in \mathbb{R}^{1+d}$, $U_{\gamma_1} \circ U_{\gamma_2} = U_{\gamma_2 + \gamma_1}$.
2. For any $G \in \text{GL}_{1+d}(\mathbb{R})$ and $\gamma \in \mathbb{R}^{1+d}$,

$$U_{\tau_{\gamma} \circ G} = V_G e^{\frac{i}{\hbar}\langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_{\hbar} = U_G U_{\gamma}$$

3. If $G \in \text{GL}_{1+d}(\mathbb{R})$ and $\gamma \in \mathbb{R}^{1+d}$, then $U_{G \circ \tau_{\gamma}} = U_G U_{G\gamma}$ is in general different from $U_{\gamma} U_G$. However, let $\mathcal{H}_{\pm} = \{X \in \mathbb{R}^{1+d}, \pm x_0 \geq 0\}$. If $G^t \sigma(\text{supp } \mathcal{F}_{\hbar} \psi) \subset \mathcal{H}_+$, then

$$U_{G \circ \tau_{\gamma}} \psi = U_{\gamma} U_G \psi;$$

and if $G^t \sigma(\text{supp } \mathcal{F}_{\hbar} \psi) \subset \mathcal{H}_-$, then

$$U_{G \circ \tau_{\gamma}} \psi = U_{S_0(\gamma)} U_G \psi,$$

where $S_0(\gamma)$ is the symmetric to γ with respect to $\mathcal{P}_0$, i.e. $S_0(\gamma) = (-\gamma_0, \gamma_1, \dots, \gamma_d)$.

Proof. Point 1 follows from the formula

$$U_\gamma = \mathcal{F}_h^{-1} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h. \quad (16)$$

We turn to point 2. Let $\mathcal{G} = \tau_\gamma \circ G = \tilde{X} \mapsto G\tilde{X} + \gamma$. We have $U_G = V_G \circ \mathcal{F}_h$ and, since $\tilde{X} = (0, \tilde{x})$, we get $\langle \tilde{X}, \sigma(\xi) \rangle = \langle \tilde{x}, \xi \rangle_d$, which gives

$$V_\gamma = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} (\langle \tilde{X}, \sigma(\xi) \rangle + \langle \gamma, \sigma(\xi) \rangle)} d\xi = \mathcal{F}_h^{-1} \circ e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \quad (17)$$

where $e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle}$ is viewed as a multiplication operator. From (16), we have

$$\begin{aligned} U_G U_\gamma &= V_G \mathcal{F}_h \mathcal{F}_h^{-1} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h \\ &= V_G e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h, \end{aligned}$$

and we notice that

$$\begin{aligned} V_G(e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \phi) &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} (\langle G\tilde{X}, \sigma(\xi) \rangle + \langle \gamma, \sigma(\xi) \rangle)} \phi(\xi) d\xi \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar} \langle \mathcal{G}\tilde{X}, \sigma(\xi) \rangle} \phi(\xi) d\xi \\ &= V_G \phi, \end{aligned}$$

so $U_G U_\gamma = V_G \mathcal{F}_h = U_G$.

Let us now consider Point 3, *i.e.* the reverse composition $U_\gamma U_G$. We have

$$[U_\gamma U_G \psi](\tilde{x}) = \mathcal{F}_h^{-1} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h \left[\tilde{y} \mapsto \int e^{\frac{i}{\hbar} \langle G\tilde{Y}, \sigma(\eta) \rangle} \mathcal{F}_h \psi(\eta) d\eta \right],$$

where $\tilde{Y} := (0, \tilde{y})$. Let us introduce the linear projection from $\mathbb{R}^{1+d}$ onto the vertical hyperplane $\mathcal{P}_0 = \{\xi_0 = 0\}$.

$$\boldsymbol{\pi} : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d,$$

so that $\forall \xi \in \mathbb{R}^d$, $\boldsymbol{\pi}(\sigma(\xi)) = \xi$. We have $\langle G\tilde{Y}, \sigma(\eta) \rangle = \langle \tilde{y}, \boldsymbol{\pi} G^t \sigma(\eta) \rangle_d$, and hence

$$[U_\gamma U_G \psi](\tilde{x}) = \int e^{\frac{i}{\hbar} \langle \tilde{x}, \xi \rangle_d} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\xi) \rangle} e^{\frac{i}{\hbar} \langle \tilde{y}, -\xi + \boldsymbol{\pi} G^t \sigma(\eta) \rangle_d} \mathcal{F}_h \psi(\eta) d\eta d\xi d\tilde{y}. \quad (18)$$

Using that

$$\int e^{\frac{i}{\hbar} \langle \tilde{y}, -\xi + \boldsymbol{\pi} G^t \sigma(\eta) \rangle_d} d\tilde{y} = \delta_{\{\xi = \boldsymbol{\pi} G^t \sigma(\eta)\}},$$

we obtain

$$U_\gamma U_G \psi(\tilde{x}) = \int e^{\frac{i}{\hbar} \langle \tilde{X}, G^t \sigma(\eta) \rangle} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\boldsymbol{\pi} G^t \sigma(\eta)) \rangle} \mathcal{F}_h \psi(\eta) d\eta$$

If $G^t\sigma(\eta) \in \mathcal{H}_+$, we have $\sigma(\pi G^t\sigma(\eta)) = G^t\sigma(\eta)$, and the integrand becomes

$$e^{\frac{i}{\hbar}\langle G(\tilde{X}+\gamma), \sigma(\eta) \rangle} \mathcal{F}_\hbar \psi(\eta),$$

which is the formula involved in $U_{G \circ \tau_\gamma} \psi$. On the other hand, if $G^t\sigma(\eta) \in \mathcal{H}_-$, then $\sigma(\pi G^t\sigma(\eta)) = S_0(G^t\sigma(\eta))$, where S_0 is the orthogonal symmetry with respect to $\mathcal{P}_0$. Hence

$$\langle \gamma, \sigma(\pi G^t\sigma(\eta)) \rangle = \langle S_0(\gamma), G^t\sigma(\eta) \rangle,$$

and therefore the integrand writes now

$$e^{\frac{i}{\hbar}\langle G(\tilde{X}+S_0(\gamma)), \sigma(\eta) \rangle} \mathcal{F}_\hbar \psi(\eta),$$

which is the formula leading to $U_{G \circ \tau_{S(\gamma)}} \psi$. □

While the computation above is only formal, it gives rise to well defined quantities under additional hypothesis, as discussed in Section 3: one may impose a compact support for $\mathcal{F}_\hbar \psi$ inside $\{\|\xi\|^2 \leq 1\}$, or demand that γ and $G\tilde{X}$ belong to the right half-space $\mathcal{H}_+$.

5.2 Geometric study of V_G

Let $\mathcal{G}(\tilde{X}) := G\tilde{X} + \gamma$, with $G \in \text{GL}_{1+d}(\mathbb{R})$ and $\gamma \in \mathbb{R}^{1+d}$. We restrict our study of V_G (15) to the region $\|\xi\| < 1$, where the map σ is smooth, and hence V_G is a usual Fourier integral operator. The phase function of V_G is

$$\phi(\tilde{x}, \xi) = \langle \tilde{x}, \pi(G^t\sigma(\xi)) \rangle_d + \langle \gamma, \sigma(\xi) \rangle, \quad (19)$$

where, in view of the general discussion (14), $\tilde{x} \in \mathbb{R}^d$, ξ plays the role of x , and there is no auxiliary variable. Hence it is automatically non-degenerate in the sense of Hörmander, and V_G is a Fourier integral operator associated with the canonical relation $(\tilde{x}, \partial_{\tilde{x}}\phi, x, -\partial_x\phi)$. However, in order to check whether it defines a (local) canonical transformation, we should compute the mixed Hessian:

$$\partial_{\tilde{x}}\partial_\xi\phi = \partial_\xi[\pi G^t\sigma(\xi)] = (\pi \circ G^t)\partial_\xi\sigma(\xi). \quad (20)$$

Notice that this Hessian is precisely the Jacobian matrix of the map g from Remark 3.1. Thus, the Hessian is non-degenerate if and only if $G^t \circ \sigma$ induces a local diffeomorphism on the hyperplane $\mathcal{P}_0 = \{0\} \times \mathbb{R}^d$.

Remark 5.3 Since $\tilde{x}$ and ξ are multidimensional variables, it may be useful to explain some of the identifications that we use in this text. For any fixed $(\tilde{x}, \xi)$, we view $\partial_{\tilde{x}}\partial_\xi\phi$ as an endomorphism of $\mathbb{R}^d$, with the usual implicit identifications. Precisely, if we view ξ as a covector (an element of the dual space

$(\mathbb{R}^d)^*$), then $\partial_{\bar{x}}\partial_{\xi}\phi$ maps the tangent space $T_x\mathbb{R}^d \simeq \mathbb{R}^d$ to $(T_{\xi}(\mathbb{R}^d)^*)^* \simeq \mathbb{R}^d$. In other words, if $(u, v) \in T_{(\bar{x}, \xi)}(T^*\mathbb{R}^d) = T_{\bar{x}}\mathbb{R}^d \times T_{\xi}(\mathbb{R}^d)^*$, we have

$$\partial_{\bar{x}}[\partial_{\xi}\phi(v)](u) = \langle u, (\pi \circ G^t) d_{\xi}\sigma \cdot v \rangle = \partial_{\xi}[\partial_{\bar{x}}\phi(u)](v).$$

△

With the following lemma, we first remark that the non-degeneracy is easy to determine at $\xi = 0$.

Lemma 5.4 *In a neighbourhood of $\xi = 0$, the Hessian (20) is non-degenerate if and only if the hyperplanes P_0 and GP_0 are not mutually perpendicular, i.e. $\langle e_0, G^{-1}e_0 \rangle \neq 0$.*

Proof. At $\xi = 0$, the map $\sigma(\xi)$ is tangent to $(1, \xi)$, therefore $\partial_{\xi}[\pi G^t\sigma(0)] = \pi G^t\pi$. Hence the Hessian is non-degenerate if and only if G^t induces an isomorphism on the hyperplane $\mathcal{P}_0 = \{0\} \times \mathbb{R}^d$. If $\langle e_0, G^{-1}e_0 \rangle = 0$, the vector $G^{t-1}e_0 \in \mathcal{P}_0$ belongs to the kernel of $\pi \circ G^t$, which is hence not injective. Conversely, if there exists a non-zero $\Xi \in \mathcal{P}_0 \cap \ker \pi \circ G^t$, then $G^t\Xi$ must be collinear to e_0 , so Ξ is collinear to $G^{t-1}e_0$, which implies $\langle G^{t-1}e_0, e_0 \rangle = 0$. □

In order to deal with the general position, we shall first give a geometric argument, and then provide a precise algebraic computation of the Hessian.

Recall that

$$g(\xi) = \pi \circ G^t \circ \sigma(\xi).$$

The map σ is a diffeomorphism from the open unit ball $\|\xi\| < 1$ to the right hemisphere in $\mathbb{R}^{1+d}$. Applying G^t on the vector $\sigma(\xi)$, we get a point on the Fourier ellipsoid $\mathcal{E} = G^t S^d$. The image $g(\xi)$ is the projection of that point onto the hyperplane $\{\xi_0 = 0\}$, see Figure 5. This map is in general not

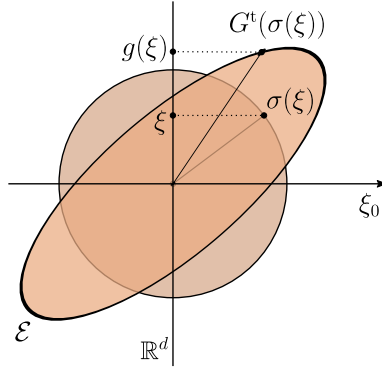


Figure 5 – The map $g = \pi \circ G^t \circ \sigma$.

injective:

$$\begin{aligned} g(\xi_1) = g(\xi_2) &\iff \pi(G^t\sigma(\xi_1) - G^t\sigma(\xi_2)) = 0 \\ &\iff G^t\sigma(\xi_1) - G^t\sigma(\xi_2) = \lambda e_0 \quad \text{for some } \lambda \in \mathbb{R}. \end{aligned}$$

Hence $G^t \sigma \xi_1$ and $G^t \sigma \xi_2$ are two points on the ellipsoid that lie on a line parallel to e_0 . In general position, a line intersects a quadric at 0 or 2 points; when we do have two distinct points, the map $\pi|_{\mathcal{E}}$ is a local diffeomorphism. At the critical points, where there is only one intersection point (this is the “apparent contour” of the ellipsoid in the direction e_0), $\pi|_{\mathcal{E}}$ cannot be a local diffeomorphism. Since $g = \pi|_{\mathcal{E}} \circ G^t \sigma$, its differential will be degenerate exactly when $\pi|_{\mathcal{E}}$ is not a local diffeomorphism. Writing

$$g(\xi_1) = g(\xi_2) \iff \sigma(\xi_1) - \sigma(\xi_2) = \lambda G^{t-1} e_0 \quad \text{for some } \lambda \in \mathbb{R},$$

we see that $\sigma(\xi_1)$ and $\sigma(\xi_2)$ are the intersection points of the sphere S^d with a line directed by $G^{t-1} e_0$. (In other words, $\sigma(\xi_1)$ and $\sigma(\xi_2)$ are images to each other under the orthogonal reflection with respect to the hyperplane $G\mathcal{P}_0$). These two points coalesce precisely when $\sigma(\xi_1) = \sigma(\xi_2)$ is orthogonal to $G^{t-1} e_0$. (See also Figure 6.) This proves the following.

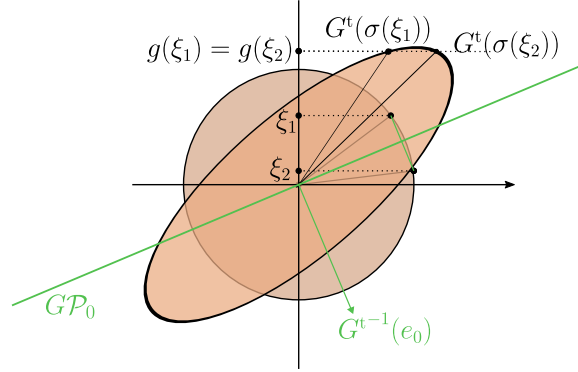


Figure 6 – The map g is not injective.

Proposition 5.5 *The Hessian (20) is degenerate (i.e. non invertible) if and only if $\sigma(\xi) \in G\mathcal{P}_0$.*

Thus, microlocally near any $(x, \xi) \in T^*\mathbb{R}^d$ such that $\|\xi\| < 1$, the operator V_G is a good Fourier integral operator with a non-degenerate phase function, and it is associated with a local symplectic diffeomorphism if and only if

$$\sigma(\xi) \notin G\mathcal{P}_0. \quad (21)$$

Remark 5.6 Physically, $\sigma(\xi)$ represents the direction of the light ray (see Section 5.4), so the condition for a non-degenerate Hessian is that the light ray (in position space) must not be parallel to the rotated hyperplane $G\mathcal{P}_0$; this is very natural: if this condition is not fulfilled, the impact of this light ray on the rotated hyperplane is not well defined. $\triangle$

Let us now turn to a more computational argument. In order to make Formula (20) more explicit, note that $(\pi \circ G^t)$ is the $d \times (1 + d)$ matrix

composed of the last d lines of G^t ; we may write

$$\pi \circ G^t = \begin{pmatrix} b & G_0^t \end{pmatrix} \quad (22)$$

where $b := \pi \circ G^t(e_0) = g(0) \in \mathbb{R}^d$, viewed as a column vector, and G_0^t is the lower-right $d \times d$ minor of the matrix G^t . Let us denote $f(\xi) := \sqrt{1 - \|\xi\|^2}$. The linear map $d_\xi \sigma = (df(\xi), d\xi_1, \dots, d\xi_d)$ is represented by the $(1+d) \times d$ matrix:

$$d_\xi \sigma = \begin{pmatrix} \frac{-\xi^t}{f(\xi)} \\ I_d \end{pmatrix}$$

where I_d is the $d \times d$ identity matrix and ξ^t is the line vector $(\xi_1, \dots, \xi_d)$. Hence

$$dg(\xi) = (\pi \circ G^t) d_\xi \sigma = \frac{-b \cdot \xi^t}{f(\xi)} + G_0^t. \quad (23)$$

In particular, we recover the conclusion of Lemma 5.4 that, if ξ is small, the non-degeneracy of the Hessian is implied by the invertibility of G_0^t (which is equivalent to the fact that the hyperplanes P_0 and GP_0 are not orthogonal). In this case, Formula (23) gives

$$\det[(\pi \circ G^t) d_\xi \sigma] = (\det G_0^t) \left(1 - \frac{\langle G_0^{-1} \xi, b \rangle}{f(\xi)} \right) \quad (24)$$

Second proof of Proposition 5.5. Let $a := G^{t-1}e_0$. Then $\sigma(\xi) \in GP_0$ if and only if $\langle \sigma(\xi), a \rangle = 0$.

Case 1. If G_0^t is not invertible, then GP_0 is perpendicular to P_0 , *i.e.* $\langle e_0, a \rangle = 0$, so $a \in P_0$. Hence $G_0^t \pi a = \pi G^t a = \pi e_0 = 0$, and on the other hand $\langle \xi, \pi a \rangle = \langle \sigma(\xi), a \rangle$. Therefore, if the latter vanishes, πa is a non-zero element of the kernel of the Hessian (23), which is hence degenerate.

Conversely, if there is a non-zero $u \in \mathbb{R}^d$ in the kernel of the Hessian, *i.e.*

$$\langle \xi, u \rangle b = G_0^t u, \quad (25)$$

we can write this as

$$\langle \xi, u \rangle \pi G^t e_0 = \pi G^t \check{u}$$

with $\check{u} := (0, u) \in \mathbb{R}^{1+d}$. Therefore $\pi G^t(\langle \xi, u \rangle e_0 - \check{u}) = 0$, which means that $G^t(\langle \xi, u \rangle e_0 - \check{u}) = \lambda e_0$ for some $\lambda \in \mathbb{R}$, so

$$\langle \xi, u \rangle e_0 - \check{u} = \lambda G^{t-1} e_0 = \lambda a.$$

Taking the scalar product with e_0 we get $\langle \xi, u \rangle = \lambda \langle a, e_0 \rangle = 0$, which implies by (25) that $G_0^t u = 0$. But the rank of G_0^t must be $d-1$ (it is less than d by assumption, and it cannot be less than $d-1$ because $\pi \circ G^t$ must have rank d). Hence, u must be collinear with πa , an other element of the kernel of G_0^t . Thus, $\langle \sigma(\xi), a \rangle = \langle \xi, \pi a \rangle = 0$.

Case 2. We now assume that G_0^t is invertible, so $\langle a, e_0 \rangle \neq 0$, and we may use Formula (24). Let us introduce the vector $u := G_0^{t-1}b \in \mathbb{R}^d$. The equality $b = G_0^t u$ writes

$$\pi G^t e_0 = G_0^t u = \pi G^t \check{u},$$

with $\check{u} := (0, u) \in \mathbb{R}^{1+d}$. Therefore, there exists $\lambda \in \mathbb{R}$ such that $G^t(e_0 - \check{u}) = \lambda e_0$, so $\check{u} = e_0 - \lambda G^{t-1} e_0 = e_0 - \lambda a$. By taking the scalar product with e_0 we get

$$\lambda \langle a, e_0 \rangle = 1, \quad (26)$$

and by applying π we get

$$u = -\lambda \pi a.$$

Therefore,

$$\langle \xi, u \rangle = -\lambda \langle \xi, \pi a \rangle = -\lambda \langle G^{-1} \check{\xi}, e_0 \rangle = -\lambda \langle G^{-1}(\sigma(\xi) - f(\xi)e_0), e_0 \rangle \quad (27)$$

$$= -\lambda \langle G^{-1} \sigma(\xi), e_0 \rangle + f(\xi) \lambda \langle e_0, a \rangle = -\lambda \langle G^{-1} \sigma(\xi), e_0 \rangle + f(\xi). \quad (28)$$

We see that the determinant (24) vanishes if and only if $\langle \xi, u \rangle = f(\xi)$, *i.e.* if and only if $\langle G^{-1} \sigma(\xi), e_0 \rangle = 0$, which is the same as $\langle \sigma(\xi), a \rangle = 0$. $\square$

Let us return to a more global viewpoint; the map $\xi \mapsto G^t \sigma(\xi)$ sends the unit ball in $\mathbb{R}^d$ to a half ellipsoid $\mathcal{E}_r \subset \mathbb{R}^{1+d}$. We can cut this half ellipsoid into two pieces (separated by the apparent contour $\mathcal{E}_0$ of the map π): $\mathcal{E}_r = \mathcal{E}_- \sqcup \mathcal{E}_0 \sqcup \mathcal{E}_+$, such that the map π is injective on each $\mathcal{E}_\pm$. Therefore, if we let $D_j := \pi(G^{t-1} \mathcal{E}_j)$, then restricted to each of the (semi-algebraic) open sets D_-, D_+ , the map g is now a smooth diffeomorphism into its image $g(D_j) = \pi(\mathcal{E}_j)$. The global lack of injectivity of g on $B(0, 1)$ is due to the fact that the images $g(D_1)$ and $g(D_2)$ will in general overlap.

Equivalently, the hyperplane in $\mathbb{R}^{1+d}$ normal to $a = G^{t-1} e_0$ intersects the right hemisphere in $\mathbb{R}^{1+d}$ into a $d - 1$ dimensional hemisphere, whose projection D_0 onto $\mathcal{P}_0$ separates the unit ball $\|\xi\| < 1$ into the two sets D_-, D_+ , that is

$$B_{\mathcal{P}_0}(0, 1) = D_- \sqcup D_0 \sqcup D_+, \quad (29)$$

with

$$D_\epsilon = \{\xi \in \mathbb{R}^d; \|\xi\| < 1; \langle \sigma(\xi), a \rangle \in \epsilon(0, +\infty)\}, \quad \epsilon \in \{0, +, -\}.$$

If $G = \text{Id}$, then $D_- = \emptyset$.

5.3 Study of $U_{\mathcal{G}}$ and canonical transformations

When the condition of Proposition 5.5 is fulfilled, $V_{\mathcal{G}}$ is a microlocally invertible FIO associated with a canonical transformation $\kappa_{V_{\mathcal{G}}}$ given implicitly by

$$\kappa_{V_{\mathcal{G}}} : (\xi, -\partial_\xi \phi) \mapsto (\tilde{x}, \partial_{\tilde{x}} \phi). \quad (30)$$

Instead of the factorisation $U_G = V_G \circ \mathcal{F}_h$, Formula (13) directly expresses the operator U_G as a unique Fourier integral operator with phase

$$\Phi(\tilde{x}, \xi, x) = \phi(\tilde{x}, \xi) - \langle x, \xi \rangle,$$

where ϕ was defined in (19). Physically speaking, U_G is more relevant than V_G , but of course, it is computationally more involved, since the number of integration variables has doubled with respect to (12). Let us compute the corresponding canonical relation. In view of the general theory (14), ξ is the auxiliary variable, and we get

$$C_\Phi = \{(\tilde{x}, x, \xi); x = \partial_\xi \phi(\tilde{x}, \xi)\}$$

which is always a smooth manifold of dimension $2d$. The canonical relation $(\tilde{x}, \tilde{\xi}) = \kappa_{U_G}(x, \xi)$ is

$$(\tilde{x}, \partial_{\tilde{x}} \Phi, x, -\partial_x \Phi)|_{C_\Phi} = (\tilde{x}, \partial_{\tilde{x}} \phi, \partial_\xi \phi(\tilde{x}, \xi), \xi).$$

As expected (in view of the fact that composition of FIOs is associated with composition of canonical relations), this is the same as the canonical transformation of (30), composed by the canonical transformation in the (x, ξ) space corresponding to the Fourier transform: $(x, \xi) \mapsto (\xi, -x)$:

$$(\tilde{x}, \tilde{\xi}) = \kappa_{U_G}(x, \xi) = \kappa_{V_G}(\xi, -x).$$

Despite the slightly more complicated phase function, it is interesting to notice that the effective computation of the canonical transformation $(\tilde{x}, \tilde{\xi}) = \kappa_\Phi(x, \xi)$ associated with U_G is actually easier than for V_G alone. Indeed, first recall from (20) that the map $g(\xi) = \pi(G^t \sigma(\xi))$ has an invertible differential, and that (from (19))

$$\phi(\tilde{x}, \xi) = \langle \tilde{x}, g(\xi) \rangle_d + \langle \gamma, \sigma(\xi) \rangle.$$

Next, we see that $\tilde{\xi} = \partial_{\tilde{x}} \phi = g(\xi)$ does not depend on x . This imposes the conjugate variable $\tilde{x}$ to be obtained, up to the addition of a closed 1-form in $\tilde{\xi}$, by the inverse adjoint of the differential of g :

$$\tilde{x} = (\mathrm{d}g^t)^{-1} x + [\text{closed}(\tilde{\xi})].$$

This is confirmed by the computation: let $S(\xi) := \langle \gamma, \sigma(\xi) \rangle$; we have $x = \partial_\xi \phi(\tilde{x}, \xi)$ and by (19), for all $v \in \mathbb{R}^d$,

$$\partial_\xi \phi(\tilde{x}, \xi) \cdot v = \langle \tilde{x}, \mathrm{d}g \cdot v \rangle_d + \mathrm{d}S \cdot v,$$

which gives $x = (\mathrm{d}g)^t(\tilde{x}) + \mathrm{d}S$. Locally, g admits a unique inverse, which we denote by g^{-1} . Let $T = (g^{-1})^* S$, so that $\mathrm{d}S = g^* \mathrm{d}T$ and thus

$$\langle \mathrm{d}S(\xi), v \rangle = \langle \mathrm{d}T(\tilde{\xi}), \mathrm{d}g(\xi) \cdot v \rangle = \langle \mathrm{d}g^t(\xi) \mathrm{d}T(\tilde{\xi}), v \rangle.$$

Hence we obtain the expected form

$$\tilde{x} = (\mathrm{d}g^{\mathfrak{t}})^{-1}x - (\mathrm{d}g^{\mathfrak{t}})^{-1}\mathrm{d}S = (\mathrm{d}g^{\mathfrak{t}})^{-1}x - \mathrm{d}T(g(\xi)). \quad (31)$$

This proves the formulas for the canonical transformations below.

Theorem 5.7 *Let $g(\xi) = \pi(G^{\mathfrak{t}}\sigma(\xi))$. The canonical relation of the Fourier integral operator U_G (obtained when $\gamma = 0$) in the region $\|\xi\| < 1$ is the set*

$$\{(x, g(\xi), \mathrm{d}g^{\mathfrak{t}}(\xi)x, \xi), \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d, \|\xi\| < 1\}.$$

Near any ξ satisfying (21), this defines a canonical transformation κ_{U_G} given by:

$$\kappa_{U_G}(x, \xi) = \left((\mathrm{d}g^{\mathfrak{t}}(\xi))^{-1}x, g(\xi) \right). \quad (32)$$

Let $S(\xi) = \langle \gamma, \sigma(\xi) \rangle$. The canonical transformation κ_{U_γ} (obtained when $G = \mathrm{Id}$) is:

$$\kappa_{U_\gamma}(x, \xi) = (x - \mathrm{d}S(\xi), \xi) \quad (33)$$

$$= \left(x + \frac{\gamma_0 \xi}{\sqrt{1 - \|\xi\|^2}} - \pi\gamma, \xi \right), \quad (34)$$

where $\gamma = (\gamma_0, \pi\gamma) \in \mathbb{R}^{1+d}$.

For a general affine transformation $\mathcal{G} = \tau_\gamma \circ G$, we have (31), which can be written:

$$\kappa_{U_{\mathcal{G}}} = \kappa_{U_G} \circ \kappa_{U_\gamma}. \quad (35)$$

Moreover, the map κ_{U_γ} is a symplectomorphism from $\mathbb{R}^d \times B(0, 1)$ onto itself, while the map κ_{U_G} extends to two injective symplectomorphisms from $\mathbb{R}^d \times D_\pm$ (see (29)), onto their respective images.

The composition formula (35) is in accordance with Lemma 5.2. In fact, another way to prove Theorem 5.7 is to use the factorisation (9), which is valid only in a microlocal sense:

$$U_G = \mathcal{F}_h^{-1} \circ J \circ (g^{-1})^* \circ \mathcal{F}_h \quad (36)$$

which gives

$$U_{\mathcal{G}} = U_G \circ U_\gamma = \mathcal{F}_h^{-1} \circ J \circ (g^{-1})^* \circ e^{\frac{i}{h}S} \circ \mathcal{F}_h;$$

it then remains to compose the corresponding canonical transformations, having in mind that multiplication by J is a pseudodifferential operator, so is associated with the identity transformation.

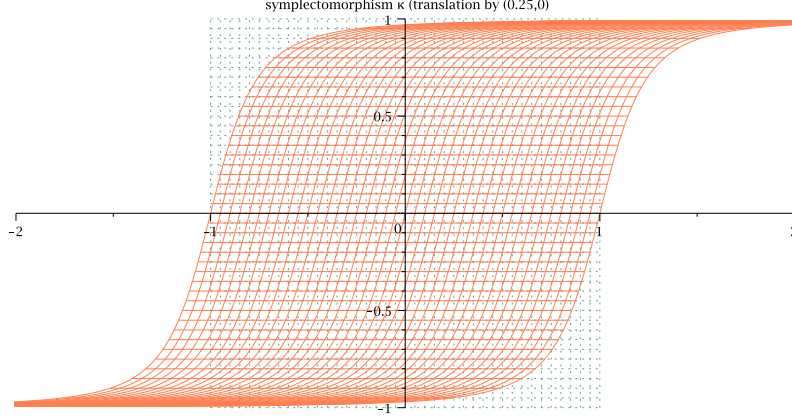


Figure 7 – The symplectomorphism κ_{U_γ} , when $\gamma = (1/4, 0)$. The dashed blue grid is the source domain $(-1, 1) \times (-1, 1)$ and the orange curves are its image under κ_{U_γ} .

The 1D case. — The canonical transformations are quite explicit when $d = 1$. For a translation of vector $\gamma = (\gamma_0, \gamma_1)$, formula (34) becomes

$$\kappa_{U_\gamma}(x, \xi) = \left(x + \frac{\gamma_0 \xi}{\sqrt{1 - \xi^2}} - \gamma_1, \xi \right), \quad (x, \xi) \in \mathbb{R} \times (-1, 1). \quad (37)$$

See Figure 7.

For a rotation $G = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$ we have $g(\xi) = -\sqrt{1 - \xi^2} \sin \alpha + \xi \cos \alpha$ and hence (32) becomes

$$\kappa_{U_G}(x, \xi) = \left(\frac{x}{\frac{\xi \sin \alpha}{\sqrt{1 - \xi^2}} + \cos \alpha}, -\sqrt{1 - \xi^2} \sin \alpha + \xi \cos \alpha \right). \quad (38)$$

See Figure 8.

Both formulas turn out to be even simpler if we switch to another set of canonical variables $(t, \theta) \in \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by

$$x = \frac{t}{\cos \theta}, \quad \xi = \sin \theta.$$

Indeed, $g(\xi) = \sin(\theta - \alpha)$ and the formulas for the corresponding canonical transformations $\tilde{\kappa}(t, \theta) = (\tilde{t}, \tilde{\theta})$ are

$$\tilde{\kappa}_{U_\gamma}(t, \theta) = (t + \gamma_0 \sin \theta - \gamma_1 \cos \theta, \theta)$$

and, if $\theta - \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\tilde{\kappa}_{U_G}(t, \theta) = \begin{cases} (t, \theta - \alpha) & \text{if } \theta - \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) \pmod{2\pi} \\ (-t, \pi + \alpha - \theta) & \text{if } \theta - \alpha \in (\frac{\pi}{2}, \frac{3\pi}{2}) \pmod{2\pi}. \end{cases}$$

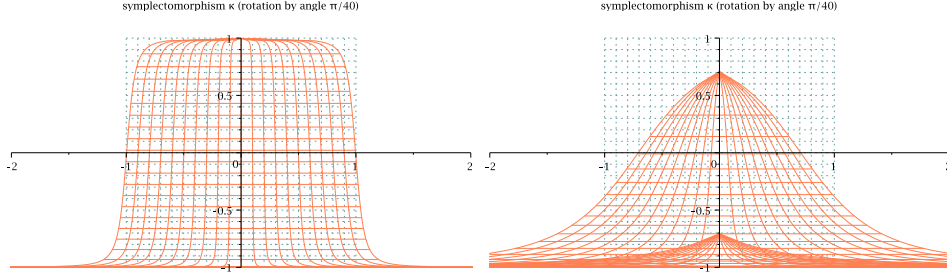


Figure 8 – The symplectomorphism κ_{U_G} , when $G \in \text{SO}(2)$ is the rotation of angle $\alpha = \pi/40$ (left) and $\alpha = \pi/4$ (right). The dashed blue grid is the source domain $(-1, 1) \times (-1, 1)$ and the orange curves are its image under κ_{U_G} (cropped to $(-2, 2)$; the true image extends to $(-\infty, +\infty) \times \{-1\}$ when $\xi \rightarrow \xi_\alpha = -\cos(\alpha)$). On the right-hand side figure, we see the non-injectivity showing up at the bottom of the picture.

The non-degeneracy condition (Proposition 5.5) is here simply $\theta - \alpha \neq \frac{\pi}{2} \bmod \pi$, which means either $\alpha = 0$ modulo π , or $\xi \neq \xi_\alpha$ with $\xi_\alpha := -\varepsilon_\alpha \cos \alpha$ and $\varepsilon_\alpha := \text{sign}(\sin \alpha)$. The injectivity domains (29) for ξ are the intervals $D_{-\varepsilon_\alpha} = (-1, \xi_\alpha)$, $D_{\varepsilon_\alpha} = (\xi_\alpha, 1)$. For instance, when $\alpha = \frac{\pi}{3}$, the points $(-\frac{1}{4}, 0)$ and $(x_2 = 1/2, \xi_2 = -\sqrt{3}/2)$ have the same image under κ_{U_G} , see Figure 9.

5.4 Geometric optics

Based on the canonical transformations of Theorem 5.7, which we obtained directly from the solution to the Helmholtz equation, we explain in this section how to recover the expected laws of geometric optics.

Lemma 5.8 *Let $(x_B, \xi_B) = \kappa_{U_G}(x_A, \xi_A)$. We consider the usual inclusion $\pi^t : \mathbb{R}^d \rightarrow \mathbb{R}^{1+d}$ defined by $\pi^t(x) = (0, x)$. Let $A = \pi^t x_A$ and $B = \mathcal{G} \pi^t x_B$. Then, $B - A$ is collinear to $\sigma(\xi_A)$.*

Proof. Because of Theorem 5.7, we may treat the cases $\mathcal{G} = G$ and $\mathcal{G} = \tau_\gamma$ separately.

Case $\mathcal{G} = G$. In view of the notation from (22), this gives

$$B = G \pi^t x_B = \begin{pmatrix} b^t \\ G_0 \end{pmatrix} x_B = \langle b, x_B \rangle e_0 + \pi^t G_0 x_B.$$

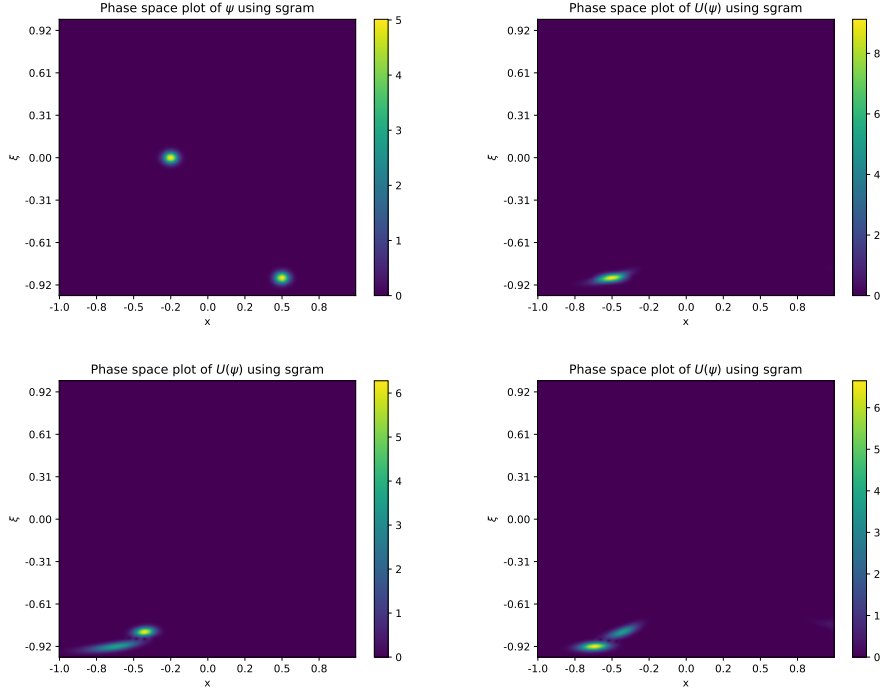


Figure 9 – The initial image (top left) is the phase space representation of a sum of two coherent states

$$\psi(x) = e^{-\frac{(x-x_0)^2}{2\hbar}} e^{\frac{i}{\hbar}x\xi_0} + e^{-\frac{(x-x_1)^2}{2\hbar}} e^{\frac{i}{\hbar}x\xi_1}$$

with $(x_0, \xi_0) = (-\frac{1}{4}, 0)$ and $(x_1, \xi_1) = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. It follows from (38) that they must superpose after a rotation G of the plane P_0 by an angle $\alpha = \frac{\pi}{3}$. Hence, according to Egorov’s theorem, the wave front set of $U_G\psi$ must contain only one point, which is confirmed by the numerical experiment (the top right image is a phase space representation of $U_G\psi$). The pictures below show the representations just before (bottom left, $\alpha = 54^\circ$) and after (bottom right, $\alpha = 66^\circ$) the “collision”. A movie showing the full rotation can be found online [16].

From (32), $x_A = (\text{dg}(\xi_A)^t)x_B$, and using (23), we have

$$x_A = -\frac{\langle b, x_B \rangle}{f(\xi_A)} \xi_A + G_0 x_B \quad (39)$$

$$\text{hence } A = -\langle b, x_B \rangle \left(\frac{\pi^t \xi_A + f(\xi_A) e_0}{f(\xi_A)} \right) + G \pi^t x_B \quad (40)$$

$$= -\frac{\langle b, x_B \rangle}{f(\xi_A)} \sigma(\xi_A) + B \quad (41)$$

which proves the lemma for G .

Case $\mathcal{G} = \tau_\gamma$. We have $A = \pi^t x_A$ and $B = \pi^t x_B + \gamma$. From Theorem 5.7, $x_B = x_A - dS(\xi_A)$, so $B - A = \gamma - \pi^t dS(\xi_A)$. Consider the map $\xi \mapsto S(\xi) - \langle \pi\gamma, \xi \rangle = \langle \gamma, \sigma(\xi) - \pi^t \xi \rangle = f(\xi) \langle \gamma, e_0 \rangle$. Its differential $dS(\xi) - \pi\gamma$ is hence equal to $-\frac{\langle \gamma, e_0 \rangle}{f(\xi)} \xi$. Hence

$$\begin{aligned} A - B &= \pi^t dS(\xi_A) - \gamma = \pi^t (dS(\xi_A) - \pi\gamma) - \langle \gamma, e_0 \rangle e_0 \\ &= -\frac{\langle \gamma, e_0 \rangle}{f(\xi_A)} (\pi^t \xi_A + f(\xi_A) e_0) \\ &= -\frac{\langle \gamma, e_0 \rangle}{f(\xi_A)} \sigma(\xi_A), \end{aligned}$$

which finishes the proof. $\square$

Thanks to this lemma, the map $\kappa_{U_{\mathcal{G}}}$ can be described as follows. To any point $(x_A, \xi_A) \in \mathbb{R}^{d+d}$, we associate the “light ray” in $\mathbb{R}^{1+d}$ emanating from the point $A = \pi^t x_A = (0, x_A) \in P_0$ with direction $\sigma(\xi_A)$. Under the condition of Proposition 5.5, this line intersects the hyperplane $\mathcal{G}P_0$ in a unique point B . Let Point B be parameterised by $x_B \in \mathbb{R}^d$ via $B = \mathcal{G}\pi^t x_B = \mathcal{G}((0, x_B))$. It follows from Lemma 5.8 that the map $x_A \mapsto x_B$ is exactly the x -component of the canonical transformation of Theorem 5.7.

It remains to interpret the transformation of the ξ component, namely $\xi_B = g(\xi_A)$. For this we simply consider a plane wave on $\mathbb{R}^{1+d}$ directed by $\sigma(\xi_A)$:

$$\psi(X) = e^{\frac{i}{\hbar} \langle X, \sigma(\xi_A) \rangle}.$$

It is a solution to the Helmholtz/Laplace/Schrödinger equation (10). Its restriction to P_0 is $\psi_0(x) = (\pi^t)^* \psi(x) = e^{\frac{i}{\hbar} \langle x, \xi_A \rangle}$, for any $x \in \mathbb{R}^d$: it is a plane wave directed by ξ_A (that is, with wave front set $\text{WF}_h(\psi_0) = \mathbb{R}^d \times \{\xi_A\}$). We now consider the transformed hologram $\psi_1 = (\pi^t)^* (\mathcal{G}^* \psi) = (\mathcal{G} \circ \pi^t)^* \psi$. First of all, since $\kappa_{U_{\mathcal{G}}}$ preserves the ξ component, we may assume that $\mathcal{G} = G \in \text{GL}_{1+d}(\mathbb{R})$. We then compute easily $\psi_1(x) = e^{\frac{i}{\hbar} \langle x, \pi G^t \sigma(\xi_A) \rangle} = e^{\frac{i}{\hbar} \langle x, g(\xi_A) \rangle}$, so ψ_1 is again plane wave directed by $\xi_B = g(\xi_A)$, as expected. Note also that, if $G^t \sigma(\xi_A)$ belongs to the right half-space, then this relation is equivalent to $\sigma(\xi_B) = G^t \sigma(\xi_A)$, which expresses the new direction of the light ray, viewed in the transformed hologram, in terms of its direction in the original hologram.

Remark 5.9 By construction, the canonical transformations of Theorem 5.7 are symplectic with respect to the usual (canonical) symplectic form $d\xi \wedge dx$ of T^*P_0 . One may wonder whether this symplectic form is natural, given the fact that the original phase space of our problem is rather $T^*\mathbb{R}^{1+d}$, and P_0 is just an embedded hyperplane. A way to see this is to consider the restriction operator $(\pi^t)^* : \psi \mapsto \psi|_{P_0} = x \mapsto \psi(\pi^t(x))$, which we implicitly use to define $U_{\mathcal{G}}$. It can be written as the Fourier integral operator

$$(\pi^t)^* \psi(x) = \frac{1}{(2\pi\hbar)^{1+d}} \int e^{\frac{i}{\hbar} (\langle \pi^t(x), \theta \rangle - \langle X, \theta \rangle)} \psi(X) dX d\theta.$$

(The difference with (14) is that we now allow X and x (which play the roles of x and $\tilde{x}$ there, respectively) to have different dimensions, but the theory remains valid.) The phase function is $\phi(x, X, \theta) = \langle \pi^t(x), \theta \rangle - \langle X, \theta \rangle$, the manifold C_ϕ is given by $X = \pi^t(x)$, and hence the Lagrangian Λ_ϕ is

$$\Lambda_\phi = \{(x, d\pi \cdot \theta, \pi^t(x), \theta), \quad x \in \mathbb{R}^d, \theta \in (\mathbb{R}^{1+d})^*\}.$$

If we write $\theta = (\theta_0, \xi)$ with $\xi \in (\mathbb{R}^d)^*$ (so that $d\pi \cdot \theta = \xi$), we see that this canonical relation sends the restriction of the canonical symplectic form $(d\theta \wedge dX)|_{P_0 \times (\mathbb{R}^{1+d})^*}$ to $d\xi \wedge dx$, which shows that the latter is the correct symplectic form to consider on T^*P_0 .

Thus, the symplectic form $d\xi \wedge dx$ is dictated by the choice of the Angular Spectrum formula. Its simplicity makes it an appealing choice for microlocal formulas; however, as mentioned before, it raises difficulties due to the non-injectivity of the canonical transformation. It could be interesting to develop a microlocal analysis of holograms directly on the co-sphere, as developed in a different context by [6].

△

6 Precised Egorov Theorem

The canonical transformation of Theorem 5.7 gives a first approximation of the propagator $U_{\mathcal{G}}$ in the sense of geometric optics: if a signal ψ is localised in phase space in some region Ω (its *wave front set* is contained in Ω), then $U_{\mathcal{G}}\psi$ is localised in $\kappa(\Omega)$, where κ is the canonical transformation associated with $U_{\mathcal{G}}$, see Figure 10. However, the notion of wave front set is not very precise: it hides the information about how ψ concentrates on the classical rays, and about the phase of ψ .

In order to obtain precise information about the propagated signal $U_{\mathcal{G}}\psi$, we need to introduce the notion of *quantum observables* (in signal processing, they would be time-frequency filters — although here the phase space is rather “position-direction”). They are selfadjoint operators P , and the “observation” of a normalised state (or signal) Ψ by P is by definition the scalar product $\langle P\Psi, \Psi \rangle$. For instance, the operator P can be the position operator $P = x_j$, the momentum operator $P = \frac{\hbar}{i}\partial_{x_j}$, or any combination, that is, a pseudodifferential operator with symbol $p(x, \xi)$. If we wish to observe the propagated signal $U_{\mathcal{G}}\psi$, we are led to compute

$$\langle PU_{\mathcal{G}}\psi, U_{\mathcal{G}}\psi \rangle = \langle U_{\mathcal{G}}^*PU_{\mathcal{G}}\psi, \psi \rangle.$$

Thus, we see that this amounts to the observation of the *initial* state ψ with the new operator $U_{\mathcal{G}}^*PU_{\mathcal{G}}$. The main tool to compute this new operator is the Egorov Theorem, whose general statement is as follows. Let P be a

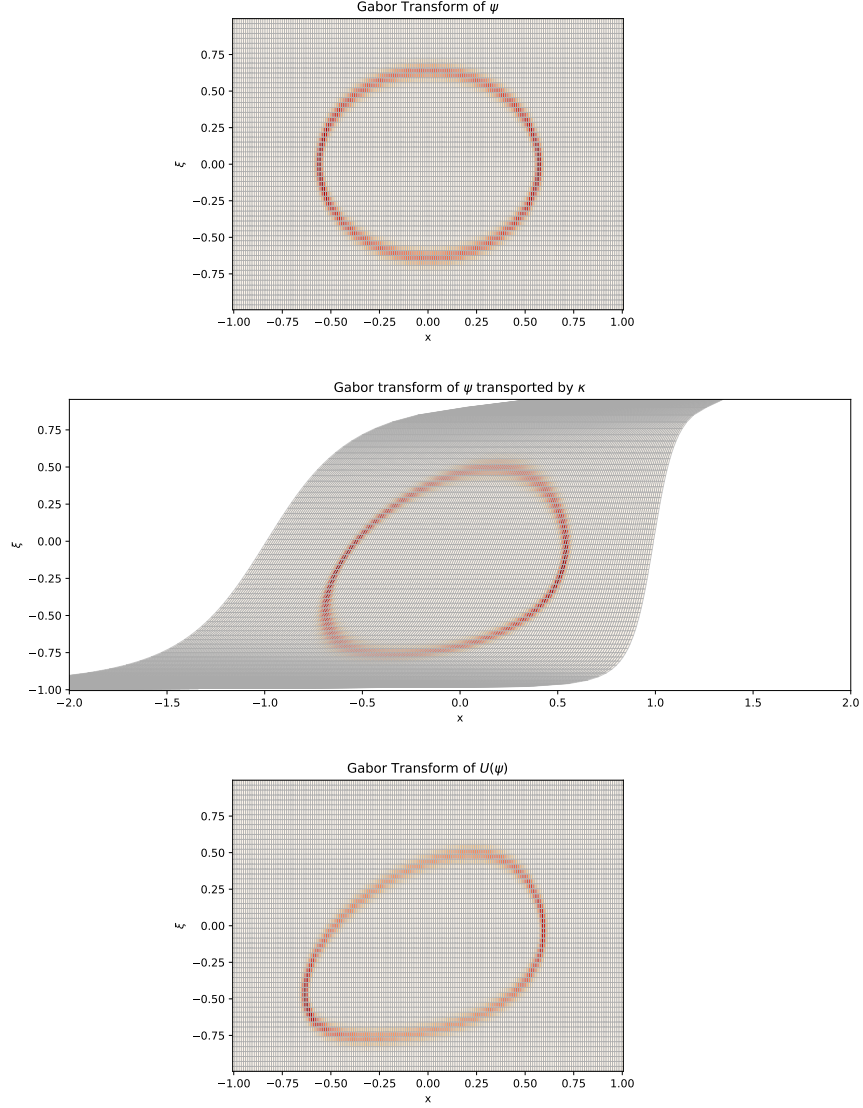


Figure 10 – Here ψ is a Hermite function (same as Figure 3). The first figure shows its semiclassical wavefront set, using a semiclassical Gabor transform. The second figure is obtained simply by applying the canonical transformations of Theorem 5.7 to Figure 1, with $\mathcal{G}X = GX + \gamma$, where G is a rotation of angle 10° and $\gamma = (\frac{3}{10}, 0)$. It should be compared to the last figure, which is the semiclassical wavefront set of the propagated signal $U_{\mathcal{G}}(\psi)$: one can see that the curves coincide quite perfectly, which is predicted by Egorov's theorem.

pseudodifferential operator with principal symbol $p(x, \xi)$. Then, when $U_{\mathcal{G}}$ is microlocally invertible,

$$U_{\mathcal{G}}^{-1} P U_{\mathcal{G}} = R$$

where R is a pseudodifferential operator with principal symbol $r = p \circ \kappa_{U_{\mathcal{G}}}$. If we fix a quantization scheme, for instance Weyl quantization, which associates to a symbol $p(x, \xi)$ a pseudodifferential operator $P = \text{Op}_h^w(p)$, we obtain

$$U_{\mathcal{G}}^{-1} \text{Op}_h^w(p) U_{\mathcal{G}} = \text{Op}_h^w(p \circ \kappa_{U_{\mathcal{G}}}) + \mathcal{O}(\hbar)$$

Suppose first that $U_{\mathcal{G}}$ is unitary: $U_{\mathcal{G}}^{-1} = U_{\mathcal{G}}^*$. Then, in order to observe the propagated signal $U_{\mathcal{G}}\psi$ with $\text{Op}_h^w(p)$, Egorov's theorem tells us that it is enough to observe the initial signal ψ with the operator $\text{Op}_h^w(p \circ \kappa_{U_{\mathcal{G}}})$, up to small errors of order $\mathcal{O}(\hbar)$. This is appealing for applications, since we can avoid computing the exact propagation $U_{\mathcal{G}}$, and content ourselves with the classical canonical transformation $\kappa_{U_{\mathcal{G}}}$.

However, if we really want to observe effects that go beyond geometric optics, we should understand the $\mathcal{O}(\hbar)$ remainder. This term is notoriously more difficult to compute, as it depends on the precise (non-geometric) formula for the quantum propagator. In this work, we obtain an explicit formula for this remainder, given in Theorem 6.4 which, to the best of our knowledge, is a new result.

Finally, in our case, $U_{\mathcal{G}}$ is not unitary, and the formula for $U_{\mathcal{G}}^{-1} P U_{\mathcal{G}}$ does not apply to $U_{\mathcal{G}}^* P U_{\mathcal{G}}$. In order to obtain the latter, we need to compute the defect of unitarity, which we perform in Section 7. As a result, this proves that the Egorov theorem for $U_{\mathcal{G}}^* P U_{\mathcal{G}}$ holds with accuracy $\mathcal{O}(\hbar^2)$ (Theorem 7.3).

In the case of the Schrodinger equation (which can be seen as an approximation of Helmholtz' equation in the paraxial regime [9]), related Egorov-type theorems have been obtained, see for instance [13] and references therein. In these works, the fact that the propagator is really the exponential of a (pseudo)differential simplifies the computation of higher order terms. Unfortunately, we can use these techniques for our Helmholtz propagator.

Remark 6.1 In the case of (microlocally) unitary Fourier integral operators U of the form (14), it was proved in [23] that Egorov's theorem holds with $\mathcal{O}(\hbar^2)$ remainder when the phase of a_0 is constant. It could be interesting to try to use their result to give a different proof of Theorem 7.3. $\triangle$

6.1 Products of V_G with pseudodifferential operators

Phase space filters are essential for holographic studies because they allow (smooth) truncation simultaneously in position and direction. For instance, the eye is naturally such a phase space filter, for not only has it a specific position in space, but also it selects a narrow beams of light rays which are directed towards it. In our phase space analysis, filter are conveniently represented by semiclassical pseudodifferential operators. Our goal in this section is to study the two cases where the filter is applied before or after

the propagation U_G . Since the action of the Fourier transform is easy to deal with, and in view of the decomposition given by Lemma 5.2, it is enough to consider products of V_G , $G \in \text{GL}_{1+d}(\mathbb{R})$, with pseudodifferential operators, where V_G is the integral operator defined in (15).

Let P, Q be semiclassical pseudodifferential operators. We consider PV_G and V_GQ . In terms of integral (*i.e.* Schwartz) kernels, we have

$$K_{V_G}(y, \eta) = \frac{e^{\frac{i}{\hbar}\phi(y, \eta)}}{(2\pi\hbar)^d}$$

$$K_P(x, y) = \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\langle x-y, \tilde{\xi} \rangle} p\left(\frac{x+y}{2}, \tilde{\xi}\right) d\tilde{\xi}$$

$$K_Q(\eta, \xi) = \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\langle \eta-\xi, \tilde{x} \rangle} q\left(\frac{\eta+\xi}{2}, \tilde{x}\right) d\tilde{x}$$

Hence

$$K_{PV_G}(x, \eta) = \frac{1}{(2\pi\hbar)^d} \iint e^{\frac{i}{\hbar}\langle x-y, \tilde{\xi} \rangle} p\left(\frac{x+y}{2}, \tilde{\xi}\right) e^{\frac{i}{\hbar}\phi(y, \eta)} d\tilde{\xi} dy$$

and

$$K_{V_GQ}(y, \xi) = \frac{1}{(2\pi\hbar)^d} \iint e^{\frac{i}{\hbar}\langle \eta-\xi, \tilde{x} \rangle} q\left(\frac{\eta+\xi}{2}, \tilde{x}\right) e^{\frac{i}{\hbar}\phi(y, \eta)} d\eta d\tilde{x}.$$

These are oscillatory integrals with respective phases

$$\Phi_{(x, \eta)}^{PV_G}(\tilde{\xi}, y) = \langle x - y, \tilde{\xi} \rangle + \phi(y, \eta) = \langle x - y, \tilde{\xi} \rangle + \langle y, \pi(G^t \sigma(\eta)) \rangle,$$

and

$$\Phi_{(y, \xi)}^{V_GQ}(\eta, \tilde{x}) = \langle \eta - \xi, \tilde{x} \rangle + \phi(y, \eta) = \langle \eta - \xi, \tilde{x} \rangle + \langle y, \pi(G^t \sigma(\eta)) \rangle.$$

The first one, $\Phi_{(x, \eta)}^{PV_G}(\tilde{\xi}, y)$, is associated with the Lagrangian manifold (its wavefront) Λ_{PV_G} given by

$$\partial_{\tilde{\xi}} \Phi_{(x, \eta)}^{PV_G} = 0; \quad \partial_y \Phi_{(x, \eta)}^{PV_G} = 0.$$

This gives $x = y$ and $\tilde{\xi} = \partial_y \phi(y, \eta) = \pi(G^t \sigma(\eta)) = g(\eta)$, and the value of the phase on Λ_{PV_G} is simply $\phi(x, \eta)$. Notice that $\Phi_{(x, \eta)}^{PV_G}(\tilde{\xi}, y)$ is polynomial of degree 2 in its variables $(\tilde{\xi}, y)$. Hence the stationary phase formula is “explicit” and yields (see for instance [20, Theorem 15.5.1] or [34, Theorem 3.13])

$$K_{PV_G}(x, \eta) \sim e^{\frac{i}{\hbar}\phi(x, \eta)} \frac{e^{i\frac{\pi}{4}\text{sgn}(\mathcal{Q}(x, \eta))}}{|\det \mathcal{Q}(x, \eta)|^{1/2}} \sum_{k \geq 0} \frac{\hbar^k}{k!} \left(\frac{\langle \mathcal{Q}(x, \eta)^{-1} D, D \rangle}{2i} \right)^k p\left(\frac{x+y}{2}, \tilde{\xi}\right)$$

taken at $y = x$, $\tilde{\xi} = \pi(G^t \sigma(\eta))$. We have denoted the quadratic form $\mathcal{Q}(x, \eta) := (\Phi_{(x, \eta)}^{PV_G})''(0)$ and $D = \frac{1}{i}(\partial_{\tilde{\xi}}, \partial_y)$.

Since the quadratic part of $\Phi_{(x,\eta)}^{PV_G}(\tilde{\xi}, y)$ is simply $-\langle y, \tilde{\xi} \rangle$, its eigenvalues are $(1, \dots, 1, -1, \dots, -1)$ and hence the signature vanishes and the determinant is 1; moreover $\mathcal{Q}^{-1} = \mathcal{Q}$, so

$$\langle \mathcal{Q}(x, \eta)^{-1} D, D \rangle = 2 \langle \partial_{\tilde{\xi}}, \partial_y \rangle$$

and

$$\begin{aligned} K_{PV_G}(x, \eta) &\sim e^{\frac{i}{\hbar} \langle x, \tilde{\xi} \rangle} \sum_{k \geq 0} \frac{(-i\hbar)^k}{k!} \langle \partial_{\tilde{\xi}}, \partial_y \rangle^k p\left(\frac{x+y}{2}, \tilde{\xi}\right)_{y=x} \\ &\sim e^{\frac{i}{\hbar} \langle x, \tilde{\xi} \rangle} \exp\left(\frac{\hbar}{2i} \langle \partial_{\tilde{\xi}}, \partial_x \rangle\right) p(x, \tilde{\xi}) \\ &= e^{\frac{i}{\hbar} \langle x, \tilde{\xi} \rangle} \left(p(x, \tilde{\xi}) - \frac{i\hbar}{2} \langle \partial_{\tilde{\xi}}, \partial_x \rangle p(x, \tilde{\xi}) + \mathcal{O}(\hbar^2) \right) \end{aligned} \quad (42)$$

Of course, this works equally well when $p = p_\hbar$ admits an asymptotic expansion of the form $p_\hbar = p_0 + \hbar p_1 + \mathcal{O}(\hbar^2)$. We obtain

Proposition 6.2 *Let $P = \text{Op}_\hbar^w(p_\hbar)$, with a symbol $p_\hbar$ of the form $p_\hbar = p_0 + \hbar p_1 + \mathcal{O}(\hbar^2)$. We have, in the C^∞ topology,*

$$K_{PV_G}(x, \eta) = e^{\frac{i}{\hbar} \langle x, \tilde{\xi} \rangle} \left(p_0(x, \tilde{\xi}) + \hbar p_1(x, \tilde{\xi}) - \frac{i\hbar}{2} \langle \partial_{\tilde{\xi}}, \partial_x \rangle p_0(x, \tilde{\xi}) + \mathcal{O}(\hbar^2) \right), \quad (43)$$

where $\tilde{\xi} = g(\eta)$.

The second composition, $V_G \mathcal{Q}$, is more complicated because the phase is not quadratic anymore. The Lagrangian manifold $\Lambda_{V_G \mathcal{Q}}$ is given by

$$\partial_{\tilde{x}} \Phi_{(y,\xi)}^{V_G \mathcal{Q}}(\eta, \tilde{x}) = 0 \quad \partial_\eta \Phi_{(y,\xi)}^{V_G \mathcal{Q}}(\eta, \tilde{x}) = 0,$$

which gives $\eta = \xi$ and $\tilde{x} = -\partial_\eta \phi(y, \eta) = -\text{dg}^t(\eta) \cdot y$. We will now Taylor expand the phase $\phi(y, \eta)$ (recall (19)) with respect to η at $\eta = \xi$ (and $\|\xi\| < 1$); as before, we write

$$\sigma(\eta) = (f(\eta), \eta), \quad f(\eta) = \sqrt{1 - \|\eta\|^2}$$

and use, with $\tilde{\eta} = \eta - \xi$:

$$f(\xi + \tilde{\eta}) = f(\xi) - \frac{\langle \xi, \tilde{\eta} \rangle}{f(\xi)} - \frac{\|\tilde{\eta}\|^2}{2f(\xi)} - \frac{\langle \xi, \tilde{\eta} \rangle^2}{2f(\xi)^3} + \mathcal{O}(\tilde{\eta}^3). \quad (44)$$

Hence

$$\sigma(\xi + \tilde{\eta}) = \sigma(\xi) + \left(-\frac{\langle \xi, \tilde{\eta} \rangle}{f(\xi)}, \tilde{\eta} \right) - \left(\frac{\|\tilde{\eta}\|^2}{2f(\xi)} + \frac{\langle \xi, \tilde{\eta} \rangle^2}{2f(\xi)^3} \right) e_0 + g_\xi(\tilde{\eta}) \quad (45)$$

$$=: \sigma(\xi) + \sigma_{1;\xi}(\tilde{\eta}) + \sigma_{2;\xi}(\tilde{\eta}) + g_\xi(\tilde{\eta}), \quad (46)$$

where $e_0 := (1, 0, \dots, 0)$ and $g_\xi = \mathcal{O}(\tilde{\eta}^3)$. Using (19), we write, accordingly:

$$\phi(y, \xi + \tilde{\eta}) = \phi(y, \xi) + \phi_{1;\xi}(y, \tilde{\eta}) + \phi_{2;\xi}(y, \tilde{\eta}) + G_\xi(y, \tilde{\eta})$$

and hence

$$K_{V_G Q}(y, \xi) = \frac{e^{\frac{i}{\hbar}\phi(y, \xi)}}{(2\pi\hbar)^d} \iint e^{\frac{i}{\hbar}\langle \tilde{\eta}, \tilde{x} \rangle} e^{\frac{i}{\hbar}(\phi_{1;\xi}(y, \tilde{\eta}) + \phi_{2;\xi}(y, \tilde{\eta}))} \tilde{q}(\xi + \frac{\tilde{\eta}}{2}, \tilde{x}) d\tilde{x} d\tilde{\eta}. \quad (47)$$

with

$$\tilde{q}(\xi + \frac{\tilde{\eta}}{2}, \tilde{x}) := e^{\frac{i}{\hbar}G_\xi(y, \tilde{\eta})} q(\xi + \frac{\tilde{\eta}}{2}, \tilde{x}).$$

Notice that $\phi_{1;\xi}$ is linear, $\phi_{2;\xi}$ is quadratic, and $G_\xi(y, \tilde{\eta}) = \mathcal{O}(\tilde{\eta}^3)$. This last estimate implies that $\tilde{q}$ is “slowly oscillating” when $\tilde{\eta}$ is small, which enables the use of the stationary phase formula with the apparent quadratic phase of (47).

The quadratic part of the phase is

$$\begin{aligned} \mathcal{R}(\tilde{x}, \tilde{\eta}) &= \langle \tilde{\eta}, \tilde{x} \rangle - \phi_{2;\xi}(\tilde{\eta}) \\ &= \langle \tilde{\eta}, \tilde{x} \rangle - \left(\frac{\|\tilde{\eta}\|^2}{2f(\xi)} + \frac{\langle \xi, \tilde{\eta} \rangle^2}{2f(\xi)^3} \right) \langle y, \pi(G^t e_0) \rangle. \end{aligned} \quad (48)$$

Its Hessian matrix (of size $2d \times 2d$) has the form (in the variables $(\tilde{x}, \tilde{\eta})$)

$$\mathcal{R} = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & -\frac{\alpha}{f} \left(\text{Id} + \frac{1}{f^2} \xi \xi^t \right) \end{pmatrix} \quad (49)$$

with $\alpha := \alpha(y) = \langle y, \pi(G^t e_0) \rangle$ and $\xi \xi^t$ is the matrix $(\xi_i \xi_j)_{1 \leq i, j \leq d}$. The signature is zero and the determinant is 1. The inverse matrix is

$$\mathcal{R}^{-1} = \begin{pmatrix} \frac{\alpha}{f} \left(\text{Id} + \frac{1}{f^2} \xi \xi^t \right) & \text{Id} \\ \text{Id} & 0 \end{pmatrix}$$

We find, with $D = \frac{1}{i}(\partial_{\tilde{x}}, \partial_{\tilde{\eta}})$,

$$\langle \mathcal{R}^{-1} D, D \rangle = -2 \langle \partial_{\tilde{x}}, \partial_{\tilde{\eta}} \rangle - \frac{\alpha(y)}{f(\xi)} \left(\Delta_{\tilde{x}} + \frac{1}{f(\xi)^2} \langle \xi, \partial_{\tilde{x}} \rangle^2 \right).$$

We apply again the quadratic stationary phase lemma:

$$K_{V_G Q}(y, \xi) \sim e^{\frac{i}{\hbar}\phi(y, \xi)} \sum_{k \geq 0} \frac{(i\hbar)^k}{k!} \left(-\frac{1}{2} \langle \mathcal{R}^{-1} D, D \rangle \right)^k \tilde{q}(\xi + \frac{\tilde{\eta}}{2}, \tilde{x})_{\tilde{\eta}=0}$$

and use the fact that the values of $\tilde{q}$ and its derivatives up to order 2 on the critical set $\Lambda_{V_G Q}$ (where $\tilde{\eta} = 0$) are equal to those of q , to obtain

Proposition 6.3 *If $Q = \text{Op}_h^w(q_h)$, with $q_h = q_0 + \hbar q_1 + \mathcal{O}(\hbar^2)$, we have*

$$\begin{aligned} K_{V_G Q}(y, \xi) = & e^{\frac{i}{\hbar} \phi(y, \xi)} [q_0(\xi, \tilde{x}) + \hbar q_1(x, \xi) \\ & + \frac{i\hbar}{2} \left(\langle \partial_{\tilde{x}}, \partial_{\xi} \rangle q_0(\xi, \tilde{x}) + \frac{\alpha \Delta_{\tilde{x}} q_0(\xi, \tilde{x})}{f} + \frac{\alpha}{f^3} \langle \xi, \partial_{\tilde{x}} \rangle^2 q_0(\xi, \tilde{x}) \right) \\ & + \mathcal{O}(\hbar^2)] , \end{aligned} \quad (50)$$

where $f = f(\xi) = \sqrt{1 - \|\xi\|^2}$, $\alpha = \alpha(y) = \langle y, \pi(G^t e_0) \rangle$, and $\tilde{x} = -\partial_{\xi} \phi(y, \xi) = -\text{d}g^t(\xi) \cdot y$.

Propositions 6.2 and 6.3 offer a way to computing the actions of phase space filters on the transformed hologram directly from the initial hologram ψ_0 ; they are also instrumental in proving the Egorov theorem mentioned earlier, as we shall see in the next section.

6.2 Conjugation by U_G

Let us now turn to the Egorov property. Assume that $Q = \text{Op}_h^w(q_h)$ and $P = \text{Op}_h^w(p_h)$ are pseudodifferential operators related by the (microlocal) equation

$$PV_G = V_G Q.$$

Let $(x, \xi) \in \mathbb{R}^{2d}$ and assume that $\sigma(\xi) \notin G\mathcal{P}_0$ (see Proposition 5.5). From Proposition 6.2 and Proposition 6.3 we get, by equating terms of order zero in $\hbar$ in $K_{PV_G}(x, \xi) = K_{V_G Q}(x, \xi)$:

$$p_0(x, \partial_x \phi(x, \xi)) = q_0(\xi, -\partial_{\xi} \phi(x, \xi)) \quad (51)$$

(indeed, the term $e^{\frac{i}{\hbar} \langle x, \tilde{\xi} \rangle}$ in (43) is equal to $e^{\frac{i}{\hbar} \phi(x, \eta)}$). In other words, $q_0 = p_0 \circ \kappa_{V_G}$, where κ_{V_G} is the canonical transformation (30). We have simply recovered the usual second statement of the Egorov theorem.

Our new result is the computation of the term of order $\hbar$. We obtain the implicit equation

$$p_1 - q_1 = \frac{i}{2} \left(\langle \partial_{\tilde{\xi}}, \partial_x \rangle p_0 + \langle \partial_{\tilde{x}}, \partial_{\xi} \rangle q_0 + \frac{\alpha(x)}{f(\xi)} \Delta_{\tilde{x}} q_0 + \frac{\alpha(x)}{f(\xi)^3} \langle \xi, \partial_{\tilde{x}} \rangle^2 q_0 \right) \quad (52)$$

where p_0, p_1 are evaluated at $(x, \tilde{\xi} = \partial_x \phi(x, \xi))$ and q_0, q_1 are evaluated at $(\xi, \tilde{x} = -\partial_{\xi} \phi(x, \xi))$.

We may now come back to the original operator $U_G = V_G \mathcal{F}_h$. In order to obtain a more pleasant formulation, we introduce a *microlocal left inverse* U_G^{-1} of U_G . In terms of wavefronts, U_G transports wave functions microlocalized near (x, ξ) to wave functions microlocalized near $\kappa_{U_G}(x, \xi)$. Since κ_{U_G} might not be injective (see Theorem 5.7), its inverse is multivalued. By definition, we call U_G^{-1} the FIO that satisfies $U_G^{-1} U_G \sim \text{Id}$ near (x, ξ) . We shall also denote by $g^{-1}(\tilde{\xi})$ a left inverse of g defined near $g(\xi)$.

Theorem 6.4 *Let P be a semiclassical pseudodifferential operator with Weyl symbol $p = p_0$ independent of $\hbar$. Then, in a phase space region where $\|\xi\| \leq 1 - \epsilon$, $\epsilon > 0$, and $\sigma(\xi) \notin G\mathcal{P}_0$, the Weyl symbol of $R = U_G^{-1}PU_G$ is $r_0(x, \xi) + \hbar r_1(x, \xi) + \mathcal{O}(\hbar^2)$ with*

$$r_0 = p_0 \circ \kappa_{U_G} \quad (53)$$

and

$$r_1 = \frac{i}{2J} \{J, r_0\} = \left[\frac{i}{2\tilde{J}} \{\tilde{J}, p_0\} \right] \circ \kappa_{U_G} \quad (54)$$

where

$$J(\xi) := \det(\mathrm{d}g(\xi))^{-1}, \quad \tilde{J}(\tilde{\xi}) = J(g^{-1}(\tilde{\xi})) = \det(\mathrm{d}g^{-1}(\tilde{\xi}))$$

and $\{\cdot, \cdot\}$ is the Poisson bracket.

Proof. Given any pseudodifferential operator R , we have

$$U_G R U_G^{-1} = V_G (\mathcal{F}_\hbar R \mathcal{F}_\hbar^{-1}) V_G^{-1} = V_G Q V_G^{-1}$$

with $Q := \mathcal{F}_\hbar R \mathcal{F}_\hbar^{-1}$. Since $\mathcal{F}_\hbar$ is a metaplectic operator, the Weyl symbols of Q and R are related by a linear change of variables, namely

$$q_\hbar(\xi, -x) = r_\hbar(x, \xi).$$

Taking $R = U_G^{-1}PU_G$, we have $P = V_G Q V_G^{-1}$, hence (51) becomes $r_0(\partial_\xi \phi, \xi) = p_0(x, \partial_x \phi)$, which gives (53). Moreover, in view of (52), we have

$$-r_1(\partial_\xi \phi, \xi) = \frac{i}{2} \left[\langle \partial_\xi, \partial_x \rangle p_0 - \langle \partial_\xi, \partial_x \rangle r_0 + \frac{\alpha(x)}{f(\xi)} \Delta_x r_0 + \frac{\alpha(x)}{f(\xi)^3} \langle \xi, \partial_x \rangle^2 r_0 \right]. \quad (55)$$

Let us now compute the first term, $\langle \partial_\xi, \partial_x \rangle p_0 - \langle \partial_\xi, \partial_x \rangle r_0$. When $T = \mathrm{Id}$, this term vanishes (and hence $p_1 = 0$, naturally, since $\alpha = 0$). In general, it can be computed in terms of r_0 , as follows.

Taking the derivative of (53) with respect to x , we obtain $\partial_x p_0 + \partial_\xi p_0 \cdot \partial_x^2 \phi = \partial_x r_0 + \partial_x \partial_\xi \phi$. Remember from (19) that ϕ is linear in x :

$$\phi(x, \xi) = \langle x, \pi G^t \sigma(\xi) \rangle_{\mathbb{R}^d},$$

and hence $\partial_x^2 \phi = 0$, which gives

$$\partial_x p_0 = \partial_x r_0 + \partial_x \partial_\xi \phi. \quad (56)$$

As usual, when $f = f(x, \xi)$ is a function on $\mathbb{R}^{2d}$, we denote by $\partial_x f$ the partial differential with respect to $x \in \mathbb{R}^d$, which is a linear map on the tangent space to the x variable (and similarly for $\partial_\xi f$). The term $\partial_x \partial_\xi \phi$ is a linear endomorphism of the tangent space $T_x \mathbb{R}^d$. Thus, (56) is an equality in the space of linear forms (covectors), *i.e.* for any $u \in T_x \mathbb{R}^d$:

$$\partial_x p_0(u) = \partial_x r_0(\partial_x [\partial_\xi \phi](u)).$$

(In this whole computation, for notation simplicity, p_0 and its derivatives are evaluated at $(x, \partial_x \phi)$ and r_0 and its derivatives are evaluated at $(\partial_\xi \phi, \xi)$, just like in (53), while ϕ and its derivatives are evaluated at (x, ξ) .) Taking now the derivative of (56) with respect to ξ , we get, for any $v \in T_\xi \mathbb{R}^d$, the following equality of linear forms:

$$\begin{aligned} \partial_\xi[\partial_x p_0] \cdot \partial_\xi[\partial_x \phi](v) &= (\partial_x^2 r_0 \cdot \partial_\xi^2 \phi(v) + \partial_\xi[\partial_x r_0](v)) \cdot \partial_x \partial_\xi \phi \\ &\quad + \partial_x r_0 \cdot (\partial_\xi[\partial_x \partial_\xi \phi](v)). \end{aligned} \quad (57)$$

If we denote by A the endomorphism $A := \partial_x \partial_\xi \phi$ of $T_x \mathbb{R}^d$, we have

$$\partial_\xi \partial_x p_0 \cdot A^t = A^t \cdot (\partial_x^2 r_0 \cdot \partial_\xi^2 \phi + \partial_\xi \partial_x r_0) + B$$

where we denote by $B = B_{x, \xi}$ the endomorphism of $T_\xi \mathbb{R}^d = (T_x \mathbb{R}^d)^*$ defined by

$$B(v) = \partial_x r_0 \cdot (\partial_\xi A(v)). \quad (58)$$

Recall that the mixed hessian matrix A was computed in Proposition 5.5; $A = dg^t(\xi)$, so it depends only on ξ , and is invertible under the conditions mentioned there. Multiplying (57) on the right by the $(A^t)^{-1}$, we get

$$\partial_\xi \partial_x p_0 = A^t \cdot (\partial_x^2 r_0 \cdot \partial_\xi^2 \phi + \partial_\xi \partial_x r_0) \cdot (A^t)^{-1} + B \cdot (A^t)^{-1}. \quad (59)$$

We now compute $\partial_\xi^2 \phi = \langle x, \pi G^t \partial_\xi^2 \sigma(\xi) \rangle$. The hessian $\partial_\xi^2 \sigma(\xi)$ was computed, as a quadratic form, in (45) (this is the term $\sigma_{2; \xi}$); and then $\partial_\xi^2 \phi$ is the lower right term of (49), *i.e.*

$$\partial_\xi^2 \phi = -\frac{\alpha}{f} \left(\text{Id} + \frac{1}{f^2} \xi \xi^t \right). \quad (60)$$

One could also write

$$\phi(x, \xi) = \langle \pi T X, \xi \rangle + f(\xi) \langle T X, e_3 \rangle = \langle \pi T X, \xi \rangle + f(\xi) \alpha(x)$$

where $X := (x, 0) \in \mathbb{R}^{1+d}$, and hence

$$\partial_\xi^2 \phi = \alpha(x) d^2 f.$$

Using (44), we obtain $df = -\frac{1}{f} \xi^t$ and $d^2 f = -\frac{1}{f} (\text{Id} + \frac{1}{f^2} \xi \xi^t)$, which gives (60) again.

Let us now take the trace of the equality (59), viewed as 2×2 matrices:

$$\text{tr} \partial_\xi \partial_x p_0 = \text{tr}(\partial_x^2 r_0 \cdot \partial_\xi^2 \phi) + \text{tr} \partial_\xi \partial_x r_0 + \text{tr}(B \cdot (A^t)^{-1}).$$

We have

$$\text{tr} \partial_\xi \partial_x p_0 = \sum_{i=1}^2 \partial_{x_i} \partial_{\xi_i} p_0 = \langle \partial_\xi, \partial_x \rangle p_0, \quad \text{tr} \partial_\xi \partial_x r_0 = \langle \partial_\xi, \partial_x \rangle r_0.$$

And

$$\partial_x^2 r_0 \cdot \partial_\xi^2 \phi = -\frac{\alpha}{f} \partial_x^2 r_0 - \frac{\alpha}{f^3} (\partial_x^2 r_0 \cdot \xi \xi^t)$$

which gives

$$\text{tr } \partial_x^2 r_0 \cdot \partial_\xi^2 \phi = -\frac{\alpha}{f} \Delta_x r_0 - \frac{\alpha}{f^3} \langle \xi, \partial_x \rangle^2 r_0.$$

Hence,

$$\langle \partial_\xi, \partial_x \rangle p_0 - \langle \partial_\xi, \partial_x \rangle r_0 = -\frac{\alpha}{f} \Delta_x r_0 - \frac{\alpha}{f^3} \langle \xi, \partial_x \rangle^2 r_0 + \text{tr} (B \cdot (A^t)^{-1}).$$

Thus, in Formula (55), several cancellations occur, we simply obtain

$$-r_1(\partial_\xi \phi, \xi) = \frac{i}{2} [\text{tr} (B \cdot (A^t)^{-1})]. \quad (61)$$

It remains to compute the trace: $\text{tr} (B \cdot (A^t)^{-1})$. Using the commutation of the derivatives with respect to ξ , the $2 \times 2 \times 2$ -tensor $\partial_\xi A = \partial_\xi \partial_x \partial_\xi \phi$ satisfies:

$$w \cdot (\partial_\xi A)(v) = v \cdot (\partial_\xi A)(w), \quad \forall v, w \in T_\xi \mathbb{R}^d.$$

Hence $B(v) = v \cdot (\partial_\xi A)(\partial_x r_0)$, which means that $B = [(\partial_\xi A)(\partial_x r_0)]^t$. Therefore $\text{tr } B \cdot (A^t)^{-1} = \text{tr } A^{-1} \cdot (\partial_\xi A)(\partial_x r_0)$. For any covector $w = (w_1, w_2)$ we have

$$A^{-1} \cdot (\partial_\xi A)(w) = \sum_j w_j A^{-1} \cdot \partial_{\xi_j} A = \sum_j w_j \partial_{\xi_j} (\log A).$$

Hence

$$\text{tr } B \cdot (A^t)^{-1} = \sum_j (\partial_{x_j} r_0) \partial_{\xi_j} (\text{tr}(\log A)) = \sum_j (\partial_{x_j} r_0) \partial_{\xi_j} (\log(\det A)).$$

Let us denote by $J = J(\xi)$ the Jacobian determinant $J = \det A^{-1} = (\det dg(\xi))^{-1}$. (Recall that $dg(\xi)$ is invertible by assumption due to Proposition 5.5.) We have

$$\text{tr } B \cdot (A^t)^{-1} = -\frac{1}{J} \sum_j (\partial_{x_j} r_0) \partial_{\xi_j} J = -\frac{1}{J} \langle \partial_x r_0, \partial_\xi J \rangle = -\frac{1}{J} \{J, r_0\}$$

(since J only depends on ξ , $\{J, r_0\} = \langle \partial_x r_0, \partial_\xi J \rangle$.) Now, using (53) and the fact that κ_{U_G} is symplectic,

$$\{J, r_0\} = \{J, p_0 \circ \kappa_{U_G}\} = \{J \circ \kappa_{U_G}^{-1}, p_0\} \circ \kappa_{U_G}$$

and $J \circ \kappa_{U_G}^{-1}(\tilde{x}, \tilde{\xi}) = J \circ g^{-1}(\tilde{\xi})$ which, together with (61), finally proves (54). $\square$

Remark 6.5 Our proof of Theorem 6.4 proceeds by direct computation; we believe that it is worth presenting here because we were not able to

find similar calculations in the literature. However, it may not shed light on the various cancellations which give rise to the simple formula (54). In Section 7 below, we present an indirect proof based on the fact that the Weyl symbol of a symmetric operator must be real valued, see Remark (7.4). It is also probable that a more conceptual proof could be derived from the microlocal formula (36), using the fact that changes of variables preserve the subprincipal symbol of pseudodifferential operators when acting on half-densities. $\triangle$

7 Lack of unitarity

When $G = \text{Id}$, we have $V_G = \mathcal{F}_h^{-1} = \frac{1}{(2\pi\hbar)^d} \mathcal{F}_h^*$, and hence $(2\pi\hbar)^{d/2} V_G$ is unitary on $L^2(\mathbb{R}^d)$. When $\mathcal{G} = \tau_\gamma$ is a translation, $(2\pi\hbar)^{d/2} V_{\mathcal{G}}$ is also unitary, due to (17). Hence, thanks to Lemma 5.2, for a general affine transformation $\mathcal{G}$, the unitarity of $V_{\mathcal{G}}$ (or, equivalently, $U_{\mathcal{G}}$) reduces to the unitarity of V_G , where G is the linear part of $\mathcal{G}$.

For a general G , the operators $(2\pi\hbar)^d V_G^* V_G$ and $(2\pi\hbar)^d V_G V_G^*$ are not the identity. Recall that the integral kernel of V_G is

$$K_{V_G}(y, \eta) = \frac{e^{\frac{i}{\hbar} \phi(y, \eta)}}{(2\pi\hbar)^d}$$

and hence the integral kernel of V_G^* is

$$K_{V_G^*}(\eta, x) = \overline{K_{V_G}(x, \eta)} = \frac{e^{-\frac{i}{\hbar} \phi(x, \eta)}}{(2\pi\hbar)^d}.$$

Therefore the integral kernel of the composition $V_G V_G^*$ is

$$K_{V_G V_G^*}(y, x) = \int K_{V_G}(y, \eta) K_{V_G^*}(\eta, x) d\eta = \frac{1}{(2\pi\hbar)^{2d}} \int e^{\frac{i}{\hbar} (\phi(y, \eta) - \phi(x, \eta))} d\eta.$$

We have

$$\phi(y, \eta) - \phi(x, \eta) = \langle y - x, \pi G^t \sigma(\eta) \rangle.$$

In general, this phase may be degenerate; however, if we restrict to the set of η such that $\sigma(\eta) \notin G\mathcal{P}_0$, then Proposition 5.5 implies that the phase is non-degenerate, and the corresponding Lagrangian submanifold is included in the diagonal. In this case, the composition is microlocally an FIO associated with the identity canonical transformation, hence a pseudodifferential operator. Let us compute it explicitly.

We consider the “change of variables” $\tilde{\eta} = g(\eta) := \pi G^t \sigma(\eta)$; as we saw in Section 5.2, this map is in general non-injective. More precisely, we use the decomposition (29); on each domain D_j , $j = \pm$, we have a smooth left inverse for $g : D_j \rightarrow g(D_j)$, which we denote by g_j^{-1} :

$$g_j^{-1}(g(\eta)) = \eta \quad \text{for } \eta \in D_j = \pi G^{t-1} \mathcal{E}_j.$$

Instead of V_G , we consider $\tilde{V} := V_G \chi$, where $\chi = \chi(\eta)$ is a smooth cut-off function compactly supported in the unit disc $\|\eta\| < 1$. Then

$$\begin{aligned} K_{\tilde{V}\tilde{V}^*}(y, x) &= \frac{1}{(2\pi\hbar)^{2d}} \int_{\|\eta\| < 1} e^{\frac{i}{\hbar}(\phi(y, \eta) - \phi(x, \eta))} \chi^2(\eta) d\eta \\ &= \frac{1}{(2\pi\hbar)^{2d}} \left[\int_{g(D_-)} e^{\frac{i}{\hbar}\langle y-x, \tilde{\eta} \rangle} J_-(\tilde{\eta}) \rho_-(\tilde{\eta}) d\tilde{\eta} + \int_{g(D_+)} e^{\frac{i}{\hbar}\langle y-x, \tilde{\eta} \rangle} J_+(\tilde{\eta}) \rho_+(\tilde{\eta}) d\tilde{\eta} \right] \end{aligned}$$

where $J_j(\tilde{\eta})$ is the Jacobian determinant of g_j^{-1} and $\rho_j(\tilde{\eta}) = \chi^2(g_j^{-1}(\tilde{\eta}))$. Hence

$$\begin{aligned} K_{\tilde{V}\tilde{V}^*}(y, x) &= \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\langle y-x, \tilde{\eta} \rangle} (J_+(\tilde{\eta}) + J_-(\tilde{\eta})) d\tilde{\eta} \\ &= \frac{1}{(2\pi\hbar)^d} \mathcal{F}_h^{-1} \tilde{J} \mathcal{F}_h, \end{aligned}$$

where $\tilde{J} = J_+ + J_-$ and

$$J_j(\tilde{\eta}) := J_j(\tilde{\eta}) \chi^2(g_j^{-1}(\tilde{\eta})) \mathbf{1}_{g(D_j)}.$$

For instance, if $\tilde{\eta} \in g(D_+) \cap g(D_-)$, then there exist $\eta_j \in D_j$ (hence $\eta_+ \neq \eta_-$), such that $g(\eta_+) = g(\eta_-) = \tilde{\eta}$. In this case $\tilde{J}(\tilde{\eta})$ really has two contributions from two different frequencies.

In general, acting on functions whose frequency variable is localised in $g(D_+) \cup g(D_-)$ we obtain the microlocal equality

$$V_G V_G^* = \frac{1}{(2\pi\hbar)^d} \mathcal{F}_h^{-1} \tilde{J} \mathcal{F}_h.$$

Consider now the operator $V_G^* V_G$; its integral kernel is

$$K_{V_G^* V_G}(\xi, \eta) = \int K_{V_G^*}(\xi, y) K_{V_G}(y, \eta) dy = \frac{1}{(2\pi\hbar)^{2d}} \int e^{\frac{i}{\hbar}(\phi(y, \eta) - \phi(y, \xi))} dy.$$

Let us consider the phase $\varphi(\xi, \eta, y) := \phi(y, \eta) - \phi(y, \xi)$. We have

$$\varphi(\xi, \eta, y) = \langle GY, \sigma(\eta) - \sigma(\xi) \rangle_{\mathbb{R}^{1+d}},$$

where we denote $Y := (0, y) = (0, y_1, \dots, y_d) \in \mathbb{R}^{1+d}$. We have

$$\sigma(\eta) - \sigma(\xi) = (f(\eta) - f(\xi), \eta - \xi).$$

On the other hand,

$$\begin{aligned} (f(\eta) - f(\xi))(f(\eta) + f(\xi)) &= f(\eta)^2 - f(\xi)^2 \\ &= \|\xi\|^2 - \|\eta\|^2 \\ &= \langle \xi - \eta, \xi + \eta \rangle. \end{aligned} \tag{62}$$

Therefore,

$$\varphi(\xi, \eta, y) = \langle GY, (0, \eta) - (0, \xi) + \frac{1}{f(\xi) + f(\eta)} \langle \xi - \eta, \xi + \eta \rangle e_0 \rangle_{\mathbb{R}^{1+d}}$$

which can be written

$$\varphi(\xi, \eta, y) = \langle GY, B \cdot (\eta - \xi) \rangle_{\mathbb{R}^{1+d}}$$

with

$$B = B(\xi, \eta) := \begin{pmatrix} \frac{-(\xi + \eta)^t}{f(\xi) + f(\eta)} \\ \text{Id} \end{pmatrix} : \mathbb{R}^d \rightarrow \mathbb{R}^{1+d}$$

Hence

$$\varphi(\xi, \eta, y) = \langle B^t GY, \eta - \xi \rangle.$$

For fixed (ξ, η) , consider the change of variables $\tilde{y} = g_{\xi, \eta}(y) := B^t GY$; we have

$$\int e^{\frac{i}{\hbar} \varphi(\xi, \eta, y)} dy = \int e^{\frac{i}{\hbar} \langle \tilde{y}, \eta - \xi \rangle} J_*(\xi, \eta, \tilde{y}) d\tilde{y}$$

where $J_*(\xi, \eta, \tilde{y})$ is the Jacobian determinant of $g_{\xi, \eta}^{-1}$. Since $g_{\xi, \eta}$ is linear in y , the Jacobian does not depend on $\tilde{y}$, and we can write

$$\int e^{\frac{i}{\hbar} \varphi(\xi, \eta, y)} dy = J_*(\xi, \eta) \int e^{\frac{i}{\hbar} \langle \tilde{y}, \eta - \xi \rangle} d\tilde{y} = (2\pi\hbar)^d J_*(\xi, \eta) \delta_{(\eta - \xi)}$$

We finally find that the operator $V_G^* V_G$ is simply the multiplication operator $u(\xi) \mapsto \frac{1}{(2\pi\hbar)^d} J_*(\xi, \xi) u(\xi)$, and

$$J_*(\xi, \xi) = \frac{1}{\det(B^t(\xi, \xi) G \pi^*)}.$$

Comparing (62) with the Taylor expansion of f at $\eta = \xi$, we see that $df(\xi) = \frac{-\xi^t}{f(\xi)}$ and hence $d\sigma(\xi) = B(\xi, \xi)$ (this was actually already computed in (45)). This proves that $J_*(\xi, \xi) = J(\xi)$; indeed:

$$\begin{aligned} \frac{1}{J(\xi)} &= \det(d_\xi(\pi G^t \sigma)) = \det(\pi G^t d_\xi \sigma) \\ &= \det(\pi G^t B(\xi, \xi)) = \det(B^t(\xi, \xi) G \pi^*) \\ &= \frac{1}{J_*(\xi, \xi)}. \end{aligned}$$

Let us summarise this in the following proposition.

Proposition 7.1 *Let $J(\xi) := (\det dg(\xi))^{-1}$. We have, microlocally near any (x, ξ) such that $\|\xi\| < 1$ and $\sigma(\xi) \notin G\mathcal{P}_0$,*

$$V_G V_G^* = \frac{1}{(2\pi\hbar)^d} \mathcal{F}_\hbar^{-1} \tilde{J} \mathcal{F}_\hbar,$$

where $\tilde{J}$ is the multiplication operator by the function

$$\tilde{J}(\xi) = \sum_{\eta \in g^{-1}(g(\xi))} J(\eta).$$

And, microlocally near any (ξ, x) such that $\|\xi\| < 1$ and $\sigma(\xi) \notin G\mathcal{P}_0$,

$$V_G^* V_G = \frac{1}{(2\pi\hbar)^d} J,$$

where J is the multiplication operator by the function $J = (\det dg(\xi))^{-1}$.

Corollary 7.2 *We have, microlocally near any (x, ξ) such that $\|\xi\| < 1$ and $\sigma(\xi) \notin G\mathcal{P}_0$,*

$$U_G U_G^* = \mathcal{F}_\hbar^{-1} \tilde{J} \mathcal{F}_\hbar$$

and

$$U_G^* U_G = \mathcal{F}_\hbar^{-1} J \mathcal{F}_\hbar.$$

We may now answer the question raised in the beginning of Section 6.

Theorem 7.3 *Let $G \in \mathrm{GL}_{1+d}(\mathbb{R})$ and $\gamma \in \mathbb{R}^{1+d}$. Let $\mathcal{G} := \tau_\gamma \circ G$ be the corresponding affine transformation. Let P be a semiclassical pseudodifferential operator with Weyl symbol $p = p_0$ independent of $\hbar$. Then, in a phase space region where $\|\xi\| \leq 1 - \epsilon$, $\epsilon > 0$, and $\sigma(\xi) \notin G\mathcal{P}_0$, the Weyl symbol of $T = U_G^* P U_G$ is $t_0 + \mathcal{O}(\hbar^2)$ (that is, the subprincipal term $\hbar t_1$ vanishes), with*

$$t_0(x, \xi) = J(\xi) p_0 \circ \kappa_{U_G}(x, \xi) \quad (63)$$

Proof. Thanks to Lemma 5.2 (item 2), it is enough to consider separately the cases $\mathcal{G} = G$ and $\mathcal{G} := \tau_\gamma$.

Case $\mathcal{G} = G$. Let $R = U_G^{-1} P U_G$. We know from Theorem 6.4, that the Weyl symbol of R is $r_0 + i\hbar \tilde{r}_1 + \mathcal{O}(\hbar^2)$, for some smooth function $\tilde{r}_1$, and $r_0 = p_0 \circ \kappa_{U_G}$. Writing $T = U_G^* U_G R$ and applying Corollary 7.2, we have $T = \hat{J} R$, where we denote by $\hat{J}$ the Fourier multiplier $\mathcal{F}_\hbar^{-1} J \mathcal{F}_\hbar$, we may derive the Weyl symbol $t_\hbar$ of T by the composition formula for pseudodifferential operators (Moyal's formula⁽¹⁾, see [34, Theorem 4.12]): if $A = \mathrm{Op}_\hbar^w(a_\hbar)$ and $B = \mathrm{Op}_\hbar^w(b_\hbar)$, and $a_\hbar$ and $b_\hbar$ admit asymptotic expansions in powers of $\hbar$, then

$$\sigma_W(A \circ B) = a_\hbar b_\hbar + \frac{\hbar}{2i} \{a_\hbar, b_\hbar\} + \mathcal{O}(\hbar^2).$$

In our situation, this gives

$$t_\hbar(x, \xi) = J(\xi) r_0(x, \xi) + i\hbar J(\xi) \tilde{r}_1(x, \xi) + \frac{\hbar}{2i} \{J, r_0\} + \mathcal{O}(\hbar^2).$$

⁽¹⁾due to Groenewold [18]

Assume that P is symmetric, so that its symbol is real valued. Then it follows from Theorem 6.4 that r_0 and $\tilde{r}_1$ are real valued. Since T is then also symmetric, t_h must be real valued as well. This implies

$$J(\xi)\tilde{r}_1(x, \xi) = \frac{1}{2}\{J, r_0\} \quad (64)$$

and hence

$$t_h(x, \xi) = J(\xi)r_0(x, \xi) + \mathcal{O}(\hbar^2),$$

which proves the theorem.

Remark 7.4 It is easy to obtain $r_h = r_0 + i\hbar\tilde{r}_1 + \mathcal{O}(\hbar^2)$, for some real valued function $\tilde{r}_1$, from Propositions 6.2 and 6.3; namely, $\tilde{r}_1 = -ir_1$ with the formula for r_1 given by (55). Hence, the above argument shows that (64) holds, and therefore

$$\tilde{r}_1(x, \xi) = \frac{1}{2J}\{J, r_0\}$$

which directly recovers (54), without further computation. $\triangle$

Case $\mathcal{G} = \tau_\gamma$. This case can be treated directly by a stationary phase argument, but let us consider here a more general route, based upon the following lemma:

Lemma 7.5 *Let $f = f(x)$ be a smooth real function and $A = \text{Op}_h^w(a)$ be a pseudodifferential operator. Then*

$$A_f := e^{-\frac{i}{\hbar}f} \circ A \circ e^{\frac{i}{\hbar}f}$$

is a pseudodifferential operator with Weyl symbol

$$a_f(x, \xi) := a(x, \xi + \text{d}f(x)) + \mathcal{O}(\hbar^2).$$

Proof of the lemma. This lemma is well known to specialists; obtaining the remainder $\mathcal{O}(\hbar)$ is standard and holds for any quantization; the vanishing of the subprincipal term is due to Weyl's quantization; we present an explicit argument (which in principle can be used for computing the expansion at any order) for the convenience of the reader. First, it is easy to see that, for any differential operator $A = \sum_{k=0}^m b_k(x)(\frac{\hbar}{i}\partial_x)^k$,

$$e^{-\frac{i}{\hbar}f} \circ A \circ e^{\frac{i}{\hbar}f} = \sum_{k=0}^m b_k(x) \left(\frac{\hbar}{i}\partial_x + \nabla f(x) \right)^k.$$

Using inductively Moyal's formula, we check that the Weyl symbol of $(\frac{\hbar}{i}\partial_x + \nabla f(x))^k$ is exactly $(\xi + \text{d}f)^k$. Hence, applying the Moyal formula

again, for the products $b_k(x) \left(\frac{\hbar}{i}\partial_x + \nabla f(x)\right)^k = \text{Op}_h^w(b_k) \circ \text{Op}_h^w((\xi + \text{d}f)^k)$, we obtain

$$a_f(x, \xi) = a_1(x, \xi + \text{d}f(x)) - \frac{\hbar}{2i} \langle \partial_x, \partial_\xi \rangle a_1(x, \xi + \text{d}f(x)) + \mathcal{O}(\hbar^2),$$

where $a_1(x, \xi) := \sum_{k=0}^m b_k(x) \xi^k$ is the left-symbol of A . By density, and localisation in phase space, this remains true for any symbol $a_1(x, \xi)$ in a good symbol class. Finally, applying the formula that relates the Weyl symbol $a_{\frac{1}{2}}$ to the left symbol a_1 :

$$a_{\frac{1}{2}}(x, \xi) = \exp\left(\frac{i\hbar}{2} \langle \partial_x, \partial_\xi \rangle\right) a_1(x, \xi) = a_1(x, \xi) - \frac{\hbar}{2i} \langle \partial_x, \partial_\xi \rangle a_1(x, \xi),$$

we obtain the required formula for a_f . $\square$

End of the proof of Theorem 7.3.

Recall from (16) that $U_\gamma = \mathcal{F}_h^{-1} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h$. Hence

$$U_\gamma^* P U_\gamma = \mathcal{F}_h^{-1} e^{-\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h P \mathcal{F}_h^{-1} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h.$$

The Weyl symbol of $\mathcal{F}_h P \mathcal{F}_h^{-1}$ is $p(-\xi, x)$. By Lemma 7.5, the Weyl symbol of $e^{-\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle} \mathcal{F}_h P \mathcal{F}_h^{-1} e^{\frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle}$ is hence $p(-\xi - \text{d}S(x), x) + \mathcal{O}(\hbar^2)$ with $S := \frac{i}{\hbar} \langle \gamma, \sigma(\cdot) \rangle$. Finally, the Weyl symbol of $U_\gamma^* P U_\gamma$ is $p(x - \text{d}S(\xi), \xi) + \mathcal{O}(\hbar^2)$, and $p(x - \text{d}S(\xi), \xi)$ which is precisely $p \circ \kappa_{U_\gamma}(x, \xi)$ according to Theorem 5.7. $\square$

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References

- [1] D. Arrivault and F. Jaillet. **ltfatpy**, Python port of the LTFAT library. <https://gitlab.lis-lab.fr/dev/ltfatpy>, 2018.
- [2] D. Blinder, A. Ahar, S. Bettens, T. Birnbaum, A. Symeonidou, H. Ottevaere, C. Schretter, and P. Schelkens. Signal processing challenges for digital holographic video display systems. *Signal Processing: Image Communication*, 70:114–130, 2019.

- [3] A.-S. Bonnet-Ben Dhia, S. Fliss, C. Hazard, and A. Tonnoir. A Rellich type theorem for the Helmholtz equation in a conical domain. *C. R. Math. Acad. Sci. Paris*, 354(1):27–32, 2016.
- [4] F. Castella. The radiation condition at infinity for the high-frequency helmholtz equation with source term: a wave-packet approach. *Journal of Functional Analysis*, 223(1):204–257, 2005.
- [5] T. Chaumont-Frelet, V. Dolean, and M. Ingreteau. Efficient approximation of high-frequency Helmholtz solutions by Gaussian coherent states. *Numer. Math.*, 156(4):1385–1426, 2024.
- [6] C. Cheverry and S. Ibrahim. The relativistic Vlasov-Maxwell system: local smooth solvability for a weak topology. *Rev. Mat. Iberoam.*, 41(2):551–602, 2025.
- [7] E. Cordero, K. Gröchenig, F. Nicola, and L. Rodino. Wiener algebras of Fourier integral operators. *Journal de Mathématiques Pures et Appliquées*, 99(2):219–233, 2013.
- [8] M. Doumic, F. Duboc, F. Golse, and R. Sentis. Simulation of laser beam propagation with a paraxial model in a tilted frame. *Journal of Computational Physics*, 228(3):861–880, 2009.
- [9] M. Doumic, F. Golse, and R. Sentis. Un modèle paraxial de propagation de la lumière : problème aux limites pour l’équation d’advection schrödinger en coordonnées obliques. *Comptes Rendus Mathématique*, 336(1):23–28, 2003.
- [10] J. J. Duistermaat. Oscillatory integrals, Lagrange immersions and unfoldings of singularities. *Comm. Pure Appl. Math.*, 27:207–281, 1974.
- [11] J. J. Duistermaat. *Fourier Integral Operators*. Progress in mathematics. Birkhäuser, 1996.
- [12] A. El Rhammad, P. Gioia, A. Gilles, M. Cagnazzo, and B. Pesquet-Popescu. View-dependent compression of digital hologram based on matching pursuit. In *Optics, Photonics, and Digital Technologies for Imaging Applications V*, volume 10679, page 106790L. International Society for Optics and Photonics, 2018.
- [13] W. Gaim and C. Lasser. Corrections to Wigner type phase space methods. *Nonlinearity*, 27(12):2951, 2014.
- [14] P. Gioia, A. Gilles, A. El Rhammad, and S. Vũ Ngọc. Phase space formulation of light propagation on tilted planes. *Photonics*, 11:1034, 2024.

- [15] P. Gioia, A. Gilles, A. Lagrange, A. El Rhammad, and S. Vũ Ngọc. Linear canonical transformations in phase space: the Gabor frames approach. In *Optics, Photonics and Digital Technologies for Imaging Applications VIII*, page 18, Strasbourg, France, 2024. SPIE.
- [16] P. Gioia and S. Vũ Ngọc. Movie of the collision of two coherent states under hologram plane rotation. <https://perso.univ-rennes1.fr/san.vu-ngoc/docs/collide.mp4>, 2025.
- [17] J. W. Goodman. *Introduction to Fourier Optics*. McGraw-Hill, second edition, 1996.
- [18] H. J. Groenewold. On the principles of elementary quantum mechanics. *Physica*, 12:405–460, 1946.
- [19] V. Guillemin and S. Sternberg. *Symplectic techniques in physics*. Cambridge University press, 1984.
- [20] V. Guillemin and S. Sternberg. *Semi-classical analysis*. International Press, Boston, MA, 2013.
- [21] V. W. Guillemin. 25 years of Fourier integral operators. In *Mathematics past and present*, pages 1–21. Springer, Berlin, 1994.
- [22] L. Hillairet and J. Wunsch. On resonances generated by conic diffraction. *Ann. Inst. Fourier (Grenoble)*, 70(4):1715–1752, 2020.
- [23] M. Hitrik and J. Sjöstrand. Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions. I. *Ann. Henri Poincaré*, 5(1):1–73, 2004.
- [24] L. Hörmander. Fourier Integral Operators I. *Acta Math.*, 127:79–183, 1971.
- [25] V. P. Maslov. *Théorie des perturbations et méthodes asymptotiques*. Dunod, Paris, 1972.
- [26] K. Matsushima. Formulation of the rotational transformation of wave fields and their application to digital holography. *Applied Optics*, 47(19):D110–D116, 2008.
- [27] K. Matsushima, H. Schimmel, and F. Wyrowski. Fast calculation method for optical diffraction on tilted planes by use of the angular spectrum of plane waves. *Journal of the Optical Society of America A*, 20(9):1755–1762, 2003.
- [28] Z. Průša, P. L. Søndergaard, N. Holighaus, C. Wiesmeyr, and P. Balazs. The Large Time-Frequency Analysis Toolbox 2.0. In *Sound, Music, and Motion*, LNCS, pages 419–442. Springer International Publishing, 2014.

- [29] D. Robert. *Autour de l'approximation semi-classique*, volume 68 of *Progress in Mathematics*. Birkhäuser, 1987.
- [30] J. Royer. Uniform resolvent estimates for a non-dissipative Helmholtz equation. *Bull. Soc. Math. Fr.*, 142(4):591–633, 2014.
- [31] M. E. Testorf, B. M. Hennelly, and J. Ojeda-Castañeda. *Phase-space optics: fundamentals and applications*. McGraw-Hill, New York, 2010.
- [32] S. Vũ Ngọc. *Systèmes intégrables semi-classiques: du local au global*. Number 22 in Panoramas et Synthèses. SMF, 2006.
- [33] X. Zeng and R. J. McGough. Optimal simulations of ultrasonic fields produced by large thermal therapy arrays using the angular spectrum approach. *The Journal of the Acoustical Society of America*, 125(5):2967–2977, 2009.
- [34] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.