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► **To cite this version:**

Guillaume Bouleux, Giacomo Kahn, He Zhihao. A Knowledge-Based Curvature for Concept Lattices. 2025. <hal-05031004>

**HAL Id: hal-05031004**

**<https://hal.science/hal-05031004v1>**

Preprint submitted on 11 Apr 2025


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# A Knowledge-Based Curvature for Concept Lattices

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**Abstract.** In this paper, we introduce the notion of geometric curvature of a concept lattice. We start by using the common definition of curvature for directed graphs, then adapt the definition to create a knowledge-based curvature of the concept lattice, using the extents and intents of the concepts.

**Keywords:** Geometry · Formal Concept Analysis · Curvature

## 1 Introduction

The curvature is a geometric quantity that quantifies how far a surface is from being *flat*. It has been described for discrete surfaces and for discrete structures such as graphs, where it is used to understand the flow of information in networks [2, 4].

In this paper, we aim at studying concept lattices with a geometric angle. We start by translating the curvature as defined for directed graphs for concept lattices, and discuss the flaws of that definition. Then, we introduce a new definition for the curvature on a concept lattice that takes into account the extents and the intents of pairs of comparable concepts. We then discuss some potential applications for that new quantity.

## 2 Preliminaries

This section is divided into two subsections. If the reader is familiar with Formal Context Analysis but not with the notion of curvature on discrete structures, we invite them to go directly to Section 2.2. If, on the contrary, they came to this paper from the field of geometry, we invite them to read the next section. In this paper, all the sets are finite.

### 2.1 Formal Concept Analysis

Formal Concept Analysis (FCA) is the mathematical frameworks that allows the data analysis community to study hierarchical knowledge structures called

concept lattices. In FCA, the starting point is the *formal context*, that is a triple  $\mathcal{O}, \mathcal{A}, I$  where  $\mathcal{O}$  and  $\mathcal{A}$  are sets, commonly called objects and attributes, while  $I$  is a binary relation between them. Usually, such a formal context is represented by a cross table where the rows represent objects of  $\mathcal{O}$ , the columns represent attributes of  $\mathcal{A}$  and the crosses are the elements of  $I$ , as in Figure 1a.

The bits of knowledge that are of interest in data analysis are the *formal concepts* of the formal context. Let  $'$  be a derivation operator that maps an object  $o$  to its description, that is  $o' = \{a \mid (o, a) \in I\}$ . The same operator can be defined on attributes, so that  $a' = \{o \mid (o, a) \in I\}$ . It is defined on sets of objects (resp. attributes) in that way :  $O' = \{a \mid (o, a) \forall o \in O\}$ .

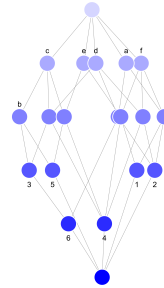
The composition of the two operators,  $''$  is a closure operator that allows to build the formal concepts of the context.

A formal concept is a couple  $(O, A)$ , where  $O \subseteq \mathcal{O}, A \subseteq \mathcal{A}, A' = O$  and  $O' = A$ . They form maximal rectangle of incidence in the crosstable that represents the formal context.

The set of objects of a formal concept is called the *extent*, while the set of attributes is called the *intent* of the formal concept.

A formal concept is a maximal rectangle full of crosses in the cross table that represents the formal context, up to permutation of rows and columns. One such concept is highlighted in Figure 1a. We will omit the brackets whenever they omission does not induce ambiguity, and will write  $(346, cd)$  for concept  $(\{3, 4, 6\}, \{c, d\})$  whenever possible.

	a	b	c	d	e	f
1	×				×	×
2	×			×		×
3		×	×	×		
4			×	×	×	×
5		×	×		×	
6	×	×	×	×		



(a) Toy context with 6 attributes from the Latin alphabet and 6 objects from context. The highlighted concept is  $(\{3, 4, 6\}, \{c, d\})$ .

(b) Concept lattice from the example formal context. The labels for the objects (resp. attributes) are only shown on the concept that introduces them in the lattice. An object is present in the concept that introduces it and all the ones above. An attribute is present in the concept that introduces it and all the ones below.

The set of all the formal concepts  $\mathcal{C}$  of a formal context  $C = (\mathcal{O}, \mathcal{A}, I)$  can be ordered by the inclusion relation on their extents. The resulting partially ordered

set is a lattice  $\mathcal{L} = (\mathcal{C}, \leq_{\mathcal{O}})$  called the *concept lattice* of the formal context. The transitive reduction of the concept lattice of the formal context from Figure 1a is shown in Figure 1b.

## 2.2 Geometry

In geometry, curvature is related to how *curved* a surface is, how much it deviates from a plane surface. For manifolds, it is computed not only at a point, but with a direction, since the curvature might be different at the same point, depending on what direction we are looking at.

Curvature on graphs was introduced in the undirected case in 2010 by Lin, Lu, and Yau [3]. The concept was then extended to directed graphs in 2018 by Yamada [6].

It is defined as follows. Let  $G = (V, E)$  be a directed graph, composed of a set  $V$  of vertices and a relation  $E$  between them, that represents arcs. The distance  $d(u, v)$  between two vertices of  $G$  is the length of a shortest path between  $u$  and  $v$ . The out-neighbourhood  $N^{out}(u)$  of a vertex  $u$  is the set  $\{v \mid (u, v) \in E\}$ . Its out-degree  $d^{out}(u)$  is the size of  $N^{out}(u)$ .

The graph  $G$  has no loops nor multi-edges, is finite, and is strongly connected, that is there is a directed path between any two vertices.

Let  $m_u^\alpha$ ,  $\alpha \in [0, 1]$  be a probability measure on  $V$  defined for each  $u \in V$  in the following way :

$$m_u^\alpha(v) = \begin{cases} \alpha & \text{if } u = v \\ \frac{1-\alpha}{d^{out}(u)} & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for any two vertices  $u$  and  $v$  in  $V$ , the  $\alpha$ -Ricci curvature  $\kappa_\alpha(u, v)$  is defined as

$$\kappa_\alpha(u, v) = 1 - \frac{W(m_u^\alpha, m_v^\alpha)}{d(u, v)}, \quad (1)$$

where  $W(m_u^\alpha, m_v^\alpha)$  is the Wasserstein distance between the two distributions  $m_u^\alpha$  and  $m_v^\alpha$ .

Curvature can be positive, indicating a very connected area or a sphere-like surface, or negative, indicating a part of the graph with low connectivity.

## 3 Curvature of the concept lattice

In this section we discuss the computation of a curvature-like quantity for a concept lattice. We start by adapting the definition of curvature on a directed graph to a poset, which gives a naive adaptation of curvature for concept lattice. Then, we give a new definition of curvature based on the extents and the intents of the elements of the concept lattice.

### 3.1 From Order Relation to Directed Graph

Let  $\mathcal{L} = (C, \leq)$  be a concept lattice on a set of concepts  $C$ . There is an immediate translation to a directed graph  $G = (C, E)$  where the vertices of the graph are the concepts and there is an edge between two concepts  $c_1$  and  $c_2$  if  $c_1 \leq c_2$ . That way, each concept has an arc linking it to every element in their filter in the lattice.

Another possible translation from a lattice to a directed graph is the representation of the order relation *via* its transitive reduction. This is the same graph that is usually depicted in order diagrams, such as the one in Figure 1b.

### 3.2 A naive definition of curvature on a concept lattice

Yamada's definition of the Ricci curvature on a directed graph considers a strongly connected graph. Indeed, the Ricci curvature is defined between any two vertices of the graph, whether they are connected or not. The computation uses the length of a shortest path between the two vertices in Equation 1. In our case, the order relation is not strongly connected: the existence of a directed path between two vertices  $u$  and  $v$  actually implies that there is no path between  $v$  and  $u$ . We avoid this problem by computing only the curvature between pairs of distinct, comparable concepts.

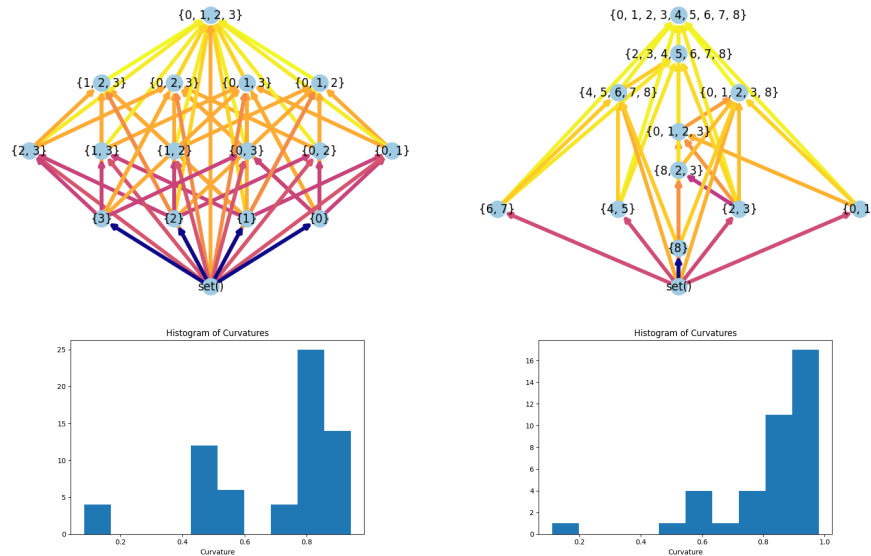


Fig. 2: Boolean lattice of dimension 4 (on the left) and concept lattice of well known dataset Planets (right) with the naive curvatures represented as a colormap on the order relation. The histogram of curvatures is displayed below.

One can visualise the distribution of the curvatures as a histogram, where each bar represents the number of edges that have a given curvature. This gives, at a glance, an idea about the overall geometry of the structure.

Figure 2 gives two examples. In both cases, we have the representation of curvature as the colour of the edges on top and the distribution of curvatures on the bottom. One can see that in both cases all the curvatures are positive, with what looks like smaller quantities in the lowest part of the lattice.

This naive definition has two crucial limitations. The first is that it only outputs non-negative curvatures : since  $\alpha$  is in the range of 0 to 1, and the out-degree of  $u$ ,  $d_{out}(u)$  is always be a positive integer, so that  $m_u^\alpha(v)$  is always positive and smaller than 1. As the result, the Wasserstein distance between distributions is always less than or equal to  $d = 1$ . In this naive setting, the distance between two comparable concepts is always 1. Thus, the curvature is always non-negative.

The second limitation is that the naive definition does not use, at all, the knowledge contained in the concept lattice and only looks at the order relation. In the next section, we give a more FCA-oriented definition of curvature.

### 3.3 A knowledge-based definition of curvature for concept lattices

As we discussed in Section 2.2, the curvature at one point and in one direction should quantify how much change there is in the surface, over a distance. For graph curvature, it quantifies the change in distribution of the neighbours over the graph metric. In this section, we aim at defining curvature over the relation of a concept lattice.

To consider a knowledge oriented definition of curvature, we have to define two things:

1. what does it mean to go "a distance" in a concept lattice,
2. what is the change that we would like to quantify.

As an added constraint, our definition should have a distance defined on one dimension of the concept and a change defined on the other. The curvature would then represent, for example, the change in the intent given the size of the change on the extent.

To tackle the first point, we need to describe a distance between concepts in a concept lattice. Since the curvature is only defined over edges, our definition needs to be defined only for comparable concepts. We chose to consider the distance between comparable concepts as the absolute value of the difference in cardinal of their extents, that is for two concepts  $C_1 = (O_1, A_1)$  and  $C_2 = (O_2, A_2)$ ,  $d(C_1, C_2) = \text{abs}(|O_1| - |O_2|)$ .

Let  $c_1$  and  $c_2$  be comparable concepts, with extent  $O_1$  and  $O_2$  and such that  $O_1 \subseteq O_2$ . Then,

- $d(c_1, c_2)$  is always non-negative,
- $d(c_1, c_2) = 0 \implies c_1 = c_2$  since  $d(x, y) = 0$  implies that  $O_1$  and  $O_2$  are the same size and  $O_1 \subseteq O_2$ ,

- $d(c_1, c_2) = d(c_2, c_1)$ , and finally
- let  $c_3$  be a third concept with extent  $O_3$ , such that  $O_1 \subseteq O_2 \subseteq O_3$ . Then  $d(c_1, c_3) = |O_3| - |O_1| = |O_3| - |O_2| + |O_2| - |O_1| = d(c_1, c_2) + d(c_2, c_3)$ , which means the triangular inequality holds.

For the second point, we need to define a measure that takes into account how much information is contained by a given concept, with respect to the attributes, to compare it to its neighbours.

Our proposition is to use a measure based on the introducer concepts of the concept lattice. For every object  $o$ , its introducer concept is the concept  $(o'', o')$  (resp.  $(a', a'')$  for attribute  $a$ ). It generalises irreducible elements for non-reduced formal contexts. For a given concept  $c$ , we denote  $intro_{\mathcal{O}}(c)$  (resp.  $intro_{\mathcal{A}}(C)$ ) the set of objects (resp. attributes) introduced by  $c$ .

Let  $c_1 = (O_1, A_1)$  be a formal concept. Let  $m_{c_1}$  be a probability measure on the set of concepts  $\mathcal{C}$  of a formal concept. We define  $m_{c_1}$  for each  $c_2 \in \mathcal{C}$  in the following way :

$$m_{c_1}(c_2) = \begin{cases} \frac{|intro_{\mathcal{A}}(c_2)|}{|A_1|} & \text{if } c_1 \leq c_2 \\ 0 & \text{otherwise.} \end{cases}$$

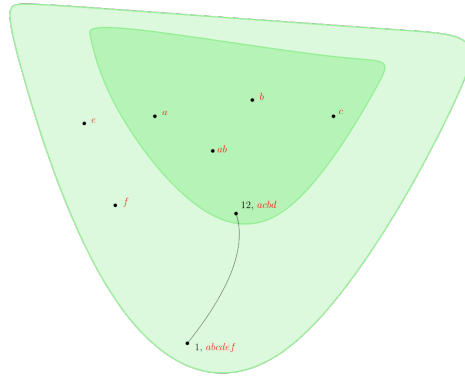


Fig. 3: Let  $c_1$  and  $c_2$  be two concepts whose intents are respectively  $abcd$  and  $abcdef$ . Their respective distributions are computed among the introducer concepts of their attributes. For  $c_1$ , the measure  $m_{c_1}$  will be non-zero only for some concepts in the deep green part. For  $m_{c_2}$ , the same concepts together with some others in the pale green part of the lattice will have a non-zero measure.

Since all the attributes in  $A_1$  are either introduced in  $c_1$  or in a concept above it,  $m_{c_1}$  is indeed a probability measure. For example, in the case of Figure 3, the measure for the concept  $c_1 = (12, abcd)$  will take non-zero values for the introducer concepts of attributes  $a$ ,  $b$ ,  $c$  and  $d$ , that will represent the portion of  $c_1$ 's intent that is introduced in each concept. On  $c_1$  the measure will be non-zero if  $c_1$  is itself an introducer concept.

For a second concept  $(1, abcdef)$ , that is below in the concept lattice, the measure will take non-zero values on the same concepts as above, but also in the introducer concepts of attributes  $e$  and  $f$ .

We will then compute the difference between those two distributions. For that, we use a variant of the Wasserstein distance on one-dimensional distribution. We compute the absolute value of the difference of the two measures on each concept. More formally, let  $\mathcal{C}$  be the set of concepts of a formal context, and  $m_{c_1}$  and  $m_{c_2}$  two probability distributions for two distinct concepts  $c_1$  and  $c_2$ . Then the distance between distributions is

$$W(m_{c_1}, m_{c_2}) = \sum_{c \in \mathcal{C}} \text{abs}(m_{c_1}(c) - m_{c_2}(c)). \quad (2)$$

Since our measure only uses the introducer concepts, it is quick – linear in the size of the context – to compute it and to compute the distance between distributions. This is a clear advantage from the naive approach that assigns non-zero values to a potentially exponential number of concepts.

We can conclude this section with the definition of a knowledge-based curvature between comparable concepts.

**Definition 1.** *For two comparable concepts  $c_1$  and  $c_2$ , the curvature  $\kappa(c_1, c_2)$  is*

$$\kappa(c_1, c_2) = 1 - \frac{W(m_{c_1}, m_{c_2})}{d(c_1, c_2)},$$

where  $W$  is the distance between distributions defined in Equation 2, and  $d$  is the distance based on the extents.

In this setting, the distance between two comparable concepts is at least 1 (since to go from one concept to its successor, one needs to add at least an object). The distance between distributions defined in Equation 2 can be greater than one, which means that the knowledge-based curvature that we just defined can be negative, unless the graph Ricci curvature adapted to concept lattices.

Figure 4 shows the knowledge-based curvature distribution of the same two concept lattices as Figure 2. One can see that the distributions are different.

## 4 Discussion and conclusion

In this short paper, we give a first definition of curvature on a concept lattice. While it incorporates the knowledge represented in a concept lattice, it might still be a flawed definition that needs refining. Indeed, this is a preliminary study, and we plan on using new distances between concepts, new probability measures, and new distances between distributions as future works.

As continuation of this work, we also plan on studying how much the distribution of curvatures of a given concept lattice constitute a signature of that lattice. To do so, we plan on comparing lattice that are supposed to be *close* [1],

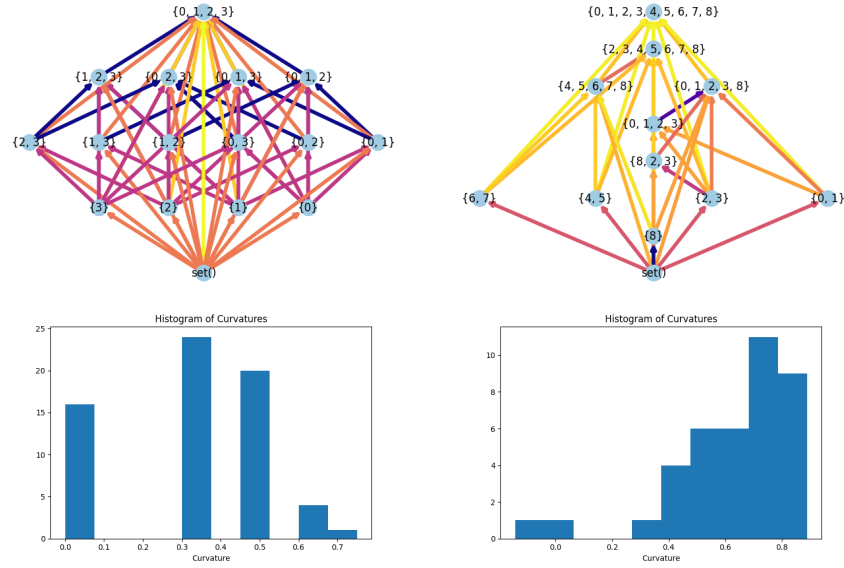


Fig. 4: Boolean lattice of dimension 4 (on the left) and concept lattice of well known dataset Planets (right) with the knowledge-based curvatures represented as a colormap on the order relation. The histogram of curvatures is displayed below.

either because they come from contexts that were modified in a small way, or because they live in the same lattices classes.

Finally, this notion seems like it would be naturally extended to the polyadic case [5]. In that case, the curvature between two comparable concepts would be computed depending on the order that is considered in the  $n$ -lattice.

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