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Fractional balanced chromatic number of signed subcubic graphs

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Abstract

A signed graph is a pair (G, σ) , where G is a graph and $\sigma : E(G) \rightarrow \{-, +\}$, called signature, is an assignment of signs to the edges. Given a signed graph (G, σ) with no negative loops, a balanced (p, q) -coloring of (G, σ) is an assignment f of q colors to each vertex from a pool of p colors such that each color class induces a balanced subgraph, i.e., no negative cycles. Let $(K_4, -)$ be the signed graph on K_4 with all edges being negative. In this work, we show that every signed (simple) subcubic graph admits a balanced $(5, 3)$ -coloring except for $(K_4, -)$ and signed graphs switching equivalent to it. For this particular signed graph the best balanced colorings are $(2p, p)$ -colorings.

Keywords: signed subcubic graphs; (p, q) -coloring; fractional balanced chromatic number

1. Introduction

Let a, b be positive integers with $a \geq b$. First introduced in [10], a *fractional $\frac{a}{b}$ -coloring* of a graph G is an assignment $f : V(G) \rightarrow \binom{[a]}{b}$ where $[a] := \{1, 2, \dots, a\}$ is a set of colors, such that $f(u) \cap f(v) = \emptyset$ for every edge uv of G . The *fractional chromatic number* of G , denoted $\chi_f(G)$, is defined to be $\chi_f(G) = \min\{\frac{a}{b} \mid G \text{ admits a fractional } \frac{a}{b}\text{-coloring}\}$. It is easy to see that $\chi_f(G) \leq \chi(G)$ as any proper k -coloring can be viewed as a fractional $\frac{k}{1}$ -coloring. Let $\Delta(G)$ denote the maximum degree of G . By Brook's theorem, for $\Delta(G) \geq 3$, if G contains no complete graph K_Δ , then $\chi_f(G) \leq \Delta$.

The fractional chromatic number of subcubic graphs (i.e., $\Delta \leq 3$) receives a great deal of attention. In particular, based on the study of independent sets in triangle-free subcubic graphs [4, 6, 8, 19], Heckman and Thomas [8] conjectured that subcubic triangle-free graphs have fractional chromatic number at most $\frac{14}{5}$ and this bound is tight. Progress towards this conjecture can be found in [5, 7, 13, 17, 18]. It is resolved by Dvořák, Sereni, and Volec in [2].

Theorem 1. [2] *Every subcubic triangle-free graph G satisfies $\chi_f(G) \leq \frac{14}{5}$.*

Recently, Dvořák, Lidický and Postle [1] showed that every subcubic triangle-free graph avoiding two exceptional graphs as subgraphs admits a fractional $\frac{11}{4}$ -coloring. This implies that every subcubic triangle-free planar graph has fractional chromatic number at most $\frac{11}{4}$. However, another conjecture of Heckman and Thomas [9] asserts that every subcubic triangle-free planar graph admits a fractional $\frac{8}{3}$ -coloring which remains open.

Following this line of study, we explore the fractional balanced coloring of signed subcubic graphs in this paper.

A *signed graph* (G, σ) is a graph $G = (V, E)$ endowed with a *signature function* $\sigma : E(G) \rightarrow \{-, +\}$ which assigns to each edge e a sign $\sigma(e)$. An edge e is called a *positive edge* (or *negative edge*, respectively) if

$\sigma(e) = +$ (or $\sigma(e) = -$, respectively). The graph G is called the *underlying graph* of (G, σ) . When the signature is clear from context, \widehat{G} is also used to denote a signed graph with the underlying graph G . Assume (G, σ) is a signed graph and v is a vertex of G . The *vertex switching* at v results in a signature σ' defined as

$$\sigma'(e) = \begin{cases} -\sigma(e), & \text{if } v \text{ is a vertex of } e \text{ and } e \text{ is not a loop;} \\ \sigma(e), & \text{otherwise.} \end{cases}$$

Two signatures σ_1 and σ_2 on the same underlying graph G are said to be *switching equivalent*, denoted by $\sigma_1 \equiv \sigma_2$, if one is obtained from the other by a sequence of vertex switchings.

Given a graph G , we denote by $(G, +)$ ($(G, -)$, respectively) the signed graph whose signature function is constantly positive (negative, respectively) on G . A signed graph (G, σ) is *balanced* if $(G, \sigma) \equiv (G, +)$. A subset X of vertices of a signed graph (G, σ) is called *balanced* if $(G[X], \sigma|_{G[X]})$ is balanced. The size of a largest balanced set in (G, σ) is denoted by $\beta(G, \sigma)$.

Definition 2. Let \widehat{G} be a signed graph. Given a positive integer p and a mapping $\phi : V(G) \rightarrow [p]$, a balanced (p, ϕ) -coloring of \widehat{G} , or simply a (p, ϕ) -coloring of \widehat{G} , is an assignment f , which assigns to each vertex v a set of $\phi(v)$ colors from the set $[p]$ of p colors in such a way that for each color i the set of vertices assigned color i is balanced.

If ϕ is the constant mapping $\phi(v) = q$ for every $v \in V(\widehat{G})$, then we write (p, q) -coloring in place of (p, ϕ) -coloring.

It is observed that a signed graph \widehat{G} admits a (p, q) -coloring for some $p \geq q$ if and only if it contains no negative loops. Hence, we assume that all signed graphs mentioned in this paper satisfy this property. On the other hand, the presence of a positive loop at a vertex does not affect the (p, q) -colorability of a signed graph. So we will always assume the signed graphs considered here have a positive loop attached to each of its vertices. On the other hand, in this work we do not allow parallel edges

Definition 3. Given a signed graph \widehat{G} , the fractional balanced chromatic number, denoted $\chi_{fb}(\widehat{G})$, is defined as

$$\chi_{fb}(\widehat{G}) = \inf \left\{ \frac{p}{q} \mid \widehat{G} \text{ admits a } (p, q)\text{-coloring} \right\}.$$

It is easily observed that the fractional balanced chromatic number is invariant under vertex switching.

For the case $q = 1$ in Definition 2, the fractional balanced coloring is reduced to a *balanced p -coloring* of \widehat{G} , a notion first studied by Zaslavsky [20] under the terminology "balanced partition". The balanced coloring has drawn more attention recently when Jimenez, McDonald, Naserasr, Nurse, and Quiroz [12] showed an equivalent formulation of the famous Hadwiger conjecture with the setting of signed graphs and balanced chromatic number. For general p and q Kuffner, Naserasr, Wang, Yu, Zhou, and Zhu [15] defined the signed analogy of the Kneser graphs, which serve as the homomorphism targets for fractional balanced coloring, and studied their balanced chromatic number. The same group showed that Hedetniemi's conjecture holds for the fractional balanced chromatic number and the categorical product of signed graphs [16]. The problem studied in the current work also follows this direction of research.

Observe that, being balanced, the color class i can be switched to induce only positive edges. Hence, in practice, we use the following refined definition: A (p, ϕ) -coloring of (G, σ) is a mapping f of vertices where each vertex v is assigned a set of $\phi(v)$ colors from the set $\pm[p] := \{\pm 1, \pm 2, \dots, \pm p\}$ such that first of all $-f(v) \cap f(v) = \emptyset$, and secondly if $\sigma(uv) = +1$, then $-f(u) \cap f(v) = \emptyset$ and if $\sigma(uv) = -1$, then $f(u) \cap f(v) = \emptyset$. For future reference, we write

$$\binom{[p]}{\pm q} := \{A \mid A \text{ is a } q\text{-subset of } \pm[p] \text{ such that } -A \cap A = \emptyset\}.$$

The first observation for (p, ϕ) -colorings is that one can permute colors and switch the role of i with $-i$. Thus we have the following.

Observation 4. *If a signed graph (G, σ) admits a (p, ϕ) -coloring, then for a fixed vertex v , any set A of $\phi(v)$ colors satisfying $-A \cap A = \emptyset$ can be selected as the color set of v .*

As a follow up to this this observation we have:

Lemma 5. *If each 2-connected block of a signed graph \widehat{G} admits a (p, ϕ) -coloring, then \widehat{G} itself admits (p, ϕ) -coloring,*

Proof. Observe that given two (signed) graphs \widehat{G}_1 and \widehat{G}_2 on distinct sets of vertices, if we identify one vertex from each, we create no new cycle, and, hence, the resulting signed graph is balanced if and only if both \widehat{G}_1 and \widehat{G}_2 are balanced. Thus merging the colorings of both \widehat{G}_1 and \widehat{G}_2 at the identified vertex, which can be done thanks to Observation 4, we have a coloring for the merged signed graph. \square

Another observation of a similar flavor is that a bridge (i.e., a cut edge) do not affect the coloring at all.

Lemma 6. *If a connected signed graph \widehat{G} contains a bridge uv , then any (p, ϕ) -colorings $\widehat{G} - uv$ is also a (p, ϕ) -coloring of \widehat{G} .*

Proof. That is simply because uv belong to no cycle, in particular to no negative cycle. \square

We denote by K_4^\bullet the graph obtained from K_4 by subdividing one edge exactly once. Let \widehat{K}_4^\bullet be the signed graph on K_4^\bullet where all edges except one edge incident to the vertex of degree 2 are negative, see Figure 1a. In the figures, negative edges are in red and solid line, while positive edges are in blue and dashed.

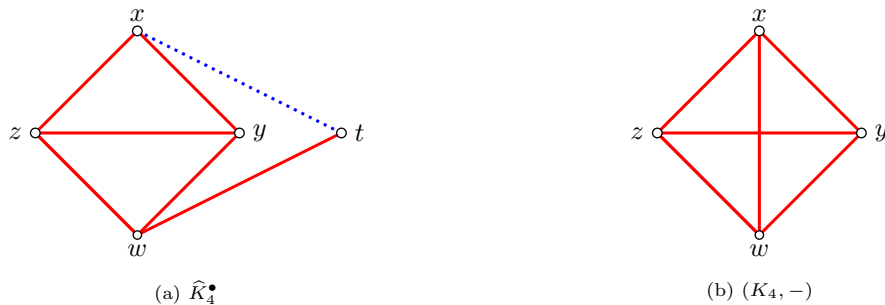


Figure 1: Subcubic graphs \widehat{K}_4^\bullet and $(K_4, -)$

In this work, we focus on the signed subcubic graphs and prove the following main result.

Theorem 7. *Every signed subcubic graph \widehat{G} not switching equivalent to $(K_4, -)$ admits a $(5, 3)$ -coloring.*

This implies that the fractional balanced chromatic number of any signed subcubic graph is at most $\frac{5}{3}$, except for $(K_4, -)$. The bound of $\frac{5}{3}$ is tight and is achieved by the signed graph \widehat{K}_4^\bullet . A proof of $\chi_{fb}(\widehat{K}_4^\bullet) = 5/3$ is given in Lemma 16.

To prove Theorem 7, we will prove a stronger statement (Theorem 8 below) for which we need the following notions. A *block* in a graph G is a maximal 2-connected subgraph of G . Let C_3^* denote a triangle xyz of G such that $d_G(x) = d_G(y) = 2$, and $d_G(z) \leq 3$. Similarly, let C_4^* denote a 4-cycle $xyzw$ of G such that $d_G(x) = d_G(y) = d_G(z) = 2$, and $d_G(w) \leq 3$.

Theorem 8. *Let \widehat{G} be a signed connected subcubic graph with no block of its underlying graph isomorphic to any graph in $\{C_3^*, C_4^*, K_4^\bullet, K_4\}$. Let $\phi(v) = 6 - d_G(v)$ for every $v \in V(G)$. Then \widehat{G} admits a $(5, \phi)$ -coloring.*

The rest of this paper is organized as follows. We give basic properties of fractional balanced colorings of signed subcubic graphs in Section 2. In Section 2.1, we prove Theorem 7 using Theorem 8. Section 3 is devoted to proving Theorem 8. Some remarks and further discussion are provided in Section 4.

2. Preliminaries

Some of the basic properties we will need are the following. Let $S = \{s_1, s_2, \dots, s_t\}$ be a set of integers. Denote by S^* the set of absolute values of the elements in S . For example, if $S = \{1, 2, 3, -3\}$, then $S^* = \{1, 2, 3\}$. We denote by $|S|$ the cardinality of S . Clearly, $|S^*| \leq |S|$.

Observation 9. *Given a positive integer p , let k_1 and k_2 be two positive integers such that $2 \leq k_i \leq p$ for $i \in [2]$. For any two sets $A_i \in \binom{[p]}{\pm k_i}$ for $i \in [2]$, there exists a proper subset $B_i \subsetneq A_i$ such that $B_i \in \binom{[p]}{\pm(k_i-1)}$ and $B_1^* \neq B_2^*$.*

Observation 10. *Let \widehat{G} be a signed graph and let $\phi_i : V(\widehat{G}) \rightarrow [p]$ for $i \in [2]$ such that $\phi_2(v) \leq \phi_1(v)$ for every $v \in V(\widehat{G})$. If \widehat{G} admits a (p, ϕ_1) -coloring f_1 , then \widehat{G} admits a (p, ϕ_2) -coloring f_2 such that $f_2(v) \subseteq f_1(v)$ for each $v \in V(\widehat{G})$.*

Observation 11. *Let \widehat{G} and \widehat{H} be signed graphs with a homomorphism ψ from \widehat{G} to \widehat{H} . Assume \widehat{H} admits a (p, ϕ) -coloring for a given integer p and a mapping $\phi : V(\widehat{G}) \rightarrow [p]$. Then \widehat{G} admits a (p, ϕ') -coloring where $\phi'(u) = \phi(\psi(u))$.*

Proposition 12. [16] *For each positive integer k with $k \geq 2$, the negative cycle C_{-k} admits a $(k, k-1)$ -coloring.*

In particular, by removing colors $\pm 6, \dots, \pm k$ when $k \geq 5$, we have:

Corollary 13. *For each positive integer k with $k \geq 5$, the negative cycle C_{-k} admits a $(5, 4)$ -coloring.*

Every balanced signed graph $(G, +)$ admits a (k, k) -coloring for any positive integer k , in particular, $\chi_{fb}(G, +) = 1$.

Lemma 14. *Let ϕ_3 be an assignment of integer 3, 3, and 4 to the three vertices of C_3 and let ϕ_4 be an assignment of 3, 3, 4, and 4 to the vertices of C_4 . We then have the following claims.*

- (C_3, σ) admits a $(5, \phi_3)$ -coloring for any σ .
- (C_4, σ) admits a $(5, \phi_4)$ -coloring for any σ .
- (K_4^\bullet, σ) admits a $(5, 3)$ -coloring for any σ .

Proof. First notice that every balanced signed graph admits a $(5, 5)$ -coloring, which is stronger than requested $(5, \phi)$ -colorings in each case. So, it is enough to consider signatures that induce some negative cycle.

For C_{-3} (C_{-4} , respectively) we assign the color set $\{1, 2, 3, 4\}$ to the vertex v with $\phi_3(v) = 4$ (the vertices with $\phi_4(x) = 4$, respectively), and color sets $\{1, 2, 5\}$ and $\{3, 4, 5\}$ to the other two vertices.

For K_4^\bullet by symmetry, beside the balanced case, there are three switching equivalent classes signed graphs: the one in Figure 1a, and the two in Figure 2.

For \widehat{K}_4^\bullet , we define $f(t) = \{3, 4, 5\}$, $f(x) = \{1, 2, 3\}$, $f(y) = \{-2, -4, -5\}$, $f(z) = \{-1, -3, 5\}$, and $f(w) = \{1, 2, 4\}$. It can be easily checked that f is a $(5, 3)$ -coloring.

For the graph in Figure 2a, contracting first xt then xw results in a C_{-3} as a homomorphic image. Similarly, for the graph in Figure 2b, contracting first xt then zw results in a C_{-3} as a homomorphic image. Both are $(5, 3)$ -colorable and we are done. \square

Lemma 15. *Every signed graph \widehat{G} on at most 4 vertices which is not switching equivalent to $(K_4, -)$, admits a $(3, 2)$ -coloring.*

Proof. Any such signed graph is a subgraph of (K_4, σ) that is not switching equivalent to $(K_4, -)$. Since each edge of (K_4, σ) is in exactly two triangles, there exist two positive triangles sharing an edge. By possibly a switching, we may assume this edge is positive. Contracting this edge results in a homomorphic image which is either C_{+3} or C_{-3} both of which admit a $(3, 2)$ -coloring. \square

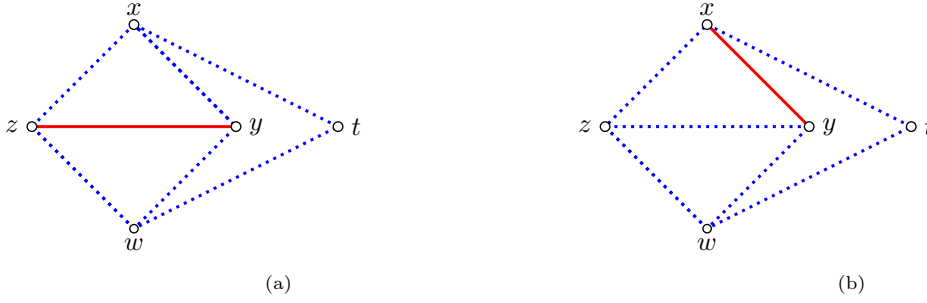


Figure 2: Two signed graphs on K_4^\bullet

Lemma 16. $\chi_{fb}(\widehat{K}_4^\bullet) = \frac{5}{3}$.

Proof. The upper bound $\chi_{fb}(\widehat{K}_4^\bullet) \leq \frac{5}{3}$ is already shown in Lemma 14.

Similar to graph cases, it was shown in [16] that the fractional balanced chromatic number of a signed graph satisfies

$$\chi_{fb}(G, \sigma) \geq \frac{|V(G)|}{\beta(G, \sigma)}$$

Thus, noticing that the size of a maximum balanced set in \widehat{K}_4^\bullet is 3, we obtain the desired lower bound. \square

2.1. The proof of Theorem 7

We first derive Theorem 7 from Theorem 8.

Proof of Theorem 7. Let \widehat{G} be a subcubic signed graph on n vertices which is a not $(K_4, -)$. We consider the following cases:

Case (1). The underlying graph of \widehat{G} is isomorphic to one of C_3, C_4 or K_4^\bullet . In this case, by Lemma 14, we are done.

Case (2). G contains no block isomorphic to any graph in $\{C_3^*, C_4^*, K_4^\bullet\}$. By Theorem 8, \widehat{G} admits a $(5, \phi)$ -coloring f where $\phi(v) = 6 - d_G(v)$. By Observation 10, since $6 - d_G(v) \geq 3$ for G being subcubic, \widehat{G} admits a $(5, 3)$ -coloring.

Case (3). G contains a block H whose underlying graph is in $\{C_3^*, C_4^*, K_4^\bullet\}$. We remove all vertices of H except the one connecting it to the rest of \widehat{G} . We color the resulting graph by an induction hypothesis and then extend the coloring to the rest of H by Observation 10. \square

2.2. Extension of partial $(5, \phi)$ -colorings

We need a further preparation to complete the proof of Theorem 8. We first have the following observation.

Observation 17. Let (G, σ) be a signed graph with a 1-vertex u whose neighbor is v and $\sigma(uv) = -$. Let $\phi : V(G) \rightarrow [p]$ be a mapping and assume f is a (p, ϕ) -coloring of $(G, \sigma) - u$. Then for any $X \in \binom{[p]}{\pm\phi(u)}$ such that $X \cap f(v) = \emptyset$, the assignment $f(u) = X$ is an extension of f to a (p, ϕ) -coloring of (G, σ) .

The following notion will be used. Let f be a $(5, \phi)$ -coloring of \widehat{G} . For any 1-vertex v and its neighbor u , we define the *available color set* of v with respect to f by $A_f(v) := \pm[5] \setminus f(u)$. Here, $|A_f^*(v)| = 5$ and $|A_f(v)| = 10 - \phi(u)$, in particular, if $\phi(u) = 6 - d_G(u)$, then $|A_f(v)| = 4 + d_G(u)$.

3. Proof of Theorem 8

In this section, we assume that \widehat{G} is a counterexample to Theorem 8 with the number of vertices being minimized. That is to say, \widehat{G} is a signed subcubic graph with no block isomorphic to any element of $\mathcal{B}_0 := \{C_3^*, C_4^*, K_4^\bullet, K_4\}$ that does not admit a $(5, \phi)$ -coloring with $\phi(v) = 6 - d_G(v)$. By minimality of \widehat{G} ,

any signed subcubic graph \widehat{H} with no block isomorphic to any member of \mathcal{B}_0 and has fewer vertices than \widehat{G} , admits a $(5, \phi')$ -coloring with $\phi'(v) = 6 - d_H(v)$. We may assume that \widehat{G} is connected. Furthermore, by Lemma 15, \widehat{G} has at least 5 vertices, and by Corollary 13, \widehat{G} must contain at least one 3-vertex.

Claim 1. \widehat{G} is 2-connected. In particular, $\delta(\widehat{G}) \geq 2$.

Proof. We first show that there is no vertex of degree 1. Suppose to the contrary and assume v is a vertex of degree 1 with u being its only neighbor. If v is a vertex of degree 1, then it is in no cycle and it can be given all the 5 colors without being involved in inducing a negative cycle. The key point is to show that $\widehat{G} - v$ admits the required coloring. If $\widehat{G} - v$ satisfies the conditions of the theorem, i.e., if $\widehat{G} - v$ contains none of $C_3^*, C_4^*, K_4^\bullet, K_4$ as a block, then we have a coloring by the minimality of G . Otherwise, either $\widehat{G} - v$ is signed graph on K_4^\bullet for which a $(5, 3)$ -coloring is required. That is provided in Lemma 14. Or, $\widehat{G} - v$ has one of C_3^*, C_4^* as a block. In this case we note two facts: 1. there should be an a cut edge connecting this C_3^* , or C_4^* to the rest of the graph, and 2. the neighbor u of v is a vertex of degree 3 in \widehat{G} and hence requires only 3 colors. We can then apply Lemma 14 to complete the coloring on v and the subgraph C_3^* or C_4^* we are working with. Then we can use the cut edge and minimality \widehat{G} to complete the coloring to the rest of the graphs.

If v is not of degree 1, then, since G is subcubic, having a vertex cut implies having an cut edge uv . Let \widehat{G}_v and \widehat{G}_u be the components of $\widehat{G} - uv$ containing v and u respectively. Adding the vertex u to \widehat{G}_v and v to \widehat{G}_u we have two proper subgraphs of \widehat{G} each satisfying the conditions of the theorem, that is to say neither contains one of $C_3^*, C_4^*, K_4^\bullet, K_4$ as a block. They each then admit a $(5, \phi)$ -coloring that can be put together thanks to Lemma 6. \square

We observe that the bad blocks that might be created after deleting or cutting are either C_3^* or C_4^* . The next claim guarantees that after deletion, we shall not create a block isomorphic to C_3^* or C_4^* .

A t -cycle in a signed graph G is referred to as a (d_1, d_2, \dots, d_t) -cycle if $d_G(v_i) = d_i$ for every i with $i \in [t]$.

Claim 2. \widehat{G} contains none of the following: a $(2, 3, 3)$ -triangle, a $(2, 3, 2, 3)$ -cycle, or a $(2, 2, 3, 3)$ -cycle.

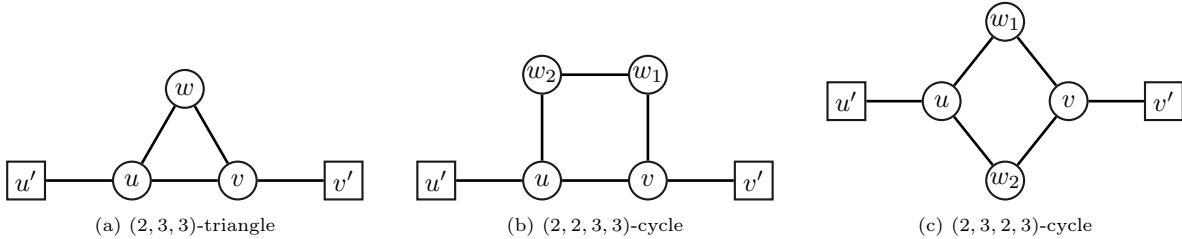


Figure 3: Configurations in Claim 2

Proof. Suppose, to the contrary, that \widehat{G} contains one of the following: a $(2, 3, 3)$ -triangle wuv , a $(2, 2, 3, 3)$ -cycle w_1w_2uv , or a $(2, 3, 2, 3)$ -cycle w_1vw_2u where $d_G(u) = d_G(v) = 3$ and $d_G(w) = d_G(w_i) = 2$ for $i \in [2]$. See Figure 3. Let u' and v' be the other neighbors of u and v , respectively. Since \widehat{G} is 2-connected (Claim 1), u' and v' are distinct. By possibly a switching, we may assume that uu', vv' are both negative and each edge of the cycle is negative except uv for Figure 3a and Figure 3b and vw_2 for Figure 3c. Let $X = \{u, v\}$.

Let \widehat{H} be the component of $\widehat{G} - X$ not containing w or w_i and let \widehat{H}_X be the subgraph of \widehat{G} by removing w or both w_i 's and the edge uv . Note that $d_{H_X}(v) = d_{H_X}(u) = 1$. Since \widehat{H}_X is a proper subgraph of \widehat{G} and, moreover, it contains no block isomorphic to any element of \mathcal{B}_0 , hence admits a $(5, \phi')$ -coloring f where $\phi'(x) = 6 - d_{H_X}(x)$ for $x \in V(H_X)$. Furthermore, since $d_{H_X}(v) = d_{H_X}(u) = 1$, $d_{H_X}(u') \geq 2$ and $d_{H_X}(v') \geq 2$ without loss of generality, assume that $\{c_1, c_2, c_3, c_4, \pm c_5\} \subseteq A_f(v)$ and $\{d_1, d_2, d_3, d_4, d_5, d_u\} \subseteq A_f(u)$ with

$|c_k| = |d_k| = k$ for each $k \in [5]$ and $d_u \in \pm[5] \setminus \{d_1, d_2, d_3, d_4, d_5\}$. In each case, we define a new mapping g as follows:

$$g(v) = \begin{cases} \{c_1, c_2, -d_5\}, & \text{if } uv \text{ is negative;} \\ \{c_1, c_2, d_5\}, & \text{if } uv \text{ is positive or } u, v \text{ are not adjacent;} \end{cases}$$

$$g(u) = \{d_3, d_4, d_5\}.$$

Observe that $g(u) \subset f(u)$, $g(v) \subset f(v)$, and $g(u), g(v) \in \binom{[5]}{\pm 3}$. We shall extend such a coloring g to be a $(5, \phi)$ -coloring with $\phi(x) = 6 - d_G(x)$ for $x \in V(G)$.

- For the $(2, 3, 3)$ -triangle wuv , recall that both of wv and wu are negative. Let $g(w) = \{-c_1, -c_2, -d_3, -d_4\}$.
- For the $(2, 2, 3, 3)$ -cycle w_1w_2uv , recall that vw_1, w_1w_2, w_2u are all negative. Let $g(w_1) = \{-c_1, -c_2, d_3, d_4\}$, and $g(w_2) = \{c_1, c_2, -d_3, -d_4\}$.
- For the $(2, 3, 2, 3)$ -cycle w_1vw_2u , recall that edges uw_1, vw_1, uw_2 are negative. Since u and v are not adjacent in this case, $g(v) = \{c_1, c_2, d_5\}$ and $g(u) = \{d_3, d_4, d_5\}$. For vertices w_1 and w_2 , we define that $g(w_1) = \{-c_1, -c_2, -d_3, -d_4\}$ and

$$g(w_2) = \begin{cases} \{-c_1, -c_2, -d_3, -d_4\}, & \text{if } vw_2 \text{ is negative;} \\ \{-c_1, -c_2, d_3, d_4\}, & \text{if } vw_2 \text{ is positive.} \end{cases}$$

In each case, it is easy to check that together with $g(x) = f(x)$ for $x \in V(H_X) \setminus \{u, v\}$, such g is a $(5, \phi)$ -coloring of \widehat{G} with $\phi(x) = 6 - d_G(x)$, a contradiction. \square

In the following claims, we aim to show that there are no consecutive 2-vertices in \widehat{G} .



Figure 4: Consecutive 2-vertices in Claim 3 and Claim 4

Claim 3. \widehat{G} contains no 2-vertex whose two neighbors are both 2-vertices.

Proof. Assume to the contrary that \widehat{G} contains a 2-vertex w which has two 2-neighbors u and v , whose neighbors are u' and v' , respectively, as depicted in Figure 4a. As \widehat{G} contains no block isomorphic to C_4^* , we have that $u' \neq v'$. By Claim 1, u' and v' cannot be in the same $(2, 3, 3, 3)$ -cycle or the same $(3, 3, 3)$ -triangle in \widehat{G} . By possible switching, we may assume that $u'u, uw, wv, vv'$ are all negative.

Let $\widehat{G}' := \widehat{G} - \{u, w, v\}$. Note that no block isomorphic to C_3^* or C_4^* is created in \widehat{G}' , as otherwise either u' or v' would be in one of $(2, 3, 3)$ -triangle, $(2, 2, 3, 3)$ -cycle or $(2, 3, 2, 3)$ -cycle in \widehat{G} , a contradiction to Claim 2. By minimality, \widehat{G}' admits a $(5, \phi')$ -coloring f where $\phi'(x) = 6 - d_{G'}(x)$ for $x \in V(G')$. Since $d_G(u') = d_{G'}(u') + 1$ and $d_G(v') = d_{G'}(v') + 1$, by definition $f(u') \in \binom{[5]}{\pm(7-d_G(u'))}$ and $f(v') \in \binom{[5]}{\pm(7-d_G(v'))}$. By Observation 9, we can choose two subsets $f'(v') \subsetneq f(v')$ and $f'(u') \subsetneq f(u')$ satisfying that $f'(v') \in \binom{[5]}{\pm(6-d_G(v'))}$, $f'(u') \in \binom{[5]}{\pm(6-d_G(u'))}$, and $f'(v')^* \neq f'(u')^*$. Let $A_{f'}(u) = \pm[5] \setminus f'(u')$ and $A_{f'}(v) = \pm[5] \setminus f'(v')$. We may assume, without loss of generality, that $\{c_1, c_2, c_3, c_4, \pm c_5\} \subseteq A_{f'}(u)$ and $\{d_1, d_2, d_3, \pm d_4, d_5\} \subseteq A_{f'}(v)$, where $|c_i| = |d_i| = i$ for every $i \in [5]$. We define a new mapping g as follows:

$$g(u) = \{c_1, c_2, c_4, d_5\}, \quad g(v) = \{d_1, d_3, c_4, d_5\}, \quad g(w) = \{-c_2, -d_3, -c_4, -d_5\},$$

$g(u') = f'(u')$, $g(v') = f'(v')$, and $g(x) = f(x)$ for $x \in V(G) \setminus \{u', v', u, v, w\}$. Noting that $g(u) \subset A_{f'}(u)$ and $g(v) \subset A_{f'}(v)$, it is easy to see that g is a $(5, \phi)$ -coloring of \widehat{G} with $\phi(x) = 6 - d_G(x)$, a contradiction. \square

Claim 4. \widehat{G} contains no adjacent 2-vertices.

Proof. Assume to the contrary that \widehat{G} contains two adjacent 2-vertices u and v , whose neighbors are u' and v' , respectively, as depicted in Figure 4b. By Claim 3, $d_G(u') = d_G(v') = 3$. As \widehat{G} contains no block isomorphic to C_3^* , we know that $u' \neq v'$. Similar as before, u' and v' cannot be in the same $(2, 3, 3, 3)$ -cycle or the same $(3, 3, 3)$ -triangle in \widehat{G} . By possibly a switching, we may assume that $u'u, uv, vv'$ are all negative. Let $\widehat{G}' := \widehat{G} - \{u, v\}$. Using similar argument in the proof of Claim 3, no block isomorphic to C_3^* or C_4^* is created in \widehat{G}' . By minimality, there exists a $(5, \phi')$ -coloring f of \widehat{G}' where $\phi'(x) = 6 - d_{G'}(x)$ for $x \in V(G')$. By the definition $f(u') \in \binom{[5]}{\pm 4}$ and $f(v') \in \binom{[5]}{\pm 4}$. By Observation 9, we can find $f'(u') \subsetneq f(u')$ and $f'(v') \subsetneq f(v')$ with $f'(u') \in \binom{[5]}{\pm 3}$ and $f'(v') \in \binom{[5]}{\pm 3}$ such that $f'(u') \neq f'(v')$. Without loss of generality, assume that $A_{f'}(u) := \pm[5] \setminus f'(u') = \{c_1, c_2, c_3, \pm c_4, \pm c_5\}$ and $A_{f'}(v) := \pm[5] \setminus f'(v') = \{d_1, d_2, \pm d_3, d_4, \pm d_5\}$ where $|c_i| = |d_i| = i$ for every $i \in [5]$. We define a mapping $g(x) \in \binom{[5]}{\pm(6-d_G(x))}$ for each $x \in V(G)$ as follows:

$$g(u) = \{c_1, c_3, -d_4, c_5\}, \quad g(v) = \{d_2, -c_3, d_4, -c_5\},$$

$g(u') = f'(u')$, $g(v') = f'(v')$, and $g(x) = f(x)$ for $x \in V(\widehat{G}) \setminus \{u, v, u', v'\}$. Noting that $g(u) \subset A_{f'}(u)$ and $g(v) \subset A_{f'}(v)$, one can easily check that g is a $(5, \phi)$ -coloring of \widehat{G} where $\phi(x) = 6 - d_G(x)$, a contradiction. \square

Next, we consider 3-vertices with neighbors of degree 2 in \widehat{G} .

Claim 5. \widehat{G} contains no 3-vertex with two 2-neighbors.

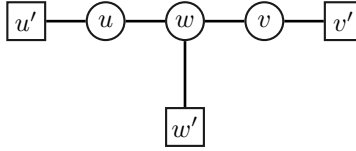


Figure 5: A 3-vertex having two 2-neighbors

Proof. Let w be such a 3-vertex in \widehat{G} with two 2-neighbors u and v with $N_G(u) = \{w, u'\}$, $N_G(v) = \{w, v'\}$, and $N_G(w) = \{u, v, w'\}$. See Figure 5. By Claim 2, there is neither a $(2, 3, 3)$ -triangle nor a $(2, 3, 2, 3)$ -cycle, so w', u' , and v' are three distinct vertices. Moreover, by Claim 4, $d_G(u') = d_G(v') = 3$. By switching, if needed, we may assume that uw, vw , and ww' are all negative.

Let $\widehat{G}' := \widehat{G} - \{w\}$. By Claim 2, no block isomorphic to C_3^* or C_4^* is created in \widehat{G}' . By minimality, there exists a $(5, \phi')$ -coloring f of \widehat{G}' where $\phi'(x) = 6 - d_{G'}(x)$ for $x \in V(G')$. As $d_{G'}(u) = d_{G'}(v) = 1$ and each of them has a neighbor of degree 3 in \widehat{G}' , by the definition of f , $|A_f(v)| = |A_f(u)| = 7$ and $|A_f(v)^*| = |A_f(u)^*| = 5$. Without loss of generality, we may assume that $A_f(v) = \{c_1, c_2, c_3, \pm 4, \pm 5\}$ and $A_f(u) = \{d_1, d_2, d_3, d_4, d_5, d_u^1, d_u^2\}$ where $|c_i| = |d_i| = i$ for every $i \in [5]$ and $d_u^j \in \pm[5] \setminus \{d_1, d_2, d_3, d_4, d_5\}$ for $j \in [2]$. Observing that $1 \leq d_{G'}(w') \leq 2$, we consider the following cases based on the degree of w' . Let $a_i \in \pm[5]$ such that $|a_i| = i$ for $i \in [5]$.

Case (1) Assume that $d_G(w') = 2$. In this case, we can assume $f(w') = \{a_1, a_2, a_3, a_4, a_5\}$. We consider two subcases:

- (1.1) If one of $|d_u^j| \in \{1, 2, 3\}$, say $d_u^j = -d_3$ (i.e., $\pm 3 \in f(u)$), then we define $g(v) = \{c_1, c_2, d_4, a_5\}$, $g(u) = \{d_1, d_2, a_3, d_4\}$, $g(w') = \{a_1, a_2, a_3, a_5\}$ and $g(w) = \{-a_3, -d_4, -a_5\}$.
- (1.2) Otherwise, $f(u) = \{d_1, d_2, d_3, \pm 4, \pm 5\}$. Now we define $g(v) = \{c_1, c_2, a_4, a_5\}$, $g(u) = \{d_1, d_2, a_4, a_5\}$, $g(w') = \{a_1, a_2, a_4, a_5\}$ and $g(w) = \{a_3, -a_4, -a_5\}$.

Case (2) Assume that $d_G(w') = 3$. In this case, $f(w') \in \binom{[5]}{\pm 4}$. As we fix $A_f(v) = \{c_1, c_2, c_3, \pm 4, \pm 5\}$, by symmetry there are only two possibilities for $f(w')$.

- (2.1) If $f(w') = \{a_1, a_2, a_3, a_4\}$, then define $g(v) = \{c_1, c_3, a_4, d_5\}$, $g(u) = \{d_1, d_2, d_3, d_5\}$, $g(w') = \{a_1, a_3, a_4\}$ and $g(w) = \{-d_2, -a_4, -d_5\}$.

(2.2) If $f(w') = \{a_1, a_2, a_4, a_5\}$, then define $g(v) = \{c_1, c_2, a_4, d_5\}$, $g(u) = \{d_1, d_2, d_3, d_5\}$, $g(w') = \{a_1, a_2, a_4\}$ and $g(w) = \{-d_3, -a_4, -d_5\}$.

In each case, let $g(x) = f(x)$ for $x \in V(\widehat{G}) \setminus \{u, w, v, w'\}$. Note that $g(u) \subset f(u)$, $g(v) \subset f(v)$, and $g(w') \subset f(w')$, and, moreover, $g(w) \cap g(y) = \emptyset$ for each $y \in \{u, v, w'\}$. Thus, g is a $(5, \phi)$ -coloring of \widehat{G} where $\phi(x) = 6 - d_G(x)$, a contradiction. \square

We now consider triangles or 4-cycles sharing edges in \widehat{G} .

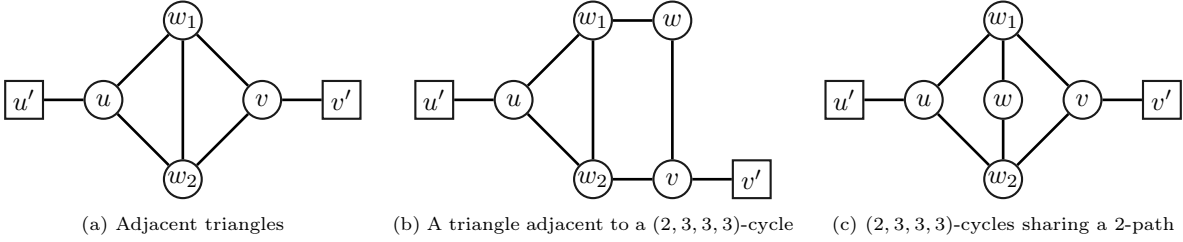


Figure 6: Configurations in Claim 6

Claim 6. \widehat{G} contains none of the following: adjacent triangles, a triangle adjacent to a $(2, 3, 3, 3)$ -cycle, or two $(2, 3, 3, 3)$ -cycles sharing a 2-path.

Proof. Assume to the contrary that \widehat{G} contains one of the said structures. We use the labeling of Figure 6a, Figure 6b and Figure 6c. In each case, since \widehat{G} has at least 5 vertices, \widehat{G} has no cut-vertex and so $u' \neq v'$. Moreover, observe that u' and v' cannot be in the same $(2, 3, 3, 3)$ -cycle or the same $(3, 3, 3)$ -triangle in \widehat{G} .

Let $X_1 = \{u, v, w_1, w_2\}$ and $X_2 = X_3 = \{u, v, w, w_1, w_2\}$. Let $\widehat{G}'_i := \widehat{G} - X_i$ for $i \in [3]$. By Claim 2 and the above observation, \widehat{G}'_i contains no block isomorphic to C_3^* or C_4^* . By minimality, there exists a $(5, \phi')$ -coloring f of \widehat{G}'_i where $\phi'(x) = 6 - d_{G'_i}(x)$ for $x \in V(G'_i)$. Since $d_G(u') = d_{G'_i}(u') + 1$ and $d_G(v') = d_{G'_i}(v') + 1$, by definition $f(u') \in \binom{[5]}{\pm(7-d_G(u'))}$ and $f(v') \in \binom{[5]}{\pm(7-d_G(v'))}$. By Observation 9, we can choose two subsets $f'(v') \subsetneq f(v')$ and $f'(u') \subsetneq f(u')$ satisfying that $f'(v') \in \binom{[5]}{\pm(6-d_G(v'))}$, $f'(u') \in \binom{[5]}{\pm(6-d_G(u'))}$, and $f'(v') \neq f'(u')$. Let $A_{f'}(u) = \pm[5] \setminus f'(u')$ and $A_{f'}(v) = \pm[5] \setminus f'(v')$. Without loss of generality, assume that $\{c_1, c_2, c_3, c_4, \pm c_5\} \subseteq A_{f'}(u)$ and $\{d_1, d_2, d_3, \pm d_4, d_5\} \subseteq A_{f'}(v)$, where $|c_i| = |d_i| = i$ for every $i \in [5]$.

We first consider \widehat{G}'_1 and define an appropriate mapping $g(x) \in \binom{[5]}{\pm(6-d_G(x))}$ for $x \in X_1$ such that $g(u) \subset A_{f'}(u)$ and $g(v) \subset A_{f'}(v)$. By possibly a switching, we may assume that uw_1, uw_2 , and vw_1 are negative. Based on the signature of two adjacent triangles, and by symmetry we have three cases to discuss.

- (1) If w_1w_2 is negative and vw_2 is positive, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, -c_4, d_5\}$, $g(w_1) = \{-c_2, -c_3, -d_5\}$, and $g(w_2) = \{c_2, c_3, -c_4\}$.
- (2) If w_1w_2 and vw_2 are both negative, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, $g(w_1) = \{-c_2, -c_3, -d_5\}$ and $g(w_2) = \{c_2, c_3, -c_4\}$.
- (3) If w_1w_2 is positive and vw_2 is negative, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, and $g(w_1) = g(w_2) = \{-c_2, -c_3, -d_5\}$.

We next consider \widehat{G}'_2 and define an appropriate mapping $g(x) \in \binom{[5]}{\pm(6-d_G(x))}$ for $x \in X_2$ such that $g(u) \subset A_{f'}(u)$ and $g(v) \subset A_{f'}(v)$. By possibly a switching, we may assume that uw_1, ww_1 , and vw_2 are negative. Based on the signatures of the triangle and the 4-cycle, noting that $\{uw_2, w_1w_2, wv\}$ is contained in an edge-cut, by possible switching we have four cases.

- (1) If uw_2, w_1w_2, wv are all negative, then we define $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, $g(w_1) = \{-c_2, -c_3, -d_5\}$, $g(w_2) = \{c_2, c_3, -c_4\}$, and $g(w) = \{-d_1, c_2, c_3, -c_4\}$.

(2) If uw_2, w_1w_2 are negative and wv is positive, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, $g(w_1) = \{-c_2, -c_3, -d_5\}$, $g(w_2) = \{c_2, c_3, -c_4\}$, and $g(w) = \{d_1, c_2, c_3, d_5\}$.

(3) If uw_2, wv are negative and w_1w_2 is positive, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, $g(w_1) = g(w_2) = \{-c_2, -c_3, -d_5\}$, and $g(w) = \{-d_1, c_2, c_3, -c_4\}$.

(4) If w_1w_2, wv are negative and uw_2 is positive, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, -c_4, d_5\}$, $g(w_1) = \{-c_2, -c_3, -d_5\}$, $g(w_2) = \{c_2, c_3, c_4\}$, and $g(w) = \{-d_1, c_2, c_3, c_4\}$.

We finally consider \widehat{G}'_3 and define an appropriate mapping $g(x) \in \binom{[5]}{\pm(6-d_G(x))}$ for $x \in X_3$ such that $g(u) \subset A_{f'}(u)$ and $g(v) \subset A_{f'}(v)$. By possible switching, we may assume that uw_1, wv_1, vw_1 , and uw_2 are negative. Based on the signatures of the two 4-cycles, by possible switching we have three cases.

(1) If wv_2 and vw_2 are negative, then we define $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, $g(w_1) = g(w_2) = \{c_2, c_3, -d_5\}$, and $g(w) = \{-c_2, -c_3, c_4, d_5\}$.

(2) If wv_2 is positive and vw_2 is negative, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, c_4, d_5\}$, $g(w_1) = \{c_2, c_3, -d_5\}$, $g(w_2) = \{-c_2, -c_3, -d_5\}$, and $g(w) = \{c_1, -c_2, -c_3, c_4\}$.

(3) If wv_2 is negative and vw_2 is positive, then let $g(u) = \{c_1, c_4, d_5\}$, $g(v) = \{d_1, -c_4, d_5\}$, $g(w_1) = \{c_2, c_3, -d_5\}$, $g(w_2) = \{c_2, c_3, -c_4\}$ and $g(w) = \{-c_2, -c_3, c_4, d_5\}$.

For each case, together with $g(u') = f'(u')$, $g(v') = f'(v')$, and $g(x) = f(x)$ for $x \in V(\widehat{G}) \setminus \{u, v, u', v', w_1, w_2\}$, each mapping g is a $(5, \phi)$ -coloring of \widehat{G} with $\phi(x) = 6 - d_G(x)$, a contradiction. \square

Claim 7. \widehat{G} contains no 3-vertex with one 2-neighbor. Consequently, \widehat{G} is cubic.

Proof. Let v be a 3-vertex of \widehat{G} and $N_G(v) = \{v_1, v_2, v_3\}$. By Claim 5, at most one of v_i is a 2-vertex and assume that $d_G(v_1) = 2$. We may apply switching such that, for $i \in [3]$, each vv_i is negative in \widehat{G} . Let $\widehat{G}' := \widehat{G} - \{v\}$. By Claim 6, no block isomorphic to C_3^* or C_4^* is created by deleting a 3-vertex, and thus \widehat{G}' contains no block isomorphic to any member of \mathcal{B}_0 .

By minimality, \widehat{G}' admits a $(5, \phi')$ -coloring f of \widehat{G}' where $\phi'(x) = 6 - d_{G'}(x)$ for $x \in V(G')$. Since $d_{G'}(v_i) = 1$ and $d_{G'}(v_i) = 2$ for $i \in \{2, 3\}$, we have $f(v_1) \in \binom{[5]}{\pm 5}$ and $f(v_i) \in \binom{[5]}{\pm 4}$ for $i \in \{2, 3\}$. Without loss of generality, assume that $\{a_1, a_2, a_3, a_4, a_5\} \subset f(v_1)$ and $f(v_2) = \{b_1, b_2, b_3, b_4\}$. By symmetry, we may assume that $f(v_3) = \{c_1, c_2, c_3, c_k\}$ for some $k \in \{4, 5\}$. Note that $|a_i| = |b_i| = |c_i| = i$ for each $i \in [5]$. We consider two cases:

Case (1) If one of $b_i = c_i$ for $i \in [3]$, say $b_1 = c_1$, then we define $g(v_1) = \{a_2, a_3, a_4, a_5\}$, $g(v_2) = \{b_1, b_2, b_3\}$, $g(v_3) = \{b_1, c_2, c_3\}$, and $g(v) = \{-b_1, -a_4, -a_5\}$.

Case (2) Otherwise, for each $i \in [3]$, $b_i \neq c_i$. So a_1 is the same as exactly one of b_1 or c_1 .

(2.1) If $a_1 = b_1$, then we define $g(v_1) = \{a_1, a_2, a_3, a_{9-k}\}$, $g(v_2) = \{a_1, b_2, b_3\}$, $g(v_3) = \{c_2, c_3, c_k\}$, and $g(v) = \{-a_1, -a_{9-k}, -c_k\}$.

(2.2) If $a_1 = c_1$, then we define $g(v_1) = \{a_1, a_2, a_3, a_5\}$, $g(v_2) = \{b_2, b_3, b_4\}$, $g(v_3) = \{a_1, c_2, c_3\}$, and $g(v) = \{-a_1, -b_4, -a_5\}$.

For each case, let $g(x) = f(x)$ for $x \in V(\widehat{G}) \setminus \{v, v_1, v_2, v_3\}$. Each mapping g is a $(5, \phi)$ -coloring of \widehat{G} with $\phi(x) = 6 - d_G(x)$, a contradiction. \square

We complete the proof with our last claim. It contradicts the fact that \widehat{G} must contain a 3-vertex. Recall that \widehat{G} has at least 5 vertices.

Claim 8. \widehat{G} contains no 3-vertices.

Proof. Let v be a 3-vertex of \widehat{G} and $N_G(v) = \{v_1, v_2, v_3\}$. By Claim 7, $d_G(v_i) = 3$ for each $i \in [3]$. We may apply switching such that each vv_i for $i \in [3]$ is negative in \widehat{G} . Let $\widehat{G}' := \widehat{G} - \{v\}$. By Claim 2 and Claim 6, \widehat{G}' contains no block isomorphic to C_3^* or C_4^* .

By minimality, \widehat{G}' admits a $(5, \phi')$ -coloring f of \widehat{G}' where $\phi'(x) = 6 - d_{G'}(x)$ for $x \in V(G')$. Since $d_{G'}(v_i) = 2$ for $i \in [3]$, we have $f(v_i) \in \binom{[5]}{\pm 4}$ for $i \in [3]$. Without loss of generality, we may assume that $f(v_1) = \{a_1, a_2, a_3, a_4\}$ with $|a_i| = i$ for each $i \in [4]$. By symmetry, we have two possibilities for $f(v_2)$: (1) $f(v_2) = \{b_1, b_2, b_3, b_4\}$; (2) $f(v_2) = \{b_\alpha, b_\beta, b_\gamma, b_5\}$ where $|b_i| = i$. Moreover, assume that $f(v_3) = \{c_h, c_j, c_k, c_\ell\}$ with $|c_i| = i$ for $i \in \{h, j, k, \ell\}$. We discuss two cases:

Case (1). $f(v_2) = \{b_1, b_2, b_3, b_4\}$.

Subcase (1.1) One of $\{c_h, c_j, c_k, c_\ell\}$ has absolute value 5, say $|c_\ell| = 5$.

In this case, $\{|c_h|, |c_j|, |c_k|\} \subset [4]$. Without loss of generality, we may assume that $\{c_h, c_j, c_k, c_\ell\} = \{c_1, c_2, c_3, c_5\}$. We define a new mapping g as follows:

$$g(v_1) = \{a_1, a_2, a_3\}, \quad g(v_2) = \{b_1, b_2, b_4\}, \quad g(v_3) = \{c_1, c_2, c_5\}, \quad g(v) = \{-a_3, -b_4, -c_5\}$$

and $g(x) = f(x)$ for each $x \in V(\widehat{G}) \setminus \{v, v_1, v_2, v_3\}$. Noting that $g(v_i) \subset f(v_i)$ for $i \in [3]$, this mapping g is a $(5, \phi)$ -coloring of \widehat{G} where $\phi(x) = 6 - d_G(x)$ for $x \in V(G)$.

In the following subcases, we know that $f(v_3) \cap \{5, -5\} = \emptyset$ and may thus assume that $\{c_h, c_j, c_k, c_\ell\} = \{c_1, c_2, c_3, c_4\}$.

Subcase (1.2) One of $\{c_1, c_2, c_3, c_4\}$, say c_1 , belongs to $f(v_1) \cap f(v_2)$.

That is to say, $a_1 = b_1 = c_1$. We define a mapping g as follows:

$$g(v_1) = \{a_1, a_2, a_3\}, \quad g(v_2) = \{b_1, b_2, b_3\}, \quad g(v_3) = \{c_1, c_2, c_3\}, \quad g(v) = \{-a_1, 4, 5\}$$

and $g(x) = f(x)$ for each $x \in V(\widehat{G}) \setminus \{v, v_1, v_2, v_3\}$. One may readily check that g is a $(5, \phi)$ -coloring of \widehat{G} where $\phi(x) = 6 - d_G(x)$ for $x \in V(G)$.

Subcase (1.3) Otherwise, each c_i for $i \in \{1, 2, 3, 4\}$ is in either $\{i, -i\} \setminus \{a_i, b_i\}$ or $(f(v_1) \setminus f(v_2)) \cup (f(v_2) \setminus f(v_1))$.

Since each set $f(v_j)$ for $j \in [3]$ is in the same form, without loss of generality, we may assume that $a_1 = b_1 = -c_1$. We define a mapping g as follows:

$$g(v_1) = \{a_1, a_2, a_3\}, \quad g(v_2) = \{b_1, b_2, b_3\}, \quad g(v_3) = \{c_2, c_3, c_4\}, \quad g(v) = \{c_1, -c_4, 5\}$$

and $g(x) = f(x)$ for each $x \in V(\widehat{G}) \setminus \{v, v_1, v_2, v_3\}$. Such a mapping g is a $(5, \phi)$ -coloring of \widehat{G} where $\phi(x) = 6 - d_G(x)$ for $x \in V(G)$.

Case (2). $f(v_2) = \{b_\alpha, b_\beta, b_\gamma, b_5\}$ for $\alpha, \beta, \gamma \in [4]$.

In this case, we first claim that $\{h, j, k, \ell\} \neq \{1, 2, 3, 4\}$ and $\{h, j, k, \ell\} \neq \{\alpha, \beta, \gamma, 5\}$. As otherwise, by the symmetries of the vertices v_1, v_2 , and v_3 , we may apply the same arguments in Subcase (1.1) and complete the proof. Recalling that $\{\alpha, \beta, \gamma\}$ is a 3-subset of $\{1, 2, 3, 4\}$, we have that $\{c_h, c_j, c_k, c_\ell\} = \{c_{\alpha'}, c_{\beta'}, c_{\gamma'}, c_5\}$ where $\alpha', \beta', \gamma' \in [4]$ and $|\{\alpha, \beta, \gamma\} \cap \{\alpha', \beta', \gamma'\}| = 2$. Without loss of generality, we assume that $\alpha = \alpha', \beta = \beta'$ and $\gamma \neq \gamma'$. We define a mapping g as follows:

$$g(v_1) = \{a_\alpha, a_\beta, a_\gamma\}, \quad g(v_2) = \{b_\alpha, b_\beta, b_5\}, \quad g(v_3) = \{c_{\alpha'}, c_{\beta'}, c_{\gamma'}\}, \quad g(v) = \{-a_\gamma, -b_5, -c_{\gamma'}\}$$

and $g(x) = f(x)$ for each $x \in V(\widehat{G}) \setminus \{v, v_1, v_2, v_3\}$. Since $\gamma \neq \gamma'$ and $\gamma, \gamma' \in [4]$, g can be checked to be a $(5, \phi)$ -coloring of \widehat{G} where $\phi(x) = 6 - d_G(x)$ for $x \in V(G)$. \square

4. Remarks and questions

Besides the example \widehat{K}_4^\bullet , it is unclear whether there exists an infinite family of subcubic signed graphs critically having fractional balanced chromatic number $\frac{5}{3}$.

We conjecture that if we further forbid \widehat{K}_4^\bullet from our graph class of Theorem 7, the fractional balanced chromatic number is bounded by $\frac{8}{5}$.

Conjecture 18. *Every signed subcubic graph not isomorphic to $(K_4, -)$ and not containing \widehat{K}_4^\bullet admits an $(8, 5)$ -coloring.*

The value of $\frac{8}{5}$ is reached by a signed 3-dimensional hypercube depicted in Figure 7. One may ask the same question of finding infinite critical examples that achieve this bound.

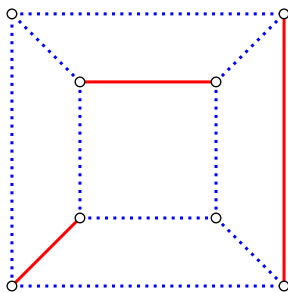


Figure 7: A 3-cube with all faces negative

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