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Performance Evaluation of a Producer/Consumer Tandem Coupled by a Buffer with On/Off Control Policy

Stéphane MOCANU*and Yves DALLERY

Laboratoire Productique Logistique
Ecole Centrale de Paris
Grande Voie des Vignes
92295 Chatenay Malabry, France
mocanu@pl.ecp.fr, dallery@pl.ecp.fr

Abstract

In the present paper we study a special case of a two unreliable machine series production line separated by an intermediate buffer. We use the continuous model of the system and we assume that the production rates of the servers are equal. The difference from the classical model is the use of a control policy intended to minimize the number of stops of the servers. The performance evaluation criteria that we use takes into account both system productivity and the number of server stops. The exact solution of the model is obtained by a singular systems technique and analytical formulae for the performance evaluation criteria are deduced.

1 Introduction

We consider here the continuous model of an producer/consumer system composed by single servers. The servers are supposed to be unreliable with failure frequencies and repair rates being distributed exponentially and there is a finite capacity intermediate buffer.

The nominal production rates of the servers are considered to be equal and constant. The main difference between the considered model and the classical two servers continuous model is the control policy of the system. We are interested here to model packaging production lines. In such production lines servers stops involves production loss. Then the control policy has to minimize the number of servers stops.

There are two situation in which a server can be stopped. Naturally, a server will be stopped when a failure occurs. We will call such a situation an *uncontrollable stop* and we will not consider these stops in the optimisation criteria, as failure rates are independent from the control policy.

A second situation when a server is stopped occurs, by example, when the buffer is empty and a failure occurs on the upstream server. Then the downstream server will be starved and then stopped. Reciprocally, if the buffer is full and a failure occurs on the downstream server, the upstream server will be blocked, and then stopped. We will call such stop situations a *controllable stop*, and the control policy will be intended to avoid such situations, but keeping however an acceptable production rate.

Then the optimisation criteria is of the form :

$$J = \alpha_t(1 - X) + \alpha_u n s_u + \alpha_d n s_d, \tag{1}$$

*The author is also with Laboratoire d'Automatique de Grenoble, LAG/ENSIEG, B.P. 46 38402 Saint Martin d'Hères, France

where ns_u and ns_d are the numbers of controllable stops on the upstream and downstream servers, and X is the productivity of the system,

$$X = \frac{P}{P_d}. \quad (2)$$

In (2), P is the system throughput, and P_d is the (nominal) throughput of the downstream server.

2 The control policy

The simplest control policy in order to avoid controllable stops, policy that will be studied in the following, consist in regenerating a minimal level of the buffer, respectively a minimal level of available places, after a starving, respectively a blocking. We use then an on/off switching policy with different start/stop levels. The production rate control curve of each server has a hysteresis shape defined by the stop and (re)start levels.

The conventional representation of the system is given in Figure 1. We notice the presence of the start and stop levels used by the command policy (Figure 2).

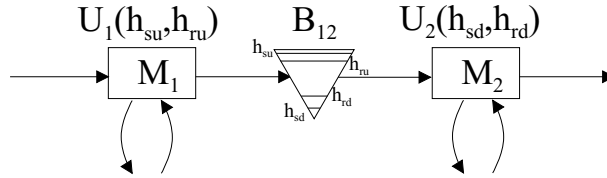


Figure 1: The conventional representation of the system.

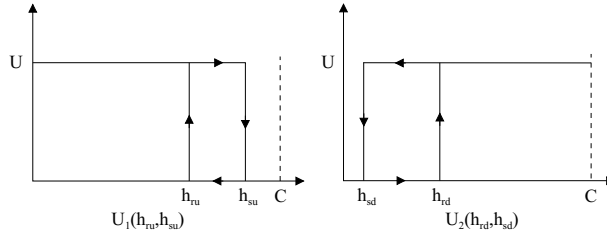


Figure 2: The control policy.

We will describe in the following the behaviour of such a system. While the level of the intermediate buffer stays between h_{su} and h_{sd} levels both servers are started and they alternate working and failure periods following the exponential distributions with rates λ_1 and λ_2 of failure frequency and μ_1 , respectively μ_2 for repair times.

When the level of the buffer equals, let say, the upper stop level h_{su} the upstream server is stopped and we are waiting that the downstream server empties the stock to the upper restart level h_{ru} . Then the upstream server is restarted and we turn back to the simultaneous functioning. Reciprocally, if the level of the intermediate buffer decreases until the lower stop level h_{sd} the downstream server is stopped while the upstream server remounts the buffer contents to the restart level h_{rd} .

In the “normal” functioning mode, as the nominal production rates are equal the buffer contents level will be constant while both server are operational. Changes of the buffer level can occur sequel to a failure of one of the servers. It follows that, for example, upper stop level can be attained only if the downstream server is down and the upstream server is operation. Reciprocally the lower stop level can be attained if the upstream server is down and the downstream server if operational. We can see that there are three possible states of a server: working, failed or stopped. We will denote the state by the symbols: 1 – working, 0 – failed and S – stopped. Then, an example of a possible state trajectory is given in Figure 3.

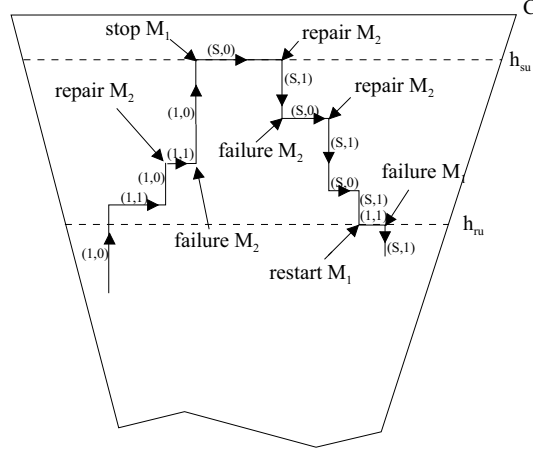


Figure 3: A possible state trajectory of the system.

The stop levels h_{su} and h_{sd} are fixed by technological constraints (in the real case a server cannot be stopped instantaneously, then there is some “accelerating” and “decelerating” curves). As we consider here only the control curve, and not the real one, we simply fix h_{su} and h_{sd} at some values in $(0, C)$, and we consider all states corresponding to buffer levels in $[0, h_{sd}]$ or $(h_{su}, C]$ being inaccessible. Then, the buffer comportement is similar to the case of a buffer of capacity $h_{su} - h_{sd}$.

3 The model

We will model the system as a Markov chain with mixed state space, the discrete part of the model characterizing the server states changes and the continuous part corresponding to the buffer status.

3.1 The state space

A state of the system will be denoted by a triplet (α_1, α_2, h) where $\alpha_{1,2} \in \{0, 1, S\}$ and h is the buffer level.

From the point of view of possible server states pairs, sequel to the description of the system functioning in the previous section the feasible set of state pairs is

$$E = \{(0, 0), (1, 1), (0, 1), (1, 0), (S, 1), (S, 0), (1, S), (0, S)\}$$

that represents the discrete state space. In order to describe the complete state space we remark that one can describe three functioning modes : the “nominal” mode $E_1 = \{(\beta_1, \beta_2) | \beta_{1,2} \in \{0, 1\}\}$ when both servers are operational, and the “single” modes $E_2 = \{(S, \beta_2) | \beta_2 \in \{0, 1\}\}$ and $E_3 = \{(\beta_1, S) | \beta_1 \in \{0, 1\}\}$ when only one server is operational, the other one being stopped.

In the nominal mode the level of the intermediate buffer can vary between minimal and maximal levels, then the mixed state space of this mode is : $\mathcal{E}_1 = \{(\beta_1, \beta_2, h) | \beta_{1,2} \in \{0, 1\}, h \in (h_{sd}, h_{su})\}$.

In single modes the level of the buffer can vary only between the upper stop level and the upper restart level for the mode (S, β_2) and between the lower stop level and the lower restart level for the mode (β_1, S) . Then the corresponding state spaces are : $\mathcal{E}_2 = \{(S, \beta_2, h) | \beta_2 \in \{0, 1\}, h \in (h_{ru}, h_{su})\}$ and $\mathcal{E}_3 = \{(\beta_1, S, h) | \beta_1 \in \{0, 1\}, h \in (h_{sd}, h_{rd})\}$.

The transitions between modes are made only from the boundary states $(1, 0, h_{su})$ (between the nominal mode and the single mode (S, β_2)) and $(S, 1, h_{rd})$ (the reciprocal transition) and symmetrically $(0, 1, h_{sd})$ (nominal mode \rightarrow single mode (β_1, S)) and $(1, S, h_{rd})$ (back to the nominal mode).

3.2 Internal dynamics

The internal dynamics characterizes the internal transitions (discrete or continuous) inside each functioning mode.

3.2.1 The nominal mode

In nominal mode the internal dynamics are given by the classical two-machines line continuous model as described in [3]. We redraw bellow the equations in matrix form :

$$A\dot{\underline{f}} = B\underline{f}, \quad (3)$$

where,

$$A = \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & -U & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\lambda_1 - \mu_2 & 0 & \lambda_2 & \mu_1 \\ 0 & -\lambda_2 - \mu_1 & \lambda_1 & \mu_2 \\ \mu_2 & \mu_1 & -\lambda_1 - \lambda_2 & 0 \\ \lambda_1 & \lambda_2 & 0 & -\mu_1 - \mu_2 \end{bmatrix}, \quad (4)$$

and f is the column vector

$$\underline{f} = [f_{10} \ f_{01} \ f_{11} \ f_{00}]^T$$

As we are interested by the stationary behaviour of the system we write directly \underline{f} for $\frac{\partial f}{\partial h}$. It has been shawn in the literature that \underline{f} is a vector of probability density functions [4, 3]. We prefer the formulation (3) with the matrices A and B as in (4) in order to apply a DAE approach to the solution of the system (this technique will be detailed in the following section).

The Markov chain corresponding to the internal dynamics (3) is presented in Figure 3.

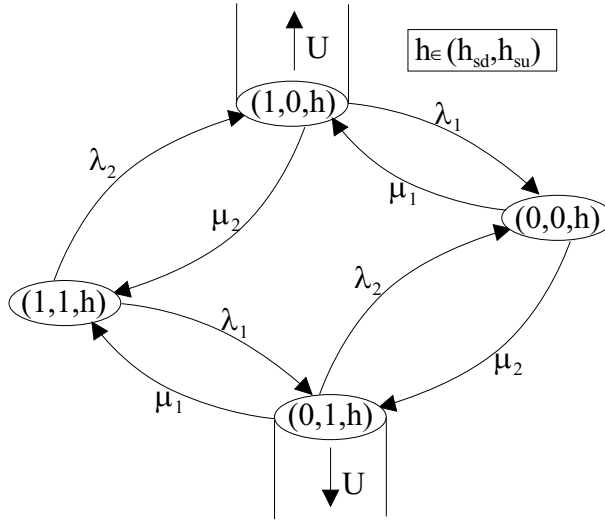


Figure 4: The Markov chain describing the nominal mode.

The equation system (3) is a DAE system with linear algebraic constant. In order to solve that system we will apply a transformation that bring the system to an ODE system. Such a technique is frequent in DAE problems and details can be found, for example in [1].

The idea is the following : by a variable change $\underline{f} = Q\underline{y}$ and a multiplication of the system by some matrix P we transform the problem into an equivalent ODE problem :

$$PAQ\dot{\underline{y}} = PBQ\underline{y}, \quad (5)$$

with

$$PAQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad PBQ = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}. \quad (6)$$

We will not comment here the existence of matrices P and Q . The proof of their existence can be found in [2, 1]. In particular P and Q can be chosen such that T is a Jordan form, but then the computation of transform matrices becomes difficult.

For our purposes, given the fact that A and B can be written in the block form :

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with A_1 and B_{22} non-singular, it was proved in [1] that we can choose

$$Q = \begin{bmatrix} A_1 & 0 \\ B_{21} & B_{22} \end{bmatrix}^{-1} \quad P = \begin{bmatrix} I & -(B_{11}Q_{12} + B_{12}Q_{22}) \\ 0 & I \end{bmatrix} \quad (7)$$

In our case

$$A_1 = \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \quad B_{22} = \begin{bmatrix} -(\lambda_1 + \lambda_2) & 0 \\ 0 & -(\mu_1 + \mu_2) \end{bmatrix} \quad (8)$$

and then

$$Q = \begin{bmatrix} U^{-1} & 0 & 0 & 0 \\ 0 & -U^{-1} & 0 & 0 \\ \frac{\mu_2}{(\lambda_1 + \lambda_2)U} & -\frac{\mu_1}{(\lambda_1 + \lambda_2)U} & -(\lambda_1 + \lambda_2)^{-1} & 0 \\ \frac{\lambda_1}{(\mu_1 + \mu_2)U} & -\frac{\lambda_2}{(\mu_1 + \mu_2)U} & 0 & -(\mu_1 + \mu_2)^{-1} \end{bmatrix} \quad (9)$$

$$P = \begin{bmatrix} 1 & 0 & \frac{\lambda_2}{\lambda_1 + \lambda_2} & \frac{\mu_1}{\mu_1 + \mu_2} \\ 0 & 1 & \frac{\lambda_1}{\lambda_1 + \lambda_2} & \frac{\mu_2}{\mu_1 + \mu_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

Then, in this case the matrix T is of the form :

$$T = \frac{\lambda_1 + \lambda_2 + \mu_1 + \mu_2}{(\lambda_1 + \lambda_2)(\mu_1 + \mu_2)U} \begin{bmatrix} -\lambda_1 \mu_2 & -\lambda_2 \mu_1 \\ \lambda_1 \mu_2 & \lambda_2 \mu_1 \end{bmatrix}, \quad (11)$$

and let $\mathcal{K} = \frac{\lambda_1 + \lambda_2 + \mu_1 + \mu_2}{(\lambda_1 + \lambda_2)(\mu_1 + \mu_2)U}$. Obviously, the solution of the equation (5) is of the form :

$$\underline{y} = [y_1 \ y_2 \ 0 \ 0]^T, \quad (12)$$

where $\underline{y}_r = [y_1 \ y_2]^T$ is the solution of the ODE :

$$\dot{\underline{y}}_r = T \underline{y}_r, \quad (13)$$

with is the of the form :

$$\underline{y}_r(h) = e^{Th} \underline{C}, \quad (14)$$

where \underline{C} is some constant vector.

In order to obtain the explicit solution, one must find the eigenvalues and eigenvectors of the matrix T . As we can easily prove T is a singular matrix, then one of the eigenvalues is 0. The second one equals $\lambda_2 \mu_1 - \lambda_1 \mu_2$ and is non-zero if $I_1 \neq I_2$. It follows that two different solutions must be considered.

3.2.2 The case $I_1 = I_2$

In this case both eigenvalues of T are zero and, if one compute the matrix exponential, it will equal

$$e^{Th} = \begin{bmatrix} 1 - \frac{\mathcal{K}\lambda_1\lambda_2}{I}h & -\frac{\mathcal{K}\lambda_1\lambda_2}{I}h \\ \frac{\mathcal{K}\lambda_1\lambda_2}{I}h & 1 + \frac{\mathcal{K}\lambda_1\lambda_2}{I}h \end{bmatrix},$$

Then

$$\underline{y}_r = \left[\left(1 - \frac{\mathcal{K}\lambda_1\lambda_2}{I}h\right) c_1 - \frac{\mathcal{K}\lambda_1\lambda_2}{I}hc_2, \frac{\mathcal{K}\lambda_1\lambda_2}{I}hc_1 + \left(1 + \frac{\mathcal{K}\lambda_1\lambda_2}{I}h\right) c_2 \right]^T$$

and

$$\underline{y} = \left[\left(1 - \frac{\mathcal{K}\lambda_1\lambda_2}{I}h\right) c_1 - \frac{\mathcal{K}\lambda_1\lambda_2}{I}hc_2, \frac{\mathcal{K}\lambda_1\lambda_2}{I}hc_1 + \left(1 + \frac{\mathcal{K}\lambda_1\lambda_2}{I}h\right) c_2, 0, 0 \right]^T.$$

Eventually, with

$$\mathcal{K}' = \frac{\mathcal{K}\lambda_1\lambda_2}{I},$$

the general solution of the nominal mode is

$$\underline{f} = \begin{bmatrix} f_{10} \\ f_{01} \\ f_{11} \\ f_{00} \end{bmatrix} = Q\underline{y} = \begin{bmatrix} \frac{(1-\mathcal{K}'h)C_1}{U} - \frac{\mathcal{K}'hC_2}{U} \\ -\frac{\mathcal{K}'hC_1}{U} - \frac{(1+\mathcal{K}'h)C_2}{U} \\ \frac{(\lambda_2-\mathcal{K}'(\lambda_1+\lambda_2)h)C_1}{I(\lambda_1+\lambda_2)U} - \frac{(\lambda_1+\mathcal{K}'(\lambda_1+\lambda_2)h)C_2}{I(\lambda_1+\lambda_2)U} \\ \frac{I(\lambda_1+\mathcal{K}'(\lambda_1+\lambda_2)h)C_1}{(\lambda_1+\lambda_2)U} - \frac{I(\lambda_2+\mathcal{K}'(\lambda_1+\lambda_2)h)C_{2,1}}{(\lambda_1+\lambda_2)U} \end{bmatrix}. \quad (15)$$

3.2.3 The case $I_1 \neq I_2$

As pointed out in the beginning of the section in this case the matrix T posses one non-zero eigenvalue equal to $\lambda_2\mu_1 - \lambda_1\mu_2$. It follows that there will be an exponential term moreover than a linear one in the expression of the general solution. After some manipulation the matrix exponential of T is obtained as :

$$e^{Th} = \begin{bmatrix} -\left(I_2 - I_1 e^{\mathcal{K}_1 h(I_1^{-1} - I_2^{-1})}\right) (I_1 - I_2)^{-1} & -\left(1 - e^{\mathcal{K}_1 h(I_1^{-1} - I_2^{-1})}\right) I_2 (I_1 - I_2)^{-1} \\ \left(1 - e^{\mathcal{K}_1 h(I_1^{-1} - I_2^{-1})}\right) I_1 (I_1 - I_2)^{-1} & \left(I_1 - I_2 e^{\mathcal{K}_1 h(I_1^{-1} - I_2^{-1})}\right) (I_1 - I_2)^{-1} \end{bmatrix}, \quad (16)$$

where

$$\mathcal{K}_1 = \frac{\lambda_1 \lambda_2 (I_1 I_2 (\lambda_1 + \lambda_2) + \lambda_1 I_2 + \lambda_2 I_1)}{(\lambda_1 + \lambda_2) (\lambda_1 I_2 + \lambda_2 I_1) U}. \quad (17)$$

After some multiplication with a constant vector and the transformation matrix Q , in the same manner as in the previous case, we obtain the general solution of the nominal mode in the case of different availabilities:

$$\begin{bmatrix} f_{10} \\ f_{01} \\ f_{11} \\ f_{00} \end{bmatrix} = \begin{bmatrix} -\frac{\left(I_2 - I_1 e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_1 I_2}}\right) C_1}{U(I_1 - I_2)} - I_2 \frac{\left(I_2 - e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_1 I_2}}\right) C_2}{U(I_1 - I_2)} \\ -I_1 \frac{\left(I_1 - e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_2 I_1}}\right) C_1}{U(I_1 - I_2)} - \frac{\left(I_1 - I_2 e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_2 I_1}}\right) C_2}{U(I_1 - I_2)} \\ -\frac{\left((\lambda_1 + \lambda_2)I_2 - (\lambda_1 I_2 + \lambda_2 I_1) e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_1 I_2}}\right) C_1}{(\lambda_1 + \lambda_2)I_2 U(I_1 - I_2)} - \frac{\left((\lambda_1 + \lambda_2)I_1 - (\lambda_1 I_2 + \lambda_2 I_1) e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_2 I_1}}\right) C_2}{(\lambda_1 + \lambda_2)I_1 U(I_1 - I_2)} \\ -I_1 I_2 \frac{\left(\lambda_1 I_2 + \lambda_2 I_1 - (\lambda_1 + \lambda_2)I_1 e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_1 I_2}}\right) C_1}{(\lambda_1 I_2 + \lambda_2 I_1) U(I_1 - I_2)} - I_1 I_2 \frac{\left(\lambda_1 I_2 + \lambda_2 I_1 - (\lambda_1 + \lambda_2)I_2 e^{-\frac{\mathcal{K}_1(I_1 - I_2)h}{I_2 I_1}}\right) C_2}{(\lambda_1 I_2 + \lambda_2 I_1) U(I_1 - I_2)} \end{bmatrix} \quad (18)$$

3.2.4 The single modes

As the behaviour of the system in the two single modes is symmetric the internal dynamics follows easily.

In each single mode the non-stopped server acts isolated, then the system dynamics corresponds to the server dynamics. For example in (S, β_2) mode the Markov chain which describes the system behaviour is the one in Figure 5.

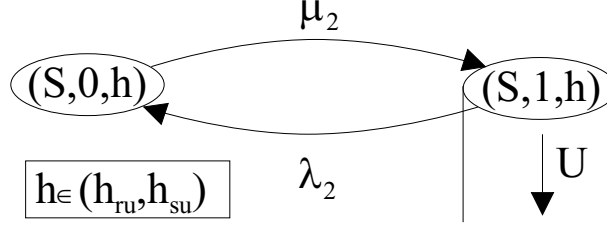


Figure 5: The Markov chain describing the (S, β_2) mode.

The corresponding dynamics are the flow conservation equations in the two discrete states :

$$\begin{cases} U f_{S1} = \mu_2 f_{S0} - \lambda_2 f_{S1} \\ 0 = \mu_2 f_{S0} - \lambda_2 f_{S1} \end{cases} \quad (19)$$

The obvious solution of (19) is then

$$f_{S1} = C_u, \quad f_{S0} = I_2 C_u, \quad (20)$$

where C_u is some constant to be determined. Symmetrically the solution of the internal dynamics of the mode (β_1, S, h) is

$$f_{1S} = C_d, \quad f_{0S} = I_1 C_d, \quad (21)$$

where C_d is the integration constant to be determined.

3.2.5 The complete chain

Consequently to the model description once the system is in the state $(1, 0, h_{su})$, the upstream server is stopped and then the unique event that can occur is a reparation of the downstream server which brings the system in the single mode (S, β_2) . When the second server empties the buffer to the level h_{ru} and the system is in state $(S, 1, h_{ru})$ the upstream server restarts and the system stays in the same state until a failure occurs on one of the servers and this brings the system back to the nominal mode. Symmetrical transitions exist between nominal mode and single (β_1, S) mode on the lower boundary. Then the complete Markov chain is the one given in Figure 6

3.3 Boundary conditions

There are 16 boundary states in the system, corresponding to buffer levels h_{su}, h_{ru}, h_{rd} and h_{sd} . Only four boundary states possess non-zero stationary probabilities. States $(0, 1, h_{su})$, $(1, 0, h_{sd})$, $(S, 1, h_{su})$ and $(1, S, h_{sd})$ are transient states as the system cannot stay in those states (for example in state $(0, 1, h_{su})$ the buffer level begins instantaneously to decrease as the only operational server is downstream server). States $(0, 0, h_{su})$, $(0, 0, h_{sd})$, $(1, 1, h_{su})$, $(1, 1, h_{sd})$, $(S, 0, h_{su})$ and $(0, S, h_{sd})$ have zero stationary probability as they can be reached only from transient states, while states $(S, 0, h_{ru})$, $(0, S, h_{rd})$ are unreachable.

We denote by $P_{\alpha, \beta, h}$ the stationary probability of the state α, β, h . In the following we will develop the boundary states equilibrium equations that will be used in order to determine the integration constants.

Firstly we will analyze a special set of interior to boundary equations that will made explicit the character of the internal dynamics of the nominal mode.

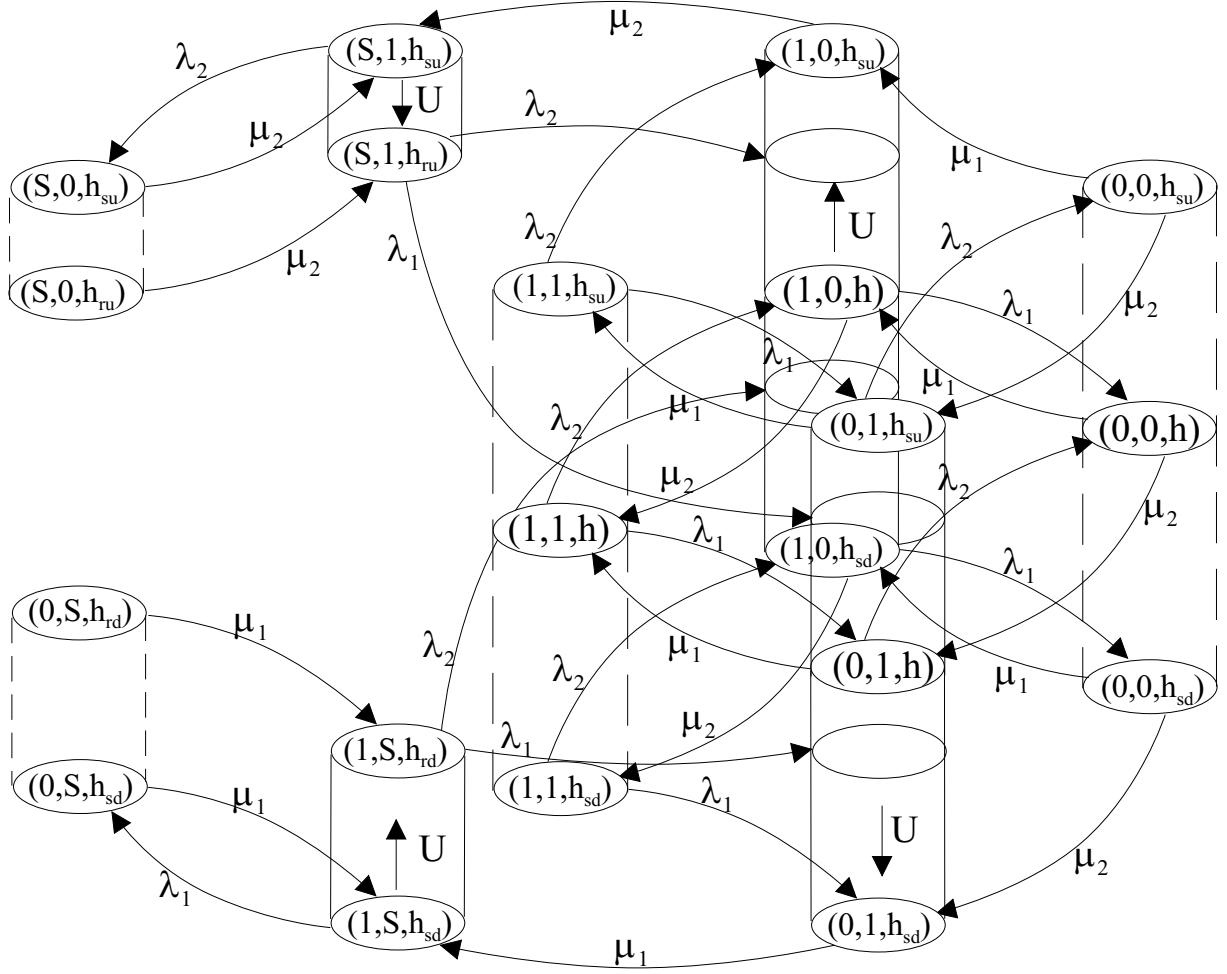


Figure 6: The complete Markov chain.

3.3.1 Interior-to-boundary equations. Discontinuity of the nominal mode dynamics.

At the upper and lower restart levels (h_{ru} and h_{rd}) the system switch back to the nominal mode under a failure of one of the servers. The particularity of these transitions is that the system will go into an **internal** state of the nominal mode. This phenomena is illustrated in Figure 7 for the transition at level h_{ru}

The state $(1, 0, h_{ru})$ is an internal state of the nominal mode and therefore its stationary probability is zero. But, if we write the flow conservation equations for this state, as the $P_{S,1,h_{ru}}$ is positive we detect a discontinuity of the internal dynamics of the nominal mode :

$$\lambda_2 P_{S,1,h_{ru}} + U f_{10}(h_{ru}^-) = U f_{10}(h_{ru}^+). \quad (22)$$

It follows that the solution \underline{f} of the internal dynamics is piece-wise continuous on the intervals (h_{sd}, h_{rd}) , (h_{rd}, h_{ru}) and $h_{ru}, h_{su})$, and on each interval, \underline{f} is of the form (15), respectively (18). Let \underline{f}^d , \underline{f}^m and \underline{f}^u denoting the continuous pieces of \underline{f} on the three intervals of continuity. Their will be, then, of the form (15) or (18), being particularized by different integration constant pairs (c_1^d, c_2^d) , (c_1^m, c_2^m) and (c_1^u, c_2^u) .

We see that the complete solution of our system is determined if we can fix the eight integration constants $(c_1^d, c_2^d, c_1^m, c_2^m, c_1^u, c_2^u, C_u, C_d)$.

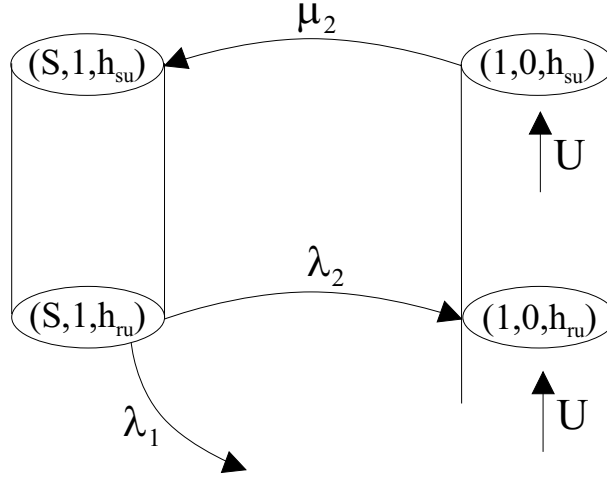


Figure 7: Transition from a boundary state of a single mode to an internal state of the nominal mode.

Boundary-to-interior equations similar to (22) will provide a first set of four equations :

$$\begin{cases} \lambda_1 P_{S,1,h_{ru}} + U f_{01}^u(h_{ru}) = U f_{01}^m(h_{ru}) \\ \lambda_2 P_{S,1,h_{ru}} + U f_{10}^m(h_{ru}) = U f_{10}^u(h_{ru}) \\ \lambda_1 P_{1,S,h_{rd}} + U f_{01}^m(h_{ru}) = U f_{01}^d(h_{ru}) \\ \lambda_2 P_{1,S,h_{rd}} + U f_{10}^d(h_{ru}) = U f_{10}^m(h_{ru}) \end{cases} \quad (23)$$

3.3.2 Interior-to-boundary and boundary-to-boundary equation

The remainder set of boundary conditions is obtained by inspection of stationary equations of boundary states.

States $(1, 0, h_{su}), (0, 1, h_{sd})$

$$\begin{cases} \mu_2 P_{1,0,h_{su}} = U f_{10}^u(h_{su}) \\ \mu_1 P_{0,1,h_{sd}} = U f_{01}^d(h_{sd}) \end{cases} \quad (24)$$

States $(0, 1, h_{su}), (1, 0, h_{sd})$

$$\begin{cases} 0 = U f_{01}^u(h_{su}) \\ 0 = U f_{10}^d(h_{sd}) \end{cases} \quad (25)$$

States $(S, 1, h_{su}), (1, S, h_{sd})$

$$\begin{cases} \mu_2 P_{1,0,h_{su}} = U f_{S1} \\ \mu_1 P_{0,1,h_{sd}} = U f_{1S} \end{cases} \quad (26)$$

States $(S, 1, h_{ru}), (1, S, h_{rd})$

$$\begin{cases} (\lambda_1 + \lambda_2) P_{S,1,h_{ru}} = U f_{S1} \\ (\lambda_1 + \lambda_2) P_{1,S,h_{rd}} = U f_{1S} \end{cases} \quad (27)$$

3.4 Constant determination

In the following we will sketch the constant determination algorithm using the equations (23)–(27).

From equation (24) and (26) one can eliminate constants C_u and C_d and identify

$$\begin{cases} f_{S1} = f_{10}^u(h_{su}) \\ f_{1S} = f_{01}^d(h_{sd}) \end{cases} . \quad (28)$$

Equation (25) can be used to express two of the constants (for example c_2^u and c_2^d) as functions of c_1^u , respectively c_1^d .

Summing the first two equations of (23) and eliminating $P_{S,1,h_{ru}}$ between the result and the first equations of (27) and (28) allows to obtain the relation

$$f_{01}^u(h_{ru}) + f_{10}^m(h_{ru}) - f_{01}^m(h_{ru}) - f_{10}^u(h_{ru}) = f_{10}^u(h_{su}). \quad (29)$$

After replacement of the expressions of \underline{f} from (15) or (18) and some simplification, (29) reduces to

$$c_1^m + c_2^m = 0.$$

The same relation can be obtained using the last two equations of (23) and the second equations of (27) and (28). This allows the elimination of c_2^m , and, after replacement in (15) and (18) one can see that these formulas becomes similar to the classical formulas in [3]. This means that in the median interval (h_{rd}, h_{ru}) where the single modes does not interact with the nominal mode, the system behaviour is similar to the free system (no control policy).

Finally, c_1^u and c_1^d can be expressed as function of c_1^m by combining the first equations of (23) and (27), respectively the third equation of (23) and the second equation of (27).

At this point all integration constant are expressed as function of c_1^m .

The constant c_1^m is computed using the normalization equations such that the total probability of the states of the system is equal to 1.

$$\begin{aligned} P_{S,1,h_{ru}} + P_{1,S,h_{rd}} + P_{1,0,h_{su}} + P_{0,1,h_{sd}} + \int_{h_{sd}}^{h_{rd}} \underline{f}^d(h) \overline{\Gamma} dh + \int_{h_{rd}}^{h_{ru}} \underline{f}^m(h) \overline{\Gamma} dh \\ + \int_{h_{ru}}^{h_{su}} \underline{f}^u(h) \overline{\Gamma} dh + (f_{S1} + f_{S0})(h_{su} - h_{ru}) + (f_{1S} + f_{0S})(h_{rd} - h_{sd}) = 1 \end{aligned} \quad (30)$$

This completely determines the solution of the system.

4 Performance evaluation criteria

We turn back our attention to our initial goal, the performance evaluation criteria J (1). The productivity of the system will be, as usual, the product between the proportion of time that the second server is producing and the production rate U of the server. The proportion of time the second server is producing pw equals the sum of probabilities of the states in which the server is working.

$$\begin{aligned} pw = P_{S,1,h_{ru}} + P_{1,S,h_{rd}} + \int_{h_{sd}}^{h_{rd}} f_{01}^d(h) + f_{11}^d(h) dh + \int_{h_{rd}}^{h_{ru}} f_{01}^m(h) + f_{11}^m(h) dh \\ + \int_{h_{ru}}^{h_{su}} f_{01}^u(h) + f_{11}^u(h) dh + f_{S1}(h_{su} - h_{ru}). \end{aligned} \quad (31)$$

The mean number of controllable stops by period of each of the servers will be equal to the proportion of time that the server is stopped divided by the mean length of a stop. As a stop occurs on a server when the other is failed and the level of the buffer attains the stop level, the mean length of a stop is equal to the residual time to repair of the other server plus the time necessary of the other server to bring the buffer level to the restart level. In the case of

exponential repair rates, the residual repair time is known to be also exponential and of the same rate [3]. The second term of the mean length of a stop is equal to the time needed by the server to produce the required quantity of work. This will represent a proportion of the working rate U equal to the availability of the server

$$E = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{1}{\mu}} = \frac{1}{1 + I}$$

In summary

$$ns_d = \frac{ps_d}{\frac{1}{\mu_2} + \frac{(1+I_2)(h_{rd}-h_{sd})}{U}}$$

$$ns_d = \frac{ps_u}{\frac{1}{\mu_1} + \frac{(1+I_1)(h_{su}-h_{ru})}{U}}$$

where

$$ps_d = P_{0,1,h_{sd}} + (f_{1S} + f_{0S})(h_{rd} - h_{sd})$$

$$ps_u = P_{1,0,h_{su}} + (f_{S1} + f_{S0})(h_{su} - h_{ru})$$

5 Conclusions

In this paper we have analyzed a case of two machines production lines with variable production rates. Actually variability of production rates is considered being driven by an on/off control policy, and nominal rates are supposed equal on both servers. The immediate continuation of our work is the analysis of the case with different nominal production rates. Further developments will include analysis of different control policies such that continuous linear variation of production rates and steppest variation with multiple controle levels (the actual on/off pilicy is a particular case of steppest variation with only two control levels).

References

- [1] K.E. Brenan, S.L. Campbell, and L.R. Petzold. *Numerical solution of initial-value problems in differential-algebraic equations*, volume 14 of *Classics in Applied Mathematics*. SIAM, Philadelphia, PA, 1996.
- [2] Felix R. Gantmacher. *Théorie des matrices*. Dunod, Paris, 1966.
- [3] S.B. Gershwin. *Manufacturing systems Engineering*. Prentice Hall, New Jersey, 1994.
- [4] Debasis Mitra. Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Adv. Appl. Probab.*, 20(3):646–676, 1988.