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# Optimal Control of an Interconnected SDE - Parabolic PDE System

Gabriel Velho<sup>1</sup>, Jean Auriol<sup>1</sup>, Islam Boussaada<sup>2</sup>, Riccardo Bonalli<sup>1</sup>

**Abstract**—In this paper, we design a controller for an interconnected system where a linear Stochastic Differential Equation (SDE) is actuated through a linear parabolic heat equation. These dynamics arise in various applications, such as coupled heat transfer systems and chemical reaction processes that are subject to disturbances. Our goal is to develop a computational method for approximating the controller that minimizes a quadratic cost associated with the state of the SDE component. To achieve this, we first perform a change of variables to shift the actuation inside the PDE domain and reformulate the system as a linear Stochastic Partial Differential Equation (SPDE). We use a spectral approximation of the Laplacian operator to discretize the coupled dynamics into a finite-dimensional SDE and compute the optimal control for this approximated system. The resulting control serves as an approximation of the optimal control for the original system. We then establish the convergence of the approximated optimal control and the corresponding closed-loop dynamics to their infinite-dimensional counterparts. Numerical simulations are provided to illustrate the effectiveness of our approach.

## I. INTRODUCTION

Interconnected systems involving parabolic Partial Differential Equations (PDEs) and Ordinary Differential Equations (ODEs) have recently attracted significant interest. In particular, systems where the coupling occurs at one boundary of the PDE and the control input is applied at the other have been a focus of recent research. These dynamics arise in various applications, such as coupled heat transfer systems and chemical reaction processes [1].

To control these coupled systems, stabilization techniques from parabolic PDE control have been developed. A widely used approach is finite-dimensional approximation [2]–[4], which discretizes the system to make control design more tractable. Another effective method is backstepping [5], which systematically transforms the system into a stable target form. These techniques have been successfully applied to parabolic PDE-ODE systems, as seen in [6]–[8] for backstepping and [1] for the dimension reduction approach.

In realistic settings, dynamical systems are often subject to disturbances arising from measurement noise,

parameter uncertainties, or external influences [9], [10]. These disturbances can significantly affect the system behavior, making their mitigation essential for ensuring reliability and safety in control applications. Stochastic Differential Equations (SDEs) provide a powerful and flexible framework for modeling such uncertainties [11], [12]. By leveraging stochastic control techniques, one can design robust stabilizing controllers that effectively mitigate random fluctuations. In particular, optimal stochastic control plays a key role in minimizing variance and improving system reliability [13].

In interconnected PDE-ODE systems, accounting for uncertainties in the ODE dynamics is often crucial for achieving both efficiency and robustness. Several methods have been proposed for hyperbolic systems, addressing bounded or model-based noise in the ODE [14], [15], as well as for stochastic differential equations (SDEs) [16], [17]. Recently, significant progress has been made in the control of parabolic Stochastic Partial differential Equations (SPDE). In particular, the spectral approximation method has proven effective in stabilizing semilinear systems [18], [19]. However, to the best of our knowledge, no existing approach specifically addresses noise mitigation in the context of coupled parabolic PDE-SDEs. This perspective enables the modeling of a broader class of dynamical systems, including processes governed by diffusion, such as heat transfer phenomena, chemical reactions, or the spread of biological populations. For such systems, backstepping techniques may not be ideal, as they introduce stochastic terms into the PDE, as shown in [17]. Furthermore, applying the method presented in [8] to time-varying gains, such as those required for optimal control, necessitates recomputing the backstepping kernels at multiple time instances, which can be computationally expensive. Furthermore, leveraging the structural properties of parabolic PDEs can lead to more effective control strategies for minimizing stochastic disturbances. For instance, the finite-dimensional approximation technique, which is well-suited for parabolic systems, may not be directly applicable to hyperbolic systems [20].

This study aims to provide the first systematic approach to controlling interconnected parabolic PDE-SDE systems. Specifically, we bridge the gap between deterministic PDE control and stochastic control by integrating methodologies from both topics. Our contributions are twofold:

- 1) Reformulating the optimal control problem for an interconnected parabolic PDE-linear SDE system

<sup>1</sup>Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France. This project has received funding from the Agence Nationale de la Recherche (ANR) via grant PANOPLY ANR-23-CE48-0001-01.

<sup>2</sup>Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France & IPSA, 94200, Ivry sur Seine, France.

as a well-posed Linear Quadratic (LQ) control problem within the SPDE framework.

- 2) Extending the finite-dimensional approximation methods used to compute optimal LQ controls for parabolic PDEs [1], [4] to the stochastic setting. Specifically, we establish the convergence of the approximated Riccati operator and the resulting controlled dynamics.

The remaining of the paper is organized as follows: After recalling some standard notations the problem under consideration is stated in Section II. In Section III it is shown that the considered coupled PDE+SDE can be recast into an SPDE. We give our control strategy in Section IV and prove convergence of our control method in Section V. Finally, some numerical results are shown in Section VI.

## II. PROBLEM FORMULATION

### A. Notations

We assume that we are given a filtered probability space  $(\Omega, \mathcal{F} \triangleq (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ , and that stochastic perturbations are due to a one-dimensional Wiener process  $W_t$ , which is adapted to filtration  $\mathcal{F}$ . Let  $T > 0$  be some given time horizon. For a Hilbert space  $H$ , we denote by  $L^2_{\mathcal{F}}([0, T]; H)$  the set of square integrable processes  $Z : [0, T] \times \Omega \rightarrow H$  that are  $\mathcal{F}$ -progressively measurable, whereas the subset  $C^2_{\mathcal{F}}([0, T]; H) \subseteq L^2_{\mathcal{F}}([0, T]; H)$  contains processes whose sample paths are continuous. The associated norm is

$$\|Z(\cdot)\|_{C^2_{\mathcal{F}}([0, T]; H)} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Z(t)\|_H^2 \right].$$

We also recall the *Burkholder-Davis-Gundy* (B-D-G) inequality for stochastic integrals [21]: for  $p \geq 1$ ,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Z(t)\|_H^p \right] \leq C_p \mathbb{E} \left[ \left( \int_0^T \|Z(t)\|_H^p dt \right)^{p/2} \right], \quad (1)$$

where  $C_p$  is a constant which depends on  $p$  uniquely. The spaces of semi-definite and definite positive symmetric matrices in  $\mathbb{R}^n$  are denoted by  $\mathcal{S}_n^+$  and  $\mathcal{S}_n^{++}$ , respectively. We denote by  $H^1(0, 1) \triangleq \{u \in L^2([0, 1]; \mathbb{R}), \text{ such that } u' \in L^2([0, 1]; \mathbb{R})\}$ . It is a Hilbert space with the scalar product

$$\langle u, v \rangle_{H^1} \triangleq \int_0^1 u(x)v(x)dx + \int_0^1 u'(x)v'(x)dx.$$

### B. Control system

We consider coupled PDE+SDE systems of the form

$$\begin{cases} u_t(t, x) = \Delta u(t, x) + cu(t, x) \\ dX_t = (AX_t + Bu(t, 0))dt + (CX_t + Du(t, 0))dW_t \\ \partial_x u(t, 1) = U(t), \quad \partial_x u(t, 0) = 0 \\ u(0, x) = u_0(x), \quad X(0) = X_0 \end{cases} \quad (2)$$

in the time-space domain  $[0, T] \times [0, 1]$ , where  $T > 0$ . The state of the system is  $(X_t, u(t, \cdot)) \in \mathbb{R}^d \times H^1(0, 1)$ . The

coefficient  $c$  is a constant modeling the rate of growth ( $c > 0$ ) or decay ( $c < 0$ ) of the solution  $u$  at each point in space. The matrices  $A \in \mathbb{R}^{q \times q}$ ,  $B \in \mathbb{R}^{q \times 1}$ ,  $C \in \mathbb{R}^{q \times q}$ ,  $D \in \mathbb{R}^{q \times 1}$  are also constant. The control input  $U$  takes values in  $\mathbb{R}$ . We denote by  $\text{Dom}(\Delta) \triangleq \{u \in H^2(0, 1), \partial_x u(0) = \partial_x u(1) = 0\}$ . Finally, the initial condition  $(X_0, u_0) \in \mathbb{R}^d \times \text{Dom}(\Delta)$  is assumed to be deterministic. We consider the state space  $H^1(0, 1)$  for the PDE solution in order to ensure continuity at the boundary, which is guaranteed by the trace theorem for  $H^1(0, 1)$ . Note that the terms  $cu(t, x)$  and  $AX_t$  can introduce instabilities in the PDE and the SDE if they are positive. The class of systems described by (2) naturally arises in scenarios where a heat equation governs one part of the system—such as a pipe transmitting heat—while it is coupled to a temperature system influenced by random perturbations. A typical example is the temperature regulation of a building, where external factors like fluctuating outdoor temperatures or varying sunlight exposure introduce stochastic disturbances. In these settings, the Neumann-type conditions model a set flow between the systems. We defer the proof of the well-posedness of system (2) to Section III.

### C. Optimal Control Problem

Our primary objective is to develop a method for computing a control that ensures the reliable regulation of our system. A natural approach is to design controls that minimize the variance of the SDE state [13]. To achieve this, we consider a quadratic cost of the form

$$J(U) \triangleq \mathbb{E} \left[ \int_0^T (X_t^T Q X_t + rU(t)^2) dt + X_T^T G X_T \right] \quad (3)$$

where  $X_t$  is subject to the dynamic (2). The matrices  $Q$  and  $G$  are in  $\mathcal{S}_q^+$ , and  $r > 0$  is a positive constant. To obtain a controller that is both computationally tractable and practical for implementation, we minimize the cost over the set of feedback controls [11]. To do so, we first perform a change of variables in the PDE, which allows the control to act within the domain rather than at the boundary. We then formulate the resulting dynamics within the SPDE framework, using the existing literature to ensure the existence of an optimal control. Finally, we approximate the optimal control using spectral methods and establish the convergence of both the control approximation and the associated cost toward their respective optimal values.

*Remark 1:* Note that the cost function (3) does not explicitly account for the PDE state  $u(t, x)$ , as our primary focus is on controlling the SDE, with the PDE modeling the medium through which the control goes through. However, it would be possible (at the cost of additional notational complexities) to incorporate an additional term of the form  $\mathbb{E} \left[ \int_0^T \langle Mu(t, \cdot), u(t, \cdot) \rangle_{H^1} dt \right]$  in (3), where  $M$  is a positive self-adjoint operator. We leave it as a future work.

### III. RECASTING INTO AN SPDE LQ PROBLEM

#### A. Change of variable

In this section, we perform a change of variable in the PDE state as in [22], which results in the input operator becoming bounded, at the cost of adding regularity on the control input  $U$ . Consider the new PDE state  $z(t, x)$  defined by

$$z(t, x) \triangleq u(t, x) - \rho(x)U(t), \quad (4)$$

where  $\rho \in H^1([0, 1])$  is the unique solution to the elliptic equation

$$\begin{cases} -\Delta\rho(x) + (\mu - c)\rho(x) = 0 \\ \partial_x\rho(0) = 0, \quad \partial_x\rho(1) = 1 \end{cases} \quad (5)$$

where  $\mu > c$ . The parameter  $\mu$  is required to guarantee the well-posedness of equation (5) through the Lax-Milgram theorem [23, Corollary 5.8]. In practice, we want to choose  $\mu$  relatively close to  $c$ , as the change of variable (6) in the control is proportional to a term  $e^{\mu t}$ . The variable  $z$  verifies the PDE system

$$dz(t, x) = \Delta z(t, x)dt + cz(t, x) + \mu\rho(x)U(t)dt - \rho(x)dU(t).$$

Therefore, by choosing  $U(t)$  such that

$$\begin{cases} dU(t) = \mu U(t)dt + V(t)dt \\ U(0) = U_0, \end{cases} \quad (6)$$

with  $V(t)$  being the new control variable, we obtain the following dynamic on  $z$  and  $X_t$

$$\begin{cases} z_t(t, x) = \Delta z(t, x) + cz(t, x) - \rho(x)V(t) \\ dX_t = (AX_t + B(z(t, 0) - \rho(0)U(t)))dt \\ \quad + (CX_t + D(z(t, 0) - \rho(0)U(t)))dW_t \\ dU(t) = \mu U(t)dt + V(t)dt \\ \partial_x z(t, 1) = 0, \quad \partial_x z(t, 0) = 0, \quad U(0) = U_0 \\ z(0, x) = u_0(x) - \rho(x)U_0, \quad X(0) = X_0. \end{cases} \quad (7)$$

Note that we can retrieve  $U$  from the new control  $V$  with the formula  $U(t) = e^{\mu t}U_0 + \int_0^t e^{\mu(t-s)}V(s)ds$ . We keep the parameter  $U_0$  undefined for the moment, as we use it to further minimize the cost function (3) later on.

#### B. Well-posedness of the system

We now prove the well-posedness of the coupled PDE-SDE system (7) and state some regularity properties of the solution.

*Proposition 2:* Let  $V(\cdot)$  be a stochastic process in  $L^2_{\mathcal{F}_t}([0, T], \mathbb{R})$ . Then, equation (7) has a unique solution. Moreover, the state  $(z, X)$  enjoy the following regularity:

$$\forall \omega \in \Omega, \quad z(\omega) \in C([0, T]; H^1(0, 1)) \cap L^2([0, T]; H^2(0, 1)), \\ X_{(\cdot)} \in C^2_{\mathcal{F}}([0, T]; \mathbb{R}^d)$$

*Proof:* We first prove the statement for the PDE state  $z$  since the SDE does not directly act on the PDE. For all realizations  $\omega \in \Omega$ , the control law  $V(t, \omega)$  is in  $L^2([0, T], \mathbb{R})$ , and therefore  $(t, x) \mapsto -\rho(x)V(t, \omega)$  is in  $L^2([0, T] \times [0, 1], \mathbb{R})$ . Using [23, Thm 10.11] and the fact that  $z(0, \cdot) \in H^2$ , we obtain the existence, uniqueness,

and the above stated regularity of  $z$ . Since  $z(t, \cdot) \in H^1$ , we have by Morrey's inequality [23, Thm 9.12] that

$$\mathbb{E} \left[ \int_0^T |z(t, 0)| dt \right] \leq C \mathbb{E} \left[ \int_0^T \|z(t, \cdot)\|_{H^1} dt \right] < \infty,$$

for some constant  $C > 0$ , implying that  $t \mapsto z(t, 0) \in L^2_{\mathcal{F}}$ , and therefore  $t \mapsto z(t, 0) - \rho(0)U(t) \in L^2_{\mathcal{F}}$ . We can now use [24, Chap. 1 - Thm 6.3] to obtain the existence, uniqueness, and the above stated regularity properties for the SDE solution  $X$ . ■

As stated in the previous proof,  $z \mapsto z(0)$  is a continuous linear form on  $H^1([0, 1])$ . By Riesz's theorem [23, Thm. 4.11], there exists a unique  $\gamma_0 \in H^1([0, 1])$  such that  $\forall z \in H^1([0, 1]), z(0) = \langle \gamma_0, z \rangle_{H^1}$ . In what follows, we use the notation  $z(0) = \gamma_0^* z$  for the sake of conciseness.

#### C. Existence of an optimal controller

By using the SPDE framework, we can now rewrite our optimal control problem as a Linear-Quadratic (LQ) problem in a Hilbert space. Let  $\mathcal{H}$  be the Hilbert space  $\mathcal{H} \triangleq \mathbb{R}^d \times \mathbb{R} \times H^1(0, 1)$ . We consider the augmented state

$$Z_t \triangleq \begin{pmatrix} X_t \\ Y_t \\ z_t \end{pmatrix} \in \mathcal{H}, \text{ where } Y_t \text{ stands for the signal } U(t).$$

Therefore, we infer the following linear SPDE

$$\begin{cases} dZ_t = [(\tilde{\Delta} + \tilde{A})Z_t + \tilde{B}V(t)]dt + \tilde{C}Z_t dW_t. \\ Z_0 = \begin{pmatrix} X_0 \\ U_0 \\ u_0 - \rho U_0 \end{pmatrix}, \end{cases} \quad (8)$$

with

$$\tilde{\Delta} \triangleq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta \end{pmatrix}, \quad \tilde{A} \triangleq \begin{pmatrix} A & -B\rho(0) & B\gamma_0^* \\ 0 & \mu & 0 \\ 0 & 0 & c \end{pmatrix}, \\ \tilde{B} \triangleq \begin{pmatrix} 0 \\ 1 \\ -\rho \end{pmatrix}, \quad \tilde{C} \triangleq \begin{pmatrix} C & -D\rho(0) & D\gamma_0^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The cost (3) can in turn be expressed as

$$J(V, U_0) \triangleq \mathbb{E} \left[ \int_0^T \langle \tilde{Q}Z_t, Z_t \rangle_{\mathcal{H}} dt + \langle \tilde{G}Z_T, Z_T \rangle_{\mathcal{H}} \right], \quad (9)$$

where  $\tilde{Q}$  and  $\tilde{G}$  are positive self-adjoint operators defined by

$$\tilde{Q} \triangleq \begin{pmatrix} Q & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{G} \triangleq \begin{pmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To guarantee that this LQ problem admits a unique solution, we need to add a regularizing term in the cost function, ensuring strong convexity with respect to the new control variable  $V$ . Although there exists a one-to-one correspondence between  $V$  and its integral  $U$ , the term  $r \int_0^T U(t)^2 dt$  in the cost function does not fully penalize variations in  $V$ , rendering the optimal control

problem ill-posed. We therefore recast the modified cost function  $J_\delta$  as

$$J_\delta(V, U_0) \triangleq J(V, U_0) + \delta \mathbb{E} \left[ \int_0^T V(t)^2 dt \right], \quad (10)$$

where  $\delta > 0$  is a parameter that can be chosen small at will. In this setting, our control problem is well-posed, and the optimal controller ensures a low-variance state, leading to reliable system regulation. We therefore focus now on finding the solution to

$$\bar{J}_\delta \triangleq \min_{U_0, V} J_\delta(V, U_0).$$

We can now use the results from [25] to obtain the existence and explicit form of the optimal feedback LQ controller.

*Proposition 3:* There exists a unique optimal controller  $\bar{V}$  that minimizes the cost function (10) under the dynamic (8). It is in the form of a feedback law:

$$\bar{V}(t) = -\delta^{-1} \tilde{B}^* \Pi(t) Z_t, \quad (11)$$

where  $\Pi(\cdot)$  is a bounded self-adjoint positive operator in  $\mathcal{H}$ , defined as the unique mild solution to the operator-valued Riccati equation

$$\begin{cases} \dot{\Pi} + \Pi(\tilde{\Delta} + \tilde{A}) + (\tilde{\Delta} + \tilde{A}^*)\Pi \\ \quad + \tilde{C}^* \Pi \tilde{C} + \tilde{Q} - \delta^{-1} \Pi \tilde{B} \tilde{B}^* \Pi = 0, \\ \Pi(T) = \tilde{G}. \end{cases} \quad (12)$$

Moreover, the minimal cost is given by

$$J(\bar{V}, U_0) = \langle \Pi(0) Z_0, Z_0 \rangle_{\mathcal{H}}. \quad (13)$$

*Proof:* As a classic result from functional analysis, the Laplacian  $\Delta$  with domain  $\text{Dom}(\Delta) = \{z \in H^2, \partial_x z(0) = \partial_x z(1) = 0\}$  generates a strongly continuous semigroup of operators on  $H^1$  [23]. Since the operators  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{C}$  are bounded, we can apply theorem [25, Thm 9.1] to obtain the desired result. ■

We can further minimize the optimal cost by choosing an optimal value for the problem parameter  $U_0$ . For conciseness, we define

$$L_\rho = \begin{pmatrix} 0 \\ 1 \\ -\rho \end{pmatrix}, \quad M_0 = \begin{pmatrix} X_0 \\ 0 \\ u_0 \end{pmatrix}, \quad (14)$$

*Proposition 4:* The cost function  $J_\delta(V, Z_0)$  is minimized for the parameters  $V = \bar{V}$  as given in (11), and

$$\bar{U}_0 = - \frac{\langle \Pi(0) L_\rho, M_0 \rangle_{\mathcal{H}}}{\langle \Pi(0) L_\rho, L_\rho \rangle_{\mathcal{H}}} \quad (15)$$

if  $\langle \Pi(0) L_\rho, L_\rho \rangle_{\mathcal{H}} \neq 0$ , or  $U_0 = 0$  otherwise.

*Proof:* Remark that  $L_\rho$  and  $M_0$  are defined such that  $Z_0 = L_\rho U_0 + M_0$ . Let us define  $F(U_0) \triangleq J_\delta(\bar{V}, L_\rho U_0 + M_0)$  the functional we seek to minimize. We have through Proposition 3 that

$$F(U_0) = \langle \Pi(0)(L_\rho U_0 + M_0), (L_\rho U_0 + M_0) \rangle_{\mathcal{H}},$$

which is a quadratic polynomial in  $U_0$ . It is straightforward to show that, if  $\langle \Pi(0) L_\rho, L_\rho \rangle_{\mathcal{H}} \neq 0$ , then  $F(U_0)$  is

minimal for  $\bar{U}_0$  as defined in (15). If  $\langle \Pi(0) L_\rho, L_\rho \rangle_{\mathcal{H}} = 0$ , then, since  $\Pi(0)$  is a non-negative self-adjoint operator,  $\Pi(0) L_\rho = 0$ . This implies that

$$\langle \Pi(0) L_\rho, M_0 \rangle = \langle L_\rho, \Pi(0) M_0 \rangle = 0$$

, meaning that  $F(U_0)$  is constant, and we can choose  $\bar{U}_0 = 0$ . ■

Numerically solving the infinite-dimensional Riccati equation requires discretization [3], and choosing an appropriate method is crucial for ensuring both convergence and feasibility. In the deterministic setting, several numerical approaches have been proposed, including spectral methods that discretize the state by projecting onto the first eigenvectors of the Laplacian operator [2]–[4], as well as more general Galerkin methods [26] based on finite element techniques. The Galerkin approach has also been extended to the stochastic setting in [27], and a splitting method was introduced in [28] to approximate the Riccati solution. In the next section, we extend the spectral method to the finite-time LQ control problem for SPDEs, to the best of our knowledge, for the first time.

#### IV. DISCRETIZATION OF THE SYSTEM

To compute an easy-to-implement numerical approximation of the optimal control (11), we employ a spectral approach [4]. We start to discretize the system dynamics onto a finite-dimensional subspace  $\mathcal{H}_N \subset \mathcal{H}$ . We then solve the optimal control problem within  $\mathcal{H}_N$  and demonstrate that the resulting finite-dimensional optimal controller provides a good approximation of the true optimal controller. Furthermore, we establish the convergence of both the approximated controller and its associated cost to the exact optimal controller and its corresponding minimal cost, respectively.

##### A. Finite-Dimensional System

We consider the Laplacian eigenfunctions  $\phi_n$  and eigenvalues  $\lambda_n$ , verifying

$$\begin{cases} -\Delta \phi_n(x) = \lambda_n \phi_n \\ \partial_x \phi_n(0) = 0, \quad \partial_x \phi_n(1) = 0, \end{cases} \quad (16)$$

and we arrange these eigenvalues in increasing order, i.e.,  $0 = \lambda_0 < \lambda_1 < \dots$ . In our case, we can explicitly solve the Sturm-Liouville problem, the solutions being given by

$$\lambda_n = \pi^2 n^2, \quad \phi_n = A_n \cos(n\pi x), \quad (17)$$

with coefficients  $A_n = \sqrt{\frac{2}{1+\pi^2 n^2}}$  chosen such that  $\phi_n$  is an orthonormal basis in  $H^1$ . In what follows, we denote the discretization of  $H^1$  with  $H_N^1 \triangleq \text{Span}(\phi_0, \dots, \phi_N)$ , and we accordingly denote by  $\mathcal{H}_N \triangleq \mathbb{R}^d \times \mathbb{R} \times H_N^1$  the discretization of  $\mathcal{H}$ . We denote by  $P_N$  the projection of  $\mathcal{H}$  onto  $\mathcal{H}_N$ , which is given by projecting onto the first  $N$  eigenvectors  $\phi_0, \dots, \phi_N$ . The explicit expression of  $P_N$  is

$$P_N \begin{pmatrix} X \\ Y \\ z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ \sum_{n=0}^N \langle z, \phi_n \rangle_{H^1} \phi_n \end{pmatrix} \quad (18)$$

This projector is orthogonal, as for  $Z \in \mathcal{H}$  and  $\phi \in \mathcal{H}_N$

$$\langle P_N Z, \phi \rangle_{\mathcal{H}_N} = \langle P_N Z, \phi \rangle_{\mathcal{H}} = \langle Z, \phi \rangle_{\mathcal{H}}.$$

The adjoint operator  $P_N^*$  is therefore the injection of  $\mathcal{H}_N$  back into  $\mathcal{H}$ . We now solve the optimal control problem for the approximation of (8) on  $\mathcal{H}_N$ . We define the matrices  $\tilde{\Delta}_N = P_N \tilde{\Delta} P_N^*$ ,  $\tilde{A}_N = P_N \tilde{A} P_N^*$ ,  $\tilde{C}_N = P_N \tilde{C} P_N^*$ , and  $\tilde{B}_N = P_N \tilde{B}$ , and consider the approximated state  $Z_t^N \in \mathcal{H}_N$  that verifies the linear SDE

$$\begin{cases} dZ_t^N = [(\tilde{\Delta}_N + \tilde{A}_N)Z_t^N + \tilde{B}_N V(t)] dt + \tilde{C}_N Z_t^N dW_t, \\ Z_0^N = P_N Z_0. \end{cases} \quad (19)$$

The different matrices can be written explicitly along the basis of vectors  $(\phi_0, \dots, \phi_N)$  as

$$\tilde{\Delta}_N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -L_N \end{pmatrix}, \quad \tilde{A}_N = \begin{pmatrix} A & -B\rho(0) & B(\gamma_0^N)^T \\ 0 & \mu & 0 \\ 0 & 0 & cI_N \end{pmatrix}$$

$$\tilde{C}_N = \begin{pmatrix} C & -D\rho(0) & D(\gamma_0^N)^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_N = \begin{pmatrix} 0 \\ 1 \\ -\rho^N \end{pmatrix}$$

with  $L_N \triangleq \text{diag}(\lambda_0, \dots, \lambda_N)$ ,  $\gamma_0^N \triangleq (\phi_0(0), \dots, \phi_N(0))^T$  and  $\rho^N \triangleq (\langle \rho, \phi_0 \rangle_{H^1}, \dots, \langle \rho, \phi_N \rangle_{H^1})^T$ .

### B. Approximation of the optimal feedback control

We now define the approximated cost  $J_\delta^N(V, Z_0^N)$  as

$$J_\delta^N(V, Z_0) \triangleq \mathbb{E} \left[ \int_0^T ((Z_t^N)^T \tilde{Q}_N Z_t^N + \delta V(t)^2) dt \right] + \mathbb{E} [(Z_T^N)^T \tilde{G}_N Z_T^N], \quad (20)$$

where  $\tilde{Q}_N \triangleq P_N \tilde{Q} P_N^*$  and  $\tilde{G}_N \triangleq P_N \tilde{G} P_N^*$ . Classic results from optimal linear quadratic control [24] yields the following optimal controller.

*Proposition 5:* There exists a unique optimal controller  $\bar{V}_N$  that minimizes the cost (20) under the dynamic (19). Moreover, this optimal controller is given by the feedback law

$$\bar{V}_N(t) = -\delta^{-1} \tilde{B}_N^T \Pi_N(t) Z_t^N, \quad (21)$$

where  $\Pi_N(\cdot)$  is a positive symmetric matrix in  $\mathcal{L}(\mathcal{H}_N)$ , and is the unique solution to the matrix-valued Riccati equation

$$\begin{cases} \dot{\Pi}_N + \Pi_N(\tilde{\Delta}_N + \tilde{A}_N) + (\tilde{\Delta}_N + \tilde{A}_N^*) \Pi_N \\ \quad + \tilde{C}_N^* \Pi_N \tilde{C}_N + \tilde{Q}_N - \delta^{-1} \Pi_N \tilde{B}_N \tilde{B}_N^* \Pi_N = 0, \\ \Pi_N(T) = \tilde{G}_N. \end{cases} \quad (22)$$

Finally, the associated minimal cost is given by

$$J(\bar{V}_N, Z_0^N) = \langle \Pi_N(0) Z_0^N, Z_0^N \rangle_{\mathcal{H}_N}. \quad (23)$$

As done in Proposition 4, one easily shows that there exists an optimal value of  $U_0$ , denoted  $\bar{U}_0^N$ , that further minimizes the cost  $J(\bar{V}_N, Z_0^N)$ . The feedback controller  $\bar{V}_N$  is straightforward to compute in practice [24]. We show in the next section that  $\bar{V}_N$  is a good approximation of the optimal control  $\bar{V}$  when  $N$  is big enough.

## V. CONVERGENCE RESULTS

### A. Convergence of the Dynamic Operators

In what follows, we denote by  $S(\cdot)$  and  $S_N(\cdot)$  the strongly continuous semi-group generated respectively by  $\tilde{\Delta}$  and  $\tilde{\Delta}_N$ . It is a known result that the spectral approximation of the Laplacian  $\tilde{\Delta}_N$  verifies the Trotter-Kato theorem [4, Thm 4.3]. Consequently, we have that

$$\forall Z \in \mathcal{H}, \quad \lim_{N \rightarrow +\infty} \sup_{0 \leq t \leq T} \|P_N^* S_N(t) P_N Z - S(t) Z\|_{\mathcal{H}} = 0, \quad (24)$$

Moreover, we can verify that all the other bounded operators converge strongly pointwise, that is

$$\begin{aligned} \forall Z \in \mathcal{H}, \quad & \|\tilde{A} Z - P_N^* \tilde{A}_N P_N Z\|_{\mathcal{H}} \rightarrow 0, \\ & \|\tilde{C} Z - P_N^* \tilde{C}_N P_N Z\|_{\mathcal{H}} \rightarrow 0, \\ & \|\tilde{B} - P_N^* \tilde{B}_N\|_{\mathcal{H}} \rightarrow 0. \end{aligned} \quad (25)$$

These convergences also hold for the adjoints of the operators  $\tilde{A}^*$ ,  $\tilde{B}^*$  and  $\tilde{C}^*$ .

### B. Convergence of the Feedback Operator

In this subsection we prove the strong convergence of the feedback operator  $\Pi_N$  towards  $\Pi$ .

*Proposition 6:* Let  $Z \in \mathcal{H}$ . We have that

$$\begin{aligned} & \|\Pi(t) Z - P_N^* \Pi_N(t) P_N Z\|_{\mathcal{H}} \rightarrow 0, \\ & \sup_{t \in [0, T]} \|\Pi(t) Z - P_N^* \Pi_N(t) P_N Z\|_{\mathcal{H}} \rightarrow 0. \end{aligned} \quad (26)$$

*Proof:* The Riccati equation (12) closely resembles the classical Riccati equation solved in the deterministic setting [4], with the main difference being the additional term  $C^* \Pi C$  which is not particularly difficult to handle. However, since we could not find a proof that allows for a direct adaptation, we provide a full proof here using a standard approach. Let  $t \in [0, T]$  and  $Z \in \mathcal{H}$ , we prove this convergence result through an estimate on the difference  $\Pi(t) Z - P_N^* \Pi_N(t) P_N Z$ . Let  $\tau \in [t, T]$ , we define by  $F(\tau) Z$  and  $F_N(\tau) Z$  the  $C^1$  functionals

$$\begin{aligned} F(\tau) Z & \triangleq S(\tau - t)^* \Pi(\tau) S(\tau - t) Z, \\ F_N(\tau) Z & \triangleq P_N^* S_N(\tau - t)^* \Pi_N(\tau) S_N(\tau - t) P_N Z. \end{aligned}$$

Through the fundamental theorem of analysis, we have

$$\begin{aligned} F(T) Z - F(t) Z & = \int_t^T \dot{F}(\tau) Z d\tau \\ & = \int_t^T S(\tau - t)^* (\tilde{\Delta}^* \Pi(\tau) + \Pi(\tau) \tilde{\Delta}) S(\tau - t) Z d\tau \\ & \quad + \int_t^T S(\tau - t)^* \dot{\Pi}(\tau) S(\tau - t) Z d\tau. \end{aligned}$$

Using the Riccati equation (12), and the fact that  $F(t) = \Pi(t)$ , we obtain

$$\begin{aligned} \Pi(t)Z &= S(T-t)^* \tilde{G} S(T-t)Z \\ &+ \int_t^T S(\tau-t)^* \left( \tilde{A}^* \Pi(\tau) + \Pi(\tau) \tilde{A} \right. \\ &\quad \left. + \tilde{Q} + \tilde{C}^* \Pi(\tau) \tilde{C} \right. \\ &\quad \left. - \delta^{-1} \Pi(\tau) \tilde{B} \tilde{B}^* \Pi(\tau) \right) S(\tau-t)Z d\tau \end{aligned}$$

Similarly, we have

$$\begin{aligned} P_N^* \Pi_N(t) P_N Z &= P_N^* S_N(T-t)^* \tilde{G}_N S_N(T-t) P_N Z \\ &+ \int_t^T S_N(\tau-t)^* \left( \tilde{A}_N^* \Pi_N(\tau) + \Pi_N(\tau) \tilde{A}_N \right. \\ &\quad \left. + \tilde{Q}_N + \tilde{C}_N^* \Pi_N(\tau) \tilde{C}_N \right. \\ &\quad \left. - \delta^{-1} \Pi_N(\tau) \tilde{B}_N \tilde{B}_N^* \Pi_N(\tau) \right) S_N(\tau-t) P_N Z d\tau. \end{aligned}$$

All operators  $S_N(\cdot), \Pi_N(\cdot), \tilde{A}_N, \tilde{B}_N, \tilde{C}_N, \tilde{G}_N, \tilde{Q}_N, P_N$  have an operator norm that is uniformly bounded (for  $\Pi_N$ , see the method proposed in [26] for a proof). Therefore, there exists a (possibly overloaded) positive constant  $K$ , depending on the bounds of the operator norms, such that

$$\begin{aligned} &\|\Pi(t)Z - P_N^* \Pi_N(t) P_N Z\|_{\mathcal{H}} \\ &\leq K \int_t^T \|\Pi(\tau)Z - P_N^* \Pi_N(\tau) P_N Z\|_{\mathcal{H}} d\tau \\ &+ K \left( \|\tilde{A}Z - P_N^* \tilde{A}_N P_N Z\|_{\mathcal{H}} + \|\tilde{C}Z - P_N^* \tilde{C}_N P_N Z\|_{\mathcal{H}} \right. \\ &\quad \left. + \|\tilde{Q}Z - P_N^* \tilde{Q}_N P_N Z\|_{\mathcal{H}} + \|\tilde{G}Z - P_N^* \tilde{G}_N P_N Z\|_{\mathcal{H}} \right. \\ &\quad \left. + \|\tilde{B} \tilde{B}^* Z - P_N^* \tilde{B}_N \tilde{B}_N^* P_N Z\|_{\mathcal{H}} \right. \\ &\quad \left. + \sup_{\tau \in [0, T]} \|S(\tau)Z - P_N^* S_N(\tau) P_N Z\|_{\mathcal{H}} \right). \end{aligned}$$

We conclude the proof thanks to Gronwall's Lemma, and (24) and (25).  $\blacksquare$

We can now show the convergence of the controller and the optimal cost.

### C. Convergence of the Controller and the Optimal Cost

We show first the strong convergence of the closed-loop state, then of the controller and finally of the optimal cost.

*Proposition 7:* Let  $\bar{Z}_t$  be the solution to the closed loop dynamic (8) with the controller  $\bar{V}$  defined in (11), and let  $\bar{Z}_t^N$  be the solution of (19) with  $\bar{V}_N$  defined in (21). Then

$$\left\| \bar{Z}_t - P_N^* \bar{Z}_t^N \right\|_{C_{\mathcal{F}}^2([0, T]; \mathcal{H})} \rightarrow 0. \quad (27)$$

*Proof:* Similarly to what has been done in the proof of Proposition 6, we first express  $\bar{Z}_t$  and  $\bar{Z}_t^N$  as integrals involving uniformly bounded operators. We then bound their differences and establish the desired result using the B-D-G inequality for the stochastic integral given in

equation (1), combined with Gronwall's Lemma. We have the following expression for  $\bar{Z}_t$ ,

$$\begin{aligned} \bar{Z}_t &= S(t)Z_0 + \int_0^t S(t-s) \tilde{C} \bar{Z}_s dW_t \\ &+ \int_0^t S(t-s) (\tilde{A} - \delta^{-1} \tilde{B} \tilde{B}^* \Pi(t)) \bar{Z}_s dt. \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{Z}_t^N &= S_N(t)Z_0^N + \int_0^t S_N(t-s) \tilde{C}_N \bar{Z}_s^N dW_t \\ &+ \int_0^t S_N(t-s) (\tilde{A}_N - \delta^{-1} \tilde{B}_N \tilde{B}_N^* \Pi_N(t)) \bar{Z}_s^N dt. \end{aligned}$$

Let us focus on the stochastic integral. Due to the B-D-G inequality (1), there exists  $K > 0$  such that

$$\begin{aligned} &\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s (S(s-\tau) \tilde{C} \bar{Z}_\tau - P_N^* S_N(s-\tau) \tilde{C}_N \bar{Z}_\tau^N) dW_\tau \right\|_{\mathcal{H}}^2 \right] \\ &\leq K \int_0^t \mathbb{E} \left[ \sup_{0 \leq \tau \leq s} \left\| S(s-\tau) \tilde{C} \bar{Z}_\tau - P_N^* S_N(s-\tau) \tilde{C}_N \bar{Z}_\tau^N \right\|_{\mathcal{H}}^2 \right] ds \\ &\leq K \int_0^t \mathbb{E} \left[ \sup_{0 \leq \tau \leq s} \left\| \bar{Z}_\tau - \bar{Z}_\tau^N \right\|_{\mathcal{H}}^2 \right] ds \\ &+ K \int_0^t \mathbb{E} \left[ \sup_{0 \leq \tau \leq s} \left\| (\tilde{C} - P_N^* \tilde{C}_N P_N) \bar{Z}_\tau \right\|_{\mathcal{H}}^2 \right] ds \\ &+ K \int_0^t \mathbb{E} \left[ \sup_{0 \leq \tau \leq s} \left\| (S(s-\tau) - P_N^* S_N(s-\tau) P_N) \bar{Z}_\tau \right\|_{\mathcal{H}}^2 \right] ds. \end{aligned}$$

Note that we can add  $P_N P_N^* = I_N$  anywhere in the expression of  $\bar{Z}_t^N$  to factorize each term appropriately. To prove the convergence to 0 of the last two integrals, we use the following lemma from [29].

*Lemma 8:* [29, Lem 5.1] Let  $\Psi(\cdot) : [0, T] \rightarrow \mathcal{L}(\mathcal{H})$  and for  $N \geq 1$ ,  $\Psi_N(\cdot) : [0, T] \rightarrow \mathcal{L}(\mathcal{H})$  such that the operator norm of  $\Psi_N(t)$  is uniformly bounded for  $N$  and  $t$ , and  $\forall Z \in \mathcal{H}$ ,  $\Psi_N(t)Z \rightarrow \Psi(t)Z$  uniformly in  $t$ . Let  $g : [0, T] \rightarrow \mathcal{H}$  be a continuous function, then

$$\sup_{0 \leq t \leq T} \left\| \Psi_N(t)g(t) - \Psi(t)g(t) \right\|_{\mathcal{H}} \rightarrow 0.$$

Since for all realizations  $\omega \in \Omega$ , the function  $t \mapsto \bar{Z}_t(\omega)$  is continuous in  $\mathcal{H}$ , we can apply the previous lemma pointwise in  $\omega$ . We then use the boundedness of the operators and of  $\bar{Z}_t$  in  $C_{\mathcal{F}}^2([0, T]; \mathcal{H})$  to apply the dominated convergence on the integral of the expectation and show convergence to 0. Using this method, we can show that

$$\begin{aligned} &\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \bar{Z}_s - \bar{Z}_s^N \right\|_{\mathcal{H}}^2 \right] \leq \Theta_N \\ &+ K \int_0^t \mathbb{E} \left[ \sup_{0 \leq \tau \leq s} \left\| \bar{Z}_\tau - \bar{Z}_\tau^N \right\|_{\mathcal{H}}^2 \right] ds \end{aligned}$$

with  $\Theta_N \rightarrow 0$ . We can now conclude using Gronwall's Lemma.  $\blacksquare$

We can now show convergence of the control and of the cost.

*Proposition 9:* The following holds as  $N \rightarrow \infty$ :

$$\begin{aligned} \|\bar{V} - \bar{V}_N\|_{C^2_T([0,T],\mathbb{R})} &\rightarrow 0, \\ \bar{U}_0^N &\rightarrow \bar{U}_0, \\ J_\delta^N(\bar{V}_N, \bar{U}_0^N) &\rightarrow J_\delta(\bar{V}, \bar{U}_0). \end{aligned} \quad (28)$$

*Proof:* Proposition 7 combined with Lemma 8 yields directly the results. ■

These convergence results confirm that the easily implementable controls  $\bar{V}_N$  serve as accurate approximations of  $\bar{V}$ , as they become sufficiently close for large  $N$ .

## VI. SIMULATIONS

To show the effectiveness of our control method, we consider an academic example within the framework of (2), with a real valued SDE. In our simulations, we chose for the SDE the parameters  $A = 2, B = 2, C = 1, D = 0.5$  and  $X_0 = 1$ . The PDE parameters are  $c = 0.5$  and  $u_0 = 1$ . Note that with this choice of parameters, both the PDE and SDE are unstable. We apply the control strategy defined in proposition 5 with  $N = 3, Q = 10, R = 1, G = 10$  and  $\delta = 0.5$ . The PDE was simulated using a Galerkin method with an order higher than  $N$ .

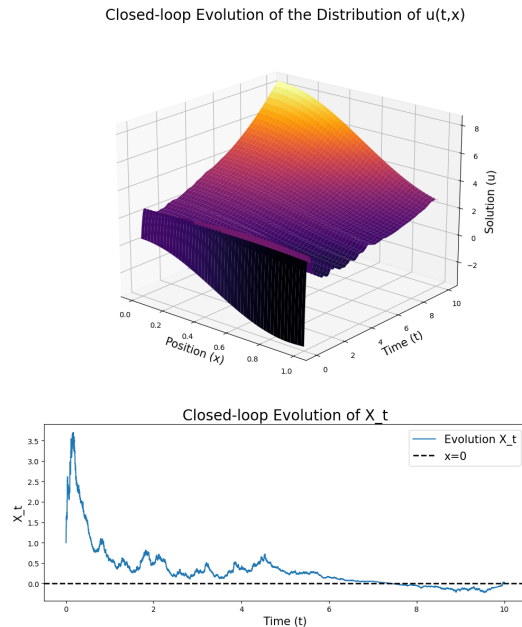


Fig. 1. Realization of a closed-loop trajectory with  $N = 3$

Figure 1 shows that despite an initial spike induced by the PDE, the controller effectively stabilizes the state, overcoming instabilities and disturbances. However, the PDE state itself remains unstabilized, which is expected since it is not penalized in the cost function. Instead, it naturally adjusts to minimize the variance of the SDE state.

## VII. CONCLUSION AND PERSPECTIVES

In this paper, we introduced a method for the optimal control of an interconnected system composed of a heat

PDE cascaded into a linear SDE system. Our approach extends spectral approximation methods developed for parabolic PDE to the stochastic setting. We established the convergence of the approximated optimal controller to the true optimal control of the original system, as well as the convergence of the corresponding closed-loop state dynamics. Future research directions include extending our framework to more general PDE operators with sufficiently fast-converging finite-dimensional approximations. Additionally, we aim to investigate the case of stochastic systems with possibly nondeterministic coefficients in the SDE, further enhancing the applicability of our results.

## REFERENCES

- [1] L. Mohammadi, I. Aksikas, S. Dubljevic, and J. F. Forbes, "Optimal boundary control of coupled parabolic PDE–ODE systems using infinite-dimensional representation," *Journal of Process Control*, vol. 33, pp. 102–111, Sept. 2015.
- [2] K. Ito, "Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces," Tech. Rep. NASA-CR-178302, May 1987.
- [3] H. T. Banks and K. Ito, "Approximation in LQR problems for infinite dimensional systems with unbounded input operators," tech. rep., North Carolina State University. Center for Research in Scientific Computation, 1994.
- [4] K. A. Morris, *Controller Design for Distributed Parameter Systems*. Communications and Control Engineering, Cham: Springer International Publishing, 2020.
- [5] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*. SIAM, Sept. 2008.
- [6] S. Tang and C. Xie, "State and output feedback boundary control for a coupled PDE–ODE system," *Systems & Control Letters*, vol. 60, pp. 540–545, Aug. 2011.
- [7] G. Antonio Susto and M. Krstic, "Control of PDE–ODE cascades with Neumann interconnections," *Journal of the Franklin Institute*, vol. 347, pp. 284–314, Feb. 2010.
- [8] J. Wang and M. Krstic, "Output Feedback Boundary Control of a Heat PDE Sandwiched Between Two ODEs," *IEEE Transactions on Automatic Control*, vol. 64, pp. 4653–4660, Nov. 2019.
- [9] R. Bonalli, T. Lew, and M. Pavone, "Sequential convex programming for non-linear stochastic optimal control," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 28, p. 64, 2022.
- [10] T. Lew, R. Bonalli, and M. Pavone, "Risk-averse trajectory optimization via sample average approximation," *IEEE Robotics and Automation Letters*, vol. 9, no. 2, pp. 1500–1507, 2023.
- [11] N. Touzi, *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE*, vol. 29 of *Fields Institute Monographs*. New York, NY: Springer, 2013.
- [12] R. Bonalli and B. Bonnet, "First-order pontryagin maximum principle for risk-averse stochastic optimal control problems," *SIAM Journal on Control and Optimization*, vol. 61, no. 3, pp. 1881–1909, 2023.
- [13] T. Lew, R. Bonalli, and M. Pavone, "Sample average approximation for stochastic programming with equality constraints," *SIAM Journal on Optimization*, vol. 34, no. 4, pp. 3506–3533, 2024.
- [14] J. Redaud, F. Briescia-Argomedo, and J. Auriol, "Output regulation and tracking for linear ODE-hyperbolic PDE–ODE systems," *Automatica*, vol. 162, p. 111503, Apr. 2024.
- [15] J. Deutscher, "Backstepping Design of Robust State Feedback Regulators for Linear  $2 \times 2$  Hyperbolic Systems," *IEEE Transactions on Automatic Control*, vol. 62, pp. 5240–5247, Oct. 2017.
- [16] G. Velho, J. Auriol, R. Bonalli, and I. Boussaada, "Stabilization and Optimal Control of Interconnected SDE–Scalar PDE System," *IEEE Control Systems Letters*, pp. 1–1, 2024.

- [17] G. Velho, J. Auriol, I. Boussaada, and R. Bonalli, “Stabilization and Optimal Control of an Interconnected  $n + m$  Hetero-directional Hyperbolic PDE - SDE System.” Submitted to TAC, Feb. 2025.
- [18] P. Wang and E. Fridman, “Predictor methods for finite-dimensional observer-based control of stochastic parabolic PDEs,” *Systems & Control Letters*, vol. 181, p. 105632, Nov. 2023.
- [19] P. Wang, R. Katz, and E. Fridman, “Constructive finite-dimensional boundary control of stochastic 1D parabolic PDEs,” *Automatica*, vol. 148, p. 110793, Feb. 2023.
- [20] J. Auriol, K. A. Morris, and F. Di Meglio, “Late-lumping backstepping control of partial differential equations,” *Automatica*, vol. 100, pp. 247–259, Feb. 2019.
- [21] C. Marinelli and M. Röckner, “On the maximal inequalities of Burkholder, Davis and Gundy,” *Expositiones Mathematicae*, vol. 34, pp. 1–26, Jan. 2016.
- [22] R. Katz and E. Fridman, “Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement,” *European Journal of Control*, vol. 62, pp. 158–164, Nov. 2021.
- [23] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, NY: Springer, 2011.
- [24] J. Yong and X. Y. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*. No. 43 in Applications of mathematics, New York: Springer, 1999.
- [25] Q. Lü and X. Zhang, “A Concise Introduction to Control Theory for Stochastic Partial Differential Equations,” Jan. 2021.
- [26] M. Krolller and K. Kunisch, “Convergence Rates for the Feedback Operators Arising in the Linear Quadratic Regulator Problem Governed by Parabolic Equations,” *SIAM Journal on Numerical Analysis*, vol. 28, pp. 1350–1385, Oct. 1991.
- [27] A. Prohl and Y. Wang, “Strong error estimates for a space-time discretization of the linear-quadratic control problem with the stochastic heat equation with linear noise,” *IMA Journal of Numerical Analysis*, vol. 42, pp. 3386–3429, Oct. 2022.
- [28] T. Damm, H. Mena, and T. Stillfjord, “Numerical solution of the finite horizon stochastic linear quadratic control problem: Numerical solution of the finite horizon stochastic linear quadratic control problem,” *Numerical Linear Algebra with Applications*, vol. 24, p. e2091, Aug. 2017.
- [29] J. S. Gibson, “The Riccati Integral Equations for Optimal Control Problems on Hilbert Spaces,” *SIAM Journal on Control and Optimization*, vol. 17, pp. 537–565, July 1979.