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Existence of stationary solutions of supercritical nonlinear Schrödinger equations on some metric graphs

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Abstract

We consider the existence of stationary wave solutions with prescribed mass to a supercritical nonlinear Schrödinger equation on a noncompact connected metric graph without a small mass assumption.

1 Introduction

In this work, we consider the existence of stationary wave solutions with prescribed mass to a supercritical nonlinear Schrödinger equation on a *noncompact connected metric graph* \mathcal{G} :

$$-u'' + \lambda u = |u|^{p-2}u, \quad (\text{NLS})$$

with mass supercritical power, that is $p > 6$, and Kirchhoff (namely current conservation) conditions

$$\sum_{e \succ v} u'_{|e}(v) = 0, \quad \forall v \in \mathcal{V}. \quad (\text{KC})$$

The first work to settle a very similar question is due to Chang, Jeanjean and Soave [8], in the case where the graph is compact. Since the graph is compact, the energy functional admits a minimum under a mass constraint and the authors considered minimizing sequences. The question of extending such a result to non-compact graphs was then considered by the first author with Chang, Jeanjean and Soave [6] in the case where the nonlinearity is localized to the compact core of a graph with finitely many edges. There infimum of the energy is not finite. Thus the authors considered the existence of critical points for the mass constrained energy through the analysis of Palais-Smale sequences. The existence of such sequences with the some information on the Morse index asked for a new abstract theory on bounded Palais-Smale sequence considered

by these authors [7]. The restriction to a localized nonlinearity and the finite number of edges was then released by Dovetta, Jeanjean and Serra [9] up to a smallness assumption on the mass together with some geometrical assumptions on the graph. Interestingly, they considered periodic graphs. The purpose of the present analysis is to release as much as possible the small mass constraint and the the geometrical assumption. We make an assumptions on the bottom of the spectrum and the absence of positive solutions with zero frequency. We provide examples of graphs fulfilling these assumptions.

2 Framework

Recall a metric graph is a couple $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, where \mathcal{E} is a set of real intervals called edges¹ and \mathcal{V} is the set of the endpoints, called vertices², of the edges and $e \succ v$ denotes all the edges $e \in \mathcal{E}$ such that v is an endpoint of e while $u|_e$ is the restriction of u to e . Any bounded edge e is identified with a closed bounded interval³, e.g. $I_e = [0, |e|]$ (where $|e|$ is the length of e), while each unbounded edge is identified with a closed half-line, e.g. $I_e = [0, +\infty)$. The notation $u'_{|e}(v)$ stands for the derivative outward to the vertex into the edge, e.g. $u'_{|e}(0)$, if the vertex v is identified with 0, $-u'_{|e}(|e|)$, if it is identified with π_e .

We prove the existence of solutions to (NLS) with prescribed norm (or mass) as critical points of the energy functional

$$E(u, \mathcal{G}) := \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \frac{1}{p} \int_{\mathcal{G}} |u|^p dx \quad (\text{E})$$

under the constraint

$$\int_{\mathcal{G}} |u|^2 dx = \mu > 0. \quad (1)$$

In other words, we prove the existence of critical points of the functional $E(u, \mathcal{G})$ constrained on the L^2 -sphere, e.g. in

$$H_{\mu}^1(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) : \int_{\mathcal{G}} |u|^2 dx = \mu \right\}.$$

If $u \in H_{\mu}^1(\mathcal{G})$ is such a critical point, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies the following problem

$$\begin{aligned} \forall e \in \mathcal{E}, \quad & -u''_{|e} + \lambda u|_e = |u|_e|^{p-2} u|_e \\ \forall v \in \mathcal{V}, \quad & \sum_{e \succ v} u'_{|e}(v) = 0. \end{aligned} \quad (2)$$

For our proof to hold, we make the two following assumptions.

Assumption 1. *The spectrum of $-\Delta_{\mathcal{G}}$, the Kirchoff laplacian, is such that*

$$\inf \sigma(-\Delta_{\mathcal{G}}) = 0.$$

¹Each interval is from a different copy of \mathbb{R} . The same interval can appear several times.

²More precisely this is a partition of the set of endpoints of \mathcal{E} again from different copies of \mathbb{R} . Elements from the same set in the partition are identified to model the connection between edges. An edge is closed if its ends are identified. Two vertices can be connected by several edges.

³There is no orientation and each edge is determined up to an isometry and hence only its length is relevant.

See Section 3.1 for examples.

Assumption 2. If $u \in H^1(\mathcal{G})$ satisfies (2) with $\lambda = 0$ and $u \geq 0$ then $u \equiv 0$.

See Section 3.2 for examples.

Remark 3. The Gagliardo-Nirenberg inequality, see (9) below, together with the ideas from the Appendix, impose some constraint on the L^∞ -norm of solutions to (2) with respect to the mass μ . From the ideas of the Appendix, on the other hand, the maximum of a positive solution is bounded by means of the supremum of the lengths of the edges. Therefore, if μ is small no positive solution to (2) is available. Together with Remark 12, below, this shows that Theorem 4, below, is valid if μ is small enough without imposing Assumptions 1 nor 2.

Here is the main result of this analysis.

Theorem 4. Let \mathcal{G} be noncompact connected metric graph such that Assumptions 1 and 2 hold then for any fixed $\mu > 0$, there exists $(u, \lambda) \in H_\mu^1(\mathcal{G}) \times \mathbb{R}^+$, $u > 0$, which solves (2).

Notations. A metric graph \mathcal{G} is naturally endowed with a metric space structure corresponding to the infimum of the path lengths between two points. The corresponding space of complex valued continuous functions is denoted $C(\mathcal{G})$ and $C_c(\mathcal{G})$ is the subspace of continuous functions with compact support.

The space $L^p(\mathcal{G})$ is the set

$$\left\{ (u_{\uparrow e})_{e \in \mathcal{E}} \in \otimes_{e \in \mathcal{E}} L^p(e), \sum_{e \in \mathcal{E}} \|u_{\uparrow e}\|_p^p < +\infty \right\}$$

endowed with the norm

$$\|u\|_{L^p(\mathcal{G})} = \left(\sum_{e \in \mathcal{E}} \|u_{\uparrow e}\|_{L^p(e)}^p \right)^{1/p}.$$

The Sobolev space $H^1(\mathcal{G})$ is the completions of $C_c(\mathcal{G})$ with respect to the norm

$$\|u\|_{H^1(\mathcal{G})} = \left(\sum_{e \in \mathcal{E}} \left(\|u'_{\uparrow e}\|_{L^2(e)}^2 + \|u_{\uparrow e}\|_{L^2(e)}^2 \right) \right)^{1/2}$$

or equivalently the subspace of $L^2(\mathcal{G})$:

$$\{u \in C(\mathcal{G}) \cap L^2(\mathcal{G}), (u'_{\uparrow e})_e \in L^2(\mathcal{G})\}.$$

3 Properties of quantum graphs and examples

A quantum graph is a metric graph endowed with a self-adjoint operator. A usual reference for quantum graphs is the monograph by G. Berkolaiko and P. Kuchment [3].

The quadratic form

$$u \rightarrow \int_{\mathcal{G}} |u'|^2$$

is non negative and closed, see [17, Section VIII.8], on $H_0^1(\mathcal{G})$ and $H^1(\mathcal{G})$. This corresponds respectively to the Dirichlet and Neumann laplacian with Kirchoff boundary conditions at the inner vertices.

The degree, $d(v)$, of a vertex v is defined as the number of edges e such that $e \succ v$. Vertices of degree 1 are endpoints of the graph. If there are no such vertex then Dirichlet and Neumann laplacians coincide. Note that vertices of degree 2 are in a sense inessential since one can replace the two edges by one having the same total length. There is one exception if the graph has one vertex and thus one edge, this then correspond to the torus of dimension 1. This could be overcome if the degree count only distinct edges then in the torus example, the one vertex is of degree one and not an end. We prefer to avoid this and work with the torus exception.

Although, in this analysis, we can consider the Dirichlet laplacian, we will focus on the Neumann one since for vertices of degree 1 they coincide with the Kirchoff condition. We now denote it $-\Delta_{\mathcal{G}}$ and call it Kirchoff laplacian.

3.1 Spectral properties of the Kirchoff laplacian

The main goal of this subsection is to discuss examples for which Assumption 1 holds.

For an account of the knowledge on the spectrum on quantum graphs, we refer to the monograph by Kurasov [14]. In particular, see [14, Remark 2.7], 0 is an eigenvalue if and only if non zero constant functions are integrable. Moreover, if the graph is unbounded, in the sense that

$$\sum_{e \in \mathcal{E}} |e| = +\infty,$$

the bottom of the spectrum is zero if the bottom of the essential spectrum is zero.

A condition ensuring that the bottom of the spectrum is zero, is, see [15, Corollary 3.18.(iii)], that intrinsic size is infinite, that is

$$\sup_{e \in \mathcal{E}} |e| = +\infty.$$

Indeed, using Rayleigh quotient,

$$\inf \sigma(-\Delta_{\mathcal{G}}) = \inf_{f \in H^1(\mathcal{G}), f \neq 0} \frac{\|f'\|_2}{\|f\|_2},$$

this follows from consider the usual spreading tent functions on sequences of edges of lengths tending to infinity.

But this not a sufficient condition, we can refer to [5, Appendix], where it is shown that periodic graphs, see [3, Definition 4.1.1.], the infimum of the Rayleigh quotient is 0.

Examples of graph such that $\inf \sigma(-\Delta_{\mathcal{G}}) > 0$ are provided by equilateral Bethe lattices (equilateral tree with vertices of constant degree larger than 3), see [14, Example 8.3], or exponentially growing antitrees, see [14, Example 8.6]. We refer to [14, Appendix] and [15, Chapter 8] for a more detailed discussion where other examples are presented.

3.2 Concave functions

In this subsection, we discuss the validity of Assumption 2. Notice that if $u \in H^2(\mathcal{G})$ and $u > 0$ then

$$u''_{\uparrow e} = -u_{\uparrow e}^{p-1}$$

and u is a concave function on each edge of the graph.

Let us recall that on \mathbb{R} a regular concave function is always below its tangents. Unless it is constant, some of the tangents vanish and hence there are no regular positive concave function on the real line. On a half-line such as $(0, +\infty)$, $x \mapsto \ln(1+x)$ provides an example of regular positive concave function. While a non constant regular positive concave function with a critical point on an open half line vanishes somewhere. In the context of metric graphs show that of there is an infinite edge then there is no positive concave function which tends to 0 at its infinite end.

On a rooted regular tree⁴ of degree 2, such that the edges of the n -th level from the root are of length 2^{n-1} , there is positive function affine on each edge of the n -th level to $2^{-n+1} - 2^{-n}x$, each is identified with $[0, 2^{n-1}]$. So we can exclude positive continuous piecewise concave functions tending to 0 and satisfying the Kirchoff condition. Note that here the function is also piecewise convex.

Let \mathcal{G} be a \mathbb{Z} -periodic graph with fundamental domain W_0 , see [3, Definition 4.1.1.]. Let us denote by τ the corresponding group action of \mathbb{Z} on \mathcal{G} . Let $W_k = \tau(k)W_0$, $k \in \mathbb{Z}$. As a set, $\mathcal{G} = \cup_{k \in \mathbb{Z}} W_k$. An edge is called entering in W_0 , if either it is in W_{-1} with a final vertex in W_0 or it is in W_0 with a starting vertex in W_{-1} . Denote the corresponding set by \mathcal{E}_0^+ . Similarly, define entering edges to W_k : $\mathcal{E}_k^+ = \tau(k)\mathcal{E}_0^+$. Define exiting edges \mathcal{E}_0^- and then \mathcal{E}_k^- . Note that $\mathcal{E}_k^+ = \mathcal{E}_{k-1}^-$ and all these sets are finite and their cardinals are uniformly bounded.

Now let $\widetilde{W}_k = \mathcal{E}_k^+ \cup W_k \cup \mathcal{E}_k^-$ as a subgraph of \mathcal{G} , in the sense that all the common vertices are identified. Hence \widetilde{W}_k includes W_k and all the edges connected to it as well as the vertices. Using Kirchoff condition, we obtain

$$\sum_{v \in W_k} \sum_{e \succ v, e \in \widetilde{W}_k} u'_{\uparrow e}(v) = 0$$

or

$$\sum_{e \in \widetilde{W}_k} \sum_{e \succ v, v \in W_k} u'_{\uparrow e}(v) = 0.$$

Note that if $u'' < 0$ then for any $v \in W_k$

$$\sum_{e \in \mathcal{E}, e \succ v} u'_{\uparrow e}(v) < 0$$

and hence

$$\sum_{v \in \widetilde{W}_k \setminus W_k} \sum_{e \succ v, e \in \widetilde{W}_k} u'_{\uparrow e}(v) > 0$$

or

$$\sum_{(e,v) \in \mathcal{E}_k^-, e \succ v} u'_{\uparrow e}(v) < \sum_{(e,v) \in \mathcal{E}_k^+, e \succ v} u'_{\uparrow e}(v).$$

⁴A tree is a connected and simply connected graph, that is a connected graph with no closed path. A path is a connected sequence of edges where two consecutive edges are distinct. A root on a tree is a distinguished degree one vertex.

Let

$$\sigma_k := \sum_{(e,v) \in \mathcal{E}_k^-, e \succ v} u'|_e(v),$$

using again $u'' < 0$, we eventually get $\sigma_{k-1} < \sigma_k$. If $u \in H^2(\mathcal{G})$ then $u' \rightarrow 0$ at infinity and so does the sequence $(\sigma_k)_{k \in \mathbb{Z}}$. This is a contradiction. Therefore \mathbb{Z} -periodic graphs satisfy Assumption 2.

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4 Some preliminary definitions and results

In the rest of this analysis, we consider \mathcal{G} to be noncompact connected metric graph such that Assumptions 1 and 2 hold.

Our strategy closely follows [6] with some important modifications to include a distributed nonlinear potential.

We first consider an auxiliary family of energy functionals $E_\rho(\cdot, \mathcal{G}) : H^1(\mathcal{G}) \rightarrow \mathbb{R}$, ρ in $[\frac{1}{2}, 1]$, given by

$$E_\rho(u, \mathcal{G}) := \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \frac{\rho}{p} \int_{\mathcal{G}} |u|^p dx, \quad \forall u \in H^1(\mathcal{G}). \quad (3)$$

Since we are focusing in ρ close to 1. The condition $\rho \geq 1/2$ is arbitrary. It is meant to emphasize that we do not consider ρ vanishing.

We make a crucial use of the abstract analysis [7] and more precisely [7, Theorem 1]. Let us recall this result. Beforehand, let us mention that the existence result we are aiming to, boils down to the convergence of Palais-Smale sequences induced by a mountain pass structure on the functional $E_\rho(u, \mathcal{G})$ from (3) constrained to $H_\mu^1(\mathcal{G})$. The lack of compactness is compensated by a bound on the corresponding sequence of Morse indices.

Let $(E, \langle \cdot, \cdot \rangle)$ and $(H, (\cdot, \cdot))$ be two *infinite-dimensional* Hilbert spaces and assume that $E \hookrightarrow H \hookrightarrow E'$, with continuous injections. For simplicity, we assume that the continuous injection $E \hookrightarrow H$ has norm at most 1 and identify E with its image in H . Set

$$\|u\|^2 := \langle u, u \rangle, \quad |u|^2 := (u, u), \quad u \in E,$$

and, for $\mu > 0$,

$$S_\mu := \{u \in E \mid |u|^2 = \mu\}.$$

In the context of this analysis, we shall have $E = H^1(\mathcal{G})$ and $H = L^2(\mathcal{G})$. Clearly, S_μ is a smooth submanifold of E of codimension 1. Its tangent space at a given point $u \in S_\mu$ can be considered as the closed codimension 1 subspace of E given by:

$$T_u S_\mu = \{v \in E \mid (u, v) = 0\}.$$

In the following definition, we denote by $\|\cdot\|_*$ and $\|\cdot\|_{**}$, respectively, the operator norm of $\mathcal{L}(E, \mathbb{R})$ and of $\mathcal{L}(E, \mathcal{L}(E, \mathbb{R}))$.

Definition 5. Let $\phi : E \rightarrow \mathbb{R}$ be a C^2 -functional on E and $\alpha \in (0, 1]$. We say that ϕ' and ϕ'' are α -Hölder continuous on bounded sets if for any $R > 0$ one can find $M = M(R) > 0$ such that, for any $u_1, u_2 \in B(0, R)$:

$$\|\phi'(u_1) - \phi'(u_2)\|_* \leq M\|u_1 - u_2\|^\alpha, \quad \|\phi''(u_1) - \phi''(u_2)\|_{**} \leq M\|u_1 - u_2\|^\alpha. \quad (4)$$

Definition 6. Let ϕ be a C^2 -functional on E . For any $u \in E$, we define the continuous bilinear map:

$$D^2\phi(u) = \phi''(u) - \frac{\phi'(u) \cdot u}{|u|^2}(\cdot, \cdot).$$

Definition 7. Let ϕ be a C^2 -functional on E . For any $u \in S_\mu$ and $\theta > 0$, we define the approximate Morse index $\tilde{m}_\theta(u)$ by

$$\sup \{ \dim L \mid L \text{ subspace of } T_u S_\mu \text{ s.t.: } D^2\phi(u)[\varphi, \varphi] \leq -\theta\|\varphi\|^2, \forall \varphi \in L \}.$$

If u is a critical point for the constrained functional $\phi|_{S_\mu}$ and $\theta = 0$, we say that this is the Morse index of u as constrained critical point.

The following abstract theorem was established in [7]. Its derivation is based on a combination of ideas from [11, 12, 13] implemented in a convenient geometric setting. Related theorems, but dealing with unconstrained functionals, are developed in [2, 16].

Theorem 8 (Theorem 1 in [7]). Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \rightarrow \mathbb{R}$ of the form

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \rho \in I,$$

where $B(u) \geq 0$ for every $u \in E$, and

$$\text{either } A(u) \rightarrow +\infty \text{ or } B(u) \rightarrow +\infty \text{ as } u \in E \text{ and } \|u\| \rightarrow +\infty. \quad (5)$$

Suppose moreover that Φ'_ρ and Φ''_ρ are α -Hölder continuous on bounded sets for some $\alpha \in (0, 1]$. Finally, suppose that there exist $w_1, w_2 \in S_\mu$ (independent of ρ) such that, setting

$$\Gamma = \{ \gamma \in C([0, 1], S_\mu) \mid \gamma(0) = w_1, \quad \gamma(1) = w_2 \},$$

we have

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi_\rho(\gamma(t)) > \max\{\Phi_\rho(w_1), \Phi_\rho(w_2)\}, \quad \rho \in I.$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \rightarrow 0^+$ such that, as $n \rightarrow +\infty$,

1. $\Phi_\rho(u_n) \rightarrow c_\rho$;
2. $\|\Phi'_\rho|_{S_\mu}(u_n)\|_* \rightarrow 0$;
3. $\{u_n\}$ is bounded in E ;
4. $\tilde{m}_{\zeta_n}(u_n) \leq 1$.

Remark 9. An immediate observation, see [7, Remarks 1.3], following from Theorem 8.(2)-(3) is that

$$\Phi'_\rho(u_n) + \lambda_n(u_n, \cdot) \rightarrow 0 \text{ in } E' \text{ as } n \rightarrow +\infty \quad (6)$$

where we have set

$$\lambda_n := -\frac{1}{\mu}(\Phi'_\rho(u_n) \cdot u_n). \quad (7)$$

The sequence $\{\lambda_n\} \subset \mathbb{R}$ defined in (7) is called sequence of almost Lagrange multipliers.

Remark 10. Theorem 8.(4) directly implies that if there exists a subspace $W_n \subset T_{u_n}S_\mu$ such that

$$D^2\Phi_\rho(u_n)[w, w] = \Phi''_\rho(u_n)[w, w] + \lambda_n(w, w) < -\zeta_n\|w\|^2, \quad \forall w \in W_n \setminus \{0\}, \quad (8)$$

then necessarily $\dim W_n \leq 1$.

5 Mountain pass solutions

To show that the family $E_\rho(\cdot, \mathcal{G})$ enters into the framework of Theorem 8 we first need to show that it has a mountain pass geometry on $H_\mu^1(\mathcal{G})$ uniformly with respect to $\rho \in [\frac{1}{2}, 1]$.

Gagliardo-Nirenberg inequalities on graphs. For every graph \mathcal{G} , recall Gagliardo-Nirenberg inequalities : for any $p \geq 2$, there exists $K_{p, \mathcal{G}} > 0$ (depending on p and \mathcal{G} only) such that for all u in $H^1(\mathcal{G})$

$$\|u\|_p^p \leq K_{p, \mathcal{G}} \|u\|_2^{\frac{p}{2}+1} \|u'\|_2^{\frac{p}{2}-1}, \quad \|u\|_\infty \leq \sqrt{2} \|u\|_2^{\frac{1}{2}} \|u'\|_2^{\frac{1}{2}}. \quad (9)$$

We refer for instance [1, Section 2]. Note that the second statement follows from the fundamental theorem of calculus (e.g. [4, Lemma 1.2.8]) while the former is an interpolation of the $L^2(\mathcal{G})$ -norm with the second. We also refer to [10] for further comments and bibliographical aspects.

Mountain pass geometry.

Lemma 11. For every $\mu > 0$, there exist $w_1, w_2 \in S_\mu$ independent of $\rho \in [\frac{1}{2}, 1]$ such that

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E_\rho(\gamma(t), \mathcal{G}) > \max\{E_\rho(w_1, \mathcal{G}), E_\rho(w_2, \mathcal{G})\}, \quad \forall \rho \in \left[\frac{1}{2}, 1\right],$$

where

$$\Gamma := \{\gamma \in C([0, 1], H_\mu^1(\mathcal{G})) \mid \gamma \text{ is continuous, } \gamma(0) = w_1, \gamma(1) = w_2\}.$$

Moreover, there exists $\kappa > 0$ such that

$$\forall \rho \in \left[\frac{1}{2}, 1\right], c_\rho \geq \kappa.$$

Proof. The proof is borrowed from [6, Lemma 3.1]. We provide it for the reader's convenience with some due adaptations since the framework is different.

For any $\mu, k > 0$, denote

$$A_{\mu,k} := \{u \in H_\mu^1(\mathcal{G}) \mid \int_{\mathcal{G}} |u'|^2 dx < k\}.$$

First note that, due to Assumption 1, $A_{\mu,k} \neq \emptyset$ whenever $k > 0$ and, since $H_\mu^1(\mathcal{G})$ is connected,

$$\partial A_{\mu,k} = \{u \in H_\mu^1(\mathcal{G}) \mid \int_{\mathcal{G}} |u'|^2 dx = k\} \neq \emptyset.$$

Now, by the Gagliardo-Nirenberg inequality (9), we obtain for all u in $\partial A_{\mu,k}$

$$\begin{aligned} E_\rho(u, \mathcal{G}) &\geq \frac{1}{2} \|u'\|_{L^2(\mathcal{G})}^2 - \rho \frac{K_{p,\mathcal{G}}}{p} \mu^{\frac{p}{4} + \frac{1}{2}} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2} - 1} \\ &\geq \frac{1}{2} k \left(1 - \left(\frac{k}{k_1} \right)^{\frac{p-6}{4}} \right) \end{aligned}$$

where

$$k_1 := \left(\frac{p}{2\rho K_{p,\mathcal{G}}} \right)^{\frac{4}{p-6}} \mu^{-\frac{p+2}{p-6}}.$$

Let

$$\kappa_\rho(k) := \frac{1}{2} k \left(1 - \left(\frac{k}{k_1} \right)^{\frac{p-6}{4}} \right)$$

then for $k_0 \in (0, k_1)$, for any $\mu > 0$ and $u \in \partial A_{\mu,k_0}$ we have

$$\inf_{u \in \partial A_{\mu,k_0}} E_\rho(u, \mathcal{G}) \geq \kappa_\rho(k) > 0.$$

Next, observe that for any $u \in A_{\mu,k}$

$$E_\rho(u, \mathcal{G}) \leq \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx \leq \frac{1}{2} k.$$

Since $A_{k,\mu} \neq \emptyset$ for all $k > 0$, it is possible to choose a $w_1 \in H_\mu^1(\mathcal{G})$ such that

$$\|w_1'\|_{L^2(\mathcal{G})}^2 < k_0 \quad \text{and} \quad E_\rho(w_1, \mathcal{G}) \leq \kappa_\rho(k_0), \quad \forall \rho \in \left[\frac{1}{2}, 1 \right]. \quad (10)$$

Moreover, we also observe that we can identify any edge, say e_1 , with the interval $[-\pi_1/2, \pi_1/2]$ if it is bounded or $[-\pi_1/2, +\infty)$ if not. It follows that any compactly supported H^1 function w on $[-\pi_1/2, \pi_1/2]$, with mass μ , can be seen as a function in $H_\mu^1(\mathcal{G})$. Defining $w_t(x) := t^{1/2}w(tx)$, with $t > 1$, we have $w_t \in H_\mu^1(\mathcal{G})$ and that

$$\begin{aligned} E_\rho(w_t, \mathcal{G}) &= \frac{t^2}{2} \int_{e_1} |w'|^2 dx - \frac{\rho t^{\frac{p-2}{2}}}{p} \int_{e_1} |w|^p dx \\ &\leq \frac{t^2}{2} \left(\int_{e_1} |w'|^2 dx - \frac{2\rho t^{\frac{p-6}{2}}}{p} \int_{e_1} |w|^p dx \right) \end{aligned}$$

for every $\rho \in [\frac{1}{2}, 1]$. Since $p > 6$, then the right-hand side tends to $-\infty$ as $t \rightarrow +\infty$, and in particular there exists $t_2 > 0$ large enough such that $\|w'_t\|_{L^2(\mathcal{G})}^2 = t^2 \|w'\|_{L^2(\mathcal{G})}^2 > 2k_0$ and $E_\rho(w_t, \mathcal{G}) < 0$ for all $t > t_2$ and $\rho \in [\frac{1}{2}, 1]$. We set $w_2 := w_{2t_2}$, and we point out that

$$\|w'_2\|_{L^2(\mathcal{G})}^2 > 2k_0 \quad \text{and} \quad E_\rho(w_2, \mathcal{G}) < 0 \quad \forall \rho \in \left[\frac{1}{2}, 1\right]. \quad (11)$$

At this point the thesis follows easily. Let Γ and c_ρ be defined as in the statement of the lemma for our choice of w_1 and w_2 ; the fact that $\Gamma \neq 0$ is straightforward, since

$$\gamma_0(t) := \frac{\mu^{1/2}}{\|(1-t)w_1 + tw_2\|_{L^2(\mathcal{G})}} [(1-t)w_1 + tw_2] \quad t \in [0, 1]$$

belongs to Γ . By (10) and (11), we note that for every $\gamma \in \Gamma$ there exists $t_\gamma \in [0, 1]$ such that $\gamma(t_\gamma) \in \partial A_{\mu, k_0}$, by continuity. Therefore, for any $\rho \in [1/2, 1]$, we have for every $\gamma \in \Gamma$

$$\max_{t \in [0, 1]} E_\rho(\gamma(t), \mathcal{G}) \geq E_\rho(\gamma(t_\gamma), \mathcal{G}) \geq \inf_{u \in \partial A_{\mu, k_0}} E_\rho(u, \mathcal{G}) \geq \kappa_\rho(k_0)$$

and thus $c_\rho \geq \kappa_\rho(k_0)$ while

$$\max\{E_\rho(w_1, \mathcal{G}), E_\rho(w_2, \mathcal{G})\} = E_\rho(w_1, \mathcal{G}) < \frac{\alpha}{2}.$$

To conclude we remark that $\kappa_\rho(k_0) \geq \kappa_1(k_0)$ for $\rho \in [1/2, 1]$. We set $\kappa := \kappa_1(k_0)$. \square

Remark 12. *If Assumption 1 is not satisfied then Lemma 11 still holds if $\mu > 0$ is small enough for $E(u, \mathcal{G}) - \sigma \|u\|_2^2$ where $\sigma := \inf \sigma(-\Delta_{\mathcal{G}})$.*

Properties of the essential spectrum.

Lemma 13. *For any $u \in H^1(\mathcal{G})$, for any $\lambda < 0$, there exists an infinite dimensional subspace L of $H^1(\mathcal{G})$ such that*

$$\int_{\mathcal{G}} [|w'|^2 + (\lambda - (p-1)\rho|u|^{p-2}) w^2] \, dx < 0, \forall w \in L.$$

Proof. Since $\inf \sigma_{\text{ess}}(-\Delta_{\mathcal{G}}) = 0$ and $V_u : w \mapsto (p-1)\rho|u|^{p-2}w$ is compact from $H^2(\mathcal{G})$ to $L^2(\mathcal{G})$ Weyl's criterion gives $0 = \inf \sigma_{\text{ess}}(-\Delta_{\mathcal{G}} - V_u)$ and so for any $\lambda < 0$ there exists an infinite dimensional subspace L of $H^1(\mathcal{G})$ such that

$$\forall w \in L, \int_{\mathcal{G}} [|w'|^2 + (\lambda - |u|^{p-2}) w^2] \, dx < 0$$

which concludes the proof. \square

An abstract compactness lemma.

Lemma 14. *Let $\{u_n\} \subset H_\mu^1(\mathcal{G})$ be a bounded nonnegative sequence. Let $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}$ and $(\rho_n)_{n \in \mathbb{N}} \in \mathbb{R}$ be convergent to λ_∞ and ρ_∞ respectively and such that*

$$E_{\rho_n}(u_n) \rightarrow c > 0 \quad \text{and} \quad E'_{\rho_n}(u_n) + \lambda_n u_n \rightarrow 0 \quad \text{in the dual of } H^1(\mathcal{G}) \quad (12)$$

and if, for each integer n , the inequality

$$E''_{\rho_n}(u_n)[\varphi, \varphi] + \lambda_n \|\varphi\|_{L^2(\mathcal{G})}^2 < 0 \quad (13)$$

holds for any $\varphi \in W_n \setminus \{0\}$, W_n a subspace of $T_{u_n} S_\mu$ then the dimension of W_n is at most 1. Then $\{u_n\}$ has a converging subsequence in $H^1(\mathcal{G})$ to some positive u_∞ such that

$$-u''_\infty + \lambda_\infty u_\infty = \rho_\infty u_\infty^{p-1} \quad (14)$$

with the Kirchhoff condition (2) at the vertices. Moreover, we have $m_0(u_\infty) \leq 1$.

Proof. Up to the extraction of subsequences, we can assume (λ_n) and (ρ_n) to be convergent to λ_∞ and ρ_∞ , respectively, and

$$u_n \rightharpoonup u_\infty \text{ in } H^1(\mathcal{G}), \quad (15)$$

$$u_n \rightarrow u_\infty \text{ in } L^r_{\text{loc}}(\mathcal{G}), r > 2, \quad (16)$$

$$u_n(x) \rightarrow u_\infty(x) \text{ for a.e. } x \in \mathcal{G}, \quad (17)$$

which implies that $u_\infty \geq 0$ and u_∞ satisfies (14) (with the Kirchhoff condition (2) at the vertices).

Let us show that $u_\infty \not\equiv 0$. First let us prove that $\lambda_\infty \geq 0$. Since the codimension of $T_{u_n} S_\mu$ is 1, we infer that if the inequality (13) holds for every $\varphi \in V_n \setminus \{0\}$ for a subspace V_n of $H^1(\mathcal{G})$, then the dimension of V_n is at most 2. If $\lambda_n < 0$, Lemma 13 provides for $n \in \mathbb{N}$ large, the existence of a subspace V_n of $H^1(\mathcal{G})$ with $\dim V_n \geq 3$ and $a_n > 0$ such that,

$$E''_{\rho_n}(u_n)[\varphi, \varphi] + \lambda_n |\varphi|^2 \leq -a_n \|\varphi\|^2, \quad \forall \varphi \in V_n \setminus \{0\}.$$

Therefore, for $n \in \mathbb{N}$ large, $\lambda_n \geq 0$. Hence $\lambda_\infty \geq 0$

Now from (12) and the fact that $\lambda_n \rightarrow \lambda_\infty$, we deduce that

$$\int_{\mathcal{G}} (u'_n \varphi' + \lambda_n u_n \varphi) \, dx - \rho_n \int_{\mathcal{G}} u_n^{p-1} \varphi \, dx = o(1) \|\varphi\|_{H^1(\mathcal{G})}.$$

Moreover, by (14),

$$\int_{\mathcal{G}} (u'_\infty \varphi' + \lambda_\infty u_\infty \varphi) \, dx - \rho_\infty \int_{\mathcal{G}} u_\infty^{p-1} \varphi \, dx = 0$$

Therefore, taking the difference, and $\varphi = u_n - u_\infty$, we infer that

$$\int_{\mathcal{G}} |(u_n - u_\infty)'|^2 \, dx + \lambda_\infty \int_{\mathcal{G}} |u_n - u_\infty|^2 \, dx \rightarrow 0 \quad (18)$$

as $n \rightarrow \infty$. If we assume that $u_\infty \equiv 0$ then we deduce that

$$\int_{\mathcal{G}} |u'_n|^2 \, dx + \lambda_\infty \int_{\mathcal{G}} |u_n|^2 \, dx \rightarrow 0. \quad (19)$$

If $\lambda_\infty > 0$, this is not possible since $\|u_n\|_{L^2(\mathcal{G})}^2 = \mu > 0$. If $\lambda_\infty = 0$, then (19) contradicts the assumption $c > 0$ in (12). Hence, $u_\infty \not\equiv 0$.

Appealing to the Kirchhoff condition (2) and the uniqueness in Cauchy-Lipschitz theorem, we have in fact that $u_\infty > 0$ in \mathcal{G} . Indeed, assume by contradiction that there exists $x_0 \in \mathcal{G}$ such that $u_\infty(x_0) = 0$. If x_0 stays in the interior of some edge, then by $u_\infty \geq 0$ on \mathcal{G} it follows that $u'_\infty(x_0) = 0$ leading to $u_\infty \equiv 0$ on the whole edge. If instead x_0 is a vertex, then by $u_\infty(x) \geq 0$, $u'_\infty(x_0)_e \geq 0$ for all $e \succ x_0$ and by the Kirchhoff condition we also get $u'_\infty(x_0) = 0$. Hence, by uniqueness, $(u_\infty)_e \equiv 0$, so $u_\infty \equiv 0$ on any edge containing x_0 ; but then, by repeating this argument from first neighbors to next ones and so iteratively (since \mathcal{G} is connected) we deduce that $u_\infty \equiv 0$ on \mathcal{G} , which is the desired contradiction.

Note that (18) implies $u_n \rightarrow u_\infty$ in $\dot{H}^1(\mathcal{G})$ and even $H^1(\mathcal{G})$ if $\lambda_\infty > 0$. Now we claim that $\lambda_\infty > 0$. If $\lambda_\infty = 0$, then from (14) contradicts Assumption 2 since $u_\infty \in H^1(\mathcal{G})$ and $u_\infty > 0$. So $\lambda_\infty > 0$ and $u_n \rightarrow u_\infty$ strongly in $H^1(\mathcal{G})$.

It remains to show that the Morse index $m(u_\infty)$ is at most 1. If not, in view of Definition 7 we may assume by contradiction that there exists a $W_0 \subset T_u S_\mu$ with $\dim W_0 = 2$ such that

$$D^2 E_{\rho_\infty}(u_\infty)[w, w] < 0, \forall w \in W_0 \setminus \{0\}. \quad (20)$$

Then, since W_0 is of finite dimension, there exists $\beta > 0$ such that

$$D^2 E_{\rho_\infty}(u_\infty)[w, w] < -\beta, \forall w \in W_0 \setminus \{0\} \quad \text{with} \quad \|w\|_{H^1(\mathcal{G})} = 1,$$

using the homogeneity of $D^2 E_{\rho_\infty}(u_\infty)$, we deduce that

$$D^2 E_{\rho_\infty}(u_\infty)[w, w] < -\beta \|w\|_{H^1(\mathcal{G})}^2, \forall w \in W_0 \setminus \{0\}.$$

Now, from [7, Corollary 1] or using directly that E'_ρ and E''_ρ are α -Hölder continuous on bounded sets for some $\alpha \in (0, 1]$, and Sobolev inequality it follows that there exists $\delta_1 > 0$ small enough such that, for any $v \in S_\mu$ satisfying $\|v - u_\infty\| \leq \delta_1$ and $|\rho - \rho_\infty| < \delta_1$,

$$D^2 E_\rho(v)[w, w] < -\frac{\beta}{2} \|w\|_{H^1(\mathcal{G})}^2, \forall w \in W_0 \setminus \{0\}. \quad (21)$$

Hence, noting that $\|u_n - u_\infty\|_{H^1(\mathcal{G})} \leq \delta_1$ and $|\rho_n - \rho_\infty| < \delta_1$ for $n \in \mathbb{N}$ large enough, we get

$$D^2 E_\rho(u_n)[w, w] < -\frac{\beta}{2} \|w\|_{H^1(\mathcal{G})}^2, \forall w \in W_0 \setminus \{0\} \quad (22)$$

for any such large n . Therefore, since $\dim W_0 > 1$ this contradicts (13). Thus we infer that $m_0(u_\infty) \leq 1$. \square

6 Main statements

Proposition 15. *For any fixed $\mu > 0$ and almost every $\rho \in [\frac{1}{2}, 1]$, there exists $(u_\rho, \lambda_\rho) \in H_\mu^1(\mathcal{G}) \times \mathbb{R}^+$ which solves*

$$\begin{cases} -u''_\rho + \lambda_\rho u_\rho = \rho u_\rho^{p-1}, & u_\rho > 0 \quad \text{in } \mathcal{G}, \\ \sum_{e \succ v} u'_\rho(v) = 0 & \text{for any vertex } v. \end{cases} \quad (23)$$

Moreover, $E_\rho(u_\rho, \mathcal{G}) = c_\rho$ and $m(u_\rho) \leq 1$.

Proof. For simplicity, we omit the dependence of the functionals $E_\rho(\cdot, \mathcal{G})$ on \mathcal{G} . We apply Theorem 8 to the family of functionals E_ρ , with $E = H^1(\mathcal{G})$, $H = L^2(\mathcal{G})$, $S_\mu = H_\mu^1(\mathcal{G})$, and Γ defined in Lemma 11. Setting

$$A(u) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx \quad \text{and} \quad B(u) = \frac{\rho}{p} \int_{\mathcal{G}} |u|^p dx,$$

assumption (5) holds, since we have that

$$u \in H_\mu^1(\mathcal{G}), \|u\| \rightarrow +\infty \implies A(u) \rightarrow +\infty.$$

Let E'_ρ and E''_ρ denote respectively the free first and second derivatives of E_ρ . Clearly, E'_ρ and E''_ρ are both of class C^1 , and hence locally Hölder continuous, on $H_\mu^1(\mathcal{G})$, which implies that Assumption (4) holds.

Thus, taking into account Lemma 11, by Theorem 8, for almost every $\rho \in [1/2, 1]$, there exist a bounded sequence $\{u_{n,\rho}\} \subset H_\mu^1(\mathcal{G})$, that we shall denote simply by $\{u_n\}$ from now on, and a sequence $\{\zeta_n\} \subset \mathbb{R}^+$ with $\zeta_n \rightarrow 0^+$, such that

$$E'_\rho(u_n) + \lambda_n u_n \rightarrow 0 \quad \text{in the dual of } H_\mu^1(\mathcal{G}), \quad (24)$$

where

$$\lambda_n := -\frac{1}{\mu} E'_\rho(u_n) u_n \quad (25)$$

and if the inequality

$$E''_\rho(u_n)[\varphi, \varphi] + \lambda_n \|\varphi\|_{L^2(\mathcal{G})}^2 < -\zeta_n \|\varphi\|_{H^1(\mathcal{G})}^2 \quad (26)$$

holds for any $\varphi \in W_n \setminus \{0\}$ in a subspace W_n of $T_{u_n} S_\mu$, then $\dim W_n \leq 1$. In addition, by diamagnetic inequality (see for example [7, Remark 1.4]) since $u \in H_\mu^1(\mathcal{G}) \implies |u| \in H_\mu^1(\mathcal{G})$, the map $u \mapsto |u|$ is continuous, and $E_\rho(u) = E_\rho(|u|)$, it is possible to choose $\{u_n\}$ with the property that $u_n \geq 0$ on \mathcal{G} .

The sequence $\{u_n\}$ being bounded, it follows by (25) that $\{\lambda_n\}$ is bounded. Then, passing to a subsequence, there exists $\lambda_\rho \in \mathbb{R}$ such that $\lim_{n \rightarrow +\infty} \lambda_n = \lambda_\rho$. Taking $\rho_n = \rho$ for all $n \in \mathbb{N}$ in Lemma 14, we conclude the proof. \square

We now turn to the case $\rho = 1$. Beforehand, let us consider the

Lemma 16. *For $\mu > 0$, for $\rho > 0$ and $u \in H_\mu^1(\mathcal{G})$, $u > 0$ such that*

$$\begin{cases} -u'' + \lambda u = \rho u^{p-1}, & \text{in } \mathcal{G}, \\ \sum_{e>v} u'(v) = 0 & \text{for any vertex } v, \end{cases} \quad (27)$$

define

$$L(u, \mathcal{G}) := \frac{1}{2} \int_{\mathcal{G}} (u')^2 dx + \frac{\rho}{p} \int_{\mathcal{G}} u^p dx - \frac{\lambda}{2} \int_{\mathcal{G}} u^2 dx$$

then

$$\left(\frac{1}{2} - \frac{1}{p}\right) L(u, \mathcal{G}) = \left(\frac{1}{2} + \frac{1}{p}\right) E(u, \mathcal{G}) + \frac{3\lambda}{2} \left(\frac{1}{p} - \frac{1}{6}\right) \mu.$$

Proof. Note that for any $e \in \mathcal{E}$, $u|_e \in C^2(e)$ and multiplying $-u'' + \lambda u_\rho - \rho u^{p-1}$ by u and integrating on \mathcal{G} provides the Pohozaev identity

$$\int_{\mathcal{G}} (u')^2 dx + \lambda \int_{\mathcal{G}} u^2 dx - \rho \int_{\mathcal{G}} u^p dx = 0$$

so that

$$E(u, \mathcal{G}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} (u')^2 dx - \frac{\lambda}{p} \int_{\mathcal{G}} u^2 dx$$

and

$$L(u, \mathcal{G}) := \left(\frac{1}{2} + \frac{1}{p}\right) \int_{\mathcal{G}} (u')^2 dx - \lambda \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} u^2 dx$$

so that

$$\begin{aligned} \left(\frac{1}{2} + \frac{1}{p}\right) E(u, \mathcal{G}) + \left(\frac{1}{2} + \frac{1}{p}\right) \frac{\lambda}{p} \int_{\mathcal{G}} u^2 dx &= \left(\frac{1}{2} - \frac{1}{p}\right) L(u, \mathcal{G}) \\ &+ \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} u^2 dx \end{aligned}$$

which concludes the proof. \square

Corollary 17. For $\mu > 0$, for $\rho > 0$ and $u \in H_\mu^1(\mathcal{G})$, $u > 0$ such that

$$\begin{cases} -u'' + \lambda u = \rho u^{p-1}, & \text{in } \mathcal{G}, \\ \sum_{e \succ v} u'(v) = 0 & \text{for any vertex } v, \end{cases} \quad (28)$$

define for any $e \in \mathcal{E}$

$$\ell_{\lambda, \rho}(u, e) := \frac{1}{2}(u')^2 + \frac{\rho}{p} u^p - \frac{\lambda}{2} u^2,$$

which is constant on e , then

$$\sum_{e \in \mathcal{E}, \ell_{\lambda, \rho}(u, e) < 0} |e| \ell_{\lambda, \rho}(u, e) \geq -\left(\frac{1}{2} + \frac{1}{p}\right) E(u, \mathcal{G}) + \frac{3\lambda}{2} \left(\frac{1}{6} - \frac{1}{p}\right) \mu.$$

Proposition 18. For any fixed $\mu > 0$ there exists $(u, \lambda) \in H_\mu^1(\mathcal{G}) \times \mathbb{R}^+$ which solves

$$\begin{cases} -u'' + \lambda u = u^{p-1}, & u > 0 \text{ in } \mathcal{G}, \\ \sum_{e \succ v} u'(v) = 0 & \text{for any vertex } v. \end{cases} \quad (29)$$

Proof. The thesis will follow as in Proposition 15 considering a sequence $\rho_n \rightarrow 1$ such that the conclusion of Theorem 8 holds and the corresponding $(u_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ from Proposition 15. In order to apply Lemma 14 and conclude, we have to prove that $(u_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ are bounded in $H^1(\mathcal{G})$ and \mathbb{R} respectively.

Note that by Pohozaev identity

$$E(u_n, \mathcal{G}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} (u_n')^2 dx - \frac{\lambda_n}{p} \int_{\mathcal{G}} u_n^2 dx$$

since $\int_{\mathcal{G}} u_n^2 dx = \mu$, $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathcal{G})$ if and only if $(\lambda_n)_{n \in \mathbb{N}}$ is bounded in \mathbb{R} .

So let us assume that, up to a subsequence extraction, $\lambda_n \rightarrow +\infty$. Recall

$$\ell_{\lambda,\rho}(u, e) = \frac{1}{2}(u')^2 + \frac{\rho}{p}u^p - \frac{\lambda}{2}u^2,$$

For each sequence of edges e_n such that $\ell_{\lambda_n, \rho_n}(u_n, e_n) < 0$, first we prove that $\ell_{\lambda_n, \rho_n}(u_n, e_n) = o(\lambda_n^{\frac{p}{p-2}})$. If it were not true then $\ell_{1,1}(\lambda_n^{-\frac{1}{p-2}}u_n, e_n) \not\rightarrow 0$. Since $\ell_{1, \rho_n}(\lambda_n^{-\frac{1}{p-2}}u_n, e_n) \in [\frac{\rho_n}{p} - \frac{1}{2}, 0]$, up to a subsequence extraction, $|u_n|$ stays away from 0 and hence $\inf_{e_n} u_n \geq \kappa \rho_n^{-\frac{1}{p-2}} \lambda_n^{\frac{1}{p-2}}$, for some $\kappa > 0$, and

$$\mu^2 \geq \kappa^2 \inf_{e \in \mathcal{E}} |e| \rho_n^{-\frac{2}{p-2}} \lambda_n^{\frac{2}{p-2}}$$

which leads to a contradiction in the limit $n \rightarrow \infty$. The same contradiction would hold if there is a subsequence of $(u_n)_n$ made of constant functions necessarily equal to $\rho^{-\frac{1}{p-2}} \lambda_n^{\frac{1}{p-2}}$.

From (31) and (32) below (see Appendix), we deduce

$$\|u_n\|_{L^2(e_n)} \sim_{n \rightarrow \infty} -|e_n| \frac{\kappa_p \lambda_n^{\frac{2}{p-2}}}{2 \ln(-\lambda_n^{-\frac{p}{p-2}} \ell_{\lambda_n, \rho_n}(u_n, e_n))}$$

Since for any $\kappa > 0$ and $C > 0$, if $X > 0$ is sufficiently small then $0 < -X \ln(X) < \kappa/C$, we deduce, for any $C > 0$ that there exists N such that for $n \geq N$

$$\|u_n\|_{L^2(e_n)} \geq C \lambda_n^{-1} |e_n| |\ell_{\lambda_n, \rho_n}(u_n, e_n)|$$

Recall Corollary 17,

$$\sum_{e \in \mathcal{E}, \ell_{\lambda_n, \rho_n}(u_n, e) < 0} |e| |\ell_{\lambda_n, \rho_n}(u_n, e)| \geq -\left(\frac{1}{2} + \frac{1}{p}\right) E(u_n, \mathcal{G}) + \frac{3\lambda_n}{2} \left(\frac{1}{6} - \frac{1}{p}\right) \mu$$

so that

$$\mu \geq C \lambda_n^{-1} \left(-\left(\frac{1}{2} + \frac{1}{p}\right) E(u_n, \mathcal{G}) + \frac{3\lambda_n}{2} \left(\frac{1}{6} - \frac{1}{p}\right) \mu \right)$$

which leads to a contradiction in the limit $n \rightarrow \infty$ if $\mu > 0$ since C can be chosen such that $C \frac{3}{2} \left(\frac{1}{6} - \frac{1}{p}\right) > 1$. This contradiction shows $(u_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ are bounded in $H^1(\mathcal{G})$ and \mathbb{R} respectively. Hence Lemma 14 applies and we conclude as in Proposition 15. \square

Theorem 4 follows from Proposition 18 \square

Appendix

This appendix is devoted to some useful properties of the ODE problem considered in this work. This is closely related to the phase space analysis done in [10, Section]. The references listed therein presents other properties of solutions.

A Local positive solutions to stationary NLS

Let us consider on \mathbb{R} , real solutions to the ODE:

$$-v'' - \rho|v|^{p-2}v + m^2v = 0, \quad \rho \in (0, 1], m \in \mathbb{R}. \quad (\text{ODE})$$

This equation is invariant by translation and symmetry with respect to any point. If v is a solution so is $-v$. The equilibria correspond to $v \equiv 0$ or constants v such that $\rho|v|^{p-2} = m^2$. We are interested in $v > 0$.

Note that if v is a solution of (ODE) then u such that $u(x) = \rho^{\frac{1}{p-2}} m^{-\frac{2}{p-2}} v(m^{-1}x)$ is a solution of

$$-u'' - |u|^{p-2}u + u = 0.$$

Let us define the Hamiltonian

$$\ell_{m,\rho}(v) := \frac{1}{2}|v'|^2 + \frac{\rho}{p}|v|^p - \frac{m^2}{2}|v|^2.$$

The Hamiltonian well. Let

$$\pi_{m,\rho}(v) := \frac{\rho}{p}|v|^p - \frac{m^2}{2}|v|^2.$$

This is an even function vanishing at 0 (an equilibria of (ODE)) and $\pm\gamma_+$ where

$$\gamma_+ := \left(\frac{p}{2\rho}\right)^{\frac{1}{p-2}} m^{\frac{2}{p-2}}.$$

Its minima are at $\pm\gamma_-$ (two equilibrium of (ODE)) where

$$\gamma_- := \rho^{-\frac{1}{p-2}} m^{\frac{2}{p-2}},$$

with value $\beta := \left(\frac{1}{p} - \frac{1}{2}\right) \rho^{-\frac{2}{p-2}} m^{\frac{2p}{p-2}}$, 0 is a local maximum (value is 0) and it tends to $+\infty$ at $\pm\infty$. The level set of $\pi_{m,\rho}$ at μ real, $\pi_{m,\rho}^{-1}\{\mu\}$,

- is empty if $\mu < \beta$,
- contains only $\{\pm\gamma_-\}$ if $\mu = \beta$,
- contains only two pairs of opposite reals if $\mu \in (\beta, 0)$,
- contains only $\pm\gamma_+$ and 0 if $\mu = 0$,
- and contains only a pair of opposite reals if $\mu > 0$.

Then note that $\pi_{m,\rho}(x) = \rho^{-\frac{2}{p-2}} m^{\frac{2p}{p-2}} \pi_{1,1}(\rho^{\frac{1}{p-2}} m^{-\frac{2}{p-2}} x)$ and we have

- $\pi_{1,1}(y) = 0$ if and only if $y = 0$ or $y = \left(\frac{p}{2}\right)^{\frac{1}{p-2}}$;
- $\pi'_{1,1}(y) = y^{p-1} - y$ and $\pi'_{1,1}(y) = 0$ if and only if $y = 0$ or $y = 1$. Note that $\pi_{1,1}(1) = \left(\frac{1}{p} - \frac{1}{2}\right)$;

Therefore $\pi_{1,1}$ is invertible from $[0, 1]$ to $\left[\left(\frac{1}{p} - \frac{1}{2}\right), 0\right]$ with inverse denoted g . Then g is continuous and $g(0) = 0$ and for $z \in \left[\left(\frac{1}{p} - \frac{1}{2}\right), 0\right]$, we have

$$\left(\frac{1}{p}g(z)^{p-2} - \frac{1}{2}\right)g(z)^2 = z \quad (30)$$

and thus

$$g(z) = \sqrt{\frac{-z}{\frac{1}{2} - \frac{1}{p}g(z)^{p-2}}} \sim_{z \rightarrow 0^-} \sqrt{-2z}.$$

Hence for $u \in \left[\left(\frac{1}{p} - \frac{1}{2}\right)\rho^{-\frac{2}{p-2}}m^{-\frac{2p}{p-2}}, 0\right]$

$$\begin{aligned} \pi_{m,\rho}(x) = u \text{ and } \rho^{\frac{1}{p-2}}m^{-\frac{2}{p-2}}x \leq 1 &\Leftrightarrow \pi_{1,1}\left(\rho^{\frac{1}{p-2}}m^{-\frac{2}{p-2}}x\right) = \rho^{\frac{2}{p-2}}m^{-\frac{2p}{p-2}}u \\ &\text{and } \rho^{\frac{1}{p-2}}m^{-\frac{2}{p-2}}x \leq 1 \\ &\Leftrightarrow x = \rho^{-\frac{1}{p-2}}m^{\frac{2}{p-2}}g\left(\rho^{\frac{2}{p-2}}m^{-\frac{2p}{p-2}}u\right) \end{aligned}$$

and if $u = o\left(\rho^{\frac{2}{p-2}}m^{\frac{2p}{p-2}}\right)$, as $m \rightarrow 0$ or $m \rightarrow +\infty$, then

$$x \sim \rho^{-\frac{1}{p-2}}m^{\frac{2}{p-2}}\sqrt{-2\rho^{\frac{2}{p-2}}m^{-\frac{2p}{p-2}}u} = m^{-1}\sqrt{-2u} = o\left(\rho^{\frac{1}{p-2}}m^{\frac{2}{p-2}}\right).$$

Rearranging (30), we have

$$\left(\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{p} \frac{g(z)^{p-2} - 1 - (p-2)(g(z)-1)g(z)^2}{(g(z)-1)^2}\right)(g(z)-1)^2 = z - \left(\frac{1}{p} - \frac{1}{2}\right)$$

which leads to

$$g(z) - 1 \sim_{z \rightarrow \left(\frac{1}{p} - \frac{1}{2}\right)^-} \sqrt{\frac{2p}{(p-2)(p-4)} \left(z - \left(\frac{1}{p} - \frac{1}{2}\right)\right)}$$

Properties of solutions. Any weak continuous solution is of class C^2 and thus there are only strong solutions and $\ell_{m,\rho}(v)$ is constant on intervals. Note that $\ell_{m,\rho}(v) \geq \pi_{m,\rho}(v)$. As $\lim_{v \rightarrow \pm\infty} \pi_{m,\rho}(v) = +\infty$, all the solutions are bounded. Moreover, we have the following properties.

1. If $m = 0$ there is no solution with $\ell_{m,\rho}(v) \equiv 0$ except $v \equiv 0$.
2. Since $|v'|^2 = 2(\ell_{m,\rho}(v) - \pi_{m,\rho}(v))$, v' is bounded thus any maximal solution is global (defined on \mathbb{R}).
3. Let v be defined on a half line and tend to k at the infinite end. Then $2|v'|^2 = \ell_{m,\rho}(v) - \pi_{m,\rho}(v)$ tends to $k' = \ell_{m,\rho}(v) - \pi_{m,\rho}(k)$. Since v' is continuous and v has a limit, $k' = 0$. Then v'' tends to $m^2k - \rho|k|^{p-2}k$ and with the same reasoning $m^2k - \rho|k|^{p-2}k = 0$ that is $k = 0$ or $k =$

$\pm \rho^{-\frac{1}{p-2}} m^{\frac{2}{p-2}}$. If $k = \pm \rho^{-\frac{1}{p-2}} m^{\frac{2}{p-2}}$, the minimum of $\ell_{m,\rho}$, then $v' \equiv 0$ and v is constant. Note that if v is maximal and v' never vanishes, v is monotonic and has limits at both infinities which are different equilibria (this is a heteroclinic orbit and $\ell_{m,\rho}(v) = 0$) which is impossible. Indeed, one of this equilibrium is γ_- or $-\gamma_-$ and therefore $\ell_{m,\rho}(v) = \pi_{m,\rho}(\pm\gamma_-) = \beta$ and v is constant.

4. If v is a maximal solution, all the critical values of v are in the same level set of $\ell_{m,\rho}$ and are either made of two values if $\ell_{m,\rho}(v) \neq 0$. If $\ell_{m,\rho}(v) < 0$, v does not vanish. If $\ell_{m,\rho}(v) > 0$, v vanishes somewhere. If $\ell_{m,\rho}(v) = 0$, either it $v \equiv 0$ or $|v|$ has a maximum $\left(\frac{\rho}{2}\right)^{\frac{1}{p-2}} \rho^{-\frac{1}{p-2}} m^{\frac{2}{p-2}}$ and tends to 0 at infinity. Indeed, if 0 is a critical point then the solution is, by uniqueness, identically zero.
5. If v is a maximal solution and has critical points then v is symmetric with respect to any of its critical point (from uniqueness of the Cauchy problem and the invariance by symmetry and translation of the equation). If v has two distinct critical values then it is periodic and
 - $\ell_{m,\rho}(v) < 0$, $v > 0$ non constant, iff its maximum $v_+ \in (\gamma_-, \gamma_+)$, then there is another critical value $v_- \in (0, \gamma_-)$, and v_- is the minimum and v is positive (and periodic). Recall $\ell_{m,\rho}(v) < 0$ and $v < 0$ iff $\ell_{m,\rho}(-v) < 0$ and $-v > 0$.
 - $\ell_{m,\rho}(v) > 0$ iff its maximum $v_+ > \gamma_+$, then $-v_+$ is its minimum (and v is periodic).
 - $\ell_{m,\rho}(v) = 0$, v non constant iff its maximum is $v_+ = \gamma_+$, then v is positive and tends to 0 at infinity. Note that $\ell_{m,\rho}(v) = 0$, v constant iff $v \equiv 0$.
 - $\ell_{m,\rho}(v) = \beta$ iff its maximum $v_+ = \gamma_-$ and then v is constant.

Period with negative Hamiltonian. We consider v maximal with $\ell := \ell_{m,\rho}(v) < 0$. Hence, v has maximal value v_+ at x_{v_+} and minimal value v_- at x_{v_-} , the smallest minimum larger than x_{v_+} so that

$$\ell = \pi_{m,\rho}(v_+) = \pi_{m,\rho}(v_-)$$

and if v is not constant then let us now consider the period $T_{m,\rho}(\ell)$. Recall that the solution is symmetric with respect to its critical points. Then, for

$m = \rho = 1$, we have

$$\begin{aligned}
\frac{1}{2}T_{1,1}(\ell) &= \int_{x_{v+}}^{x_{v-}} dx = \int_{v-}^{v+} \frac{dv}{\sqrt{2\ell - \frac{2}{p}v^p + v^2}} = 2 \int_{v-}^1 \frac{dv}{\sqrt{2\ell - \frac{2}{p}v^p + v^2}} \\
&= \sqrt{2} \int_{v-}^1 \frac{dv}{\sqrt{\ell - \pi_{1,1}(v)}} = \sqrt{2} \int_{\pi_{1,1}(1)}^{\pi_{1,1}(v-)} \frac{1}{\sqrt{\ell - p}} \frac{dp}{\pi'_{1,1}(g(p))} \\
&= \sqrt{2} \int_{\frac{1}{p} - \frac{1}{2}}^{\ell} \frac{1}{\sqrt{\ell - p}} \frac{1}{g(p)} \frac{dp}{g(p)^{p-2} - 1} \\
&\sim_{\ell \rightarrow 0-} -\sqrt{2} \int_{\frac{1}{p} - \frac{1}{2}}^{\ell} \frac{1}{\sqrt{\ell - p}} \frac{1}{\sqrt{-2p}} dp = \int_1^{\frac{p-2}{2p(-\ell)}} \frac{1}{\sqrt{p-1}} \frac{1}{\sqrt{p}} dp \\
&\sim_{\ell \rightarrow 0-} -\ln(-\ell).
\end{aligned}$$

We thus infer

$$T_{m,\rho}(\ell) = m^{-1}T_1(\rho^{\frac{1}{p-2}} m^{-\frac{2p}{p-2}} \ell) \sim_{\ell \rightarrow 0-} -2m^{-1} \ln(-\rho^{\frac{1}{p-2}} m^{-\frac{2p}{p-2}} \ell). \quad (31)$$

Norm with negative Hamiltonian. Now let us consider the square of the L^2 norms over a period $N_{m,\rho}(\ell)$. If $m = \rho = 1$ then

$$\begin{aligned}
\frac{1}{2}N_{1,1}(\ell) &:= \int_{x_{v+}}^{x_{v-}} v(x)^2 dx = \int_{v-}^{v+} \frac{v^2 dv}{\sqrt{2\ell - \frac{2}{p}v^p + v^2}} = 2 \int_{v-}^1 \frac{v^2 dv}{\sqrt{2\ell - \frac{2}{p}v^p + v^2}} \\
&= \sqrt{2} \int_{\frac{1}{p} - \frac{1}{2}}^{\ell} \frac{1}{\sqrt{\ell - p}} \frac{g(p)}{g(p)^{p-2} - 1} dp
\end{aligned}$$

which is integrable and has a finite limit κ_p as $\ell \rightarrow 0-$. We thus infer

$$N_{m,\rho}(\ell) = \rho^{-\frac{2}{p-2}} m^{\frac{4}{p-2}-1} N_1(\rho^{\frac{1}{p-2}} m^{-\frac{2p}{p-2}} \ell) \sim_{\ell \rightarrow 0-} 2\kappa_p \rho^{-\frac{2}{p-2}} m^{\frac{4}{p-2}-1}. \quad (32)$$

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