



HAL
open science

Three-body interactions in Rabi-coupled Bose gases: a perturbative approach

S Tiengo, R Eid, Thomas Bourdel

► **To cite this version:**

S Tiengo, R Eid, Thomas Bourdel. Three-body interactions in Rabi-coupled Bose gases: a perturbative approach. *Physical Review A : Atomic, molecular, and optical physics [1990-2015]*, 2025, 111, pp.053319. <hal-05006058>

HAL Id: hal-05006058

<https://hal.science/hal-05006058v1>

Submitted on 26 Mar 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Three-body interactions in Rabi-coupled Bose gases: a perturbative approach

S. Tiengo, R. Eid, and T. Bourdel*

*Université Paris-Saclay, Institut d'Optique Graduate School,
CNRS, Laboratoire Charles Fabry, 91127 Palaiseau, France*

(Dated: March 25, 2025)

In ultracold atomic gases, radio-frequency coupling between two spin states can lead to atoms being in a stable coherent superposition of the two states (dressed states). When the two-body interactions (scattering lengths) are different between the two states, it offers the possibility not only to control the interatomic interaction but also to modify the equation of states with emerging three-body terms with different signs and scalings with the parameters. We derive these terms in a unified perturbation framework and show that they correspond to genuine three-body processes found at different orders in perturbation theory. Our work justifies the introduction of contact three-body interactions in dressed-state systems when the spin degree of freedom is integrated out.

I. INTRODUCTION

Ultracold quantum gases are model many-body systems that permit one to access a variety of physical phenomena, such as superfluidity or phase transitions [1]. A key simplification comes from the fact that they are dilute systems with a dominating two-body interaction that, in addition, can be modeled by a contact interaction [2]. In this context, the scattering length a is the only quantity that fully characterizes the interaction strength. Importantly, the scattering lengths can be modified through diffusion (Fano-Feshbach) resonances by tuning the magnetic field [3]. This degree of control and the versatility of the external potential allows for versatile and quantitative studies (or quantum simulations) of complex many-body problems such as the BEC-BCS transition for fermions [4–7], the Mott-transition [8], the Berenzinski-Kosterlitz-Thouless superfluid transition in 2D [9] or the generalized hydrodynamics in 1D [10] for bosons.

Physics can be made even richer when mixtures are used such that different two-body interactions may coexist [11]. For bosons, this has led to the observation of phase separation for repulsive interspecies interaction [12] and quantum droplets for attractive interspecies interaction [13, 14]. Coupling between two spin states (for example, using a radio-frequency field) adds another degree of freedom. Competition between spin interaction and coupling can lead to bistability [15] or ferromagnetism phenomena [16] in the spin degree of freedom. Coherent coupling also leads to an alternative way of controlling the two-body interaction [17]. Indeed, the effective interaction then varies with the spin composition of the dressed state, which can be controlled by the detuning of the coupling field. In this configuration, it was also shown that three-body interaction terms either positive or negative emerge in the equation of state and may play a dominant role in the dynamics of Bose-Einstein condensates. These terms were found to appear either from mean-field shift of the energy difference between the two spin states [18], or in complex calculations beyond the mean field in the limit of large coupling [19]. These experimental demonstrations of controllable three-body interactions without strong enhancement of losses in ultra-cold dilute gases came in a context where such interactions have been considered in numerous theoretical works [20–35].

In this paper, we perform perturbation theory as a function of the interaction strengths and up to the third order. We recover the three-body interaction terms for a Bose gas with coherent coupling between two spin states in a unified framework at different orders in perturbation. We first treat the problem of a homogeneous condensate in the dressed state and calculate the modification of its ground-state energy. This calculation directly links to previous results on the equation of states [18, 19]. Second, few-body scattering processes [28] are considered and calculated. Our results justify the introduction of a contact three-body interaction term in an effective Hamiltonian in the reduced Hilbert space, in which the spin excitations are no longer considered. Finally, a generalization of the calculations to one dimension (1D) is presented.

In this paper, we perform perturbation theory as a function of the interaction strengths and up to the third order. We recover the three-body interaction terms for a Bose gas with coherent coupling between two spin states in a unified framework at different orders in perturbation. We first treat the problem of a homogeneous condensate in the dressed state and calculate the modification of its ground-state energy. This calculation directly links to previous results on the equation of states [18, 19]. Second, few-body scattering processes [28] are considered and calculated. Our results justify the introduction of a contact three-body interaction term in an effective Hamiltonian in the reduced Hilbert space, in which the spin excitations are no longer considered. Finally, a generalization of the calculations to one dimension (1D) is presented.

II. HAMILTONIAN IN THE DRESSED-STATE BASIS

In the second quantization formalism, the single particle and the interaction Hamiltonians H_0 and H_{int} in a coupled two-component homogeneous Bose system of volume V read:

* thomas.bourdel@institutoptique.fr

$$H_0 = \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2m} + \frac{\hbar\delta}{2} \right) a_{\uparrow\mathbf{p}}^\dagger a_{\uparrow\mathbf{p}} + \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2m} - \frac{\hbar\delta}{2} \right) a_{\downarrow\mathbf{p}}^\dagger a_{\downarrow\mathbf{p}} + \sum_{\mathbf{p}} \frac{\hbar\Omega}{2} (a_{\uparrow\mathbf{p}}^\dagger a_{\downarrow\mathbf{p}} + a_{\downarrow\mathbf{p}}^\dagger a_{\uparrow\mathbf{p}}), \quad (1)$$

$$H_{\text{int}} = \frac{1}{2V} \sum_{\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4} (g_{\uparrow\uparrow} a_{\uparrow\mathbf{p}_3}^\dagger a_{\uparrow\mathbf{p}_4}^\dagger a_{\uparrow\mathbf{p}_1} a_{\uparrow\mathbf{p}_2} + g_{\downarrow\downarrow} a_{\downarrow\mathbf{p}_3}^\dagger a_{\downarrow\mathbf{p}_4}^\dagger a_{\downarrow\mathbf{p}_1} a_{\downarrow\mathbf{p}_2} + 2g_{\uparrow\downarrow} a_{\uparrow\mathbf{p}_3}^\dagger a_{\downarrow\mathbf{p}_4}^\dagger a_{\uparrow\mathbf{p}_1} a_{\downarrow\mathbf{p}_2}), \quad (2)$$

where m is the atomic mass, \hbar the reduced Planck constant, δ the coupling detuning, and Ω the Rabi frequency of the coupling between the two spin states. $a_{\sigma\mathbf{p}}^\dagger$ and $a_{\sigma\mathbf{p}}$ are the creation and annihilation operators of a particle with a certain spin σ in the $\{\uparrow, \downarrow\}$ basis and momentum \mathbf{p} . In the interaction Hamiltonian appear the three coupling constants $g_{\sigma\sigma'} = 4\pi\hbar^2 a_{\sigma\sigma'}/m$ for the $\sigma - \sigma'$ contact interactions with scattering lengths $a_{\sigma\sigma'}$.

In order to perform perturbation theory, we first diagonalize the single particle Hamiltonian H_0 by a rotation to the dressed-state basis $|+\rangle, |-\rangle$, thus introducing the related operators

$$\begin{aligned} a_{+\mathbf{p}} &= \cos\theta a_{\downarrow\mathbf{p}} + \sin\theta a_{\uparrow\mathbf{p}}, \\ a_{-\mathbf{p}} &= \sin\theta a_{\downarrow\mathbf{p}} - \cos\theta a_{\uparrow\mathbf{p}}, \end{aligned} \quad (3)$$

where θ is the mixing angle such that $\cotan(2\theta) = \delta/\Omega$. In this description, θ evolves from 0 to $\pi/2$ and is $\pi/4$ at resonance (that is, for $\delta = 0$). In this rotated basis H_0 reads:

$$\begin{aligned} H_0 &= \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} (a_{+\mathbf{p}}^\dagger a_{+\mathbf{p}} + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}) \\ &+ \sum_{\mathbf{p}} \frac{\hbar\tilde{\Omega}}{2} (a_{+\mathbf{p}}^\dagger a_{+\mathbf{p}} - a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}), \end{aligned} \quad (4)$$

where $\tilde{\Omega} = \sqrt{\Omega^2 + \delta^2}$. In the same basis, the interaction Hamiltonian [36] is

$$\begin{aligned} H_{\text{int}} &= \frac{1}{2V} \sum_{\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4} \\ &\left(g_1 a_{+\mathbf{p}_3}^\dagger a_{+\mathbf{p}_4}^\dagger a_{+\mathbf{p}_1} a_{+\mathbf{p}_2} \right. \\ &+ g_2 a_{-\mathbf{p}_3}^\dagger a_{-\mathbf{p}_4}^\dagger a_{-\mathbf{p}_1} a_{-\mathbf{p}_2} \\ &+ g_3 a_{+\mathbf{p}_3}^\dagger a_{-\mathbf{p}_4}^\dagger a_{+\mathbf{p}_1} a_{-\mathbf{p}_2} \\ &+ g_4 (a_{+\mathbf{p}_3}^\dagger a_{+\mathbf{p}_4}^\dagger a_{-\mathbf{p}_1} a_{-\mathbf{p}_2} + a_{-\mathbf{p}_3}^\dagger a_{-\mathbf{p}_4}^\dagger a_{+\mathbf{p}_1} a_{+\mathbf{p}_2}) \\ &+ g_5 (a_{+\mathbf{p}_3}^\dagger a_{+\mathbf{p}_4}^\dagger a_{+\mathbf{p}_1} a_{-\mathbf{p}_2} + a_{+\mathbf{p}_3}^\dagger a_{-\mathbf{p}_4}^\dagger a_{+\mathbf{p}_1} a_{+\mathbf{p}_2}) \\ &\left. + g_6 (a_{+\mathbf{p}_3}^\dagger a_{-\mathbf{p}_4}^\dagger a_{-\mathbf{p}_1} a_{-\mathbf{p}_2} + a_{-\mathbf{p}_3}^\dagger a_{-\mathbf{p}_4}^\dagger a_{+\mathbf{p}_1} a_{-\mathbf{p}_2}) \right). \end{aligned} \quad (5)$$

The first three terms of the Hamiltonian correspond to elastic collisions between two atoms in the dressed states, with, respectively, g_1, g_2, g_3 as the interaction strength coefficients. The last three terms with coupling constants g_3, g_4 and g_5 , instead describe scattering events

that do not preserve the particle dressed state (inelastic collisions).

The dependence of the coupling coefficients on the mixing angle (or equivalently on the detuning δ/Ω) is:

$$\begin{aligned} g_1 &= g_{\downarrow\downarrow} \cos^4\theta + g_{\uparrow\uparrow} \sin^4\theta + \frac{1}{2} g_{\uparrow\downarrow} \sin^2 2\theta \\ g_2 &= g_{\downarrow\downarrow} \sin^4\theta + g_{\uparrow\uparrow} \cos^4\theta + \frac{1}{2} g_{\uparrow\downarrow} \sin^2 2\theta \\ g_3 &= (g_{\downarrow\downarrow} + g_{\uparrow\uparrow}) \sin^2 2\theta + 2g_{\uparrow\downarrow} \cos^2 2\theta \\ g_4 &= \frac{1}{4} (g_{\downarrow\downarrow} + g_{\uparrow\uparrow} - 2g_{\uparrow\downarrow}) \sin^2 2\theta \\ g_5 &= \sin 2\theta (g_{\uparrow\uparrow} \sin^2\theta - g_{\downarrow\downarrow} \cos^2\theta + g_{\uparrow\downarrow} \cos 2\theta) \\ g_6 &= \sin 2\theta (g_{\uparrow\uparrow} \cos^2\theta - g_{\downarrow\downarrow} \sin^2\theta - g_{\uparrow\downarrow} \cos 2\theta). \end{aligned} \quad (6)$$

The precise description of higher-order interaction pro-

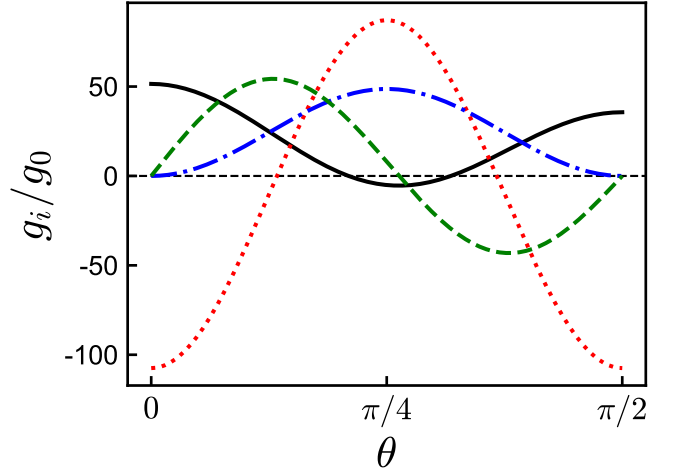


FIG. 1. g_2 (black solid line), g_3 (red dotted line), g_4 (blue dash-dotted line), g_6 (green dashed line) as a function of the mixing angle θ and normalized to $g_0 = 4\pi\hbar^2 a_0/m$, where a_0 is the atomic Bohr radius. $a_{\uparrow\uparrow} = 51.5 a_0$, $a_{\downarrow\downarrow} = 35.6 a_0$, $a_{\uparrow\downarrow} = -53.7 a_0$.

cesses is especially relevant in conditions where the usual two-body collisions is reduced thanks to the compensation between positive and repulsive bare coupling constants. In such a situation, higher-order interactions will play an important role. It occurs in particular when the intraspecies coupling constant $g_{\uparrow\uparrow}$ and $g_{\downarrow\downarrow}$ are positive whereas $g_{\uparrow\downarrow}$ is negative, conditions that can be met in potassium 39 [37]. As an example, the coupling constants g_2, g_3, g_4, g_6 are plotted for the second and third

lowest states of ^{39}K at a magnetic field of 55.6 G in Fig.1. If $\theta = 0$ or $\theta = \pi/2$ (i.e. $\delta/\Omega \rightarrow \pm\infty$), we find only elastic collisions and the original Hamiltonian in the non-rotated internal state basis. In the vicinity of the resonance $\delta \approx \pi/4$, $|-\rangle|-\rangle$ interactions are reduced as g_2 is close to zero. Note that $g_6 \propto \frac{\partial g_2}{\partial \theta}$ such that g_6 vanishes when g_2 is minimal.

III. PERTURBATIVE EXPANSION OF THE CONDENSATE ENERGY

We first perform perturbation theory in order to calculate the ground state energy corresponding to a Bose-Einstein condensate in state $|-, \mathbf{p} = \mathbf{0}\rangle$. The unperturbed ground state is $|\psi_0\rangle = \frac{1}{\sqrt{N!}}(a_{-, \mathbf{p}=\mathbf{0}}^\dagger)^N |0\rangle$, where N is the atom number. At the first order in perturbation theory, the correction to the energy is given by the expectation value of H_{int} in the unperturbed ground state

$$E^{(1)} = \langle \psi_0 | H_{\text{int}} | \psi_0 \rangle. \quad (7)$$

Among all the terms of H_{int} , only the term multiplied by the coupling constant g_2 allows two colliding atoms to remain in the dressed state $|-\rangle$. There is therefore a single term in the sum that contributes at the first order

$$\begin{aligned} E^{(1)} &= \frac{g_2}{2V} \langle \psi_0 | a_{-, \mathbf{p}=\mathbf{0}}^\dagger a_{-, \mathbf{p}=\mathbf{0}}^\dagger a_{-, \mathbf{p}=\mathbf{0}} a_{-, \mathbf{p}=\mathbf{0}} | \psi_0 \rangle \\ &= g_2 \frac{N(N-1)}{2V} \simeq g_2 \frac{N^2}{2V}. \end{aligned} \quad (8)$$

The first order correction (eq. 8) corresponds to the standard mean-field interaction energy with a coupling constant g_2 that varies with the detuning δ/Ω .

The energy correction at the second order is given by

$$E^{(2)} = \sum_i \frac{\langle \psi_0 | H_{\text{int}} | \psi_i \rangle \langle \psi_i | H_{\text{int}} | \psi_0 \rangle}{E_0 - E_i}, \quad (9)$$

where the sum is taken over all possible intermediate states $|\psi_i\rangle \neq |\psi_0\rangle$. We can recognize that the terms of H_{int} that contribute to the second order are those multiplied by the coupling constants g_2 , g_4 , and g_6 . At the second order the energy correction is given by

$$\begin{aligned} E^{(2)} &= -\frac{g_2^2 N^2}{2V^2} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{m}{\mathbf{p}^2} - \frac{g_4^2 N^2}{2V^2} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{2\hbar\tilde{\Omega} + \mathbf{p}^2/m} \\ &\quad - \frac{g_6^2 N^2}{4V^2} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{\hbar\tilde{\Omega} + \mathbf{p}^2/m} - \frac{g_6^2 N^3}{4V^2 \hbar\tilde{\Omega}} \end{aligned} \quad (10)$$

Note that we have omitted the terms that are not extensive ($\propto N$) in the thermodynamic limit (i.e. for large N and V , with constant density $n = N/V$). The last term originates from the term with all momenta equal to 0 and scales as N^3 because there are three a_{-0} or a_{-0}^\dagger in the corresponding Hamiltonian term. It corresponds to the negative three-body energy contribution calculated at the

mean field level, taking into account a density-dependent shift of the detuning [18].

The first three terms are divergent at large \mathbf{p} . This is a consequence of the unphysical nature of the contact potential in 3D, which needs to be regularized, for example, by adding a momentum cutoff in H_{int} . An alternative way is to renormalize the coupling constant, which is equivalent to removing the divergent parts in the sums [38]. After such a procedure, the first term disappears, whereas the two other terms are renormalized

$$E^{(2)} = \frac{(2\sqrt{2}g_4^2 + g_6^2)N^2 \sqrt{\hbar\tilde{\Omega}} m^{3/2}}{2V 8\pi\hbar^3} - \frac{g_6^2 N^3}{4V^2 \hbar\tilde{\Omega}}. \quad (11)$$

In the specific case, where g_2 is minimal and zero ($g_6 = 0$), we exactly recover the renormalization of the two-body interaction due to beyond-mean-field effects calculated from a Bogoliubov type approach in the limit of large coupling as compared to the interaction energy $\hbar\Omega \gg gN/V$ [19].

Let us now consider third-order perturbation theory. Generally, it reads:

$$E^{(3)} = \sum_i \sum_j \frac{\langle \psi_0 | H_{\text{int}} | \psi_i \rangle \langle \psi_i | H_{\text{int}} | \psi_j \rangle \langle \psi_j | H_{\text{int}} | \psi_0 \rangle}{(E_i - E_0)(E_j - E_0)}. \quad (12)$$

In principle, the sum includes terms with g_4 or g_6 . Since, if g_6 is non zero, there is already a three-body contribution at the second-order of perturbation, we focus on the case where $g_6 = 0$. In that case, the only non-zero terms are of the form

$$\begin{aligned} (g_4 a_{-0}^\dagger a_{-0}^\dagger a_{+\mathbf{p}_1} a_{+(-\mathbf{p}_1)}) (g_3 a_{+\mathbf{p}_2}^\dagger a_{-\mathbf{p}_2}^\dagger a_{+\mathbf{p}_2} a_{-\mathbf{p}_2}) \\ (g_4 a_{+\mathbf{p}_3}^\dagger a_{+(-\mathbf{p}_3)}^\dagger a_{-0} a_{-0}), \end{aligned} \quad (13)$$

where the $\mathbf{p} = 0$ are imposed by the fact that there is no atoms in $|-, \mathbf{p} \neq 0\rangle$ in $|\psi_0\rangle$.

The specific case of $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_3 = 0$ can be neglected in the thermodynamic limit as it scales as $(N/V)^3$. For $\mathbf{p}_1 \neq 0$, \mathbf{p}_2 and \mathbf{p}_3 have to take the same value up to a sign. There are thus 4 equivalent terms and

$$\begin{aligned} E^{(3)} &= \frac{g_4^2 g_3 N^3}{2V^3} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{(2\hbar\tilde{\Omega} + \mathbf{p}^2/m)^2} \\ E^{(3)} &= \frac{g_4^2 g_3 n^3}{16\pi \sqrt{2\hbar\tilde{\Omega}}} \frac{Vm^{3/2}}{\hbar^3}. \end{aligned} \quad (14)$$

Again, it is exactly the positive value found in the Bogoliubov approach in the limit of large coupling $\hbar\Omega \gg gn$ [19]. The perturbation theory permits us to find in a unified framework two-body and three-body interaction terms in the equation of state that were previously found in mean-field and beyond mean-field approaches. However, the perturbative expansion is less general than the Bogoliubov approach; it does not resolve the problem when $\hbar\Omega$ is not large compared to gn . In particular, perturbation theory cannot recover the nonanalytical Lee-Huang-Yang $N^{5/2}$ energy scaling for vanishing Ω [13].

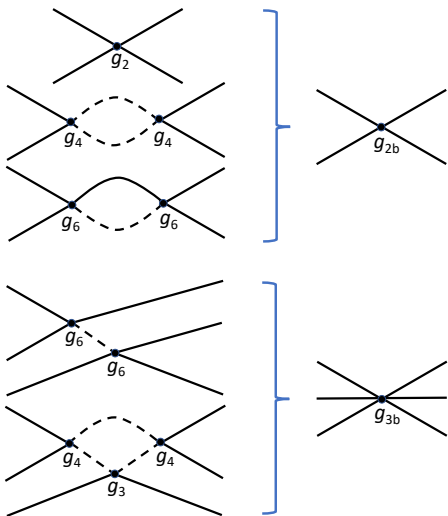


FIG. 2. Feynman diagrams corresponding to the possible scattering events. The plain and dashed lines represent the $|-\rangle$ and $|+\rangle$ state, respectively. The three first diagrams correspond to two-body scattering at first and second order in perturbation theory and contributes to g_{2b} in the effective interaction Hamiltonian. The two last diagrams correspond to three-body scattering at the second and third-order and contributes to g_{3b} in the effective interaction Hamiltonian.

IV. THREE-BODY SCATTERING IN PERTURBATION THEORY

The above calculation shows that the two- and three-body terms in the equation of state of a coherently dressed condensate can be found in perturbation theory. In this section, we show that they actually correspond to few-particle collisions, whose strength can be deduced from perturbative scattering theory. The corresponding Feynman diagrams are shown in Fig. 2.

Let us first consider two-body elastic collisions from the initial state $|\psi_i^{2b}\rangle = a_{-p_1}^\dagger a_{-p_2}^\dagger |0\rangle$ to the final state $|\psi_f^{2b}\rangle = a_{-p_3}^\dagger a_{-p_4}^\dagger |0\rangle$. Scattering theory tells that we should consider the matrix elements $\langle \psi_f^{2b} | T | \psi_i^{2b} \rangle$, where T is called the scattering matrix and can be expressed as

$$T = H_{\text{int}} + H_{\text{int}}G_0H_{\text{int}} + H_{\text{int}}G_0H_{\text{int}}G_0H_{\text{int}} + H_{\text{int}}G_0H_{\text{int}}G_0H_{\text{int}}G_0H_{\text{int}} + \dots, \quad (15)$$

$$\text{where } G_0 = \frac{1}{H_0 - E_0}, \quad (16)$$

a form that is suited in order to apply perturbation theory.

If we restrict to the first order (i.e. the first diagram in Fig. 2 or Born approximation), only the g_2 term of the Hamiltonian contributes, and the matrix element has the same non-zero value for any value of the momenta that ensure total momentum conservation. It corresponds to the usual two-body s -wave scattering amplitude that leads to an isotropic scattering halo. The second-order

terms include as before contributions originating from g_2 , g_4 , and g_6 , and the sums need to be regularized at high momenta with the usual renormalization of the coupling constant. Remarkably, for collisions at low momenta, i.e. such that $p^2/m \ll \hbar\Omega$, one can neglect the corresponding kinetic terms in the denominator such that the exact same type of terms as in the previous energy calculation appears. We thus find two additional terms up to the second-order in perturbation (corresponding to the second and third diagrams in Fig 2)

$$\langle \psi_f^{2b} | T | \psi_i^{2b} \rangle \approx 2g_2/V + \frac{(2\sqrt{2}g_4^2 + g_6^2)\sqrt{\hbar\Omega}}{4\pi V}. \quad (17)$$

This result can be simply interpreted as a modification of the scattering length due to the coupling. This correction can also be found from an exact calculation of two-body scattering in the presence of coupling (see supplemental material in [19])

We now consider three particles in the $|-\rangle$ dressed state with momenta p_1, p_2, p_3 and ask the question if they can elastically scatter to other momenta p'_1, p'_2, p'_3 . We thus consider $\langle -p'_1, -p'_2, -p'_3 | T | -p_1, -p_2, -p_3 \rangle$. For simplicity, we consider the case where all momenta are different. At the second order in perturbation, the only terms in H_{int} that allow us to change the three momenta are g_2^2 and g_6^2 because they need to annihilate two atoms in $|-\rangle$ and create at least one atom in $|-\rangle$. The g_2^2 term disappears with the renormalization of the coupling constant. The term $\propto g_6^2$ (fourth diagram in Fig. 2) first annihilates two $|-\rangle$ particles among the three, creates one in $|+\rangle$ and one in $|-\rangle$ with one of the final momenta, second annihilates the last initial $|-\rangle$ particle and the $|+\rangle$ particle, and creates the 2 other $|-\rangle$ particles with the final momenta. There are 9 possible intermediate states. If all momenta are small $p^2/m \ll \hbar\Omega$, the matrix element reduces to

$$\langle \psi_f^{3b} | H_{\text{int}}G_0H_{\text{int}} | \psi_i^{3b} \rangle = \frac{9g_6^2}{V^2\hbar\Omega} \quad (18)$$

with no momentum dependence.

At the third order in perturbation theory, there are several possible terms, including g_4 or g_6 . Similarly to before, we consider the case $g_6 = 0$, so that there is no three-body scattering at the second order. In that case, there is a single possible diagram (the fifth in Fig. 2). There remains one integration over a momentum which is not diverging and we find (after careful counting of the number of terms that contributes)

$$\langle \psi_f^{3b} | H_{\text{int}}G_0H_{\text{int}}G_0H_{\text{int}} | \psi_i^{3b} \rangle = \frac{9g_4^2g_3}{4\pi V^2\sqrt{2\hbar\Omega}}. \quad (19)$$

Interestingly, for symmetric conditions ($g_{\uparrow\uparrow} = g_{\downarrow\downarrow}$ and $\delta = 0$), the latter term can be previously found in a exact three-body calculation [28].

The above results show that two- and three-body interactions are also found in few-body scattering with the initial Hamiltonian. In the perturbative framework, the na-

ture of the three-body interactions is clear. They correspond to higher-order scattering with intermediate states that have a different spin.

Given the success of the perturbative expansion, it is tempting to continue the expansion to higher orders. However, we expect that the fourth order will not be accurate as corrections from short-range physics are then expected [20, 34]. Moreover, three-body recombination losses typically scale as a^4 [39] and also have to be considered at this level of perturbative expansion.

V. EFFECTIVE HAMILTONIAN

The above perturbative results justify an effective interaction Hamiltonian H_{eff} in an Hilbert space reduced to the dressed state $|-\rangle$ with contact three-body interaction

$$H_{\text{eff}} = \frac{g_{2b}}{2V} \sum_{\mathbf{p}'_1 + \mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2} a_{p'_2}^\dagger a_{p'_1}^\dagger a_{p_2} a_{p_1} \quad (20)$$

$$+ \frac{g_{3b}}{3V^2} \sum_{\mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3} a_{p'_3}^\dagger a_{p'_2}^\dagger a_{p'_1}^\dagger a_{p_3} a_{p_2} a_{p_1}, \quad (21)$$

where $g_{2b} = V \langle \psi_f^{2b} | T | \psi_i^{2b} \rangle / 2$ and $g_{3b} = V^2 \langle \psi_f^{3b} | T | \psi_i^{3b} \rangle / 12$. The numerical factors come from the number of permutations of the different momenta, and the corresponding two-body and three-body diagrams are shown on the right of Fig. 2. This effective Hamiltonian is constructed to give the same matrix elements as the ones found previously, thus leading to the same low-energy physics as the real system with the dressing field and possible virtual spin excitations. It also leads to previously calculated ground-state energy shifts, when applied to a condensate in the $|-\rangle$ state in the Born approximation. Higher-order terms originating from the effective Hamiltonian can be neglected at the level of accuracy of the initial third-order perturbation theory.

VI. GENERALIZATION TO 1D SYSTEMS

In an elongated trap, the confinement may be such that the radial excitations of the gas correspond to the highest energy scale of the problem. In the simple case of a radial harmonic trap with frequency $\omega_\perp / 2\pi$, it requires $g_{\sigma\sigma'} n \ll \hbar\omega_\perp$ as well as $\Omega \ll \omega_\perp$. In the absence of coppling ($\Omega = 0$), 1D quantum droplets have been predicted [40]. With coupling, 1D perturbation calculations can be done similarly to the 3D ones. The only difference is that the sums over the momenta are 1D rather than 3D. It is actually simpler because there is no divergence, and no renormalization of the coupling constants is necessary. The 1D coupling constants for a harmonic trap can be

approximated by $g_{\sigma\sigma'}^{1D} = 2\hbar\omega_\perp a_{\sigma\sigma'}$ in the limit $a_{\sigma\sigma'} \ll \sqrt{\hbar/m\omega_\perp}$ [41].

The three-body interaction arising at the second-order in perturbation (and also found at the mean-field level) is not modified. This is expected as the calculation involves only spin excitations with no momentum change. On the contrary, the renormalization of the two-body interaction and the emergence of a three-body interaction are modified. In the simple case where $g_6 = 0$:

$$E^{(2)} = - \frac{(g_4^{1D})^2 N^2 \sqrt{m}}{4L \sqrt{2\hbar\tilde{\Omega}\hbar}}, \quad (22)$$

$$E^{(3)} = \frac{(g_4^{1D})^2 g_3^{1D} N^3}{32\sqrt{2}L^2} \frac{1}{(\hbar\tilde{\Omega})^{3/2}} \frac{\sqrt{m}}{\hbar}, \quad (23)$$

where L is the size of the 1D system. The sign of the dominant two-body term is now negative, in contrast to the 3D case. This may be expected since for $\Omega = 0$, the beyond-mean-field correction is negative in 1D [40]. In addition, the scaling with Ω is modified as compared to the 3D case. These corrections can also be found in a beyond-mean-field approach in the presence of a coupling similarly to what was done in 3D [19].

VII. CONCLUSION

In this paper, we have performed a perturbative treatment of the interactions in a Bose dressed system with competing interactions. This has been done not only for the energy of a condensate but also for the two-body and three-body scattering problems. Our approach permits us to nicely explain the origin of the different kinds of three-body interactions in a unified framework at different orders in perturbation theory. It justifies the use of an effective Hamiltonian that includes a contact three-body interaction term when conditions for a valid perturbative expansion are met. The control of three-body interactions through radio-frequency-dressing appears as an important addition to the atomic gas toolbox. Precise calculations of the values of g_{2b} and g_{3b} for specific experimental conditions ($a_{\uparrow\uparrow}$, $a_{\uparrow\downarrow}$, $a_{\downarrow\downarrow}$, δ , Ω) are straightforward. Interestingly, losses and short-range physics corrections would appear only at higher-order order in the expansion and are thus expected to be negligible compared to the three-body interactions found in the present work.

For compensating two-body interactions (with scattering lengths of opposite signs between the different spins), the dressing permits one to reach situations where the two-body interactions are reduced, such that higher-order three-body interaction terms are sizable and may even dominate the equation of state. This is in contrast to single-component dilute Bose gases where three-body interactions are usually negligible and/or associated with losses when the scattering length is increased [34]. Previous experimental works on three-body interactions in dressed Bose gases have focused on the modification of

the equation of states [18, 19]. Future works may directly detect the specific collision halos that would originate from three-body collisions. This could be done by studying the collision between two condensates under conditions such that there are no two-body collisions ($g_{2b} = 0$) but only three-body collisions ($g_{3b} = 0$).

ACKNOWLEDGMENTS

This research was supported by CNRS, Ministère de l'Enseignement Supérieur et de la Recherche, Quantum Paris-Saclay, Région Ile-de-France in the framework of Domaine d'Intérêt Majeur Quantip (H3B), France 2030 in the framework of PEPR Quantique (ANR-23-PETQ-0001), the Simons Foundation (Award No. 563916, localization of waves), France and Chicago collaborating in the sciences.

-
- [1] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, *Rev. Mod. Phys.* **80**, 885 (2008).
- [2] C. W. M. Inguscio, S. Stringari, ed., *Bose-Einstein Condensation in Atomic Gases*, Proceedings of the International School of Physics "Enrico Fermi", Vol. 140 (IOS Press, Amsterdam, 1999).
- [3] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Feshbach resonances in ultracold gases, *Rev. Mod. Phys.* **82**, 1225 (2010).
- [4] C. A. Regal, M. Greiner, and D. S. Jin, Observation of resonance condensation of fermionic atom pairs, *Phys. Rev. Lett.* **92**, 040403 (2004).
- [5] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Crossover from a molecular Bose-Einstein condensate to a degenerate Fermi gas, *Phys. Rev. Lett.* **92**, 120401 (2004).
- [6] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Condensation of pairs of fermionic atoms near a Feshbach resonance, *Phys. Rev. Lett.* **92**, 120403 (2004).
- [7] T. Bourdel, L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, Experimental study of the BEC-BCS crossover region in lithium 6, *Phys. Rev. Lett.* **93**, 050401 (2004).
- [8] M. Greiner, O. Mandel, T. Esslinger, T. Hänsch, and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms, *Nature* **415**, 39 (2002).
- [9] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, Berezinskii-Kosterlitz-Thouless crossover in a trapped atomic gas, *Nature* **441**, 1118 (2006).
- [10] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, Generalized hydrodynamics on an atom chip, *Phys. Rev. Lett.* **122**, 090601 (2019).
- [11] C. W. M. Inguscio, S. Stringari, ed., *Quantum Mixtures with Ultra-cold Atoms*, Proceedings of the International School of Physics "Enrico Fermi", Vol. 211 (IOS Press, Amsterdam, 2025).
- [12] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Dynamics of component separation in a binary mixture of Bose-Einstein condensates, *Phys. Rev. Lett.* **81**, 1539 (1998).
- [13] D. S. Petrov, Quantum mechanical stabilization of a collapsing Bose-Bose mixture, *Phys. Rev. Lett.* **115**, 155302 (2015).
- [14] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Quantum liquid droplets in a mixture of Bose-Einstein condensates, *Science* **359**, 301 (2018).
- [15] T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler, Classical bifurcation at the transition from Rabi to Josephson dynamics, *Phys. Rev. Lett.* **105**, 204101 (2010).
- [16] R. Cominotti, C. Rogora, A. Zenesini, G. Lamporesi, and G. Ferrari, Ultracold atomic spin mixtures in ultrastable magnetic field environments, *Europhysics Letters* **146**, 45001 (2024).
- [17] J. Sanz, A. Frölian, C. S. Chisholm, C. R. Cabrera, and L. Tarruell, Interaction control and bright solitons in coherently coupled Bose-Einstein condensates, *Phys. Rev. Lett.* **128**, 013201 (2022).
- [18] A. Hammond, L. Lavoine, and T. Bourdel, Tunable three-body interactions in driven two-component Bose-Einstein condensates, *Phys. Rev. Lett.* **128**, 083401 (2022).
- [19] L. Lavoine and T. Bourdel, Beyond-mean-field crossover from one dimension to three dimensions in quantum droplets of binary mixtures, *Phys. Rev. A* **103**, 033312 (2021).
- [20] T. T. Wu, Ground state of a Bose system of hard spheres, *Phys. Rev.* **115**, 1390 (1959).
- [21] A. Gammal, T. Frederico, L. Tomio, and P. Chomaz, Atomic Bose-Einstein condensation with three-body interactions and collective excitations, *Journal of Physics B: Atomic, Molecular and Optical Physics* **33**, 4053 (2000).
- [22] A. Gammal, T. Frederico, L. Tomio, and F. Abdullaev, Stability analysis of the d-dimensional nonlinear Schrödinger equation with trap and two- and three-body interactions, *Physics Letters A* **267**, 305 (2000).
- [23] T. Köhler, Three-body problem in a dilute Bose-Einstein condensate, *Phys. Rev. Lett.* **89**, 210404 (2002).
- [24] A. Bulgac, Dilute quantum droplets, *Phys. Rev. Lett.* **89**, 050402 (2002).
- [25] S. Tan, Three-boson problem at low energy and implications for dilute Bose-Einstein condensates, *Phys. Rev. A* **78**, 013636 (2008).
- [26] V. Kumar, R. Radha1, and M. Wadati, Phase engineering and solitons of Bose-Einstein condensates with two- and three-body interactions, *J. Phys. Soc. Jpn* **79**, 074005 (2010).
- [27] A. F. M. Crosta, S. Trillo, The Whitham approach to dispersive shocks in systems with cubic-quintic nonlinearities, *New J. Phys.* **14**, 093019 (2012).
- [28] D. S. Petrov, Three-body interacting bosons in free space,

- Phys. Rev. Lett. **112**, 103201 (2014).
- [29] D. S. Petrov, Elastic multibody interactions on a lattice, Phys. Rev. A **90**, 021601 (2014).
- [30] K. Zlochastiev, Stability and metastability of trapless Bose-Einstein condensates and quantum liquids, Zeitschrift für Naturforschung A **72**, 677 (2017).
- [31] R. Killip, T. Oh, O. Pocovnicu, and M. Vişan, Solitons and scattering for the cubic–quintic nonlinear Schrödinger equation on \mathbb{R}^3 , Arch. Rational Mech. Anal **225**, 469 (2017).
- [32] J. D’Incao, Few-body physics in resonantly interacting ultracold quantum gases, J. Phys. B: At. Mol. Opt. Phys. **51**, 043001 (2018).
- [33] Y. Sekino and Y. Nishida, Quantum droplet of one-dimensional bosons with a three-body attraction, Phys. Rev. A **97**, 011602 (2018).
- [34] P. M. A. Mestrom, V. E. Colussi, T. Secker, and S. J. J. M. F. Kokkelmans, Scattering hypervolume for ultracold bosons from weak to strong interactions, Phys. Rev. A **100**, 050702 (2019).
- [35] W. Zwerger, Quantum-unbinding near a zero temperature liquid–gas transition, J. Stat. Mech. , 103104 (2019).
- [36] C. P. Search and P. R. Berman, Manipulating the speed of sound in a two-component Bose-Einstein condensate, Phys. Rev. A **63**, 043612 (2001).
- [37] C. D’Errico, M. Zaccanti, M. Fattori, G. Roati, M. Inguscio, G. Modugno, and A. Simoni, Feshbach resonances in ultracold 39 K, New Journal of Physics **9**, 223 (2007).
- [38] C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, 2008).
- [39] E. Braaten and H.-W. Hammer, Universality in few-body systems with large scattering length, Physics Reports **428**, 259 (2006).
- [40] D. S. Petrov and G. E. Astrakharchik, Ultradilute low-dimensional liquids, Phys. Rev. Lett. **117**, 100401 (2016).
- [41] M. Olshanii, Atomic scattering in the presence of an external confinement and a gas of impenetrable bosons, Phys. Rev. Lett. **81**, 938 (1998).