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# Open-loop control design for contraction in affine nonlinear systems

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**Abstract**—This paper presents two main contributions. First, it is shown that if a nonlinear time-varying system is contractive, then it is incrementally exponentially stable. Second, building on this result, a method is proposed, for designing feedforward inputs that induce contraction or incremental exponential stability in affine-in-control systems. The effectiveness of the approach is demonstrated through illustrative examples.

**Index Terms**—Affine-in-control systems, contractive systems, feedforward inputs, incremental exponential stability.

## I. INTRODUCTION

THE notions of contraction and incremental stability are concepts that aim to study the behavior of the trajectories of a given system toward each other, without needing an attractor such as an equilibrium point. These concepts were found useful and natural in studying many complex biological and technical systems [10], [15].

Contraction theory for the analysis of dynamical systems was first introduced in the work [4]. Then, a region of contraction is a domain of the state space, on which the symmetric part of the Jacobian matrix of the system is uniformly negative definite with respect to some Riemannian metric [13]. It has been shown that the trajectories that start and stay in a contraction region remain close to each other, relative to the distance between initial conditions, and converge exponentially towards each other.

Incremental stability notions, asymptotic and exponential, were formally introduced in the work [5], where sufficient Finsler Lyapunov conditions, which are milder than the uniform negativity property of the Jacobian, have been provided.

For instance, for a nonlinear system

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad (1)$$

where  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a class  $\mathcal{C}^1$  vector field, the contraction conditions aim at proving that the origin of the displacement dynamics (the linearized dynamics of (1) along its trajectories):

$$\dot{\delta x}(t) = J_f(t, x(t))\delta x(t), \quad (2)$$

where  $\delta x(t) \in \mathbb{R}^n$  is the displacement vector,  $J_f(t, x) := \frac{\partial f}{\partial x}(t, x)$  is the Jacobian of  $f$  in the state, and  $x(\cdot)$  is a solution of (1), is uniformly globally asymptotically/exponentially stable. Uniformity is understood in the sense that the solution upper estimate is independent of the trajectory along which the system (1) is linearized. In [5] and [13], it has been shown that the uniform negativity assumption on the Jacobian and the existence of an exponential Finsler Lyapunov function are both, under some smoothness condition on the vector field, sufficient conditions for the incremental exponential stability of the system (1).

The uniform negativity assumption on the Jacobian with respect to a Riemannian metric  $M(t, x)$  means that

$$(t, x, \delta x) \mapsto \delta x^\top M(t, x)\delta x \quad (3)$$

is a Lyapunov function for the system (2), where  $M(\cdot, \cdot)$  is a symmetric uniformly positive definite function [13], that is  $\exists m > 0$  such that  $M(t, x) \geq mI_n, \forall t \in \mathbb{R}, x \in \mathbb{R}^n$ . The Finsler Lyapunov condition (can also be called a contraction metric) generalizes the quadratic Lyapunov function (3) to a generic one [5]. Moreover, notice that the system (1) and the system:  $\dot{x} = f(t, x) + g(t)$ , where  $g : \mathbb{R} \rightarrow \mathbb{R}^n$  is any piecewise continuous function, have the same Jacobian.

It has been proven in [1] that, in the case of autonomous systems (1) with a uniformly bounded Jacobian, and with the system (2) admitting a uniformly globally exponentially stable origin, the system (1) is, in fact, incrementally exponentially stable. In this paper, we prove this property for a generic time-varying system without any boundedness condition on the Jacobian.

In terms of control synthesis, some recent works have addressed the problem of designing a feedback control that forces the systems to contract, using different approaches; forwarding [7], linear matrix inequalities [6], control metric [14], and backstepping [18]. In all these results, the existence of a metric  $M(t, x)$  satisfying the Demidovich condition:

$$\dot{M}(t, x) + J_f(t, x)^\top M(t, x) + M(t, x)J_f(t, x) \leq -\lambda I_n \quad (4)$$

for some positive real  $\lambda$ , all  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ , is required. The sufficiency of such a condition for contraction has been shown in early works [4], [13], but finding this metric is not trivial. Therefore, in the present study we aim at providing an open-loop control design method that is based on the analysis of the Jacobian directly.

One of the main motivations for such a choice is that open-loop control can be used to *entrain* several weakly coupled systems [16]. This phenomenon of entrainment is observed in biological oscillations modulated by external stimuli and underlies many physiological processes, including the maintenance of circadian rhythms, propagation of neural signals, and somitogenesis [8], [17]. This makes identifying conditions that ensure entrainment to be a central and long-standing question in mathematical and computational biology [9]. Motivated by this result, we address the problem of designing open-loop control laws that ensure contraction, and we propose a practical method for such a design. As explained above, incremental exponential stability can be analyzed either by directly estimating the distance between two solutions of (1), or by proving contraction via the uniform global exponential stability of the origin of (2) along arbitrary trajectories of (1). In this paper, using the second approach, we provide a sufficient condition for the existence of a feedforward input forcing a nonlinear system to be contractive/incrementally exponentially stable. We also provide a design method for such an input, and explain the intuition behind it.

#### Notation

- $\mathbb{R}$  is the set of real numbers, and  $\mathbb{R}_+^*$  is the set of strictly positive real numbers.
- $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^n$ .
- For a symmetric matrix  $P^\top = P \in \mathbb{R}^{n \times n}$ , the minimal and maximal eigenvalues are denoted  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$ , respectively.
- A function  $u : \mathbb{R} \rightarrow \mathbb{R}$  is said to be of class  $\mathcal{C}^p$  on a compact set  $S \subset \mathbb{R}$  if its restriction on  $S$  can be extended to a class  $\mathcal{C}^p$  function on an open neighborhood of  $S$ .
- For a set  $D \subset \mathbb{R}^n$ ,  $\text{int}(D)$  denotes its interior.
- For two symmetric matrices  $P, Q \in \mathbb{R}^{n \times n}$ , we denote by  $P \geq Q$  the property:  $x^\top P x \geq x^\top Q x$ ,  $\forall x \in \mathbb{R}^n$ .
- $I_n$  denotes the identity matrix of order  $n$ .

## II. PRELIMINARIES

Consider a nonlinear system (1) where  $f$  is of class  $\mathcal{C}^2$ . Since we are studying incremental stability, we need to further assume that the system (1) is forward complete, i.e.,  $\forall t_0 \in \mathbb{R}$ ,  $\forall x_{t_0} \in \mathbb{R}^n$ , a solution  $\phi_{t_0}(t, x_{t_0})$  of (1) exists for all  $t \geq t_0$ .

*Remark 1:* In the literature concerning incremental stability, see e.g. [5], the vector field  $f$  is often taken of class  $\mathcal{C}^2$ .

*Definition 1:* Given a number  $T > 0$ , we say that the vector field  $f$  is  $T$ -periodic if it holds that:

$$f(t+T, x) = f(t, x), \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^n. \quad (5)$$

*Definition 2:* The system (1) is contractive if there exist  $k, \lambda > 0$  such that for any solution  $x(\cdot)$  of (1), and any solution  $\delta x(\cdot)$  of the linearized dynamics (2):

$$|\delta x(t)| \leq k |\delta x(t_0)| e^{-\lambda(t-t_0)}, \quad \forall t_0 \in \mathbb{R}, \forall t \geq t_0. \quad (6)$$

Notice that for this definition, a contraction region is not specified, since global exponential stability of the origin of (2) can be satisfied even when there is no quadratic metric of the form (3) decreasing along solutions globally.

*Definition 3:* The system (1) is called incrementally exponentially stable (IES) if there exist  $k, \lambda > 0$  such that for any two solutions  $x_1(\cdot), x_2(\cdot)$  of (1) and  $\forall t_0 \in \mathbb{R}, \forall t \geq t_0$ :

$$|x_1(t) - x_2(t)| \leq k |x_1(t_0) - x_2(t_0)| e^{-\lambda(t-t_0)}. \quad (7)$$

## III. PROBLEM STATEMENT

This paper studies the incremental exponential stability and stabilization of a nonlinear time-varying system. First, we show that a contractive system is in fact IES. Second, we aim at designing a feedforward input that makes a nonlinear system contractive.

For these purposes we will consider a nonlinear affine in control system

$$\dot{x}(t) = f(t, x(t)) + u(t)G(t, x(t)), \quad (8)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $t \in \mathbb{R}$ ,  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $G : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $u : \mathbb{R} \rightarrow \mathbb{R}$  are of class  $\mathcal{C}^2$ . Define the linearized dynamics along a solution  $x(\cdot)$  of (8):

$$\dot{\delta x}(t) = (J_f(t, x(t)) + u(t)J_G(t, x(t)))\delta x(t), \quad (9)$$

where  $\delta x(t) \in \mathbb{R}^n$  is the displacement vector,  $J_f(\cdot, \cdot) := \frac{\partial f}{\partial x}(\cdot, \cdot)$  and  $J_G(\cdot, \cdot) := \frac{\partial G}{\partial x}(\cdot, \cdot)$  are the Jacobian matrices of  $f$  and  $G$  in the state  $x$ , respectively.

*Remark 2:* The reason for considering a nonlinear affine in control system instead of a generic nonlinear system of the form:

$$\dot{x}(t) = F(t, x(t), u(t))$$

where  $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is of class  $\mathcal{C}^2$ , is that we aim at controlling the linearized dynamics, therefore, we need an understanding of how the input  $u$  influences the Jacobian matrix of the vector field  $F$ .

Finally, we denote for  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^n$  the symmetric parts of these Jacobians as:

$$\begin{aligned} A(t, x) &= J_f(t, x)^\top + J_f(t, x), \\ R(t, x) &= J_G(t, x)^\top + J_G(t, x). \end{aligned} \quad (10)$$

## IV. MAIN RESULTS

### A. Connection between definitions 2 and 3

In the paper [1], the authors showed that for autonomous systems, if the Jacobian of the vector field is bounded then contraction implies IES. Our first result shows that this implication holds true for time-varying systems, even when the Jacobian is not assumed to be bounded:

*Theorem 1:* If the system (1) is contractive, then it is IES.

*Proof:* Let  $t_0 \in \mathbb{R}$  and  $x_0, x_1 \in \mathbb{R}^n$ . Denote by  $\Gamma(x_0, x_1)$  the set of  $\mathcal{C}^1$  curves  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  such that  $\gamma(0) = x_0$  and  $\gamma(1) = x_1$ . Since the square path length between  $x_0$  and  $x_1$  is minimized by the path  $\theta(s) := s x_0 + (1-s)x_1$ , then:

$$\inf_{\gamma \in \Gamma(x_0, x_1)} \int_0^1 \left| \frac{\partial \gamma}{\partial s}(s) \right|^2 ds = \int_0^1 \left| \frac{\partial \theta}{\partial s}(s) \right|^2 ds = |x_0 - x_1|^2. \quad (11)$$

From (11) we have  $\forall t \geq t_0$ :

$$|\phi_{t_0}(t, x_0) - \phi_{t_0}(t, x_1)|^2 = \inf_{\gamma \in \mathcal{S}(t, t_0, x_0, x_1)} \int_0^1 \left| \frac{\partial \gamma}{\partial s}(s) \right|^2 ds. \quad (12)$$

where  $\mathcal{S}(t, t_0, x_0, x_1) = \Gamma(\phi_{t_0}(t, x_0), \phi_{t_0}(t, x_1))$ . Fixing a  $t \geq t_0$ , the map  $s \in [0, 1] \mapsto \phi_{t_0}(t, \theta(s))$  is a  $\mathcal{C}^1$  curve in  $\mathcal{S}(t, t_0, x_0, x_1)$ , thus:

$$|\phi_{t_0}(t, x_0) - \phi_{t_0}(t, x_1)|^2 \leq \int_0^1 \left| \frac{\partial}{\partial s} \phi_{t_0}(t, \theta(s)) \right|^2 ds. \quad (13)$$

Since  $f$  is of class  $\mathcal{C}^2$ , then  $\phi_{t_0}(\cdot, \cdot)$  is also  $\mathcal{C}^2$  ([3], Theorem 4.1), as such, for a  $\mathcal{C}^2$  curve  $\gamma \in \Gamma(x_0, x_1)$ , the map  $(t, s) \in [t_0, +\infty) \times [0, 1] \mapsto \phi_{t_0}(t, \gamma(s))$  satisfies:

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial}{\partial s} \phi_{t_0}(t, \gamma(s)) &= \frac{\partial}{\partial s} \frac{\partial}{\partial t} \phi_{t_0}(t, \gamma(s)) \\ &= \frac{\partial}{\partial s} f(t, \phi_{t_0}(t, \gamma(s))) \\ &= J_f(t, \phi_{t_0}(t, \gamma(s))) \frac{\partial}{\partial s} \phi_{t_0}(t, \gamma(s)), \end{aligned} \quad (14)$$

therefore the map  $t \in [t_0, +\infty) \mapsto \frac{\partial}{\partial s} \phi_{t_0}(t, \gamma(s))$  is a solution of (2), consequently:

$$\left| \frac{\partial}{\partial s} \phi_{t_0}(t, \theta(s)) \right| \leq k \left| \frac{\partial}{\partial s} \phi_{t_0}(t_0, \theta(s)) \right| e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0, \quad (15)$$

since  $\phi_{t_0}(t_0, \theta(s)) = \theta(s) \forall s \in [0, 1]$ , then:

$$\left| \frac{\partial}{\partial s} \phi_{t_0}(t, \theta(s)) \right| \leq k \left| \frac{\partial \theta}{\partial s}(s) \right| e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0, \quad (16)$$

combining this with (13) we obtain  $\forall t \geq t_0$ :

$$\begin{aligned} |\phi_{t_0}(t, x_0) - \phi_{t_0}(t, x_1)| &\leq k \left( \int_0^1 \left| \frac{\partial \theta}{\partial s}(s) \right|^2 ds \right)^{\frac{1}{2}} e^{-\lambda(t-t_0)} \\ &= k |x_0 - x_1| e^{-\lambda(t-t_0)} \end{aligned} \quad (17)$$

thus, the system (1) is IES.  $\blacksquare$

The techniques in this proof have been inspired by the proofs of the main results of the papers [5], [11].

## B. Control design for contraction/incremental exponential stability

The second result concerns an open-loop control design for the contraction/incremental exponential stability of the system (8). Since our goal is to make the origin of the system (9) uniformly globally exponentially stable, and the dynamics of the squared norm of the system (9) is:

$$\frac{\partial}{\partial t} |\delta x(t)|^2 = \delta x(t)^\top (A(t, x(t)) + u(t)R(t, x(t))) \delta x(t), \quad (18)$$

then we need to find a sufficiently smooth input  $u(t)$  such that the matrix  $A(t, x) + u(t)R(t, x)$  is uniformly negative definite (uniform in state and time). However, this cannot be achieved in many cases. For instance, when  $A(t, x) > 0$  and  $R(t, x)$  is sign-indefinite for some values of arguments. In these cases, overshoots (i.e., an increase in the norm of  $\delta x$

and therefore increase in the distance between solutions of (8)) are inevitable independently in the choice of  $u$ . Fortunately, exhibiting overshoots does not imply lack of exponential total decay as long as the overshoots are compensated in other regions where  $A(t, x) + u(t)R(t, x) < 0$ . To handle these scenarios we need the following:

*Assumption 1:*  $\exists m > 0$  and a strictly increasing radially unbounded sequence of time instants  $(a_i)_{i \in \mathbb{Z}}$  such that:

$$[a_{2i}, a_{2i+1}] \subset \{t \in \mathbb{R} : \forall x \in \mathbb{R}^n : R(t, x) \geq mI_n\}, \quad (19)$$

or:

$$[a_{2i}, a_{2i+1}] \subset \{t \in \mathbb{R} : \forall x \in \mathbb{R}^n : R(t, x) \leq -mI_n\}, \quad (20)$$

and there exist a positive real  $M > 0$ ,  $\varepsilon : \mathbb{R} \rightarrow \mathbb{R}_+^*$  such that:

$$A(t, x) \leq MI_n, \quad \forall t \in D^1, x \in \mathbb{R}^n, \quad (21)$$

$$A(t, x) \leq \varepsilon(t)I_n, \quad \forall t \in D^2, x \in \mathbb{R}^n, \quad (22)$$

where  $D^1 = \cup_{i \in \mathbb{Z}} [a_{2i+1}, a_{2i+2}]$ , and  $D^2 = \cup_{i \in \mathbb{Z}} [a_{2i}, a_{2i+1}]$ . And there exist positive reals  $k, L > 0$  such that  $\forall i \in \mathbb{Z}$ :

$$a_{2i+1} - a_{2i} \geq k, \quad (23)$$

$$a_{2i+2} - a_{2i+1} \leq L. \quad (24)$$

This assumption captures the idea that the system's Jacobian does not exhibit undesirable behavior indefinitely. Specifically, any time interval  $[a_{2i+1}, a_{2i+2}]$ , during which the Jacobian cannot be rendered negative definite by the input  $u$ , is bounded in length due to (24). As a result, although the norm of the displacement vector  $|\delta x(t)|$  may fail to decrease—and can even grow—over such time intervals, it remains possible to compensate for these temporary overshoots in subsequent time segments. Inequality (22) indicates that, with an appropriate choice of input, the term  $u(t)R(t, x)$  can be used to counteract the effect of  $A(t, x)$ . Moreover, inequality (21) ensures that even within intervals of time where overshoots are unavoidable, these overshoots remain bounded and can thus be compensated for by achieving sufficiently strong convergence in the time intervals where  $R(t, x)$  is sign-definite. Finally, inequality (23) quantifies the minimum rate of monotonous convergence attained within each time interval where  $R(t, x)$  maintains a definite sign.

*Remark 3:* This assumption does not take into account whether  $A(t, x)$  is uniformly negative (uniform in the state) for some instants of time. Intuitively, one can see that in such a case, a small  $u$  for those times can work. Moreover, we can require the fulfillment of all above conditions only for the time instants when  $A(t, x) > -\alpha I_n$  for some  $\alpha > 0$  defining the decay rate, and this requirement is omitted for brevity.

*Remark 4:* Assumption 1 is restrictive since it tabulates the properties of  $R(t, x)$  for all  $x \in \mathbb{R}^n$ . A more general and natural setting would allow  $R(t, x)$  to be sign-definite only in certain regions of the state space, provided that solutions can be steered into those regions. This idea underpins the whole intuition of Riemannian metrics in contraction analysis, however, constructing such a metric often requires a deep understanding of the system's behavior throughout the state

space (see [2] for a scalar example whose Riemannian metric is rather complex).

**Theorem 2:** Under Assumption 1, there exists a class  $\mathcal{C}^2$  function  $u : \mathbb{R} \rightarrow \mathbb{R}$  that forces the system (8) to be contractive/IES.

*Proof:* Denote  $D_{2i} = [a_{2i}, a_{2i+1}]$  the intervals where  $R(t, x)$  is sign-definite satisfying either (19) or (20),  $D_{2i+1} = [a_{2i+1}, a_{2i+2}]$ , and  $k_{2i} = \max\{\varepsilon(t) : t \in D_{2i}\}$  for  $i \in \mathbb{Z}$ . Let  $\alpha > 0$ ,  $c > \alpha + \frac{e^{(M+\alpha)L}}{k}$ .

For the design, we distinguish two cases.

**1<sup>st</sup> case:**  $\exists t \in D_{2i+1}$ ,  $\exists x \in \mathbb{R}^n$  such that  $\lambda_{\min}(R(t, x)) = 0$  or  $\lambda_{\max}(R(t, x)) = 0$ .

If  $R(t, x) \geq mI_n$  on  $D_{2i}$ , then we denote:

$$T_1^{2i+1} := \min\{t \in D_{2i+1} : \exists x \in \mathbb{R}^n : \lambda_{\min}(R(t, x)) = 0\}, \quad (25)$$

if  $R(t, x) \leq -mI_n$  on  $D_{2i}$ , then:

$$T_1^{2i+1} := \min\{t \in D_{2i+1} : \exists x \in \mathbb{R}^n : \lambda_{\max}(R(t, x)) = 0\}, \quad (26)$$

if  $R(t, x) \geq mI_n$  on  $D_{2i+2}$ , then:

$$T_2^{2i+1} := \max\{t \in D_{2i+1} : \exists x \in \mathbb{R}^n : \lambda_{\min}(R(t, x)) = 0\}, \quad (27)$$

if  $R(t, x) \leq -mI_n$  on  $D_{2i+2}$ , then:

$$T_2^{2i+1} := \max\{t \in D_{2i+1} : \exists x \in \mathbb{R}^n : \lambda_{\max}(R(t, x)) = 0\}. \quad (28)$$

These bounds exist due to the continuity of  $R(\cdot, \cdot)$ .

We define  $u(\cdot)$  on each  $D_{2i+1}$  as follows:

- on  $[a_{2i+1}, T_1^{2i+1}]$ ,  $u$  is any class  $\mathcal{C}^2$  function satisfying  $u(T_1^{2i+1}) = 0$ ,  $u'(T_1^{2i+1}) = 0$ ,  $u''(T_1^{2i+1}) = 0$ , and in the case of  $R(t, x) \geq mI_n$  on  $D_{2i}$ :

$$\begin{cases} u(a_{2i+1}) = -c_{2i+1}m, \\ -c_{2i+1}m \leq u(t) \leq 0, \quad \forall t \in [a_{2i+1}, T_1^{2i+1}], \end{cases} \quad (29)$$

in the case of  $R(t, x) \leq -mI_n$  on  $D_{2i}$ :

$$\begin{cases} u(a_{2i+1}) = c_{2i+1}m, \\ 0 \leq u(t) \leq c_{2i+1}m, \quad \forall t \in [a_{2i+1}, T_1^{2i+1}], \end{cases} \quad (30)$$

where  $c_{2i+1} > \max\left(\frac{c+k_{2i}}{m^2}, \frac{c+k_{2i+2}}{m^2}\right)$  has to be large to guarantee that  $u$  satisfies inequality (33), which will subsequently allow for a good design on  $D_{2i+2}$ .

A function  $u$  satisfying these properties clearly exists, since  $R(t, x)$  is sign-definite  $\forall t \in (a_{2i+1}, T_1^{2i+1})$  and all  $x \in \mathbb{R}^n$ .

- on  $[T_1^{2i+1}, T_2^{2i+1}]$ ,  $u(t) = 0$ .
- on  $[T_2^{2i+1}, a_{2i+2}]$ ,  $u$  is any class  $\mathcal{C}^2$  function satisfying  $u(T_2^{2i+1}) = 0$ ,  $u'(T_2^{2i+1}) = 0$ ,  $u''(T_2^{2i+1}) = 0$ , and in the case of  $R(t, x) \geq mI_n$  on  $D_{2i+2}$ :

$$\begin{cases} u(a_{2i+2}) = -c_{2i+1}m, \\ -c_{2i+1}m \leq u(t) \leq 0, \quad \forall t \in [T_2^{2i+1}, a_{2i+2}], \end{cases} \quad (31)$$

in the case of  $R(t, x) \leq -mI_n$  on  $D_{2i+2}$ :

$$\begin{cases} u(a_{2i+2}) = c_{2i+1}m, \\ 0 \leq u(t) \leq c_{2i+1}m, \quad \forall t \in [T_2^{2i+1}, a_{2i+2}]. \end{cases} \quad (32)$$

**2<sup>nd</sup> case:**  $\forall t \in D_{2i+1}$ ,  $\forall x \in \mathbb{R}^n$ ,  $R(t, x)$  is sign-definite. Since  $R(\cdot, \cdot)$  is continuous, either  $\lambda_{\min}(R(t, x)) > 0$ ,  $\forall t \in D_{2i+1}$ ,  $x \in \mathbb{R}^n$ , or  $\lambda_{\max}(R(t, x)) < 0$ ,  $\forall t \in D_{2i+1}$ ,  $x \in \mathbb{R}^n$  is satisfied. We only treat the first sub-case, the second follows the same reasoning.

The first sub-case can be true only if  $R(t, x) \geq mI_n$ ,  $\forall t \in D_{2i} \cup D_{2i+2}$ ,  $\forall x \in \mathbb{R}^n$ , therefore, we define  $u$  on  $D_{2i+1}$  as any  $\mathcal{C}^2$  function satisfying:  $u(a_{2i+1}) = u(a_{2i+2}) = -c_{2i+1}m$  and  $u(t) \leq 0$ ,  $\forall t \in D_{2i+1}$ .

Now let us define the control  $u$  on the intervals  $D_{2i}$ , which only needs to be a  $\mathcal{C}^2$  function with sufficiently large norm to counterbalance the effect of  $A(t, x)$ . Notice that  $\forall i \in \mathbb{Z}$ :

$$\min(|u(a_{2i+1})|, |u(a_{2i+2})|) > \max\left(\frac{c+k_{2i}}{m}, \frac{c+k_{2i+2}}{m}\right), \quad (33)$$

therefore, to complete  $u$  on  $D_{2i}$ , choose any  $\mathcal{C}^2$  function satisfying:

$$\begin{cases} u(t) \leq -\frac{c+k_{2i}}{m}, \quad \forall t \in D_{2i}, \text{ if } R(\cdot, \cdot) \geq mI_n \text{ on } D_{2i}, \\ u(t) \geq \frac{c+k_{2i}}{m}, \quad \forall t \in D_{2i}, \text{ if } R(\cdot, \cdot) \leq -mI_n \text{ on } D_{2i}, \end{cases} \quad (34)$$

an example of such a function would be a polynomial in parts. The obtained  $u$  is  $\mathcal{C}^2$  on  $\mathbb{R}$ , and satisfies:

$$\begin{cases} u(t)R(t, x) \leq 0, \quad \forall t \in D^1, \quad x \in \mathbb{R}^n \\ u(t) \leq -\frac{c+k_{2i}}{m}, \quad \forall t \in D_{2i}, \text{ if } R(\cdot, \cdot) \geq mI_n \text{ on } D_{2i}, \\ u(t) \geq \frac{c+k_{2i}}{m}, \quad \forall t \in D_{2i}, \text{ if } R(\cdot, \cdot) \leq -mI_n \text{ on } D_{2i}. \end{cases} \quad (35)$$

Let us prove that this  $u(\cdot)$  makes the system IES. Consider a Finsler Lyapunov candidate:  $V(t, \delta x) = g(t)\delta x^\top \delta x$ , where:

$$g(t) = \begin{cases} e^{-(M+\alpha)(t-a_{2i+1})}, & \forall t \in D_{2i+1} \\ \xi_{2i+2}(t-a_{2i+2}) + \zeta_{2i+2}, & \forall t \in D_{2i+2}. \end{cases} \quad (36)$$

for each  $i \in \mathbb{Z}$ , where  $\xi_{2i+2} := \frac{1-\zeta_{2i+2}}{a_{2i+3}-a_{2i+2}} \leq \frac{1}{k}$  and  $\zeta_{2i+2} := e^{-(M+\alpha)(a_{2i+2}-a_{2i+1})}$ .

We have:  $e^{-(M+\alpha)L} \leq g(t) \leq 1$ ,  $\forall t \in \mathbb{R}$ , therefore,  $\forall t \in \mathbb{R}$ ,  $\forall \delta x \in \mathbb{R}^n$ :

$$e^{-(M+\alpha)L}|\delta x|^2 \leq V(t, \delta x) \leq |\delta x|^2 \quad (37)$$

and, denoting  $g'(t) := \limsup_{s \rightarrow t^+} \frac{g(s)-g(t)}{s-t}$ , we have:

$$\begin{cases} g'(t) \leq (-M-\alpha)g(t), \quad \forall t \in \text{int}(D^1), \\ g'(t) \leq \frac{1}{k}, \quad \forall t \in D^2. \end{cases} \quad (38)$$

The derivative of  $V$  along a trajectory  $\delta x(\cdot)$  of system (9) is:

$$\begin{aligned} \dot{V}(t, \delta x(t)) &:= \limsup_{s \rightarrow t^+} \frac{V(s, \delta x(s)) - V(t, \delta x(t))}{s-t} \\ &= g'(t)\delta x(t)^\top \delta x(t) + g(t)\delta x(t)^\top (A(t, x) \\ &\quad + u(t)R(t))\delta x(t). \end{aligned} \quad (39)$$

We have  $\forall t \in D_{2i}$ :

- If  $R(t, x) \geq mI_n$  on  $D_{2i}$ :

$$u(t)R(t) \leq mu(t)I_n \leq (-c - k_{2i})I_n, \quad (40)$$

- If  $R(t, x) \leq -mI_n$  on  $D_{2i}$ :

$$u(t)R(t) \leq -mu(t)I_n \leq (-c - k_{2i})I_n \quad (41)$$

and:

$$A(t, x) \leq k_{2i}, \quad \forall t \in D_{2i}. \quad (42)$$

Therefore,  $\forall t \in \text{int}(D^1)$ :

$$\dot{V}(t, \delta x(t)) \leq (-M - \alpha)V(t, \delta x(t)) + MV(t, \delta x(t)), \quad (43)$$

and  $\forall t \in D^2$ :

$$\dot{V}(t, \delta x(t)) \leq \frac{e^{(M+\alpha)L}}{k}V(t, \delta x(t)) - cV(t, \delta x(t)). \quad (44)$$

We conclude that for any solution  $\delta x(\cdot)$  of (9), and  $\forall t \in \mathbb{R}$ :

$$\dot{V}(t, \delta x(t)) \leq -\alpha V(t, \delta x(t)). \quad (45)$$

Combining (37) and (45), we conclude from Theorem 4.10 of [12] that the system (8) is contractive and therefore IES by Theorem 1.  $\blacksquare$

*Remark 5:* Notice that the rate  $\alpha$  can be chosen arbitrarily. That is, a system satisfying Assumption 1 can be forced to be contractive with any desired rate of exponential convergence.

*Remark 6:* To verify whether a system satisfies Assumption 1, it suffices to analyze the maximum and minimum eigenvalues of the matrices  $R(t, x)$  and  $A(t, x)$ .

*Remark 7:* In [16], the authors show that if the vector field is  $T$ -periodic and the Jacobian is uniformly negative definite with respect to some matrix norm, then the system is contractive. This condition is sufficient for contraction/IES, as shown in [13] and Theorem 1 of [16]. Moreover, [2] provides an example demonstrating that infinitesimal contraction is not necessary for contraction.

When the vector fields in (8) are periodic, Assumption 1 can be significantly relaxed, and we obtain the following entrainment result:

*Corollary 1:* If  $f$  and  $G$  of system (8) are  $T$ -periodic, and there exist  $m > 0$  and a segment  $[a, b] \subset [0, T]$  such that  $R(t, x) \geq mI_n$ ,  $\forall t \in [a, b]$ ,  $x \in \mathbb{R}^n$  or  $R(t, x) \leq -mI_n$ ,  $\forall t \in [a, b]$ ,  $x \in \mathbb{R}^n$ , then there exists a  $T$ -periodic input  $u$  that forces the system (8) to be contractive/IES, and approaching a  $T$ -periodic solution, i.e.,  $\exists k, \lambda > 0$  such that  $\forall t_0 \in \mathbb{R}$ , there exists a unique  $T$ -periodic solution  $\eta : [t_0, \infty) \rightarrow \mathbb{R}^n$  of (8) such that for any solution  $x$  of (8) defined on  $[t_0, \infty)$ ,  $\forall t \geq t_0$ :

$$|x(t) - \eta(t)| \leq ke^{-\lambda(t-t_0)}|x(t_0) - \eta(t_0)|. \quad (46)$$

*Proof:* Since  $f$  and  $G$  are  $T$ -periodic, then  $A$  and  $R$  of the linearized dynamics (9) are also  $T$ -periodic, and therefore Assumption 1 is clearly satisfied. Furthermore, using the same design method as in the proof of Theorem 2, we determine a class  $C^2$  input  $u$  on  $[0, T]$ , such that the first-order derivatives of  $u$  at  $0^+$  and  $T^-$  are equal, as well as the second-order derivatives, so that we extend it to the whole  $\mathbb{R}$  to be a  $T$ -periodic  $C^2$  input by definition, that forces the system to be

contractive/IES. We conclude from the proof of Theorem 2 of [16], that the system is contractive/IES to a  $T$ -periodic solution.  $\blacksquare$

## V. EXAMPLES

### A. Illustration for Corollary 1

Consider a simple scalar nonlinear control system of the form (8):

$$\dot{x} = x + u(t) \sin(t)x + g(t) \quad (47)$$

where  $g : \mathbb{R} \rightarrow \mathbb{R}$  can be any  $C^2$  function. Then  $A(t, x) = 2$  and  $R(t, x) = 2 \sin(t)$  clearly satisfy Assumption 1. In such a case, take as an example of  $D_{2i}$  the following:  $|\sin(t)| \geq \frac{1}{2}$ ,  $\forall t \in [\frac{\pi}{6} + 2i\pi, \frac{5\pi}{6} + 2i\pi] \cup [\frac{7\pi}{6} + 2i\pi, \frac{11\pi}{6} + 2i\pi]$ ,  $\forall i \in \mathbb{Z}$ . Therefore, using the reasoning of the proof of Theorem 2, we choose the input  $u$  satisfying inequality (34) on  $\cup_{i \in \mathbb{Z}} [\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi] \cup [\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi]$  and with a different sign to  $\sin(\cdot)$  on its complement. Notice that in the design of a suitable  $u$ , we ignore  $g$ , thus, once  $u$  is found, we can choose  $g$  in any way we want, which in turn forces a sort of entrainment [16]: trajectories will follow a certain shape that depends on  $u$  and  $g$ .

Performing simulations, Fig. 1 illustrates the behavior of this system for  $u(t) = -3 \sin(t)$  and  $g(t) = 0$ . Although there are overshoots, they are compensated enough in the other regions to attain contraction/incremental exponential stability.

Once a suitable  $u$  is designed, one can choose any  $g$ , and the incremental exponential stability continues to hold, but the trajectory to which all solutions converge obviously changes, as shown in Fig. 2.

### B. Conservativeness of conditions

Although Assumption 1 is very natural, it is clearly not generic. Among the excluded cases, some still allow the reasoning of the proof of Theorem 2 to remain applicable, while others render that reasoning ineffective.

For the first case, consider another simple nonlinear control system of the form (8):

$$\begin{cases} \dot{z} = z - \frac{z^3}{3} - y + u(t)z \\ \dot{y} = -y + z, \end{cases} \quad (48)$$

with  $x = \begin{bmatrix} z \\ y \end{bmatrix}$ , then  $A(t, x) = 2 \begin{bmatrix} 1 - z^2 & 0 \\ 0 & -1 \end{bmatrix}$ , and  $R(t, x) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ . This example showcases a scenario that is not included in Assumption 1; the case where  $R(t, x)$  is always positive (or negative) semi-definite. If the eigenvalues of  $A(t, x)$  corresponding to the 0 eigenvalues of  $R(t, x)$  are always negative definite, then we can still apply the reasoning of the proof of Theorem 2. Noticing that  $1 - z^2 < 0$ ,  $\forall |z| > 1$ , and  $0 \leq 1 - z^2 \leq 1$ ,  $\forall |z| \leq 1$ , in order to force  $A(t, x) + u(t)R(t, x)$  to be negative definite, we use a simple constant input  $u(t) = -2$ .

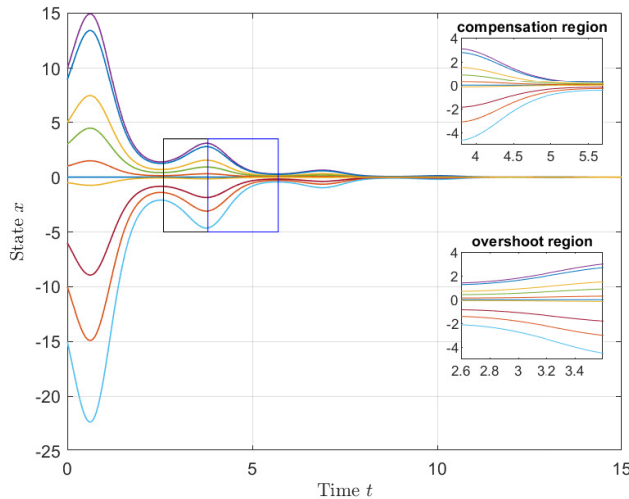


Fig. 1. Trajectories of the system (47) with  $u(t) := -3 \sin(t)$  and  $g(t) \equiv 0$ .

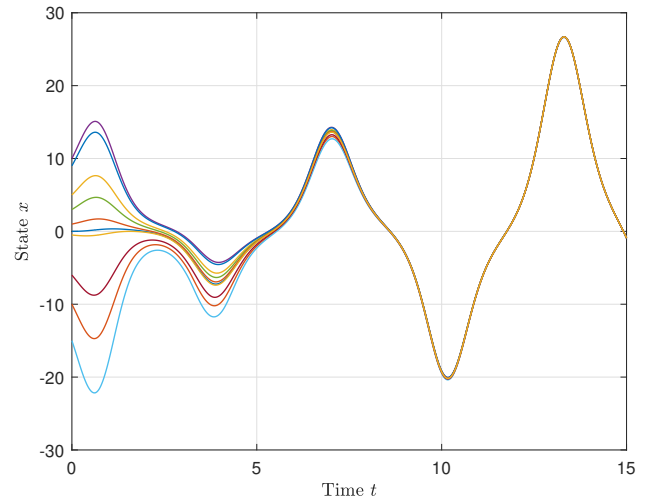


Fig. 2. Trajectories of the system (47) with  $u(t) := -3 \sin(t)$  and  $g(t) = t \cos(t)$ .

For the case where that reasoning is no longer applicable, consider a scalar system  $\dot{x} = x - \frac{x^3}{3} + u(t)$ . Since  $u$  does not appear in the Jacobian, the reasoning of the proof of Theorem 2 is no longer valid. However, it was shown in [2] that this model is IES. This suggests that, in certain cases, it may be more straightforward to design a controller directly for IES rather than relying on contraction-based methods.

## VI. CONCLUSION

This paper presents two main contributions. First, it establishes that for nonlinear time-varying systems, contraction implies incremental exponential stability. This naturally raises the question of whether the converse holds. Second, it introduces an open-loop control design method for achieving contraction and incremental exponential stability under mild assumptions. The control design method is practical since it relies solely on the analysis of explicit system data, mainly the function of maximum and minimum eigenvalues of  $R(t, x)$ . While the assumptions appear to be natural, they are not generic allowing for refinement in future research. For example, the condition on  $R(t, x)$  can be relaxed to account for the actual trajectory behavior and the specific regions of the state space, where  $R(t, x)$  is sign-definite, rather than requiring uniform definiteness across all states. The influence of the matrix  $A(t, x)$  can be additionally analyzed utilizing Finsler's lemma.

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