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# Multivariate Self-Exciting Processes with Dependencies

Caroline Hillairet\*      Thomas Peyrat†      Anthony Réveillac‡

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## Abstract

This paper introduces the class of multidimensional self-exciting processes with dependencies (MSPD), which is a unifying writing for a large class of processes: counting, loss, intensity, and also shifted processes. The framework takes into account dynamic dependencies between the frequency and the severity components of the risk, and therefore induces theoretical challenges in the computations of risk valuations. We present a general method for calculating different quantities related to these MSPDs, which combines the Poisson imbedding, the pseudo-chaotic expansion and Malliavin calculus. The methodology is illustrated for the computation of explicit general correlation formula.

**Keywords:** Multidimensional Hawkes Processes; Dynamic cross dependencies; Poisson imbedding; Malliavin calculus, Pseudo-chaotic expansion.

**Mathematics Subject Classification (2020):** 60G55; 60G57; 60H07.

## 1 Introduction

Risk analysis for credit or actuarial portfolios is usually based on the study of properties of the so-called cumulative loss process  $(L_t)$  over a period of time  $[0, T]$  where  $T > 0$  denotes the maturity of a contract or the time-horizon:

$$L_T = \sum_{i=1}^{N_T} Y_i, \quad T \geq 0.$$

$(N_t)_{t \geq 0}$  is a counting process that models the occurrence of the claims (as the defaults for a credit portfolio, or the losses for an insurance portfolio), while the random variables  $(Y_i)_i$  model the claims amounts. In the classic Cramer Lundberg model,  $(N_t)_{t \geq 0}$  is a Poisson process and  $(Y_i)$  is a family of positive *iid* random variables independent of  $(N_t)_{t \geq 0}$ . However, these assumptions prevent this model from being used on certain risks where contagion or dependency phenomena have been observed, as e.g. in credit risk (see [7, 15, 6] among others) or cyber risk (see [2, 3]). In contrast, Hawkes processes, introduced in the 1970s by A. Hawkes, turned out to be relevant for capturing excitation phenomena, and are now used

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in many fields (earthquake modeling, neuroscience, Limit order books in finance, credit risk, cyber risk, etc...). A Hawkes process  $(H_t)_{t \geq 0}$  is characterized by a stochastic intensity process  $(\lambda_t)_{t \geq 0}$  which is a deterministic functional of the past trajectory of the counting process itself, explicitly given as follows

$$\lambda_t = \mu + \int_0^t \Phi(t-s) dH_s, \quad t \geq 0$$

where  $\mu > 0$  is the baseline intensity and  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is the self-excitation kernel. In this standard linear Hawkes model, all claims have the same contagiousness pattern, which may not be usually the case. In particular, it would be interesting to modulate this contagion effect by the size of the claims, which seems to be quite natural for example in credit risk (a default with a large loss given default will have a higher probability to generate other cascading defaults) and also in cyber or health insurance. This motivates us to propose in this paper an extension of a Hawkes process in which the intensity (more precisely the excitation kernel  $\Phi$ ) is affected by the claims sizes, and thus introducing dependencies between the severity components (the  $(Y_i)$ ) and the frequency components (the counting process). This process can be seen as a system of weakly SDEs with respect to a Poisson measure  $N(dt, d\theta, dy)$  on  $(\mathbb{R}^+)^3$ . This representation is known as Poisson imbedding, with an extra dimension (here  $dy$ ) to take into account the impact of the claims sizes  $Y_i$  :

$$\begin{cases} \lambda_T := \mu + \int_{(0,T] \times \mathbb{R}_+^2} \varphi(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t\}} N(dt, d\theta, dy), \\ H_T = \int_{(0,T] \times \mathbb{R}_+^2} \mathbf{1}_{\{\theta \leq \lambda_t\}} N(dt, d\theta, dy), \\ L_T = \int_{(0,T] \times \mathbb{R}_+^2} y \mathbf{1}_{\{\theta \leq \lambda_t\}} N(dt, d\theta, dy) \quad T \geq 0. \end{cases}$$

Such model has been first introduced in Khabou [12] in the case  $\varphi(T-t, y) = \Phi(T-t)b(y)$ . We study here a multivariate version to better capture the different components of the risk and their cross dependencies. Thus, the purpose of this article is to present theoretical results on a general  $d$ -dimensional stochastic process called *Multivariate Self-Exciting Processes with Dependencies (MSPD)*  $\mathbf{Z}$ , which is a unifying writing for a large class of processes: counting, loss, intensity, and also shifted processes that will be defined hereafter. It is useful for several applications in finance and insurance, taking into account dynamic dependencies between the frequency and the severity components of the risk. A MSPD  $\mathbf{Z}$  is defined through the imbedding procedure: for each component  $\mathbf{Z}_T^i$ ,  $i = 1, \dots, d$

$$\begin{cases} \mathbf{Z}_T^i = \int_{\{1, \dots, d\} \times (0, T] \times \mathbb{R}_+^2} \zeta^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \\ \lambda_T^i := \boldsymbol{\mu}^i(t) + \int_{\{1, \dots, d\} \times (0, T] \times \mathbb{R}_+^2} \varphi^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \end{cases} \quad T \geq 0 \quad (1.1)$$

where  $\zeta$  and  $\varphi$  are two  $d \times d$  matrices : for  $1 \leq i, k \leq d$ ,  $\varphi^{i,k} : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  (impact of the  $k^{\text{th}}$ -dimension on the  $i^{\text{th}}$ -dimension) and  $\boldsymbol{\mu}^i : [0, T] \rightarrow \mathbb{R}_+$  are deterministic baseline intensities. A formal definition of this process will be given in Definition 2.9, and if  $\zeta^{i,k}(t, y) = y \mathbf{1}_{\{i=k\}}$  (resp.  $\zeta^{i,k}(t, y) = \mathbf{1}_{\{i=k\}}$ ), the process  $\mathbf{Z}$  coincides with the Loss process  $L$  (resp. the counting process  $H$ ).

Although a Multivariate Self-Exciting Processes with Dependencies (MSPD) models more accurately the risk, the loss of the independency assumptions, compared to the tractable Cramer Lundberg model, induces theoretical challenges in the computations of risk valuations. Therefore our objective is to present a general method for calculating different quantities useful in risk assessment. The methodology relies on the Poisson imbedding and Malliavin calculus, in the spirit of [10], which enables us to transform the computation of some expectations as follows (known as Mecke formula):

$$\mathbb{E} \left[ F \int_{\mathbf{X}} h d\mathbf{N} \right] = \int_{\mathbf{X}} h(\mathbf{x}) \mathbb{E} [F \circ \varepsilon_{\mathbf{x}}^+] \mathbf{m}(d\mathbf{x}). \quad (1.2)$$

Here  $\mathbf{x} = (k, t, \theta, y) \in \mathbf{X}$  and  $\mathbf{m}$  is the intensity measure of  $\mathbf{N}$ , and the notation  $F \circ \varepsilon_{\mathbf{x}}^+$  denotes the functional on the Poisson space where a deterministic jump is added to the paths of  $\mathbf{N}$  at time  $t$ . This expression turns out to be particularly interesting from an actuarial point of view since adding a jump at some time  $t$  corresponds to computing a stress scenario by adding artificially a claim at time  $t$ . Such processes  $F \circ \varepsilon_{\mathbf{x}}^+$  will be called *shifted processes* in the sequel. They are related to the vertical derivatives in the functional Itô calculus of Cont and Fournié in [4]. In particular, our methodology uses the pseudo-chaotic expansion for a counting process, which has been recently developed in [9].

The paper is organized as follows. Section 2 provides a description of the Poisson space and introduces the MSPD processes. Section 3 recalls some elements of Malliavin calculus and the Mecke formula. Section 4 develops the pseudo-chaotic expansion for MSPD process. As an illustration, Section 5 demonstrates how the methodology can be applied to compute the expectation and correlation of MSPD. Section 6 concludes the paper by outlining potential directions for future work and development.

## 2 Multivariate Self-Exciting Processes with Dependencies

This section introduces the theoretical framework of the Poisson measure and defines the main mathematical objects of the paper.

### 2.1 Configuration space and the Poisson measure

We first introduce the configuration space and the Poisson measure, taking the main elements from [13] and [14, Chapter 6].

We set  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$  the set of positive integers. We fix  $\mathbb{X} := \mathbb{R}_+^3$  equipped with the Borelian  $\sigma$ -field and we make use of the notation  $x := (t, \theta, y) \in \mathbb{X}$ . In this paper we fix  $\mathbf{X} := \{1, \dots, d\} \times \mathbb{R}_+^3$  equipped with the Borelian  $\sigma$ -field  $\mathcal{X}$ . Thus, elements of  $\mathbf{X}$  will be written as  $\mathbf{x} := (k, x); k \in \llbracket 1, d \rrbracket; x = (t, \theta, y) \in \mathbb{X}$ .  $\mathbf{m}$  the  $\sigma$ -finite measure on  $(\mathbf{X}, \mathcal{X})$  is defined as follows: for  $f : \mathbf{X} \rightarrow \mathbb{R}_+$  a measurable and bounded function

$$\int_{\mathbf{X}} f(\mathbf{x}) \mathbf{m}(d\mathbf{x}) := \sum_{k=1}^d \int_{\mathbb{X}} f(k, t, \theta, y) \nu^k(dy) d\theta dt$$

where for all  $1 \leq k \leq d$ ,  $\nu^k$  are independent probability measures on  $\mathbb{R}_+$  and  $dx$  denotes the Lebesgue measure on  $\mathbb{X}$ .

We define  $\Omega$  the space of configurations on  $\mathbf{X}$  as

$$\Omega := \left\{ \boldsymbol{\omega} = \sum_{j=1}^n \delta_{\mathbf{x}_j}, \mathbf{x}_j \in \mathbf{X}, j = 1, \dots, n, n \in \mathbb{N} \cup \{+\infty\} \right\}. \quad (2.1)$$

Let  $\mathbb{P}$  the Poisson measure on  $\Omega$  under which the canonical evaluation  $\mathbf{N}$  defines a Poisson random measure with intensity measure  $\mathbf{m}$ . To be more precise given any element  $\mathbf{A}$  in  $\mathcal{X}$  with  $\mathbf{m}(\mathbf{A}) > 0$ , the random variable

$$(\mathbf{N}(\boldsymbol{\omega}))(\mathbf{A}) := \boldsymbol{\omega}(\mathbf{A})$$

is a Poisson random variable with intensity  $\rho(\mathbf{A})$ . We denote by  $\mathcal{F}^{\mathbf{N}} := \sigma\{\mathbf{N}(\mathbf{A}); \mathbf{A} \in \mathcal{X}\}$  the  $\sigma$ -field generated by  $\mathbf{N}$ . We set  $\mathbb{F}^{\mathbf{N}} := (\mathcal{F}_t^{\mathbf{N}})_{t \geq 0}$  the natural history of  $\mathbf{N}$  given by  $\mathcal{F}_t^{\mathbf{N}} := \sigma\{\mathbf{N}(\mathcal{T} \times \mathbf{A}), \mathcal{T} \subset \mathcal{B}([0, t]), \mathbf{A} \in \mathcal{B}(\{1, \dots, d\} \times \mathbb{R}_+^2)\}$  where  $\mathcal{B}$  denotes the Borelian  $\sigma$ -field on the corresponding set. In the case of processes defined up to a fixed horizon  $T$ , we use the restrictions  $\mathbb{X}_T := [0, T] \times \mathbb{R}_+^2$  and  $\mathbf{X}_T := \{1, \dots, d\} \times \mathbb{X}_T$ .

## 2.2 Kernels and convolutions

This section introduces some notations, definitions and hypothesis on the matrices  $\boldsymbol{\zeta}$  and  $\boldsymbol{\varphi}$  involved in the Poisson imbedding procedure (1.1) for the construction of a Multivariate Self-Exciting Process with Dependencies (MSPD).

**Convention 2.1** (Matrices and scalars). In order to distinguish between scalars and matrices, we adopt the convention that matrices will be written in bold. If  $\mathbf{M}$  represents a matrix,  $\mathbf{M}^{i,k}$  corresponds to the element of  $\mathbf{M}$  positioned in the  $i$ -th row and  $k$ -th column and  $\mathbf{M}^{i\cdot}$  corresponds to  $i$ -th row of the matrix  $\mathbf{M}$ . If  $\mathbf{V}$  represents a vector, then  $\mathbf{V}^i$  will be the  $i$ -th component of the vector. Moreover we design by  $\mathbf{Id}_d$  the  $d$ -dimensional identity matrix and  $\mathcal{M}_{d,d}^+$  denotes the set of  $d \times d$  matrices with entries in  $\mathbb{R}_+$ .

**Definition 2.2** ( $d$ -kernel). Let  $d \in \mathbb{N}_+$  and  $(\boldsymbol{\nu}^k)_{1 \leq k \leq d}$  a family of probability densities on  $\mathbb{R}^+$ , then  $\boldsymbol{\zeta}$  is a  $d$ -kernel if  $\boldsymbol{\zeta}$  is a  $\mathcal{M}_{d,d}^+$ -valued map such that each component  $\zeta^{i,k}(t, y) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  satisfies  $\int_{\mathbb{R}_+} \zeta^{i,k}(t, y) \boldsymbol{\nu}^k(dy) < +\infty$ , for  $(t, i, k) \in \mathbb{R}_+ \times \llbracket 1, d \rrbracket^2$ .

**Definition 2.3** (Separable  $d$ -kernel). A  $d$ -kernel  $\boldsymbol{\zeta}$  is said to be separable if  $\forall (i, k) \in \llbracket 1, d \rrbracket^2$  there exists two functions  $\phi^{i,k}, b^{i,k} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\zeta^{i,k}(t, y) := \phi^{i,k}(t) b^{i,k}(y)$ . In this case, we denote  $\boldsymbol{\zeta}(t, y) = \boldsymbol{\Phi}(t) \star \mathbf{B}(y)$  where  $\star$  is the Hadamard (that is elementwise) product.

**Notation 2.4** ( $\boldsymbol{\nu}$ -mean of a  $d$ -kernel). Let  $\boldsymbol{\zeta}$  a  $d$ -kernel associated to the measure  $\boldsymbol{\nu} = (\boldsymbol{\nu}^k)_{1 \leq k \leq d}$ . We denote by  $\bar{\boldsymbol{\zeta}} : \mathbb{R}_+ \rightarrow \mathcal{M}_{d,d}^+$  the matrix in which each coefficient  $\bar{\zeta}^{i,k}$  is the mean of  $\zeta^{i,k}$  with respect to the measure  $\boldsymbol{\nu}^k$ . That is,

$$\bar{\boldsymbol{\zeta}}(t) := \left( \int_0^{+\infty} \zeta^{i,k}(t, y) \boldsymbol{\nu}^k(dy) \right)_{1 \leq i, k \leq d}. \quad (2.2)$$

**Definition 2.5** (Non-explosive  $d$ -kernel). A  $d$ -kernel  $\boldsymbol{\zeta}$  is said to be non-explosive if its spectral radius  $\mathcal{R}(\mathbf{K}) < 1$  and

$$\mathbf{K} := \int_0^{+\infty} \bar{\boldsymbol{\zeta}}(t) dt < +\infty.$$

As it is usual, the intensity of a counting process should be a predictable process. This requires to add a property on the kernel  $\varphi$  that defines the intensity  $\lambda$  in (1.1).

**Definition 2.6** (Self-excitation  $d$ -kernel). Let  $\varphi$  a non-explosive  $d$ -kernel.  $\varphi$  is said to be a self-excitation kernel if in addition, for all  $(i, k) \in \llbracket 1, d \rrbracket^2$ ,  $\varphi^{i,k}(0, y) := 0$ . By convention, self-excitation kernel will be noted  $\varphi$ , and will be used to define the intensity of a stochastic process.

Note that the condition  $\varphi^{i,k}(0, y) = 0$  implies that

$$\begin{aligned} & \int_{\{1, \dots, d\} \times (0, T] \times \mathbb{R}_+^2} \varphi^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \\ &= \int_{\{1, \dots, d\} \times (0, T) \times \mathbb{R}_+^2} \varphi^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy), \end{aligned}$$

which ensures the predictability of the intensity. In addition, we adopt the convention that for any  $\tau < 0$  we have  $\varphi(\tau, \cdot) := 0$ .

**Definition 2.7** (See *e.g.* [1]). Let  $\varphi$  a self-excitation  $d$ -kernel. Let define the sequence of iterated convolutions of  $\bar{\varphi}$  such that

$$\begin{cases} \bar{\varphi}_0 \text{ denotes the Dirac distribution in } 0, \\ \bar{\varphi}_1 := \bar{\varphi}, \quad \bar{\varphi}_n := \int_0^t \bar{\varphi}(t-s) \bar{\varphi}_{n-1}(s) ds, \quad t \in \mathbb{R}_+, n \in \mathbb{N}^*. \end{cases} \quad (2.3)$$

Since  $\|\bar{\Psi}\|_1 = (\mathbf{Id} - \mathbf{K})^{-1} - \mathbf{Id} = (\mathbf{Id} - \mathbf{K})^{-1} \mathbf{K}$ , the mapping  $\bar{\Psi}$ ,

$$\bar{\Psi} := \sum_{n=1}^{+\infty} \bar{\varphi}_n, \quad (2.4)$$

is well-defined as a limit in  $L_1(\mathbb{R}_+; dt)$ , and by definition,

$$\int_0^t \bar{\Psi}(t-s) \bar{\varphi}(s) ds = \bar{\Psi}(t) - \bar{\varphi}(t). \quad (2.5)$$

The intensity process  $\lambda$  is defined in (1.1) by an implicit equation. For sake of completeness, the following proposition details an iterative procedure for the construction of this process.

**Proposition 2.8** (Intensity process). *Consider a vector  $\mu := (\mu^i)_{1 \leq i \leq d}$  where each component  $\mu^i$  is a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and let  $\varphi$  a self-excitation  $d$ -kernel. Then the system of SDEs*

$$\lambda_T^i := \mu^i(T) + \int_{\mathbf{X}_T} \varphi^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy); \quad T \geq 0, \quad i = 1, \dots, d \quad (2.6)$$

*admits a unique  $\mathbb{F}^{\mathbb{N}}$ -predictable solution  $\lambda$ .*

*Proof.* The proof follows [5] which we extend to the multivariate case. The proof consists in iteratively constructing  $\lambda$ , starting from the deterministic baseline intensity  $\mu$  in which we add successively the excitation component each time the corresponding aggregated counting process  $\mathcal{L}$  jumps (the aggregation of jumps from each component). More precisely, for  $T \geq 0$ , let us initiate the procedure by considering a constant  $d$ -dimensional intensity and the corresponding cumulated one-dimensional counting process :

$${}^{(1)}\lambda_T := \mu(T) \quad \text{and} \quad {}^{(1)}\mathcal{L}_T := \int_{\mathbf{X}_T} \mathbf{1}_{\{\theta \leq (1)\lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy).$$

Then the sequence  $(^{(n)}\boldsymbol{\lambda}, ^{(n)}\mathcal{Z})_{n \in \mathbb{N}^*}$  is defined by induction as follows: for all  $i = 1, \dots, d$

$$\begin{aligned} ^{(n+1)}\boldsymbol{\lambda}_T^i &:= \boldsymbol{\mu}^i(T) + \int_{\mathbf{X}_T} \mathbf{1}_{\{t \leq \tau_n^{\mathcal{Z}}\}} \boldsymbol{\varphi}^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq ^{(n)}\boldsymbol{\lambda}_t^k\}} \mathbf{N}(dk, dt, d\theta, dy), \\ &\text{with } \tau_n^{\mathcal{Z}} := \inf \left\{ \tau > 0 \mid ^{(n)}\mathcal{Z}_\tau = n \right\}, \\ &\text{and } ^{(n+1)}\mathcal{Z}_T := \int_{\mathbf{X}_T} \mathbf{1}_{\{\theta \leq ^{(n)}\boldsymbol{\lambda}_t^k\}} \mathbf{N}(dk, dt, d\theta, dy). \end{aligned}$$

Remark that  $^{(n)}\mathcal{Z} = ^{(n+1)}\mathcal{Z}$  and  $^{(n)}\boldsymbol{\lambda} = ^{(n+1)}\boldsymbol{\lambda}$  on  $[0, \tau_n^{\mathcal{Z}}]$ , and  $^{(n)}\boldsymbol{\lambda} \leq ^{(n+1)}\boldsymbol{\lambda}$  *P.a.s.*, thus  $\boldsymbol{\lambda} := \lim ^{(n)}\boldsymbol{\lambda}$  is well defined and let  $\mathcal{Z}_T := \int_{\mathbf{X}_T} \mathbf{1}_{\{\theta \leq \boldsymbol{\lambda}_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) = \sum_{n \geq 1} \mathbf{1}_{\{\tau_n^{\mathcal{Z}} \leq T\}}$ . Let us prove that the increasing sequence  $(\tau_n^{\mathcal{Z}})_n$  converges to  $+\infty$ . Indeed, for any  $T > 0$ , by monotone convergence

$$\mathbb{P}\left(\lim_{n \rightarrow +\infty} \tau_n^{\mathcal{Z}} < T\right) = \lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n^{\mathcal{Z}} < T) = \lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{Z}_T \geq n).$$

Moreover by Markov inequality

$$\mathbb{P}(\mathcal{Z}_T \geq n) \leq \frac{\mathbb{E}[\mathcal{Z}_T]}{n} = \frac{\mathbb{E}[\int_0^T \sum_i \boldsymbol{\lambda}_s^i ds]}{n}$$

As  $\boldsymbol{\varphi}$  is a non explosive  $d$ -kernel,  $\mathbb{E}[\int_0^T \sum_i \boldsymbol{\lambda}_s^i ds] < \infty$  and  $\lim_{n \rightarrow +\infty} \mathbb{P}(\mathcal{Z}_T \geq n) = 0$ . Therefore  $\mathbb{P}(\lim_{n \rightarrow +\infty} \tau_n^{\mathcal{Z}} < T) = 0$  for any  $T$  and  $\lim_{n \rightarrow +\infty} \tau_n^{\mathcal{Z}} = +\infty$  *P.a.s.*. Moreover, since the intensity is constructed iteratively and  $^{(n)}\boldsymbol{\lambda} = ^{(n+1)}\boldsymbol{\lambda}$  on  $[0, \tau_n^{\mathcal{Z}}]$ , this guarantees the unicity of  $\boldsymbol{\lambda}$  when  $\tau_n^{\mathcal{Z}}$  tends to infinity.  $\square$

### 2.3 Definition of a Multidimensional Self-exciting Process with Dependencies (MSPD)

We define below a generic multidimensional process with self-exciting cross dependencies, a concept that encompasses several quantities useful in finance and insurance.

**Definition 2.9** (Multidimensional Self-exciting Process with Dependencies : MSPD). Let  $(\Omega, \mathcal{F}, \mathbb{P} \otimes \boldsymbol{\nu}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0})$  be a filtered probability space,  $\boldsymbol{\zeta}$  a  $d$ -kernel and  $\boldsymbol{\varphi}$  a self-excitation  $d$ -kernel. Moreover, let  $\boldsymbol{\mu} := (\boldsymbol{\mu}^i)_{1 \leq i \leq d}$  a family of functions  $\boldsymbol{\mu}^i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  representing the baseline intensity. A  $(\boldsymbol{\zeta}, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD  $\mathbf{Z} := (\mathbf{Z}^i)_{1 \leq i \leq d}$  is an  $\mathbb{R}^d$ -valued stochastic process  $(\mathbf{Z}_t)_{t \geq 0}$  where for every  $i \in \{1, \dots, d\}$ , the pair  $(\mathbf{Z}^i, \boldsymbol{\lambda}^i)$  is solution of the two-dimensional SDEs driven by the Poisson measure  $\mathbf{N}$ ,

$$\begin{cases} \mathbf{Z}_T^i := \int_{\mathbf{X}_T} \boldsymbol{\zeta}^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \boldsymbol{\lambda}_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \\ \boldsymbol{\lambda}_T^i := \boldsymbol{\mu}^i(T) + \int_{\mathbf{X}_T} \boldsymbol{\varphi}^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \boldsymbol{\lambda}_t^k\}} \mathbf{N}(dk, dt, d\theta, dy). \end{cases} \quad T \geq 0 \quad (2.7)$$

In order to easily define an  $(\boldsymbol{\zeta}, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD, we always adopt the following notation: the first component  $\boldsymbol{\zeta}$  refers to the  $d$ -kernel, the second  $\boldsymbol{\mu}$  to the baseline intensity, and the third component  $\boldsymbol{\varphi}$  to the self-excitation  $d$ -kernel.

**Remark 2.10.** Let us emphasize that the process  $\lambda^i$  is predictable, while  $Z^i$  is not. Indeed, when integrating with respect to  $\mathbf{N}(dt)$ ,  $Z_T^i$  can be charged by a point  $\mathbf{x}_n$  having an arrival time  $t_n = T$  (assuming  $\mathbf{1}_{\{\theta_n \leq \lambda_T^{k_n}\}} = 1$ ,  $Z_T^i = Z_{T-}^i + \zeta(0, y_n)$ ) whereas  $\lambda_t^i$  can not since  $\varphi$  is a self-excitation  $d$ -kernel (therefore satisfying  $\varphi(0, y) = 0$ ). One standard kernel in the literature is the exponential kernel defined by  $\phi(u) := \alpha e^{-\beta u}$ . This kernel can be adapted in order to fit under the framework of this paper by using  $\phi(u) := \alpha e^{-\beta u} \mathbf{1}_{\{u > 0\}}$ .

**Remark 2.11** ( $\lambda$  as an MSPD). (2.7) implies that  $\lambda_T - \mu(T)$  is also an  $(\varphi, \mu, \varphi)$ -MSPD, where  $\varphi$  is a self-excitation  $d$ -kernel.

**Remark 2.12** ((Compound) Hawkes process as an MSPD). As discussed after (1.1) in the introduction, this model contains the Hawkes process (for  $\zeta(t, y) = \mathbf{Id}_d$ ) and the compound Hawkes process (for  $\zeta(t, y) = y \mathbf{Id}_d$ ) as particular cases. If the excitation kernel  $\varphi$  does not depend on  $y$ , then there is no impact of the claims sizes (severity component) on the time-arrivals of claims (frequency component).

Our objective is to deploy a general methodology for the computation of different quantities in the space of random Poisson measures. This methodology is based on two essential tools, namely Mecke's formula and the pseudo-chaotic expansion [9], which requires the introduction of some operators on this space. In the following section, we define elements of Malliavin's calculus in order to exploit Mecke's formula through the pseudo-chaotic expansion.

### 3 Elements of Malliavin's calculus

#### 3.1 Spaces and the add point operator

We introduce some elements of Malliavin's calculus on Poisson processes. We set

$$L^0(\Omega) := \{F : \Omega \rightarrow \mathbb{R}, \mathcal{F}^{\mathbf{N}} - \text{measurable}\},$$

$$L^2(\Omega) := \{F \in L^0(\Omega), \mathbb{E}[|F|^2] < +\infty\}.$$

Fix  $n \in \mathbb{N}^*$ . We set  $\mathbf{m}^{\otimes n}$  the extension of  $\mathbf{m}$  on  $(\mathbf{X}^n, \mathfrak{X}^{\otimes n})$ . Let

$$L^0(\mathbf{X}^n) := \{f : \mathbf{X}^n \rightarrow \mathbb{R}, \mathfrak{X}^{\otimes n} - \text{measurable}\}$$

and for  $p \in \{1, 2\}$ ,

$$L^p(\mathbf{X}^n) := \left\{ f \in L^0(\mathbf{X}^n), \int_{\mathbf{X}^n} |f(\mathbf{x}_1, \dots, \mathbf{x}_n)|^p \mathbf{m}^{\otimes n}(d\mathbf{x}_1 \dots d\mathbf{x}_n) < +\infty \right\}. \quad (3.1)$$

Besides,

$$L_s^p(\mathbf{X}^n) := \{f \in L^p(\mathbf{X}^n) \text{ and } f \text{ is symmetric}\} \quad (3.2)$$

is the space of square-integrable symmetric mappings where we recall that  $f : \mathbf{X}^n \rightarrow \mathbb{R}$  is said to be symmetric if for any element  $\sigma$  in  $\mathcal{S}_n$  (the set of all permutations of  $\{1, \dots, n\}$ ),

$$f(\mathbf{x}_1, \dots, \mathbf{x}_n) = f(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(n)}), \quad \forall (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n.$$

The main ingredient we will make use of are the add-points operators on the Poisson space  $\Omega$ . For any finite set  $J$ , we set  $|J|$  its cardinal.

**Definition 3.1.** [Add-points operators] Given  $n \in \mathbb{N}^*$ , and  $J := \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathbf{X}$  a subset of  $\mathbf{X}$  with  $|J| = n$ , we set the measurable mapping :

$$\begin{aligned} \varepsilon_J^{+,n} : \Omega &\longrightarrow \Omega \\ \omega &\longmapsto \omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}}. \end{aligned}$$

Note that by definition

$$\omega + \sum_{\mathbf{x} \in J} \delta_{\mathbf{x}} \mathbf{1}_{\{\omega(\{\mathbf{x}\})=0\}} = \omega + \sum_{m=1}^n \delta_{\mathbf{x}_m} \mathbf{1}_{\{\omega(\{\mathbf{x}_m\})=0\}}$$

that is we add the atoms  $\mathbf{x}_j$  to the path  $\omega$  unless they already were part of it (which is the meaning of the term  $\mathbf{1}_{\{\omega(\{\mathbf{x}_j\})=0\}}$ ). Note that since  $\mathbf{m}$  is assumed to be atomless, given a set  $J$  as above,  $\mathbb{P}[\mathbf{N}(J) = 0] = 1$  hence in what follows we will simply write  $\omega + \sum_{j=1}^n \delta_{\mathbf{x}_j}$  for  $\varepsilon_{\mathbf{x}}^{+,n}(\omega)$ .

### 3.2 The Malliavin derivative

In the context of Malliavin's calculation, a tool that is often mentioned is the associated derivative.

**Definition 3.2.** For  $F$  in  $L^2(\Omega)$ ,  $n \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$ , we set

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F \circ \varepsilon_J^{+,|J|}, \quad (3.3)$$

where we recall that  $\emptyset \subset \mathbf{X}$ . For instance when  $n = 1$ , we write  $\mathbf{D}_{\mathbf{x}} F := \mathbf{D}_{\mathbf{x}}^1 F = F(\cdot + \delta_{\mathbf{x}}) - F(\cdot)$  which is the difference operator (also called add-one cost operator<sup>1</sup>). Relation (3.2) rewrites as

$$\mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F(\omega) = \sum_{J \subset \{1, \dots, n\}} (-1)^{n-|J|} F \left( \omega + \sum_{j \in J} \delta_{\mathbf{x}_j} \right), \quad \text{for a.e. } \omega \in \Omega.$$

Note that with this definition, for any  $\omega$  in  $\Omega$ , the mapping

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbf{D}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F(\omega)$$

belongs to  $L_s^0(\mathbf{X}^n)$  defined as (3.2).

**Remark 3.3.** If  $F$  is deterministic, then by definition  $\mathbf{D}^n F = 0$  for any  $n \geq 1$ .

The chaotic-chaotic expansion developed in [9] involves deterministic operators  $\mathcal{T}^n$  which correspond to iterated Malliavin derivatives under the specific event  $\{\mathbf{N}(\mathbf{X}) = 0\}$ .

**Definition 3.4.** For  $F \in L^0(\Omega)$ , we define the deterministic operators:

$$\mathcal{T}^0 F := F(\emptyset),$$

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<sup>1</sup>see [13, p. 5]

$$\begin{aligned} \mathcal{T}^n : L^0(\Omega) &\rightarrow L_s^0(\mathbf{X}^n) \\ F &\mapsto \mathcal{T}^n F \end{aligned}$$

where for any  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$ ,  $\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n$  is defined by

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F := \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} F(\varpi_J),$$

with  $\varpi_J := \sum_{y_i \in J} \delta_{y_i} \in \Omega$  for  $J = \{y_1, \dots, y_k\}$ .

In particular, even though  $F$  is a random variable,  $\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F$  is in  $\mathbb{R}$  as each term  $F(\varpi_J)$  is the evaluation of  $F$  at the outcome  $\varpi_J$ . Moreover, this operator belongs to  $L_s^1(\mathbf{X}^n)$ .

**Convention 3.5** (Ordered points). Since the operator  $\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n F$  is symmetric, we will assume that the points  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$  are always taken ordered with respect to the time component  $t$ .

### 3.3 Factorial measures and iterated integrals

**Proposition 3.6.** (*Factorial measures; See e.g. [13, Prop 1]*). *There exists a unique sequence of counting random measures  $(\mathbf{N}^{(n)})_{n \in \mathbb{N}^*}$  where for any  $n$ ,  $\mathbf{N}^{(n)}$  is a counting random measure on  $(\mathbf{X}^n, \mathfrak{X}^{\otimes n})$  with*

$$\begin{aligned} \mathbf{N}^{(1)} &:= \mathbf{N} \quad \text{and for } A \in \mathfrak{X}^{\otimes(n+1)}, \\ \mathbf{N}^{(n+1)}(A) &:= \int_{\mathbf{X}^n} \left[ \int_{\mathbf{X}} \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_{n+1}) \in A\}} \mathbf{N}(d\mathbf{x}_{n+1}) - \sum_{j=1}^n \mathbf{1}_{\{(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_j) \in A\}} \right] \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n); \end{aligned}$$

With this definition at hand we introduce the notion of iterated integrals. In particular for  $A \in \mathfrak{X}$ ,

$$\mathbf{N}^{(n)}(A^{\otimes n}) = \mathbf{N}(A)(\mathbf{N}(A) - 1) \times \dots \times (\mathbf{N}(A) - n + 1).$$

Note that by definition  $\mathbf{N}^{(n)}(A) \mathbf{1}_{\{\mathbf{N}(A) < n\}} = 0$ . We now turn to the definition of iterated integrals with respect to the counting measure  $\mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n)$ .

**Definition 3.7.** Let  $n \in \mathbb{N}^*$  and  $f_n \in L_s^1(\mathbf{X}^n)$ .  $\mathcal{I}_n(f_n)$  the  $n^{\text{th}}$  iterated integral of  $f_n$  with respect to the Poisson measure  $\mathbf{N}$  is defined as

$$\mathcal{I}_n(f_n) := \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{N}^{(n)}(d\mathbf{x}_1, \dots, d\mathbf{x}_n),$$

where each of the integrals above is well-defined pathwise for  $\mathbb{P}$ -a.e. for each  $\omega \in \Omega$ , as a *Stieltjes integral*.

### 3.4 The Mecke formula

We end this section by recalling a particular case of Mecke's formula (see e.g. [13, Relation (11)]).

**Lemma 3.8** (A particular case of Mecke's formula). *Let  $F \in L^0(\Omega)$ ,  $n \in \mathbb{N}$  and  $h \in L^0(\mathbf{X}^n)$  such that*

$$\int_{\mathbf{X}^n} |h(\mathbf{x}_1, \dots, \mathbf{x}_n)| \mathbb{E} [ |F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_n}^{+,n}| ] \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) < +\infty.$$

Then

$$\mathbb{E} \left[ F \int_{\mathbf{X}^n} h d\mathbf{N}^{(n)} \right] = \int_{\mathbf{X}^n} h(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbb{E} [ F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_n}^{+,n} ] \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n). \quad (3.4)$$

By taking  $F = 1$  we get

$$\begin{aligned} \mathbb{E} [\mathcal{I}_n(f_n)] &= \int_{\mathbf{X}^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\ &= \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \int_{([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)^n} f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \prod_{i=1}^n \nu^{k_i}(dy_i) d\theta_i dt_i. \end{aligned} \quad (3.5)$$

## 4 Pseudo-chaotic expansion for counting processes

The second ingredient in our analysis relies on the pseudo-chaotic expansion. This section presents some essential results around this expansion, with a focus on the process  $\mathbf{Z}$ .

### 4.1 Around the pseudo-chaotic expansion of $\mathbf{Z}$

The following theorem provides the pseudo-chaotic expansion for a  $(\zeta, \mu, \varphi)$ -MSPD, following results from [9].

**Theorem 4.1.** *Let  $\mathbf{Z}$  a  $(\zeta, \mu, \varphi)$ -MSPD, we recall that this process is given by the SDE(2.7):*

$$\begin{cases} \mathbf{Z}_T^i := \int_{\mathbf{X}_T} \zeta^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \\ \lambda_T^i := \mu^i(T) + \int_{\mathbf{X}_T} \varphi^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy). \end{cases} \quad T \geq 0$$

Then  $\mathbf{Z}_T^i$  admits the pseudo-chaotic representation

$$\mathbf{Z}_T^i = \lim_{M \rightarrow \infty} \sum_{n=1}^{+\infty} \frac{1}{n!} \mathcal{I}_n \left( \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbf{1}_{\{([0,T] \times [0,M])^n\}} \right) \quad (4.1)$$

with for all  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n$ ,

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i = \zeta^{i,k_n}(T-t_n, y_n) \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})}^{n-1} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}\}}. \quad (4.2)$$

*Proof.* Equation (4.1) follows from [9] which gives the pseudo-chaotic expansion of any random linear functional of  $\mathbf{N}$  restricted to a bounded domain (say  $[0, T] \times [0, M]$ , for  $T, M > 0$ ) of  $\mathbb{R}_+^2$ ; with a focus on random variables of the form  $H_t$  where  $H$  is a counting process with bounded intensity (we refer the reader to [9] for a complete exposition). Even though the intensity of  $\mathbf{Z}_T^i$  is unbounded, it is proved in [9] that marginals of Hawkes processes admit a pseudo-chaotic expansion. Then to prove (4.2), recalling that  $\mathbf{x} = (k, t, \theta, y)$ ,

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i = \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} \left( \int_{\mathbf{X}_T} \zeta^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \right) (\varpi_J)$$

$$\begin{aligned}
&= \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} (-1)^{n-|J|} \sum_{\mathbf{x}_m \in J} \zeta^{i, k_m}(T - t_m, y_m) \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_J)\}} \\
&= \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} \sum_{\mathbf{x}_m \in J} (-1)^{n-|J|} \zeta^{i, k_m}(T - t_m, y_m) \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_J)\}} \\
&= \sum_{m=1}^n \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_n\}; \mathbf{x}_m \in J} (-1)^{n-|J|} \zeta^{i, k_m}(T - t_m, y_m) \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_{J+\delta_{\mathbf{x}_m}})\}} \\
&= \sum_{m=1}^n \zeta^{i, k_m}(T - t_m, y_m) \sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n\}} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_{J+\delta_{\mathbf{x}_m}})\}}
\end{aligned}$$

Then, for  $m \in \{1, \dots, n\}$  we have that

$$\begin{aligned}
&\sum_{J \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{m-1}, \mathbf{x}_{m+1}, \dots, \mathbf{x}_n\}} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_{J+\delta_{\mathbf{x}_m}})\}} \\
&= \sum_{J^- \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{m-1}\}} \sum_{J^+ \subset \{\mathbf{x}_{m+1}, \dots, \mathbf{x}_n\}} (-1)^{n-1-|J^-|-|J^+|} \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_{J^- \cup J^+ + \delta_{\mathbf{x}_m}})\}} \\
&= \sum_{J^- \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{m-1}\}} \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\varpi_{J^-})\}} (-1)^{n-1-|J^-|} \sum_{J^+ \subset \{\mathbf{x}_{m+1}, \dots, \mathbf{x}_n\}} (-1)^{|J^+|}
\end{aligned}$$

where the last equality follows from the predictability of  $\lambda$ . Moreover, if  $\mathbf{x}_m \neq \mathbf{x}_n$  then,

$$\sum_{J^+ \subset \{\mathbf{x}_{m+1}, \dots, \mathbf{x}_n\}} (-1)^{|J^+|} = 0.$$

And if  $\mathbf{x}_m = \mathbf{x}_n$  then  $J^+ = \emptyset$  which implies,

$$\sum_{J^+ \subset \{\mathbf{x}_{m+1}, \dots, \mathbf{x}_n\}} (-1)^{|J^+|} = 1.$$

Thus,

$$\begin{aligned}
\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i &= \zeta^{i, k_n}(T - t_n, y_n) \sum_{U \subset \{\mathbf{x}_1, \dots, \mathbf{x}_{n-1}\}} (-1)^{n-1-|U|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}(\varpi_U)\}} \\
&= \zeta^{i, k_n}(T - t_n, y_n) \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})}^{n-1} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}\}}.
\end{aligned}$$

□

**Remark 4.2.** As  $\mu^i(T)$  is deterministic,  $\mathcal{T}^n \mu^i(T) = 0$  hence from (4.2) we deduce that

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \lambda_T^i = \varphi^{i, k_n}(T - t_n, y_n) \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})}^{n-1} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}\}}.$$

**Remark 4.3.** Consider a set  $J = (\mathbf{x}_1, \dots, \mathbf{x}_n)$  of ordered points in  $\mathbf{X}_T$  (see Convention 3.5). Then  $\mathbf{Z}_T^i$  evaluated on  $J$  satisfies, for any  $\omega \in \Omega$

$$\mathbf{Z}_T^i \left( \sum_{\mathbf{x}_m \in J} \delta_{\mathbf{x}_m} \right) = \sum_{\mathbf{x}_m \in J} \zeta^{i, k_m}(T - t_m, y_m) \mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\sum_{\mathbf{x}_j \in J} \delta_{\mathbf{x}_j})\}}$$

$$\leq \sum_{m \in J} \zeta^{i,k_m}(T - t_m, y_m) \leq \mathbf{Z}_T^i \left( \omega + \sum_{\mathbf{x}_m \in J} \delta_{\mathbf{x}_m} \right).$$

Since for all  $(i, k) \in \{1, \dots, d\}^2$ ,  $\zeta^{i,k}$  is positive,  $\mathbf{Z}_T^i(\sum_{j \in J} \delta_{\mathbf{x}_j})$  reaches its maximum when every point in  $J$  is accepted by the thinning criteria  $\mathbf{1}_{\{\theta_m \leq \lambda_{t_m}^{k_m}(\sum_{j \in J} \delta_{\mathbf{x}_j})\}}$ . In fact, if we assume that every point in  $J$  verifies the thinning criteria we have for all  $\mathbf{x}_m \in J$ ,

$$\theta_m \leq \mu^{k_m} + \sum_{j=1}^{m-1} \varphi^{k_m, k_j}(t_m - t_j, y_j) = \lambda_{t_m}^{k_m}(\sum_{j \in J} \delta_{\mathbf{x}_j}) \leq \lambda_{t_m}^{k_m}(\omega + \sum_{j \in J} \delta_{\mathbf{x}_j}), \quad (4.3)$$

where the sum stops at  $j = m - 1$  because of the predictability of  $\lambda_T^i$  (see Definition 2.6).

This brings us to introduce the following lemma which establishes a compatibility condition on a set of points  $\mathbf{x}_j$  such that  $\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_t^i \neq 0$ .

**Lemma 4.4** (Compatibility condition). *Let  $\mathbf{Z}$  a  $(\zeta, \mu, \varphi)$ -MSPD. Fix  $T \geq 0$ , let  $n \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbf{X}_T)^n$  following Convention 3.5. It holds that,*

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i = \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbf{1}_{\{\theta_1 \leq \mu^{k_1}(t_1)\}} \prod_{m=2}^n \mathbf{1}_{\{\theta_m \leq \mu^{k_m}(t_m) + \sum_{j=1}^{m-1} \varphi^{k_m, k_j}(t_m - t_j, y_j)\}}.$$

$(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is said to satisfy the compatibility condition if

$$\mathbf{1}_{\{\theta_1 \leq \mu^{k_1}\}} \prod_{m=2}^n \mathbf{1}_{\{\theta_m \leq \mu^{k_m} + \sum_{j=1}^{m-1} \varphi^{k_m, k_j}(t_m - t_j, y_j)\}} = 1. \quad (4.4)$$

*Proof.* By applying (4.2) from Theorem 4.1 to  $\mathbf{Z}$  and by definition of the operator  $\mathcal{T}$  we have that,

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i = \zeta^{i, k_n}(T - t_n, y_n) \sum_{J \subset \{1, \dots, n-1\}} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}(\sum_{p \in J} \delta_{\mathbf{x}_p})\}}.$$

Hence, inequality (4.3) (with  $m = n$ ) implies

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbf{1}_{\{\theta_n > \mu^{k_n} + \sum_{m=1}^{n-1} \varphi^{k_n, k_m}(t_n - t_m, y_m)\}} = 0.$$

Let  $\ell \in \{2, \dots, n\}$ ,

$$\begin{aligned} & \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbf{1}_{\{\theta_\ell > \mu^{k_\ell}(t_\ell) + \sum_{m=1}^{\ell-1} \varphi^{k_\ell, k_m}(t_\ell - t_m, y_m)\}} \\ &= \zeta^{i, k_n}(T - t_n, y_n) \mathbf{1}_{\{\theta_\ell > \mu^{k_\ell}(t_\ell) + \sum_{m=1}^{\ell-1} \varphi^{k_\ell, k_m}(t_\ell - t_m, y_m)\}} \sum_{J \subset \{1, \dots, n-1\}} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}(\sum_{p \in J} \delta_{\mathbf{x}_p})\}} \\ &= \zeta^{i, k_n}(T - t_n, y_n) \mathbf{1}_{\{\theta_\ell > \mu^{k_\ell}(t_\ell) + \sum_{m=1}^{\ell-1} \varphi^{k_\ell, k_m}(t_\ell - t_m, y_m)\}} \\ & \left( \sum_{J \subset \{1, \dots, n-1\}; \ell \in J} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}(\sum_{p \in J} \delta_{\mathbf{x}_p})\}} + \sum_{J \subset \{1, \dots, n-1\}; \ell \notin J} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}(\sum_{p \in J} \delta_{\mathbf{x}_p})\}} \right) \end{aligned}$$

$$\begin{aligned}
&= \zeta^{i,kn}(T-t_n, y_n) \mathbf{1}_{\{\theta_\ell > \mu^{k_\ell}(t_\ell) + \sum_{m=1}^{\ell-1} \varphi^{k_\ell, k_m}(t_\ell - t_m, y_m)\}} \\
&\left( \sum_{J \subset \{1, \dots, \ell-1, \ell+1, \dots, n-1\}} (-1)^{n-2-|J|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{kn}(\delta_{x_\ell} + \sum_{p \in J} \delta_{x_p})\}} + \sum_{J \subset \{1, \dots, \ell-1, \ell+1, \dots, n-1\}} (-1)^{n-1-|J|} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{kn}(\sum_{p \in J} \delta_{x_p})\}} \right).
\end{aligned}$$

Since on the domain  $\{\theta_\ell > \mu^{k_\ell}(t_\ell) + \sum_{m=1}^{\ell-1} \varphi^{k_\ell, k_m}(t_\ell - t_m, y_m)\}$  we have that,

$$\lambda_{t_n}^{kn}(\delta_\ell + \sum_{p \in J} \delta_{x_p}) = \lambda_{t_n}^{kn}(\sum_{p \in J} \delta_{x_p}),$$

therefore for any  $\ell \in \{2, \dots, n\}$

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbf{1}_{\{\theta_\ell > \mu^{k_\ell}(t_\ell) + \sum_{m=1}^{\ell-1} \varphi^{k_\ell, k_m}(t_\ell - t_m, y_m)\}} = 0.$$

For  $\ell = 1$

$$\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i = \mathbf{1}_{\{\theta_1 \leq \lambda_{t_1}^{k_1}(\delta_{\mathbf{x}_1})\}} = \mathbf{1}_{\{\theta_1 \leq \mu^{k_1}(t_1)\}}$$

which concludes the proof.  $\square$

## 4.2 Shifted processes

Combining Mecke's formula (3.4) and the pseudo-chaotic expansion (Theorem 4.1) involves processes of the form  $F \circ \varepsilon_{\mathbf{x}_1, \dots, \mathbf{x}_n}^{+,n}$  that we call shifted processes. An in-depth study of these processes for  $F = \mathbf{Z}^i$ , is of interest, as they can be interpreted as stressed scenarios.

**Remark 4.5** (A consequence of the compatibility condition). We would like to enhance an important consequence of the compatibility condition when shifting the intensity of an MSPD. Let  $\mathbf{Z}$  a  $(\zeta, \mu, \varphi)$ -MSPD. Let  $n \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbf{X}_T)^n$  satisfying the compatibility condition (4.4), then for any  $\omega \in \Omega$ , the shifted processes are given by

$$\begin{aligned}
\mathbf{Z}_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}(\omega) &= \left( \int_{\mathbf{X}_T} \zeta^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k\}} \mathbf{N}(dk, dt, d\theta, dy) \right) \left( \omega + \sum_{j=1}^n \delta_{\mathbf{x}_j} \right) \\
&= \left( \int_{\mathbf{X}_T} \zeta^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k(\omega + \sum_{j=1}^n \delta_{\mathbf{x}_j})\}} \mathbf{N}(dk, dt, d\theta, dy) \right) (\omega) \\
&\quad + \sum_{j=1}^n \zeta^{i,k_j}(T-t_j, y_j) \mathbf{1}_{\{\theta_j \leq \lambda_{t_j}^{k_j}(\omega + \sum_{j=1}^n \delta_{\mathbf{x}_j})\}}.
\end{aligned}$$

Since  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbf{X}_T)^n$  satisfy the compatibility condition, Remark 4.3 implies

$$\mathbf{Z}_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n} = \sum_{j=1}^n \zeta^{i,k_j}(T-t_j, y_j) + \int_{\mathbf{X}_T} \zeta^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}\}} \mathbf{N}(dk, dt, d\theta, dy).$$

Moreover, from Remark 2.11,  $\lambda - \mu$  can be seen as an MSPD which gives us,

$$\lambda_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n} = \mu^i(T) + \sum_{j=1}^n \varphi^{i,k_j}(T-t_j, y_j) + \int_{\mathbf{X}_T} \varphi^{i,k}(T-t, y) \mathbf{1}_{\{\theta \leq \lambda_t^k \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}\}} \mathbf{N}(dk, dt, d\theta, dy).$$

From the results above, we can remark that a shifted process  $\mathbf{Z}_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}$  can be seen as a MSPD-process whose baseline intensity is impacted by  $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ . This leads to the following definition.

**Definition 4.6** (Compensated shift of an MSPD). Let  $\mathbf{Z}$  a  $(\zeta, \mu, \varphi)$ -MSPD and let  $n \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbf{X}_T)^n$  satisfy the compatibility condition (4.4). Then its compensated shift  $\mathbf{Z} \odot \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}$  given by

$$\mathbf{Z}_T^i \odot \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n} := \mathbf{Z}_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n} - \sum_{j=1}^n \zeta^{i,k_j}(T - t_j, y_j), \quad (4.5)$$

is a  $(\varphi, \mu_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}, \varphi)$ -MSPD where  $\mu_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}$  is such that,

$$\mu_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^i(T) = \mu^i(T) + \sum_{j=1}^n \varphi^{i,k_j}(T - t_j, y_j).$$

In other terms we have that the compensated shift of a  $(\zeta, \mu, \varphi)$ -MSPD noted  $\mathbf{Z}_T \odot \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}$ , is a  $(\zeta, \mu_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}, \varphi)$ -MSPD.

Remark 2.11 highlighted that  $\lambda_T - \mu(T)$  is a  $(\varphi, \mu, \varphi)$ -MSPD. This stays true in the case of the compensated shift, in fact we have

$$\lambda_T^i \odot \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n} := \lambda_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n} - \mu^i(T) - \sum_{j=1}^n \varphi^{i,k_j}(T - t_j, y_j)$$

which is a  $(\varphi, \mu_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}, \varphi)$ -MSPD.

## 5 Expectation and correlations

This section illustrates how the previous results enable us to develop a new methodology for calculating quantities related to this MSPD process  $\mathbf{Z}$  (in particular risk valuation). To start with, we develop the computations for the expectation and the correlations of  $\mathbf{Z}$ . For the expectation, a well-known method consists of exploiting the fact that the expectation of the intensity satisfies a Volterra equation. However, this method is not available for the calculation of higher-order moments. Therefore, for the computation of the covariance, we will combine the pseudo-chaotic expansion with Mecke's formula to obtain expressions involving the expectation of shifted processes that can be obtained as solution of a Volterra type equation.

### 5.1 Expectation of the process $\mathbf{Z}$ and its shifted version

We start with the following result that extends [1, Theorem 2] and [8, Theorem 2.4].

**Proposition 5.1.** *Let  $\mathbf{Z}$  a  $(\zeta, \mu, \varphi)$ -MSPD. Let  $p \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_p) \in (\mathbf{X}_T)^p$  satisfying the compatibility condition (4.4). Then we have*

(i) *Expectation of  $\mathbf{Z}_T$  :*

$$\mathbb{E}[\mathbf{Z}_T] = \int_0^T \bar{\zeta}(T - v) \left( \mu(v) + \int_0^v \bar{\Psi}(v - w) \mu(w) dw \right) dv.$$

(ii) Expectation of  $\mathbf{Z}_T$  shifted by  $(\mathbf{x}_1, \dots, \mathbf{x}_p)$  :

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_T \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p}] &= \mathbb{E}[\mathbf{Z}_T] + \sum_{j=1}^p \zeta^{\cdot, k_j}(T - t_j, y_j) \\ &\quad + \sum_{j=1}^p \int_{t_j}^T \bar{\zeta}(T - v) \left( \varphi^{\cdot, k_j}(v - t_j, y_j) + \int_{t_j}^v \bar{\Psi}(v - w) \varphi^{\cdot, k_j}(w - t_j, y_j) dw \right) dv. \end{aligned}$$

*Proof.* For (i), by taking the expectation of the process  $\mathbf{Z}_T^i$  we have that,

$$\mathbb{E}[\mathbf{Z}_T^i] = \sum_{k=1}^d \int_{\mathbf{X}_T} \bar{\zeta}^{i,k}(T - t) \mathbb{E}[\boldsymbol{\lambda}_t^k] dt.$$

Hence in order to compute  $\mathbb{E}[\mathbf{Z}_T^i]$  we first need  $\mathbb{E}[\boldsymbol{\lambda}_t^k]$ , for  $k = 1, \dots, d$ . It satisfies the linear Volterra ODE which reads in a matrix form as

$$\mathbb{E}[\boldsymbol{\lambda}_T] = \boldsymbol{\mu}(T) + \int_0^T \bar{\varphi}(T - t) \mathbb{E}[\boldsymbol{\lambda}_t] dt,$$

whose solution is given by,

$$\mathbb{E}[\boldsymbol{\lambda}_T] = \boldsymbol{\mu}(T) + \int_0^T \bar{\Psi}(T - w) \boldsymbol{\mu}(w) dw.$$

Thus,

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_T] &= \int_0^T \bar{\zeta}(T - t) \mathbb{E}[\boldsymbol{\lambda}_t] dt \\ &= \int_0^T \bar{\zeta}(T - t) \left( \boldsymbol{\mu}(t) + \int_0^t \bar{\Psi}(t - w) \boldsymbol{\mu}(w) dw \right) dt. \end{aligned}$$

For (ii), we use Definition 4.6 to deduce

$$\begin{aligned} \mathbb{E}[\mathbf{Z}_T \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p}] &= \int_0^T \bar{\zeta}(T - t) \left( \boldsymbol{\mu}(t) + \sum_{j=1}^p \varphi^{\cdot, k_j}(t - t_j, y_j) \right) dt \\ &\quad + \int_0^T \bar{\zeta}(T - t) \left( \int_0^t \bar{\Psi}(t - w) \boldsymbol{\mu}(w) dw + \int_0^t \bar{\Psi}(t - w) \sum_{j=1}^p \varphi^{\cdot, k_j}(w - t_j, y_j) dw \right) dt \\ &= \mathbb{E}[\mathbf{Z}_T] \\ &\quad + \int_0^T \bar{\zeta}(T - t) \left( \sum_{j=1}^p \varphi^{\cdot, k_j}(t - t_j, y_j) + \int_0^t \bar{\Psi}(t - w) \sum_{j=1}^p \varphi^{\cdot, k_j}(w - t_j, y_j) dw \right) dt. \end{aligned}$$

□

In (ii) of Proposition 5.1, the expectation  $\mathbb{E}[\mathbf{Z}_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p}]$  of a  $(\boldsymbol{\zeta}, \boldsymbol{\mu}_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}, \boldsymbol{\varphi})$ -MSPD is expressed as the sum of the expectations of a  $(\boldsymbol{\zeta}, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD and a  $(\boldsymbol{\zeta}, \boldsymbol{\mu}_{(\mathbf{x}_1, \dots, \mathbf{x}_p)} - \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD. This indicates that the expectation of the processes exhibits some linearity property with respect to the baseline intensity. This observation is made formal in the following remark.

**Remark 5.2** (Expectation's linearity with respect to the baseline intensity).

Let  $\mathbf{Z}$  a  $(\zeta, \boldsymbol{\mu}, \varphi)$ -MSPD. Let  $p \in \mathbb{N}^*$ , and consider a family of  $d$ -dimensional vectors  $(\boldsymbol{\Lambda}_j)_{1 \leq j \leq p}$  such that  $\forall j \in \llbracket 1, p \rrbracket, \forall i \in \llbracket 1, d \rrbracket \boldsymbol{\Lambda}_j^i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Consider the baseline intensity given by

$$\tilde{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}(t) + \sum_{j=1}^p \boldsymbol{\Lambda}_j(t),$$

and the associated processes such that  $\tilde{\mathbf{Z}}$  is a  $(\zeta, \tilde{\boldsymbol{\mu}}, \varphi)$ -MSPD and  $\tilde{\tilde{\mathbf{Z}}}$  is a  $(\zeta, \sum_{j=1}^p \boldsymbol{\Lambda}_j(T), \varphi)$ -MSPD. We have that

$$\mathbb{E}[\tilde{\mathbf{Z}}_T] = \int_0^T \bar{\zeta}(T-v) \left( \tilde{\boldsymbol{\mu}}(v) + \int_0^v \bar{\Psi}(v-w) \tilde{\boldsymbol{\mu}}(w) dw \right) dv = \mathbb{E}[\mathbf{Z}_T] + \mathbb{E}[\tilde{\tilde{\mathbf{Z}}}_T].$$

**Remark 5.3.** Proposition 5.1 provides as a by product a generalization of the Wald identity which in dimension 1 reduces to

$$\mathbb{E}[L_T] = \mathbb{E}[N_T] \mathbb{E}[Y_1]; \quad \text{for } L_T := \sum_{j=1}^{N_T} Y_j$$

where  $N$  a counting process independent from the iid random variables  $(Y_j)_{j \geq 1}$ . In our setting, for  $1 \leq i \leq d$ , we prove that

$$\mathbb{E}[\mathbf{L}_T^i] = \mathbb{E}[Y^i] \mathbb{E}[\mathbf{H}_T^i] \quad \text{for } L_T^i := \sum_{j=1}^{H_T^i} Y_j^i$$

where  $(Y_j^i)_{j \geq 1}$  are iid random variables with probability density  $\nu^i$ , but in which  $\mathbb{E}[\mathbf{H}_T^i]$  is impacted by the distributions of the claims sizes  $(Y^1, \dots, Y^d)$ .

For computing the expectation of the intensity, we apply Proposition 5.1 to  $(\boldsymbol{\lambda} - \boldsymbol{\mu})$  which can be considered as a  $(\varphi, \boldsymbol{\mu}, \varphi)$ -MSPD (see Remark 2.11). Since the  $d$ -kernel  $\zeta$  coincides with the self-excitation  $d$ -kernel  $\varphi$ , the expression for the expectation simplifies using (2.5).

**Corollary 5.4** (Intensity expectation). *The expectation of the intensity  $\boldsymbol{\lambda}_T^i$  (see (2.7)) is given by*

$$\mathbb{E}[\boldsymbol{\lambda}_T] = \boldsymbol{\mu}(T) + \int_0^T \bar{\Psi}(T-w) \boldsymbol{\mu}(w) dw$$

and, for  $p \in \mathbb{N}^*$ ,  $(\mathbf{x}_1, \dots, \mathbf{x}_p) \in (\mathbf{X}_T)^p$ , its shifted version is given by

$$\mathbb{E}[\boldsymbol{\lambda}_T^i \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p}] = \mathbb{E}[\boldsymbol{\lambda}_T^i] + \sum_{j=1}^p \left( \varphi^{i,k_j}(T-t_j, y_j) + \int_{t_j}^T \bar{\Psi}^{i,\cdot}(T-w) \varphi^{\cdot,k_j}(w-t_j, y_j) dw \right).$$

## 5.2 A general correlation formula

As presented at the beginning of this section, our methodology allows one to compute more general functionals of the process  $\mathbf{Z}$  which would not be possible by using existing methodologies (as the one presented in [1] with Volterra equations or in [11] with the moment measures). In particular one would like to compute the expectation of building blocks having the form of a product  $\mathbf{Z}\boldsymbol{\Gamma}$  where  $\mathbf{Z}$  is an  $(\zeta, \boldsymbol{\mu}, \varphi)$ -MSPD and  $\boldsymbol{\Gamma} := (\boldsymbol{\Gamma}^\ell)_{1 \leq \ell \leq d} \in L^2(\Omega)$  and whose expectation of the shifted version can be written as

$$\mathbb{E} \left[ \boldsymbol{\Gamma}^\ell \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p} \right] = \mathbb{E} \left[ \boldsymbol{\Gamma}^\ell \right] + \sum_{j=1}^p \boldsymbol{\rho}_\ell(k_j, t_j, y_j),$$

with  $\boldsymbol{\rho} := (\boldsymbol{\rho}_\ell)_{1 \leq \ell \leq d} \in \mathcal{M}_{d,1}^+$ ,  $\boldsymbol{\rho}_\ell : \{1, \dots, d\} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  and such that

$$\int_{\mathbb{R}_+} \zeta^{i,k}(T-t, y) \boldsymbol{\rho}_\ell(k, t, y) \nu^k(dy) < +\infty, \quad \forall(i, k, t). \quad (5.1)$$

We define for  $T > 0$ ,  $(\zeta \boldsymbol{\rho}_\ell)^T : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathcal{M}_{d,d}^+$  and  $\overline{(\zeta \boldsymbol{\rho}_\ell)^T} : \mathbb{R}^+ \rightarrow \mathcal{M}_{d,d}^+$  as

$$\begin{aligned} (\zeta \boldsymbol{\rho}_\ell)^T(t, y) &:= \left( \zeta^{i,k}(T-t, y) \boldsymbol{\rho}_\ell(k, t, y) \right)_{1 \leq i, k \leq d}, \\ \overline{(\zeta \boldsymbol{\rho}_\ell)^T}(t) &:= \left( \int_{\mathbb{R}_+} \zeta^{i,k}(T-t, y) \boldsymbol{\rho}_\ell(k, t, y) \nu^k(dy) \right)_{1 \leq i, k \leq d}. \end{aligned}$$

Moreover, since  $(\zeta \boldsymbol{\rho}_\ell)^T$  is a  $d$ -kernel (due to the integrability Condition (5.1)) and  $\mathbf{Z}^{(\zeta \boldsymbol{\rho}_\ell)^T}$  is a  $((\zeta \boldsymbol{\rho}_\ell)^T, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD, its expectation is given by Proposition 5.1 as

$$\mathbb{E}[\mathbf{Z}_T^{(\zeta \boldsymbol{\rho}_\ell)^T}] = \int_0^T \overline{(\zeta \boldsymbol{\rho}_\ell)^T}(u) \left( \int_0^u \overline{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du.$$

**Convention:** To lighten the notations, we write  $\zeta \boldsymbol{\rho}_\ell := (\zeta \boldsymbol{\rho}_\ell)^T$  when there is no ambiguity in the context.

In the result below,  $T > 0$  is a fixed horizon.

**Theorem 5.5.** *Let  $\mathbf{Z}$  a  $(\zeta, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD and  $(\mathbf{x}_1, \dots, \mathbf{x}_p) \in (\mathbf{X}_T)^p$  satisfying the compatibility condition (4.4). Let  $\boldsymbol{\Gamma} \in \mathcal{M}_{d,1}^+$  such that for  $1 \leq \ell \leq d$ ,  $\boldsymbol{\Gamma}^\ell \in L^2(\Omega)$  and*

$$\mathbb{E} \left[ \boldsymbol{\Gamma}^\ell \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p} \right] = \mathbb{E} \left[ \boldsymbol{\Gamma}^\ell \right] + \sum_{j=1}^p \boldsymbol{\rho}_\ell(k_j, t_j, y_j), \quad (5.2)$$

where  $\boldsymbol{\rho}_\ell : \{1, \dots, d\} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is a deterministic function (specific to  $\boldsymbol{\Gamma}^\ell$ ) satisfying (5.1). Then for  $1 \leq i, \ell \leq d$  we have

$$\mathbb{E} \left[ \mathbf{Z}_T^i \boldsymbol{\Gamma}^\ell \right] = \mathbb{E} \left[ \mathbf{Z}_T^i \right] \mathbb{E} \left[ \boldsymbol{\Gamma}^\ell \right] + \mathbb{E} \left[ (\mathbf{Z}_T^{\zeta \boldsymbol{\rho}_\ell})^i \right] + \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \left( \int_0^v \overline{\Psi}(v-w) \mathbb{E} \left[ \mathbf{Z}_w^{\boldsymbol{\rho}_\ell} \right] dw + \mathbb{E} \left[ \mathbf{Z}_v^{\boldsymbol{\rho}_\ell} \right] \right) dv,$$

where  $\mathbf{Z}^{\zeta \boldsymbol{\rho}_\ell}$  is a  $(\zeta \boldsymbol{\rho}_\ell, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD and  $\mathbf{Z}^{\boldsymbol{\rho}_\ell}$  is a  $(\boldsymbol{\rho}_\ell, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD.

The demonstration of Theorem 5.5 relies on the following lemma, whose proof is postponed in the appendix.

**Lemma 5.6.** *Let  $\mathbf{Z}$  a  $(\zeta, \boldsymbol{\mu}, \boldsymbol{\varphi})$ -MSPD, and  $\boldsymbol{\rho}_\ell : \{1, \dots, d\} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  such that  $\int_{\mathbb{R}_+} \boldsymbol{\rho}_\ell(k, t, y) \nu^k(dy) < +\infty$ , then for  $1 \leq i \leq d$*

$$\begin{aligned} & \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\mathbf{X}_T^n} \left( \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \right) \sum_{j=1}^n \boldsymbol{\rho}_\ell(k_j, t_j, y_j) \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\ &= \mathbb{E} \left[ (\mathbf{Z}_T^{\zeta \boldsymbol{\rho}_\ell})^i \right] + \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \int_0^v (\overline{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\boldsymbol{\rho}_\ell}] + \mathbb{E}[\mathbf{Z}_v^{\boldsymbol{\rho}_\ell}]) dw dv. \end{aligned}$$

Moreover if  $\sum \boldsymbol{\rho}_\ell \equiv 1$ , then

$$\sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\mathbf{X}_T^n} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) = \mathbb{E} \left[ \mathbf{Z}_T^i \right].$$

*Proof of Theorem 5.5.* Using successively the pseudo-chaotic expansion for  $\mathbf{Z}_T^i$ , the Mecke formula and Lemma 5.6 we have

$$\begin{aligned}
\mathbb{E}[\mathbf{Z}_T^i \mathbf{\Gamma}^\ell] &= \lim_{M \rightarrow \infty} \sum_{n=1}^{+\infty} \frac{1}{n!} \mathbb{E}[\mathcal{I}_n(\mathcal{T}^n \mathbf{Z}_T^i \mathbf{1}_{\{([0,T] \times [0,M])^n\}}) \mathbf{\Gamma}^\ell] \\
&= \lim_{M \rightarrow \infty} \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{(\mathbf{X}_T^M)^n} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \mathbb{E}[\mathbf{\Gamma}^\ell \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^{+,n}] \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\
&= \mathbb{E}[\mathbf{Z}_T^i] \mathbb{E}[\mathbf{\Gamma}^\ell] + \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\mathbf{X}_T^n} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \sum_{j=1}^n \rho_\ell(t_j, y_j, k_j) \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\
&= \mathbb{E}[\mathbf{Z}_T^i] \mathbb{E}[\mathbf{\Gamma}^\ell] + \mathbb{E}[(\mathbf{Z}_T^{\zeta \rho_\ell})^i] + \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \left( \int_0^v \bar{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \rho_\ell}] dw + \mathbb{E}[\mathbf{Z}_v^{\varphi \rho_\ell}] \right) dv
\end{aligned}$$

where  $X_T^M := \{1, \dots, d\} \times [0, T] \times [0, M] \times \mathbb{R}_+$ .  $\square$

Theorem 5.5 is now applied to compute the covariance of two MSPDs having different kernels.

### 5.3 Correlations of MSPDs

By combining Proposition 5.1 and Theorem 5.5, we deduce the following proposition.

**Theorem 5.7** (Correlations of two MSPDs). *Let  $\mathbf{Z}$  a  $(\zeta, \mu, \varphi)$ -MSPD and  $\tilde{\mathbf{Z}}$  a  $(\tilde{\zeta}, \tilde{\mu}, \tilde{\varphi})$ -MSPD, then for  $0 \leq T \leq S$ , their covariance (for  $1 \leq i, \ell \leq d$ ) is*

$$\text{Cov}(\mathbf{Z}_T^i, \tilde{\mathbf{Z}}_S^\ell) = \mathbb{E}[(\mathbf{Z}_T^{\zeta \tilde{\rho}_\ell})^i] + \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \left( \int_0^v \bar{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \tilde{\rho}_\ell}] dw + \mathbb{E}[\mathbf{Z}_v^{\varphi \tilde{\rho}_\ell}] \right) dv, \quad (5.3)$$

where  $\zeta \tilde{\rho}_\ell := (\zeta \tilde{\rho}_\ell)^T$  and  $\mathbf{Z}^{\zeta \tilde{\rho}_\ell}$  is a  $(\zeta \tilde{\rho}_\ell, \mu, \varphi)$ -MSPD ;  
 $\varphi \tilde{\rho}_\ell := (\varphi \tilde{\rho}_\ell)^T$  and  $\mathbf{Z}^{\varphi \tilde{\rho}_\ell}$  a  $(\varphi \tilde{\rho}_\ell, \mu, \varphi)$ -MSPD with

$$\tilde{\rho}_\ell(k, t, y) := \tilde{\zeta}^{\ell, k}(S-t, y) + \int_t^S \bar{\zeta}^{\ell, \cdot}(S-v) \left( \tilde{\varphi}^{\cdot, k}(v-t, y) + \int_t^v \bar{\Psi}(v-w) \tilde{\varphi}^{\cdot, k}(w-t, y) dw \right) dv.$$

*Proof.* Let  $p \in \mathbb{N}^*$ ,  $(\mathbf{x}_1 \dots, \mathbf{x}_p) \in (\mathbf{X}_T)^p$  satisfying the compatibility condition (4.4), then from Proposition 5.1 we have that

$$\begin{aligned}
\mathbb{E}[\tilde{\mathbf{Z}}_S^\ell \circ \varepsilon_{(\mathbf{x}_1, \dots, \mathbf{x}_p)}^{+,p}] &= \mathbb{E}[\tilde{\mathbf{Z}}_S^\ell] + \sum_{j=1}^p \tilde{\zeta}^{\ell, k_j}(S-t_j, y_j) \\
&\quad + \sum_{j=1}^p \int_{t_j}^S \bar{\zeta}^{\ell, \cdot}(S-v) \left( \tilde{\varphi}^{\cdot, k_j}(v-t_j, y_j) + \int_{t_j}^v \bar{\Psi}(v-w) \tilde{\varphi}^{\cdot, k_j}(w-t_j, y_j) dw \right) dv \\
&= \mathbb{E}[\tilde{\mathbf{Z}}_S^\ell] + \sum_{j=1}^p \tilde{\rho}_\ell(k_j, t_j, y_j).
\end{aligned}$$

Thus applying Theorem 5.5 with  $\mathbf{\Gamma}^\ell := \tilde{\mathbf{Z}}^\ell$  gives

$$\text{Cov}(\mathbf{Z}_T^i, \tilde{\mathbf{Z}}_S^\ell) = \mathbb{E}[(\mathbf{Z}_T^{\zeta \tilde{\rho}_\ell})^i] + \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \left( \int_0^v \bar{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \tilde{\rho}_\ell}] dw + \mathbb{E}[\mathbf{Z}_v^{\varphi \tilde{\rho}_\ell}] \right) dv,$$

where  $\mathbf{Z}^{\zeta \tilde{\rho}_\ell}$  is a  $(\zeta \tilde{\rho}_\ell, \mu, \varphi)$ -MSPD and  $\mathbf{Z}^{\varphi \tilde{\rho}_\ell}$  a  $(\varphi \tilde{\rho}_\ell, \mu, \varphi)$ -MSPD.  $\square$

Remark that the second term in the right hand side of (5.3) can also be written as the expectation of a  $(\zeta, \mathbb{E}[\mathbf{Z}^{\varphi \rho_\ell}], \varphi)$ -MSPD. More generally, this procedure, used here to compute the covariance, can be easily iterated to compute moments of further orders, to the price of cumbersome expressions.

#### 5.4 Case of a counting process with separable kernel

In the case of counting process with separable  $d$ -kernel  $\varphi(t, y) = \Phi(t) \star \mathbf{B}(y)$ , the expression of the covariance of the process at two different dates can be simplified, thanks to an extra convolution in the final expression. For  $1 \leq i, \ell \leq d$ , we introduce the transposed  $d$ -dimensional vector  $\mathbf{c}$

$$(\mathbf{c}^k)_{1 \leq k \leq d} := \left( \frac{\int_{\mathbb{R}_+} \mathbf{B}^{i,k}(y) \mathbf{B}^{\ell,k}(y) \nu^k(dy)}{\int_{\mathbb{R}_+} \mathbf{B}^{i,k}(y) \nu^k(dy) \int_{\mathbb{R}_+} \mathbf{B}^{\ell,k}(y) \nu^k(dy)} \right)_{1 \leq k \leq d} = \left( \frac{\mathbb{E}(\mathbf{B}^{i,k}(Y^k) \mathbf{B}^{\ell,k}(Y^k))}{\mathbb{E}(\mathbf{B}^{i,k}(Y^k)) \mathbb{E}(\mathbf{B}^{\ell,k}(Y^k))} \right)_{1 \leq k \leq d}. \quad (5.4)$$

Remark that  $\mathbf{c}^k = 1$  means that  $\text{Cov}(\mathbf{B}^{i,k}(Y^k), \mathbf{B}^{\ell,k}(Y^k)) = 0$ .

**Proposition 5.8.** *Let  $\mathbf{H}$  a  $(\mathbf{Id}_d, \mu, \varphi)$ -MSPD counting process and  $\varphi$  a separable  $d$ -kernel such that  $\varphi(t, y) = \Phi(t) \star \mathbf{B}(y)$ , then for  $0 \leq T \leq S$  and  $1 \leq i, \ell \leq d$  we have*

$$\begin{aligned} & \text{Cov}(\mathbf{H}_T^i, \mathbf{H}_S^\ell) \\ &= \int_0^T \int_u^T \overline{\Psi}^{i,\cdot}(t-u) dt \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{c} \star \int_u^S \overline{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \overline{\Psi}(u-v) \mu(v) dv + \mu(u) \right) du \\ &+ \int_0^T \mathbf{Id}_d^{i,\cdot} \star \left( \mathbf{Id}_d^{\ell,\cdot} + \int_u^S \overline{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \overline{\Psi}(u-v) \mu(v) dv + \mu(u) \right) du, \end{aligned}$$

where the transposed  $d$ -dimensional vector  $\mathbf{c}$  is given in (5.4).

Moreover, if  $\mathbf{c}^k = 1$  for all  $1 \leq k \leq d$  the expression simplifies as

$$\begin{aligned} & \text{Cov}(\mathbf{H}_T^i, \mathbf{H}_S^\ell) = \\ & \int_0^T \left( \mathbf{Id}_d^{i,\cdot} + \int_u^T \overline{\Psi}^{i,\cdot}(y-u) dy \right) \star \left( \mathbf{Id}_d^{\ell,\cdot} + \int_u^S \overline{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \overline{\Psi}(u-v) \mu(v) dv + \mu(u) \right) du. \end{aligned}$$

*Proof.* Applying Theorem 5.7 for  $\zeta = \mathbf{Id}_d$  ( $\mathbf{H}$  is a counting process)

$$\text{Cov}(\mathbf{H}_T^i, \mathbf{H}_S^\ell) = \mathbb{E}[(\mathbf{Z}_T^{\zeta \rho_\ell})^i] + \int_0^T \mathbf{Id}_d^{i,\cdot} \left( \int_0^v \overline{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \rho_\ell}] dw + \mathbb{E}[\mathbf{Z}_v^{\varphi \rho_\ell}] \right) dv, \quad (5.5)$$

where  $\zeta \rho_\ell := (\zeta \rho_\ell)^T$  is the diagonal matrix whose diagonal elements are the components of the vector  $\rho_\ell$  given below,  $\mathbf{Z}^{\zeta \rho_\ell}$  is a  $(\zeta \rho_\ell, \mu, \varphi)$ -MSPD,  $\mathbf{Z}^{\varphi \rho_\ell}$  is a  $(\varphi \rho_\ell, \mu, \varphi)$ -MSPD and

$$\begin{aligned} \rho_\ell(k, t, y) &= \mathbf{1}_{\{\ell=k\}} + \int_t^S \mathbf{Id}_d^{\ell,\cdot} \left( \varphi^{\cdot,k}(v-t, y) + \int_t^v \overline{\Psi}(v-w) \varphi^{\cdot,k}(w-t, y) dw \right) dv \\ &= \mathbf{1}_{\{\ell=k\}} + \int_t^S \left( \varphi^{\ell,k}(v-t, y) + \int_t^v \overline{\Psi}^{\ell,\cdot}(v-w) \varphi^{\cdot,k}(w-t, y) dw \right) dv. \end{aligned}$$

The proof is divided in four steps, first calculating in steps 1 and 2 the expectation of the corresponding MSPDs  $\mathbb{E}[\mathbf{Z}_T^{\zeta \rho_\ell}]$  and  $\mathbb{E}[\mathbf{Z}_w^{\varphi \rho_\ell}]$ , then step 3 computes  $\int_0^v \overline{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \rho_\ell}] dw$

and finally step 4 gathers the previous expressions to give the final result.

Steps 1 and 2 rely on the assumption of a separable excitation kernel  $\varphi(t, y) = \Phi(t) \star B(y)$ . We recall the notation  $\overline{B}^{\ell, k} := \int_{\mathbb{R}_+} B^{\ell, k}(y) \nu^k(dy)$ . In what follows, the ratio  $\frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}}$  represents the relative value of the outcome  $B^{\ell, k}(y)$  with respect to its mean value  $\overline{B}^{\ell, k}$ .

Step 1: Computing  $\mathbb{E}[(Z_T^{\zeta \rho \ell})^i]$ .

By putting the relative quantity  $\frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}}$  in factor (thanks to the separability of the kernel) and then by using the convolution relation (2.5)

$$\begin{aligned} \zeta \rho \ell^{i, k}(u, y) &= \mathbf{1}_{\{i=k\}} \left( \mathbf{1}_{\{\ell=k\}} + \int_u^S \left( \varphi^{\ell, k}(v-u, y) + \int_u^v \overline{\Psi}^{\ell, \cdot}(v-w) \varphi^{\cdot, k}(w-u, y) dw \right) dv \right) \\ &= \mathbf{1}_{\{i=k\}} \left( \mathbf{1}_{\{\ell=k\}} + \int_u^S \left( \overline{\varphi}^{\ell, k}(v-u) \frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}} + \left[ \int_u^v \overline{\Psi}(v-w) \overline{\varphi}(w-u) dw \right]^{\ell, k} \frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}} \right) dv \right) \\ &= \mathbf{1}_{\{i=k\}} \left( \mathbf{1}_{\{\ell=k\}} + \frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}} \int_u^S \overline{\Psi}^{\ell, k}(v-u) dv \right); \quad u \leq T. \end{aligned}$$

Integrating with respect to  $\nu^k(dy)$  then yields

$$\overline{\zeta \rho \ell}^{i, k}(u) = \mathbf{1}_{\{i=k\}} \left( \mathbf{1}_{\{\ell=k\}} + \int_u^S \overline{\Psi}^{\ell, k}(v-u) dv \right), \quad u \leq T,$$

and using Proposition 5.1 for a  $(\zeta \rho \ell, \mu, \varphi)$ -MSPD

$$\mathbb{E}[(Z_T^{\zeta \rho \ell})^i] = \int_0^T \mathbf{Id}_d^{i, \cdot} \star \left( \mathbf{Id}_d^{\ell, \cdot} + \int_u^S \overline{\Psi}^{\ell, \cdot}(v-u) dv \right) \left( \int_0^u \overline{\Psi}(u-v) \mu(v) dv + \mu(u) \right) du.$$

Step 2: Computing  $\mathbb{E}[(Z_T^{\varphi \rho \ell})^i]$ .

By repeating the same methodology as in Step 1, we have

$$\varphi \rho \ell^{i, k}(u, y) = \varphi^{i, k}(T-u, y) \left( \mathbf{1}_{\{\ell=k\}} + \frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}} \int_u^S \overline{\Psi}^{\ell, k}(v-u) dv \right).$$

Integrating with respect to  $\nu^k(dy)$  yields

$$\begin{aligned} \overline{\varphi \rho \ell}^{i, k}(u) &= \overline{\varphi}^{i, k}(T-u) \mathbf{1}_{\{\ell=k\}} + \int_{\mathbb{R}_+} \Phi^{i, k}(T-u) B^{i, k}(y) \frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}} \int_u^S \overline{\Psi}^{\ell, k}(v-u) dv \nu^k(dy) \\ &= \overline{\varphi}^{i, k}(T-u) \mathbf{1}_{\{\ell=k\}} + \overline{\varphi}^{i, k}(T-u) \int_u^S \overline{\Psi}^{\ell, k}(v-u) dv \int_{\mathbb{R}_+} \frac{B^{i, k}(y)}{\overline{B}^{i, k}} \frac{B^{\ell, k}(y)}{\overline{B}^{\ell, k}} \nu^k(dy) \\ &= \overline{\varphi}^{i, k}(T-u) \left( \mathbf{1}_{\{\ell=k\}} + \mathbf{e}^k \int_u^S \overline{\Psi}^{\ell, k}(v-u) dv \right) \end{aligned}$$

that is

$$\overline{\varphi \rho}^{i, \cdot}(u) = \overline{\varphi}^{i, \cdot}(T-u) \star \left( \mathbf{Id}_d^{\ell, \cdot} + \mathbf{e} \star \int_u^S \overline{\Psi}^{\ell, \cdot}(v-u) dv \right),$$

and using Proposition 5.1 for a  $(\varphi \rho \ell, \mu, \varphi)$ -MSPD

$$\mathbb{E}[(Z_T^{\varphi \rho \ell})^i] = \int_0^T \overline{\varphi}^{i, \cdot}(T-u) \star \left( \mathbf{Id}_d^{\ell, \cdot} + \mathbf{e} \star \int_u^S \overline{\Psi}^{\ell, \cdot}(v-u) dv \right) \left( \int_0^u \overline{\Psi}(u-v) \mu(v) dv + \mu(u) \right) du.$$

Step 3: Computing  $\int_0^t \bar{\Psi}(t-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \rho \ell}] dw$ .

Using Step 2 and again the convolution relation (2.5), we have

$$\begin{aligned}
& \sum_{k=1}^d \int_0^t \bar{\Psi}^{j,k}(t-w) \mathbb{E}[(\mathbf{Z}_w^{\varphi \rho \ell})^k] dw \\
&= \sum_{k=1}^d \int_0^t \bar{\Psi}^{j,k}(t-w) \int_0^w \bar{\varphi}^{k,\cdot}(w-u) \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) dudw \\
&= \int_0^t \left[ \int_u^t \bar{\Psi}(t-w) \bar{\varphi}(w-u) dw \right]^{j,\cdot} \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du \\
&= \int_0^t \left( \bar{\Psi}^{j,\cdot}(t-u) - \bar{\varphi}^{j,\cdot}(t-u) \right) \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du \\
&= \int_0^t \bar{\Psi}^{j,\cdot}(t-u) \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du - \mathbb{E}[(\mathbf{Z}_t^{\varphi \rho \ell})^j].
\end{aligned}$$

Step 4: Final result.

Coming back to (5.5), and integrating the results of the previous steps,

$$\begin{aligned}
& Cov(\mathbf{H}_T^i, \mathbf{H}_S^\ell) - \mathbb{E}[(\mathbf{Z}_T^{\zeta \rho \ell})^i] = \int_0^T \mathbf{Id}_d^{i,\cdot} \left( \int_0^t \bar{\Psi}(t-w) E[\mathbf{Z}_w^{\varphi \rho \ell}] dw + \mathbb{E}[\mathbf{Z}_t^{\varphi \rho \ell}] \right) dt \\
&= \sum_{j=1}^d \int_0^T \mathbf{Id}_d^{i,j} \int_0^t \bar{\Psi}^{j,\cdot}(t-u) \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) dudt \\
&= \int_0^T \int_0^t \sum_{j=1}^d \mathbf{1}_{\{i=j\}} \bar{\Psi}^{j,\cdot}(t-u) \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) dudt \\
&= \int_0^T \int_u^T \bar{\Psi}^{i,\cdot}(t-u) dt \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du.
\end{aligned}$$

By replacing the expression of  $\mathbb{E}[(\mathbf{Z}_T^{\zeta \rho \ell})^i]$  computed in Step 1, we get

$$\begin{aligned}
& Cov(\mathbf{H}_T^i, \mathbf{H}_S^\ell) \\
&= \int_0^T \int_u^T \bar{\Psi}^{i,\cdot}(t-u) dt \star \left( \mathbf{Id}_d^{\ell,\cdot} + \mathbf{e} \star \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du \\
&+ \int_0^T \mathbf{Id}_d^{i,\cdot} \star \left( \mathbf{Id}_d^{\ell,\cdot} + \int_u^S \bar{\Psi}^{\ell,\cdot}(v-u) dv \right) \left( \int_0^u \bar{\Psi}(u-v) \boldsymbol{\mu}(v) dv + \boldsymbol{\mu}(u) \right) du.
\end{aligned}$$

□

## 6 Conclusion

In this paper, we presented a general method for calculating different quantities related to multidimensional self-exciting processes with dependencies (MSPD). This class of processes

encompasses several quantities useful for risk assessment, such as counting, loss, intensity processes and their shifted versions, in a framework of cross-dependencies and impact of the severity component on the frequency component. The methodology relies on the Poisson imbedding, the pseudo-chaotic expansion and Malliavin calculus, and is illustrated here to compute explicit formula for the correlations. A forthcoming companion paper will be dedicated to further developments such as the computation of moments of higher order, as well as the pricing of insurance contracts with underlying asset a MSPD (e.g. stop-loss contracts).

## Appendix

*Proof of Lemma 5.6.* The aim is to compute the following quantity

$$\begin{aligned} A &:= \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\mathbf{X}_T^n} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \sum_{j=1}^n \boldsymbol{\rho}_\ell(t_j, y_j, k_j) \mathbf{m}^{\otimes n}(d\mathbf{x}_1, \dots, d\mathbf{x}_n) \\ &= \sum_{n=1}^{+\infty} \int_{\Delta_T^n} \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \int_{\mathbb{R}_+^2} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i \sum_{j=1}^n \boldsymbol{\rho}_\ell(t_j, y_j, k_j) dt_1 d\theta_1 \boldsymbol{\nu}^{k_1}(dy_1) \cdots dt_n d\theta_n \boldsymbol{\nu}^{k_n}(dy_n). \end{aligned}$$

where  $\mathbf{x}_i := (k_i, t_i, \theta_i, y_i)$ ,  $\Delta_T^n$  is the simplex

$$\Delta_T^n := \{(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{X}^n, t_1 < \dots < t_i < t_{i+1} < \dots < t_n < T\}$$

so that for any symmetric map  $f$ ,  $\frac{1}{n!} \int_{[0, T]^n} f(t_1, \dots, t_n) dt_1 \cdots dt_n = \int_{\Delta_T^n} f(t_1, \dots, t_n) dt_1 \cdots dt_n$ . The proof is divided in 5 steps, by successively integrating with respect to the  $\theta_i$  (Step 1), then the  $y_i$  (Step 2), followed by the  $k_i$  (Step 3), and finally with respect to the  $t_i$  (Step 4). The last step consists in summing all the previous terms depending on  $n$  (Step 5).

Step 1 : Integration with respect to  $\theta$ .

Since  $\boldsymbol{\rho}_\ell$  does not depend on  $\theta$ , we integrate first  $\mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i$  with respect to the  $\theta_i$  variables by making use of (4.2) and Remark 4.2 we have

$$\begin{aligned} &\int_{\mathbb{R}_+^n} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i d\theta_1 \cdots d\theta_n \\ &= \int_{\mathbb{R}_+^n} \zeta^{i, k_n}(T - t_n, y_n) \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})}^{n-1} \mathbf{1}_{\{\theta_n \leq \lambda_{t_n}^{k_n}\}} d\theta_1 \cdots d\theta_n \\ &= \zeta^{i, k_n}(T - t_n, y_n) \int_{\mathbb{R}_+^{n-1}} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-1})}^{n-1} \lambda_{t_n}^{k_n} d\theta_1 \cdots d\theta_{n-1} \\ &= \zeta^{i, k_n}(T - t_n, y_n) \int_{\mathbb{R}_+^{n-1}} \varphi^{k_n, k_{n-1}}(t_n - t_{n-1}, y_{n-1}) \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-2})}^{n-2} \mathbf{1}_{\{\theta_{n-1} \leq \lambda_{t_{n-1}}^{k_{n-1}}\}} d\theta_1 \cdots d\theta_{n-1} \\ &= \zeta^{i, k_n}(T - t_n, y_n) \varphi^{k_n, k_{n-1}}(t_n - t_{n-1}, y_{n-1}) \int_{\mathbb{R}_+^{n-2}} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_{n-2})}^{n-2} \lambda_{t_{n-1}}^{k_{n-1}} d\theta_1 \cdots d\theta_{n-2}. \end{aligned}$$

Hence by induction

$$\int_{\mathbb{R}_+^n} \mathcal{T}_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}^n \mathbf{Z}_T^i d\theta_1 \cdots d\theta_n = \zeta^{i, k_n}(T - t_n, y_n) \prod_{m=2}^n \varphi^{k_m, k_{m-1}}(t_m - t_{m-1}, y_{m-1}) \boldsymbol{\mu}^{k_1}(t_1) =: F(\mathbf{k}, \mathbf{t}, \mathbf{y})$$

with  $\mathbf{k} := k_1, \dots, k_n$ ,  $\mathbf{t} := t_1, \dots, t_n$ ,  $\mathbf{y} := y_1, \dots, y_n$ , and

$$A = \sum_{n=1}^{+\infty} \int_{\Delta_T^n} \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \int_{\mathbb{R}_+} F(\mathbf{k}, \mathbf{t}, \mathbf{y}) \sum_{j=1}^n \rho_\ell(t_j, y_j, k_j) dt_1 \nu^{k_1}(dy_1) \cdots dt_n \nu^{k_n}(dy_n)$$

Step 2: Integration with respect to  $y$ .

We separately treat the cases where  $\sum \rho_\ell \equiv 1$  and  $\sum \rho_\ell \not\equiv 1$ , and we denote the corresponding values respectively  $A^\emptyset$  and  $A^\rho$ .

• If  $\sum \rho_\ell \equiv 1$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left[ \zeta^{i, k_n}(T - t_n, y_n) \prod_{m=2}^n \varphi^{k_m, k_{m-1}}(t_m - t_{m-1}, y_{m-1}) \mu^{k_1}(t_1) \right] \nu^{k_1}(dy_1) \cdots \nu^{k_n}(dy_n) \\ &= \bar{\zeta}^{i, k_n}(T - t_n) \prod_{m=2}^n \bar{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \mu^{k_1}(t_1). \end{aligned}$$

So in that case

$$A^\emptyset = \sum_{n=1}^{+\infty} \int_{\Delta_T^n} \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \bar{\zeta}^{i, k_n}(T - t_n) \prod_{m=2}^n \bar{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \mu^{k_1}(t_1) dt_1 \cdots dt_n.$$

• If  $\sum \rho_\ell \not\equiv 1$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \left[ \zeta^{i, k_n}(T - t_n, y_n) \prod_{m=2}^n \varphi^{k_m, k_{m-1}}(t_m - t_{m-1}, y_{m-1}) \mu^{k_1}(t_1) \right] \sum_{j=1}^n \rho_\ell(t_j, y_j, k_j) \nu^{k_1}(dy_1) \cdots \nu^{k_n}(dy_n) \\ &= \int_{\mathbb{R}_+^n} \left[ \zeta^{i, k_n}(T - t_n, y_n) \rho_\ell(t_n, y_n, k_n) \prod_{m=2}^n \varphi^{k_m, k_{m-1}}(t_m - t_{m-1}, y_{m-1}) \mu^{k_1}(t_1) \right] \nu^{k_1}(dy_1) \cdots \nu^{k_n}(dy_n) \\ &+ \int_{\mathbb{R}_+^n} \left[ \zeta^{i, k_n}(T - t_n, y_n) \prod_{m=2}^n \varphi^{k_m, k_{m-1}}(t_m - t_{m-1}, y_{m-1}) \mu^{k_1}(t_1) \right] \sum_{j=1}^{n-1} \rho_\ell(t_j, y_j, k_j) \nu^{k_1}(dy_1) \cdots \nu^{k_n}(dy_n) \\ &= \bar{\zeta} \rho_\ell^{i, k_n}(t_n) \prod_{m=2}^n \bar{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \mu^{k_1}(t_1) \\ &+ \sum_{j=1}^{n-1} \int_{\mathbb{R}_+^n} \zeta^{i, k_n}(T - t_n, y_n) \varphi^{k_{j+1}, k_j}(t_{j+1} - t_j, y_j) \rho_\ell(t_j, y_j, k_j) \\ &\left( \prod_{m=2, m \neq j+1}^n \varphi^{k_m, k_{m-1}}(t_m - t_{m-1}, y_{m-1}) \right) \mu^{k_1}(t_1) \nu^{k_1}(dy_1) \cdots \nu^{k_n}(dy_n) \\ &= \bar{\zeta} \rho_\ell^{i, k_n}(t_n) \prod_{m=2}^n \bar{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \mu^{k_1}(t_1) \\ &+ \sum_{j=1}^{n-1} \bar{\zeta}^{i, k_n}(T - t_n) \bar{\varphi} \rho_\ell^{k_{j+1}, k_j}(t_{j+1} - t_j) \left( \prod_{m=2, m \neq j+1}^n \bar{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \right) \mu^{k_1}(t_1) \end{aligned}$$

$$=: F^\rho(\mathbf{k}, \mathbf{t})$$

where by abuse of notation

$$\overline{\varphi \rho_\ell}^{k_{j+1}, k_j}(t_{j+1} - t_j) := \int \varphi^{k_{j+1}, k_j}(t_{j+1} - t_j, y_j) \rho_\ell(k_j, t_j, y_j) \nu^{k_j}(dy_j).$$

So in that case

$$A^\rho = \sum_{n=1}^{+\infty} \int_{\Delta_T^n} \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d F^\rho(\mathbf{k}, \mathbf{t}) dt_1 \cdots dt_n.$$

Step 3: Integration with respect to  $k$ .

This step lies essentially in the identification of matrix products.

- If  $\sum \rho_\ell \equiv 1$ ,

$$\begin{aligned} & \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \overline{\zeta}^{i, k_n}(T - t_n) \prod_{m=2}^n \overline{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \boldsymbol{\mu}^{k_1}(t_1) \\ &= \sum_{k_n=1}^d \overline{\zeta}^{i, k_n}(T - t_n) \left( \prod_{m=2}^n \overline{\varphi}(t_m - t_{m-1}) \boldsymbol{\mu}(t_1) \right)^{k_n} \\ &= \overline{\zeta}^{i, \cdot}(T - t_n) \prod_{m=2}^n \overline{\varphi}(t_m - t_{m-1}) \boldsymbol{\mu}(t_1). \end{aligned}$$

Hence

$$A^\rho = \sum_{n=1}^{+\infty} \int_{\Delta_T^n} \overline{\zeta}^{i, \cdot}(T - t_n) \prod_{m=2}^n \overline{\varphi}(t_m - t_{m-1}) \boldsymbol{\mu}(t_1) dt_1 \cdots dt_n.$$

- If  $\sum \rho_\ell \neq 1$ ,

$$\begin{aligned} & \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \overline{\zeta \rho_\ell}^{i, k_n}(t_n) \prod_{m=2}^n \overline{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \boldsymbol{\mu}^{k_1}(t_1) \\ &+ \sum_{k_1=1}^d \cdots \sum_{k_n=1}^d \sum_{j=1}^{n-1} \overline{\zeta}^{i, k_n}(T - t_n) \overline{\varphi \rho_\ell}^{k_{j+1}, k_j}(t_{j+1} - t_j) \left( \prod_{m=2, m \neq j+1}^n \overline{\varphi}^{k_m, k_{m-1}}(t_m - t_{m-1}) \right) \boldsymbol{\mu}^{k_1}(t_1) \\ &= \overline{\zeta \rho_\ell}^{i, \cdot}(t_n) \prod_{m=2}^n \overline{\varphi}(t_m - t_{m-1}) \boldsymbol{\mu}(t_1) \\ &+ \sum_{j=1}^{n-1} \overline{\zeta}^{i, \cdot}(T - t_n) \prod_{m=j+2}^n \overline{\varphi}(t_m - t_{m-1}) \overline{\varphi \rho_\ell}(t_{j+1} - t_j) \prod_{m=2}^j \overline{\varphi}(t_m - t_{m-1}) \boldsymbol{\mu}(t_1) \\ &=: G^\rho(\mathbf{t}). \end{aligned}$$

Hence,

$$A^\rho = \sum_{n=1}^{+\infty} \int_{\Delta_T^n} G^\rho(\mathbf{t}) dt_1 \cdots dt_n.$$

Step 4: Integration with respect to  $t$ .

The step consists in identifying the convolution products that appear in the expressions.

- If  $\sum \rho_\ell \equiv 1$ ,

$$\int_{\Delta_T^n} \bar{\zeta}^{i,\cdot}(T-t_n) \prod_{m=2}^n \bar{\varphi}(t_m-t_{m-1}) \boldsymbol{\mu}(t_1) dt_1 \dots dt_n = \int_0^T \int_{t_1}^T \bar{\zeta}^{i,\cdot}(T-t_n) \bar{\varphi}_{n-1}(t_n-t_1) \boldsymbol{\mu}(t_1) dt_n dt_1.$$

Hence,

$$A^\rho = \sum_{n=1}^{+\infty} \int_0^T \int_w^T \bar{\zeta}^{i,\cdot}(T-v) \bar{\varphi}_{n-1}(v-w) \boldsymbol{\mu}(w) dv dw.$$

- If  $\sum \rho_\ell \neq 1$ ,

the first term in  $G^\rho(\mathbf{t})$  integrates as above:

$$\int_{\Delta_T^n} \bar{\zeta}^{\rho_\ell, i,\cdot}(t_n) \prod_{m=2}^n \bar{\varphi}(t_m-t_{m-1}) \boldsymbol{\mu}(t_1) dt_1 \dots dt_n = \int_0^T \int_w^T \bar{\zeta}^{\rho_\ell, i,\cdot}(v) \bar{\varphi}_{n-1}(v-w) \boldsymbol{\mu}(t_1) dv dw$$

where we recall that

$$\bar{\zeta}^{\rho_\ell, k_{j+1}, k_j}(v) := \int \zeta^{k_{j+1}, k_j}(T-v, y_j) \rho_\ell(k_j, v, y_j) \boldsymbol{\nu}^{k_j}(dy_j).$$

For the second term

$$\begin{aligned} & \int_{\Delta_T^n} \sum_{j=1}^{n-1} \bar{\zeta}^{i,\cdot}(T-t_n) \prod_{m=j+2}^n \bar{\varphi}(t_m-t_{m-1}) \bar{\varphi} \rho_\ell(t_{j+1}-t_j) \prod_{m=2}^j \bar{\varphi}(t_m-t_{m-1}) \boldsymbol{\mu}(t_1) dt_1 \dots dt_n \\ &= \sum_{j=1}^{n-1} \int_0^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_0^{t_n} \dots \int_0^{t_{j+2}} \prod_{m=j+2}^n \bar{\varphi}(t_m-t_{m-1}) \int_0^{t_{j+1}} \bar{\varphi} \rho_\ell(t_{j+1}-t_j) \int_0^{t_j} \dots \\ & \quad \int_0^{t_2} \prod_{m=2}^j \bar{\varphi}(t_m-t_{m-1}) \boldsymbol{\mu}(t_1) dt_1 \dots dt_n \\ &= \sum_{j=1}^{n-1} \int_0^T \int_{t_j}^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_{t_j}^{t_n} \bar{\varphi}_{n-j-1}(t_n-t_{j+1}) \bar{\varphi} \rho_\ell(t_{j+1}-t_j) \int_0^{t_j} \bar{\varphi}_{j-1}(t_j-t_1) \boldsymbol{\mu}(t_1) dt_1 dt_{j+1} dt_n dt_j. \end{aligned}$$

Hence, denoting

$$E_n^\rho :=$$

$$\begin{aligned} & \int_0^T \int_w^T \bar{\zeta}^{\rho_\ell, i,\cdot}(v) \bar{\varphi}_{n-1}(v-w) \boldsymbol{\mu}(t_1) dv dw \\ & + \sum_{j=1}^{n-1} \int_0^T \int_{t_j}^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_{t_j}^{t_n} \bar{\varphi}_{n-j-1}(t_n-t_{j+1}) \bar{\varphi} \rho_\ell(t_{j+1}-t_j) \int_0^{t_j} \bar{\varphi}_{j-1}(t_j-t_1) \boldsymbol{\mu}(t_1) dt_1 dt_{j+1} dt_n dt_j \end{aligned}$$

$$A^\rho = \sum_{n=1}^{+\infty} E_n^\rho.$$

Step 5 : Summing in  $n$ .

- If  $\sum \rho_\ell \equiv 1$ , using that  $\bar{\varphi}_0$  is the Dirac distribution in 0,

$$\begin{aligned} A^\rho &= \sum_{n=1}^{+\infty} \int_0^T \int_w^T \bar{\zeta}^{i,\cdot}(T-v) \bar{\varphi}_{n-1}(v-w) \boldsymbol{\mu}(w) dv dw \\ &= \int_0^T \int_w^T \bar{\zeta}^{i,\cdot}(T-v) (\bar{\Psi}(v-w) + \bar{\varphi}_0(v-w)) \boldsymbol{\mu}(w) dv dw \\ &= \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \left( \boldsymbol{\mu}(v) + \int_0^v \bar{\Psi}(v-w) \boldsymbol{\mu}(w) dw \right) dv = \mathbb{E}[\mathbf{Z}_T^i]. \end{aligned}$$

- If  $\sum \rho_\ell \neq 1$ , the first term in  $E_n^\rho$  sums as above

$$\sum_{n=1}^{+\infty} \int_0^T \int_w^T \bar{\zeta}^{\rho_\ell, i,\cdot}(v) \bar{\varphi}_{n-1}(v-w) \boldsymbol{\mu}(w) dv dw = \mathbb{E} \left[ (\mathbf{Z}_T^{\zeta \rho_\ell})^i \right]$$

and the second term sums as, denoting  $\mathbf{Z}^{\zeta \rho_\ell}$  a  $(\zeta \rho_\ell, \boldsymbol{\mu}, \varphi)$ -MSPD

$$\begin{aligned} &\sum_{n=1}^{+\infty} \sum_{j=1}^{n-1} \int_0^T \int_{t_j}^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_{t_j}^{t_n} \bar{\varphi}_{n-j-1}(t_n-t_{j+1}) \bar{\varphi}_{\rho_\ell}(t_{j+1}-t_j) \int_0^{t_j} \bar{\varphi}_{j-1}(t_j-t_1) \boldsymbol{\mu}(t_1) dt_1 dt_{j+1} dt_n dt_j \\ &= \sum_{j=1}^{+\infty} \sum_{n=j+1}^{+\infty} \int_0^T \int_{t_j}^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_{t_j}^{t_n} \bar{\varphi}_{n-j-1}(t_n-t_{j+1}) \bar{\varphi}_{\rho_\ell}(t_{j+1}-t_j) \int_0^{t_j} \bar{\varphi}_{j-1}(t_j-t_1) \boldsymbol{\mu}(t_1) dt_1 dt_{j+1} dt_n dt_j \\ &= \int_0^T \int_{t_j}^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_{t_j}^{t_n} (\bar{\Psi}(t_n-t_{j+1}) + \bar{\varphi}_0(t_n-t_{j+1})) \bar{\varphi}_{\rho_\ell}(t_{j+1}, t_j) \int_0^{t_j} (\bar{\Psi}(t_j-t_1) + \bar{\varphi}_0(t_j-t_1)) \boldsymbol{\mu}(t_1) dt_1 dt_{j+1} dt_n dt_j \\ &= \int_0^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_0^{t_n} (\bar{\Psi}(t_n-t_{j+1}) + \bar{\varphi}_0(t_n-t_{j+1})) \int_0^{t_{j+1}} \bar{\varphi}_{\rho_\ell}(t_{j+1}-t_j) \left( \int_0^{t_j} \bar{\Psi}(t_j-t_1) \boldsymbol{\mu}(t_1) dt_1 + \boldsymbol{\mu}(t_j) \right) dt_j dt_{j+1} dt_n \\ &= \int_0^T \bar{\zeta}^{i,\cdot}(T-t_n) \int_0^{t_n} (\bar{\Psi}(t_n-t_{j+1}) + \bar{\varphi}_0(t_n-t_{j+1})) \mathbb{E}[\mathbf{Z}_{t_{j+1}}^{\varphi \rho_\ell}] dt_{j+1} dt_n \\ &= \int_0^T \bar{\zeta}^{i,\cdot}(T-v) \left( \mathbb{E}[\mathbf{Z}_v^{\varphi \rho_\ell}] + \int_0^v \bar{\Psi}(v-w) \mathbb{E}[\mathbf{Z}_w^{\varphi \rho_\ell}] dw \right) dv \end{aligned}$$

which concludes the proof.  $\square$

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