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DEEP-WATER LIMIT OF THE INTERMEDIATE LONG WAVE EQUATION IN L^2

ANDREIA CHAPOUTO, GUOPENG LI, TADAHIRO OH, AND DIDIER PILOD

ABSTRACT. We study the well-posedness issue of the intermediate long wave equation (ILW) on both the real line and the circle. By applying the gauge transform for the Benjamin-Ono equation (BO) and adapting the L^2 well-posedness argument for BO by Molinet and the fourth author (2012), we prove global well-posedness of ILW in L^2 on both the real line and the circle. In the periodic setting, this provides the first low regularity well-posedness of ILW. We then establish convergence of the ILW dynamics to the BO dynamics in the deep-water limit at the L^2 -level.

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1. INTRODUCTION

In this paper, we study the intermediate long wave equation (ILW):

$$\begin{cases} \partial_t u - \mathcal{G}_\delta \partial_x^2 u = \partial_x(u^2) \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathcal{M}, \quad (1.1)$$

where $\mathcal{M} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The equation (1.1), introduced in [27, 36], models the internal wave propagation of the interface in a stratified fluid of finite depth $\delta > 0$, and the unknown $u : \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}$ denotes the amplitude of the internal wave at the interface. See also Remark 1.1 in [39]. The operator \mathcal{G}_δ in (1.1) is defined by

$$\mathcal{G}_\delta = \mathcal{T}_\delta - \delta^{-1} \partial_x^{-1}, \quad (1.2)$$

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where \mathcal{T}_δ is given by

$$\widehat{\mathcal{T}_\delta f}(\xi) = -i \coth(\delta\xi) \widehat{f}(\xi), \quad \xi \in \widehat{\mathcal{M}}. \quad (1.3)$$

Here, $\widehat{\mathcal{M}}$ denotes the Pontryagin dual of \mathcal{M} , i.e. $\widehat{\mathcal{M}} = \mathbb{R}$ if $\mathcal{M} = \mathbb{R}$, and $\widehat{\mathcal{M}} = \mathbb{Z}$ if $\mathcal{M} = \mathbb{T}$, and \coth denotes the usual hyperbolic cotangent function:

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}, \quad x \in \mathbb{R} \setminus \{0\} \quad (1.4)$$

with the convention $\coth(x) - \frac{1}{x} = 0$ for $x = 0$. Note that \mathcal{T}_δ is the inverse of the so-called Tilbert transform (with an extra $-$ sign), appearing in the study of water waves of finite depth [6, 20, 2]. In particular, \mathcal{T}_δ is related to the Dirichlet-to-Neumann map in a two-dimensional finite depth two-layer fluid, and in the deep-water limit ($\delta \rightarrow \infty$), it converges to the Hilbert transform \mathcal{H} (with the Fourier multiplier $-i \operatorname{sgn}(\xi)$), corresponding to the Dirichlet-to-Neumann map in infinite depth, which indicates possible convergence of ILW (1.1) to the Benjamin-Ono equation (BO):

$$\partial_t u - \mathcal{H} \partial_x^2 u = \partial_x(u^2). \quad (1.5)$$

Indeed, ILW (1.1) is an important physical model, providing a natural connection between the deep-water regime (= the Benjamin-Ono regime) and the shallow-water regime (= the KdV regime). As such, it has been studied extensively from both the applied and theoretical points of view. See, for example, a recent book [33, Chapter 3] by Klein and Saut for an overview of the subject and the references therein. See also a survey [51]. While the ILW equation has attracted increasingly more attention in recent years (see, for example, [47, 31, 48, 37, 38, 39, 22, 12]), these two references indicate that the rigorous mathematical study of ILW is still widely open, especially as compared to BO and KdV. In particular, one of the fundamental, but challenging questions is the convergence properties of ILW in the deep-water limit ($\delta \rightarrow \infty$) and in the shallow-water limit ($\delta \rightarrow 0$). See [37, 38, 39, 12] for recent progress on convergence problems from both the deterministic and statistical points of view.

Our main goal in this paper is to present a unified argument for proving L^2 global well-posedness of ILW (1.1) on both the real line and the circle and then to establish convergence of the ILW dynamics to the BO dynamics in the deep-water limit at the L^2 -level. Before stating our main result, let us recall a few conservation laws for ILW (1.1):

$$\begin{aligned} \text{mean: } & \int_{\mathcal{M}} u dx, & \text{mass: } M(u) &= \int_{\mathcal{M}} u^2 dx, \\ \text{Hamiltonian: } & E_\delta(u) = \frac{1}{2} \int_{\mathcal{M}} u \mathcal{G}_\delta \partial_x u dx + \frac{1}{3} \int_{\mathcal{M}} u^3 dx. \end{aligned} \quad (1.6)$$

In particular, ILW (1.1) is a Hamiltonian PDE. Furthermore, it is known to be completely integrable. We, however, do not make use of the completely integrable structure in this paper.

We now state our first result.

Theorem 1.1. *Let $\mathcal{M} = \mathbb{R}$ or \mathbb{T} and $0 < \delta < \infty$. Then, given any $s \geq 0$, ILW (1.1) is globally well-posed in $H^s(\mathcal{M})$. Moreover, when $\mathcal{M} = \mathbb{T}$, by restricting to the subspace $H_0^s(\mathbb{T})$ consisting of mean-zero functions, the solution map for (1.1) is locally Lipschitz continuous from $H_0^s(\mathbb{T})$ to $C(\mathbb{R}; H_0^s(\mathbb{T}))$, where the latter is endowed with the compact-open topology (in time).*

In [46], it was shown that both BO and ILW are quasilinear in the sense that the solution map is not smooth; see also [35]. This in particular implies that we can not use a contraction argument for constructing solutions, making the well-posedness question rather challenging. Nonetheless, the well-posedness issue of the BO equation (1.5), corresponding to the infinite depth case ($\delta = \infty$), has been studied extensively on both the real line and the circle; see [26, 50, 34, 52, 24, 42, 10, 43, 44, 47, 23, 17, 32]. On the other hand, there are only a handful of known well-posedness results for ILW (1.1); see [1, 47, 45, 22]. In [47], Molinet and Vento proved local well-posedness of (1.1) in $H^{\frac{1}{2}}(\mathcal{M})$, $\mathcal{M} = \mathbb{R}$ or \mathbb{T} , via the improved energy method, which immediately extended to global well-posedness thanks to the conservation laws in (1.6). In the real line case, Molinet, Vento, and the fourth author [45] refined the improved energy method in [47] (with the short-time Strichartz estimates as in [34, 28, 40] and a generalized Coifman-Meyer theorem) and proved local well-posedness of (1.1) in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. In a recent work [22], Ifrim and Saut studied (1.1) on the real line and proved its global well-posedness in $L^2(\mathbb{R})$ (along with long-time behavior of small solutions), based on an adaptation of the argument developed for the BO equation (1.5) by Ifrim and Tataru [23] which combines the BO gauge transform and the (quasilinear) norm form method. We point out that the argument in [22] relies on dispersive estimates and thus is not applicable to the periodic setting.

Our main strategy is to view (1.1) as a perturbation¹ of (1.5) as in [22] and apply the BO gauge transform, originally introduced by Tao [52]. We then adapt the argument in [44], where Molinet and the fourth author introduced a simplified and unified approach to the L^2 well-posedness of the BO equation (1.5) on both the circle and real line, to the current ILW problem. See Sections 3 and 4 for details.

Next, we state our second result on convergence in the deep-water limit.

Theorem 1.2. *Let $\mathcal{M} = \mathbb{R}$ or \mathbb{T} and $s \geq 0$. Given $u_0 \in H^s(\mathcal{M})$, let u be the global solution to the BO equation (1.5) with $u|_{t=0} = u_0$ constructed in [44]. Let $\{u_{0,\delta}\}_{1 \leq \delta < \infty} \subset H^s(\mathcal{M})$ such that $u_{0,\delta}$ converges to u_0 in $H^s(\mathcal{M})$ as $\delta \rightarrow \infty$, and let u_δ be the global solution to (1.1) with $u_\delta|_{t=0} = u_{0,\delta}$ constructed in Theorem 1.1. Then, as $\delta \rightarrow \infty$, u_δ converges to u in $C(\mathbb{R}; H^s(\mathcal{M}))$, endowed with the compact-open topology (in time).*

Theorem 1.2 follows from a minor adaptation of the proof of Theorem 1.1. See Section 5.

In [37], the second author proved the deep-water convergence on both the real line and the circle for $s > \frac{1}{2}$. In view of the known results [47, 37], we restrict our attention to the range $0 \leq s \leq \frac{1}{2}$ in the remaining part of the paper. Furthermore, we present the proofs of Theorems 1.1 and 1.2 only in the periodic case, which is closely related to the study of invariant measures; see Remark 1.5. The required modification for the real line case is minor and straightforward in view of the presentation of the current paper and [44], where the latter treated the real line case. In the periodic setting, in order to avoid the problem at the zeroth frequency, it is crucial to work with mean-zero functions. In view of the conservation of the spatial mean, we perform a change of unknowns via the following Galilean transform [14] (with $\mu(u_0) = \frac{1}{2\pi} \int_{\mathbb{T}} u_0(x) dx$):

$$\tilde{u}(t, x) = \Gamma_{u_0}(u)(t, x) = u(t, x - 2\mu(u_0)t) - \mu(u_0) \quad (1.7)$$

¹after applying the Galilean transform (3.1) to remove the term $\delta^{-1} \partial_x^{-1}$ in (1.2). See (3.2).

to reduce the problem to the mean-zero case in proving Theorem 1.1. Lastly, for simplicity of the presentation, we restrict our attention to positive times in the remaining part of the paper.

Remark 1.3. (i) The local Lipschitz continuity the solution map for (1.1) on \mathbb{T} claimed in Theorem 1.1 indeed holds on any hyperplane of $H^s(\mathbb{T})$ of functions with a fixed mean. In [43, Theorem 1.2], Molinet proved that the solution map to (1.5) on $H_0^s(\mathbb{T})$ is in fact real-analytic for $s \geq 0$ (and fails to be smooth for $s < 0$). While we expect the same results to hold for ILW (1.1), we do not pursue this issue in this paper.

(ii) Proceeding as in [44], we can prove unconditional uniqueness of ILW (1.1) in $H^s(\mathbb{R})$ for $\frac{1}{4} < s \leq \frac{1}{2}$ and $H^{\frac{1}{2}}(\mathbb{T})$. (For $s > \frac{1}{2}$, it was shown in [47] on both the real line and the circle.) Since the required modification is straightforward, we omit details.

Remark 1.4. Given $\delta > 0$, consider the following scaling transformation [1]:

$$v(t, x) = \frac{3}{\delta} u\left(\frac{3}{\delta}t, x\right), \quad (1.8)$$

which transforms (1.1) to the following scaled ILW equation:

$$\partial_t v - \frac{3}{\delta} \mathcal{G}_\delta \partial_x^2 v = \partial_x(v^2). \quad (1.9)$$

Then, under suitable assumptions, the scaled ILW (1.9) is known to converge to the KdV equation in the shallow-water limit ($\delta \rightarrow 0$) [1, 37, 39]:

$$\partial_t v + \partial_x^3 v = \partial_x(v^2).$$

It would be of interest to study the shallow-water convergence of ILW (1.1) at the L^2 -level. As mentioned above, our argument for proving Theorem 1.1 relies on the Galilean transform (3.1) to remove $\delta^{-1} \partial_x^{-1}$ in (1.2) (which is crucial in the shallow-water limit) which is not amenable to the shallow-water limit since the spatial translation by $\delta^{-1}t$ diverges as $\delta \rightarrow 0$.

We point out that the scaling transformation (1.8) is consistent with the physical setting. The ILW equation (1.1) describes the motion of the fluid interface in a stratified fluid of depth $\delta > 0$, where u denotes the amplitude of the internal wave at the interface. As $\delta \rightarrow 0$, the entire fluid depth tends to 0 and, in particular, the amplitude of the internal wave at the interface is $O(\delta)$, which also tends to 0. Hence, if we want to observe any meaningful limiting behavior, we need to magnify the fluid motion by a factor $\sim \frac{1}{\delta}$, which is exactly what the scaling transformation (1.8) does.

Remark 1.5. The ILW equation (1.1) and the BO equation (1.5) are known to be completely integrable. In particular, they possess infinite sequences of conservation laws $E_{\delta, \frac{k}{2}}(u)$ and $E_{\text{BO}, \frac{k}{2}}(u)$ at regularities $\frac{k}{2}$, $k \in \mathbb{N} \cup \{0\}$. In [39, 12], the first three authors (with Zheng) constructed the (formally invariant) measures $\rho_{\delta, \frac{k}{2}}$ (with a density of the form $\exp(-E_{\delta, \frac{k}{2}}(u))$) associated with these conservation laws for the periodic ILW on \mathbb{T} , $k \in \mathbb{N}$, and proved their convergence to the corresponding measures for the periodic BO, constructed by Tzvetkov and Visciglia [54], in the deep-water limit.² Moreover, when $k \geq 3$, they established invariance of these measures $\rho_{\delta, \frac{k}{2}}$ under the ILW dynamics (for each fixed $0 < \delta < \infty$) and convergence of

²In [39, 12], convergence in the shallow-water limit was also studied, but we restrict our discussion to the deep-water limit here.

the associated invariant ILW dynamics to the corresponding invariant BO dynamics [55, 56] in the deep-water limit.

It is known that $\rho_{\delta, \frac{k}{2}}$ is supported on $H^{\frac{k}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\frac{k}{2}}(\mathbb{T})$ for any $\varepsilon > 0$. In particular, when $k = 2$, the measure $\rho_{\delta, 1}$ is supported slightly outside $H^{\frac{1}{2}}(\mathbb{T})$, where Theorem 1.1 now provides globally well-defined dynamics. Therefore, the $k = 2$ case is now within reach. In a forthcoming work, we will treat the $k = 2$ case: invariance of the measure $\rho_{\delta, 1}$ under the ILW dynamics (for each fixed $0 < \delta < \infty$) and deep-water convergence of the invariant $\rho_{\delta, 1}$ -dynamics to the corresponding invariant BO dynamics due to Deng, Tzvetkov, and Visciglia [15].

Remark 1.6. In a recent preprint [11], with J. Forlano, we further exploited the idea of viewing ILW (1.1) as a perturbation of BO (1.5) and established a global-in-time a priori bound on the H^s -norm of a solution to (1.1) for $-\frac{1}{2} < s < 0$. More precisely, by making use of the complete integrability of the BO equation, we showed that, given $-\frac{1}{2} < s < 0$, there exists an (explicit) non-decreasing function $C_s : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|u(t)\|_{H^s} \leq C_s(|t|, \|u_0\|_{H^s})$$

for any smooth solution u to (1.1) on $\mathcal{M} = \mathbb{R}$ or \mathbb{T} and any $t \in \mathbb{R}$. In the same work, we also proved ill-posedness of (1.1) in $H^s(\mathcal{M})$ for $s < -\frac{1}{2}$, thus establishing sharpness (of the regularity range) of the aforementioned a priori bound (modulo the endpoint $s = -\frac{1}{2}$). Despite the lack of scaling invariance, these results allow us to identify $s = -\frac{1}{2}$ as a “critical” regularity for ILW; see Remark 4.2 below for a discussion on the scaling property of ILW.

We also mention a recent work [41], where the authors studied long time behavior of solutions to ILW (1.1) on the real line, in the same spirit as the work [16] on long time decaying properties of solutions to the BO equation (1.5).

2. NOTATIONS, FUNCTION SPACES, AND PRELIMINARY ESTIMATES

By $A \lesssim B$, we mean $A \leq CB$ for some constant $C > 0$. We use $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$, if there is some small $c > 0$, such that $A \leq cB$.

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define the L^2 -based Sobolev space $H^s(\mathbb{T})$ by the norm:

$$\|f\|_{H^s} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell_n^2},$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$, and we set $H_0^s(T) = \{f \in H^s(\mathbb{T}) : \widehat{f}(0) = 0\}$ as the subspace consisting of mean-zero functions. We also define the L^p -based Sobolev space $W^{s,p}(\mathbb{T})$ by the norm:

$$\|f\|_{W^{s,p}} = \|\langle D \rangle^s f\|_{L^p} = \|\mathcal{F}^{-1}(\langle n \rangle^s \widehat{f}(n))\|_{L^p},$$

where $D = -i\partial_x$ (with the Fourier multiplier n) and \mathcal{F}^{-1} denotes the inverse Fourier transform. When $p = 2$, we have $H^s(\mathbb{T}) = W^{s,2}(\mathbb{T})$. In dealing with space-time function spaces, we use short-hand notations such as $L_T^p H_x^s$ for $L^p([0, T]; H^s(\mathbb{T}))$.

Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a smooth bump function supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\eta \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. For $\xi \in \mathbb{R}$, we set $\varphi_0(\xi) = \eta(|\xi|)$ and

$$\varphi_j(\xi) = \eta\left(\frac{|\xi|}{2^j}\right) - \eta\left(\frac{|\xi|}{2^{j-1}}\right)$$

for $j \in \mathbb{N}$. Then, for $j \in \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$, we define the Littlewood-Paley projector \mathbf{Q}_j as the Fourier multiplier operator with a symbol φ_j . Note that we have

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$$

for each $\xi \in \mathbb{R}$. Thus, we have the following Littlewood-Paley decomposition:

$$f = \sum_{j=0}^{\infty} \mathbf{Q}_j f.$$

Define the projectors \mathbf{P}_+ and \mathbf{P}_- by setting

$$\mathbf{P}_+ f = \mathcal{F}^{-1}(\mathbf{1}_{\{n \geq 1\}} \widehat{f}(n)) \quad \text{and} \quad \mathbf{P}_- f = \mathcal{F}^{-1}(\mathbf{1}_{\{n \leq -1\}} \widehat{f}(n)).$$

We also define \mathbf{P}_{hi} , \mathbf{P}_{HI} , \mathbf{P}_{lo} , and \mathbf{P}_{LO} by setting

$$\mathbf{P}_{\text{hi}} = \sum_{j=1}^{\infty} \mathbf{Q}_j, \quad \mathbf{P}_{\text{HI}} = \sum_{j=3}^{\infty} \mathbf{Q}_j, \quad \mathbf{P}_{\text{lo}} = \text{Id} - \mathbf{P}_{\text{hi}}, \quad \text{and} \quad \mathbf{P}_{\text{LO}} = \text{Id} - \mathbf{P}_{\text{HI}}, \quad (2.1)$$

and set $\mathbf{P}_{\pm, \text{hi}} = \mathbf{P}_{\pm} \mathbf{P}_{\text{hi}}$, etc.

Next, we recall the $X^{s,b}$ -spaces introduced by Bourgain [7, 8] and their variants defined by the following norms:

$$\begin{aligned} \|u\|_{X^{s,b}} &= \|\langle n \rangle^s \langle \tau - |n|n \rangle^b \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^2}, \\ \|u\|_{Z^{s,b}} &= \|\langle n \rangle^s \langle \tau - |n|n \rangle^b \widehat{u}(\tau, n)\|_{\ell_n^2 L_\tau^1}, \\ \|u\|_{\widetilde{Z}^{s,b}} &= \left(\sum_{j=0}^{\infty} \|\mathbf{Q}_j u\|_{Z^{s,b}} \right)^{\frac{1}{2}}, \\ \|u\|_{Y^{s,b}} &= \|u\|_{X^{s,b}} + \|u\|_{\widetilde{Z}^{s,b-\frac{1}{2}}}. \end{aligned}$$

See [7, 8, 30, 13] for basic properties. Note that we have $\widetilde{Z}^{s,b} \subset Z^{s,b}$. For $T > 0$, we define the space $X^{s,b}(T)$ to be the restriction of the $X^{s,b}$ -space onto the time interval $[0, T]$ via the norm:

$$\|u\|_{X^{s,b}(T)} := \inf \{ \|v\|_{X^{s,b}} : v|_{[0,T]} = u \}. \quad (2.2)$$

We also define the restriction spaces $\widetilde{Z}^{s,b}(T)$ and $Y^{s,b}(T)$ in an analogous manner. Note that $Y^{s,b}(T)$ is complete. Recall the following embedding:

$$Y^{s, \frac{1}{2}}(T) \subset Z^{s,0}(T) \subset C([0, T]; H^s(\mathbb{T})). \quad (2.3)$$

Let $S(t)$ denote the linear propagator for the BO equation (1.5) defined by

$$\widehat{S(t)f}(n) = e^{it|n|^n} \widehat{f}(n).$$

Then, we have the following linear estimates; see [7, 30, 53, 44].

Lemma 2.1. *Let $s \in \mathbb{R}$. Then, we have*

$$\begin{aligned} \|\eta(t)S(t)f\|_{Y^{s, \frac{1}{2}}} &\lesssim \|f\|_{H^s}, \\ \left\| \eta(t) \int_0^t S(t-t')F(t')dt' \right\|_{Y^{s, \frac{1}{2}}} &\lesssim \|F\|_{Y^{s, -\frac{1}{2}}}. \end{aligned}$$

Next, we recall the periodic L^4 -Strichartz estimate due to Bourgain [7]; see also [42, (6)].

Lemma 2.2. *The following estimate holds:*

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,\frac{3}{8}}} \lesssim \|u\|_{Y^{0,\frac{1}{2}}}.$$

For $0 < \delta \leq \infty$, define \mathcal{Q}_δ by

$$\mathcal{Q}_\delta = (\mathcal{T}_\delta - \mathcal{H})\partial_x \quad (2.4)$$

with the understanding that $\mathcal{Q}_\infty = 0$ (corresponding to the BO equation (1.5)), where \mathcal{T}_δ is as in (1.3). The operator \mathcal{Q}_δ naturally appears in viewing ILW as a perturbation of the BO equation; see (3.2) and (3.6) below. The next lemma establishes a smoothing property of the operator \mathcal{Q}_δ ; see also Lemma 2.2 in [22].

Lemma 2.3. *Let $s \geq 0$. Then, we have*

$$\|\mathcal{Q}_\delta f\|_{H^s} \lesssim \delta^{-1}(1 + \delta^{-s})\|f\|_{L^2}. \quad (2.5)$$

In particular, we have

$$\|\mathcal{Q}_\delta f\|_{H^s} \lesssim \|f\|_{L^2},$$

uniformly in $\delta \geq 1$.

Proof. With a slight abuse of notation, let $\widehat{\mathcal{Q}}_\delta(n)$ denotes the multiplier for the operator \mathcal{Q}_δ :

$$\widehat{\mathcal{Q}}_\delta(n) = n(\coth(\delta n) - \operatorname{sgn}(n)). \quad (2.6)$$

From (1.4), we have

$$\coth(\delta n) - \operatorname{sgn}(n) = \operatorname{sgn}(n) \cdot \frac{2}{e^{2|\delta n|} - 1} \quad (2.7)$$

for $n \neq 0$. Then, by noting that $x^{s+1} \leq C_s(e^{2x} - 1)$ for any $x \geq 0$, it follows from (2.6) and (2.7) that

$$\sup_{n \in \mathbb{Z} \setminus \{0\}} \langle n \rangle^s |\widehat{\mathcal{Q}}_\delta(n)| \lesssim \delta^{-1}(1 + \delta^{-s}),$$

from which (2.5) follows. \square

We now recall the following fractional Leibniz rule in the periodic setting; see [5]. See also [25, Lemma 9.A.2] and [18, Lemma 3.4].

Lemma 2.4. *Let $s > 0$ and $1 < p_j, q_j, r \leq \infty$, $j = 1, 2$, such that $\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}$. Then, we have*

$$\|J^s(fg)\|_{L^r(\mathbb{T})} \lesssim \|J^s f\|_{L^{p_1}(\mathbb{T})} \|g\|_{L^{q_1}(\mathbb{T})} + \|f\|_{L^{p_2}(\mathbb{T})} \|J^s g\|_{L^{q_2}(\mathbb{T})}, \quad (2.8)$$

where $J^s = \langle \partial_x \rangle^s$ denotes the Bessel potential of order $-s$.

Note that the estimate (2.8) holds for $\frac{1}{2} \leq r \leq 1$ as well (with additional assumptions such as $s > \frac{1}{r} - 1$ or $s \in 2\mathbb{N}$); see [5].

We conclude this section by recalling the basic property of the translation operator τ defined by

$$\tau_h(u)(t, x) = \tau(h, u)(t, x) := u(t, x + ht). \quad (2.9)$$

Lemma 2.5. *Let $T > 0$ and $s \geq 0$. Then, τ defined in (2.9) is continuous from $\mathbb{R} \times C([0, T]; H^s(\mathbb{T}))$ into $C([0, T]; H^s(\mathbb{T}))$, but is not uniformly continuous in every neighborhood of the origin $(0, 0)$. For fixed $h \in \mathbb{R}$, τ_h is an isometry on $C([0, T]; H^s(\mathbb{T}))$ (in particular, Lipschitz continuous).*

See [21, Propositions 3.2.1 and 3.2.2] for the first two claims. The second claim is clear from $\|\tau_h u(t) - \tau_h v(t)\|_{H_x^s} = \|u(t) - v(t)\|_{H_x^s}$ for any $t \in \mathbb{R}$.

3. GAUGE TRANSFORM AND NONLINEAR ESTIMATES

In this section, by viewing ILW (1.1) as a perturbation of BO (1.5), we first apply the gauge transform for BO to ILW. We then follow the argument in [44] and establish a priori estimates.

3.1. Gauge transform. By applying the Galilean transform

$$v(t, x) = \tau_{\delta^{-1}}(u)(t, x) = u(t, x + \delta^{-1}t), \quad (3.1)$$

where $\tau_{\delta^{-1}} = \tau(\delta^{-1}, \cdot)$ is as in (2.9), we can rewrite (1.1) as the following renormalized ILW:

$$\partial_t v - \mathcal{T}_\delta \partial_x^2 v = \partial_x(v^2) \quad (3.2)$$

with $v|_{t=0} = u_0$, where \mathcal{T}_δ is as in (1.3).

Following the recent work [22], our main strategy is to view (3.2) as a perturbation of the BO equation (1.5) as follows:

$$\partial_t v - \mathcal{H} \partial_x^2 v = \mathcal{Q}_\delta \partial_x v + \partial_x(v^2), \quad (3.3)$$

where \mathcal{Q}_δ is as in (2.4), and apply the gauge transform for the BO equation (1.5) (see (3.10) below) to (3.3). Lemma 2.3 shows that \mathcal{Q}_δ has a smoothing property of any order, which allows us to indeed view the first term on the right-hand side of (3.3) as a perturbation in a suitable sense.

In a seminal work [52], Tao introduced a gauge transform, analogous to the Hopf-Cole transform for the heat equation, in studying low regularity well-posedness of BO (1.5). Let us now recall the BO gauge transform in the periodic setting [43, 44] and apply it to (3.3). Let $F = F[v]$ be the mean-zero primitive of v defined by

$$F = F[v] = \partial_x^{-1} \mathbf{P}_{\neq 0} v, \quad (3.4)$$

where ∂_x^{-1} denotes the Fourier multiplier operator with symbol $\mathbf{1}_{\{n \neq 0\}} \cdot (in)^{-1}$ and $\mathbf{P}_{\neq 0}$ denotes the projection onto the non-zero Fourier modes. In particular, for a mean-zero function v , we have

$$v = \mathbf{P}_{\neq 0} v = \partial_x F. \quad (3.5)$$

By noting $\partial_x^{-1} \partial_x = \mathbf{P}_{\neq 0}$, it is easy to see that, if v is a solution to (3.3), then $F = F[v]$ satisfies the following equation:

$$\partial_t F - \mathcal{H} \partial_x^2 F = \mathcal{Q}_\delta v + v^2 - \mathbf{P}_0(v^2), \quad (3.6)$$

where $\mathbf{P}_0 = \text{Id} - \mathbf{P}_{\neq 0}$ denotes the projection onto the zeroth Fourier coefficient.

Remark 3.1. For $j = 1, 2$, let v_j be a function $C([0, T]; L^2(\mathbb{T}))$ such that $\widehat{v}_j(t, 0) = 0$ for any $t \in [0, T]$. Then, by letting F_j be the mean-zero primitive of v_j defined by $F_j = \partial_x^{-1} \mathbf{P}_{\neq 0} v_j$, it follows from Bernstein's inequality that

$$\|F_1(0) - F_2(0)\|_{L^\infty} \lesssim \|v_1(0) - v_2(0)\|_{L^2} \quad (3.7)$$

and

$$\|F_1 - F_2\|_{L_{T,x}^\infty} \lesssim \|v_1 - v_2\|_{L_T^\infty L_x^2}. \quad (3.8)$$

These bounds allow us to prove local Lipschitz continuity of the solution map to (3.2) in $H_0^s(\mathbb{T})$. We point out that the bounds (3.7) and (3.8) do not hold on the real line due to the low-frequency issue. In order to overcome this difficulty in the real line case, an extra assumption on the low-frequency part of initial data was imposed in Section 4 of [44], namely,

$$\mathbf{P}_{\text{LO}} v_1(0) = \mathbf{P}_{\text{LO}} v_2(0), \quad (3.9)$$

See [44, Lemma 4.1], where the bounds (3.7) and (3.8) are established in the real line case under the extra assumption (3.9).

We now introduce the gauge transform (for the periodic BO equation):³

$$W = \mathbf{P}_{+, \text{hi}}(e^{iF}). \quad (3.10)$$

By noting

$$\text{Id} = \mathbf{P}_{+, \text{hi}} + \mathbf{P}_{\text{lo}} + \mathbf{P}_{-, \text{hi}} \quad \text{and} \quad \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{-, \text{hi}} f \cdot \mathbf{P}_{-g}) = 0,$$

a direction computation with (3.10), (3.6), and (3.5) yields

$$\begin{aligned} \partial_t W - \mathcal{H} \partial_x^2 W &= \partial_t W + i \partial_x^2 W \\ &= i \mathbf{P}_{+, \text{hi}}(e^{iF}(\partial_t F + i \partial_x^2 F - (\partial_x F)^2)) \\ &= -2 \mathbf{P}_{+, \text{hi}}(W \mathbf{P}_{-} \partial_x v) - 2 \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{\text{lo}} e^{iF} \mathbf{P}_{-} \partial_x v) \\ &\quad + i \mathbf{P}_{+, \text{hi}}(e^{iF} \mathcal{Q}_\delta v) - i \mathbf{P}_0(v^2) W. \end{aligned} \quad (3.11)$$

Finally, we define $w = w(v)$ by setting

$$w = \partial_x W = \partial_x \mathbf{P}_{+, \text{hi}}(e^{iF}) = i \mathbf{P}_{+, \text{hi}}(e^{iF} v). \quad (3.12)$$

By differentiating (3.11), we then obtain

$$\begin{aligned} \partial_t w - \mathcal{H} \partial_x^2 w &= -2 \partial_x \mathbf{P}_{+, \text{hi}}(W \mathbf{P}_{-} \partial_x v) - 2 \partial_x \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{\text{lo}} e^{iF} \mathbf{P}_{-} \partial_x v) \\ &\quad + i \partial_x \mathbf{P}_{+, \text{hi}}(e^{iF} \mathcal{Q}_\delta v) - i \mathbf{P}_0(v^2) w \\ &=: \mathcal{N}_\delta(w, v). \end{aligned} \quad (3.13)$$

As compared to [44, (3-4)] (modulo constants), we have two extra terms on the right-hand side of (3.13). The third term appears from the difference between the ILW and BO equations, whereas the fourth term appears due to the current periodic setting.

³In the current periodic setting, it is possible to define a gauge transform with \mathbf{P}_+ instead of $\mathbf{P}_{+, \text{hi}}$ (as in [42, 43]) and carry out the analysis presented in this paper. We, however, keep the definition of the gauge transform (3.10) with $\mathbf{P}_{+, \text{hi}}$ so that it is easier to compare the discussion in this paper with the presentation in [44].

Let us conclude this discussion by expressing v in terms of the gauged function $w = w(v)$. From (3.5) and (3.12) with we can write v as

$$\begin{aligned} v &= \partial_x F = e^{-iF} \cdot e^{iF} \partial_x F = -ie^{-iF} \cdot \partial_x e^{iF} \\ &= -ie^{-iF} w + e^{-iF} \mathbf{P}_{\text{lo}}(e^{iF} v) - ie^{-iF} \partial_x \mathbf{P}_{-, \text{hi}} e^{iF}. \end{aligned}$$

By applying $\mathbf{P}_{+, \text{HI}}$, we then obtain

$$\begin{aligned} \mathbf{P}_{+, \text{HI}} v &= -i\mathbf{P}_{+, \text{HI}}(e^{-iF} w) + \mathbf{P}_{+, \text{HI}}(\mathbf{P}_{+, \text{hi}} e^{-iF} \cdot \mathbf{P}_{\text{lo}}(e^{iF} v)) \\ &\quad - i\mathbf{P}_{+, \text{HI}}(\mathbf{P}_{+, \text{HI}} e^{-iF} \cdot \partial_x \mathbf{P}_{-, \text{hi}} e^{iF}), \end{aligned} \tag{3.14}$$

where we have used $\mathbf{P}_{+, \text{HI}}((\text{Id} - \mathbf{P}_{+, \text{hi}})f \cdot \mathbf{P}_{\text{lo}}g) = \mathbf{P}_{+, \text{HI}}((\text{Id} - \mathbf{P}_{+, \text{HI}})f \cdot \mathbf{P}_{-, \text{hi}}f) = 0$.

3.2. Product estimates. We first recall the following product estimate; see [42, Lemma 3.2].

Lemma 3.2. *Let $s, s_1, s_2 \geq 0$ and $1 < q, q_1, q_2 < \infty$ such that $s_1 \geq s$, $1 + s = s_1 + s_2$, and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, we have*

$$\|D^s \mathbf{P}_+(f \mathbf{P}_- \partial_x g)\|_{L^q} \lesssim \|D^{s_1} f\|_{L^{q_1}} \|D^{s_2} g\|_{L^{q_2}}.$$

Next, we recall the following lemma on the multiplication with $e^{\pm iF}$ in Sobolev spaces; see [44, Lemma 2.7]. See also [42, Lemma 3.1].

Lemma 3.3. *Let $2 \leq q < \infty$ and $0 \leq s \leq \frac{1}{2}$. For $j = 1, 2$, let F_j denote the mean-zero primitive of a mean-zero real-valued function f_j on \mathbb{T} such that $\partial_x F_j = f_j$. Then, we have*

$$\|J^s(e^{\pm iF_j} g)\|_{L^q} \lesssim (1 + \|f_j\|_{L^2}) \|J^s g\|_{L^q}$$

and

$$\begin{aligned} &\|J^s((e^{\pm iF_1} - e^{\pm iF_2})g)\|_{L^q} \\ &\lesssim \left(\|f_1 - f_2\|_{L^2} + \|e^{\pm iF_1} - e^{\pm iF_2}\|_{L^\infty} (1 + \|f_1\|_{L^2}) \right) \|J^s g\|_{L^q}. \end{aligned}$$

In [44, Lemma 2.7], Lemma 3.3 was proven for $0 \leq s \leq \frac{1}{q}$ but their proof easily extends to the range $0 \leq s \leq \frac{1}{2}$. In [44, the last line on p. 372], the condition $s = \frac{1}{q} - \frac{1}{q_1}$ was used to control $\|g\|_{L^{q_1}}$ by $\|J^s g\|_{L^q}$, which yielded the restriction $s \leq \frac{1}{q}$. We can, however, use Sobolev's inequality with $s \geq \frac{1}{q} - \frac{1}{q_1}$; see [3]. In repeating the proof of [44, Lemma 2.7], the restriction $s \leq \frac{1}{2}$ appears in the bound $\|D^{s+\frac{1}{2}} \mathbf{P}_{\text{hi}} e^{\pm iF_j}\|_{L^2} \lesssim \|\partial_x e^{\pm iF_j}\|_{L^2}$ in [44, p. 373].

3.3. Nonlinear estimates. We now state nonlinear estimates which allow us to establish a priori bounds on a solution v to (3.2) and the associated gauged function w in (3.12). The following results follow from slight modifications of the corresponding results for the $\delta = \infty$ case; see Propositions 3.2 and 3.4 in [44]. The first lemma controls a solution v to (3.2) in terms of the gauged function w defined in (3.12).

Lemma 3.4. *Let $0 \leq s \leq 1$ and $0 \leq \kappa \leq 1$. Then, there exists $\theta > 0$ such that*

$$\|v\|_{X^{s-\kappa, \kappa}(T)} \lesssim T^\theta (1 + \delta^{-1}) \|v\|_{L_T^\infty H_x^s} + \|v\|_{L_{T,x}^4} \|J^s v\|_{L_{T,x}^4} \tag{3.15}$$

for any $0 < \delta \leq \infty$, $0 < T \leq 1$, and any smooth solution v to (3.2) on the time interval $[0, T]$. Moreover if $0 \leq s \leq \frac{1}{2}$, we have

$$\begin{aligned} \|J^s v\|_{L_T^p L_x^q} &\lesssim \|v(0)\|_{L^2} + (1 + \|v\|_{L_T^\infty L_x^2}) \|w\|_{Y^{s, \frac{1}{2}}(T)} \\ &\quad + (1 + \|v\|_{L_T^\infty L_x^2}) \|v\|_{L_T^\infty L_x^2} \|v\|_{L_T^\infty H_x^s} + T\delta^{-1} \|v\|_{L_T^\infty L_x^2} \end{aligned} \quad (3.16)$$

for $(p, q) = (\infty, 2)$ or $(4, 4)$, where $w = w(v)$ is as in (3.12).

Proof. Proceeding as in the proof of Proposition 3.2 in [44], we have

$$\|\tilde{v}\|_{X^{s-\kappa, \kappa}} \lesssim \|\partial_t v - \mathcal{H}\partial_x^2 v\|_{L_T^2 H_x^{s-1}} + T^\theta \|v\|_{L_T^\infty H_x^s} \quad (3.17)$$

for any $0 \leq \kappa \leq 1$, where $\tilde{v}(t) = \eta(t)v(t)$ is an extension of v from $[0, T]$ to \mathbb{R} . By substituting (3.3) to the first term on the right-hand side of (3.17) and applying Lemma 2.3 and the fractional Leibniz rule (Lemma 2.4), we have

$$\begin{aligned} \|\partial_t v - \mathcal{H}\partial_x^2 v\|_{L_T^2 H_x^{s-1}} &\lesssim \|\mathcal{Q}_\delta v\|_{L_T^2 H_x^s} + \|v\|_{L_{T,x}^4} \|J^s v\|_{L_{T,x}^4} \\ &\lesssim T^\theta \delta^{-1} \|v\|_{L_T^\infty H_x^s} + \|v\|_{L_{T,x}^4} \|J^s v\|_{L_{T,x}^4}. \end{aligned} \quad (3.18)$$

Hence, (3.15) follows from (3.17) and (3.18).

Note that the identity (3.14) does not involve the depth parameter δ (or an equation) and is the same as (3-6) in [44] (modulo constants). Thus, provided that (i) $(p, q) = (\infty, 2)$ and $0 \leq s \leq \frac{1}{2}$ or (ii) $(p, q) = (4, 4)$ and $0 \leq s \leq \frac{1}{4}$, we can proceed as in the proof of Proposition 3.2 in [44]. Note, however, that in [44], the low-frequency contribution was estimated by studying the Duhamel formulation of the BO equation; see [44, (3-17) and (3-18)]. In our case, we instead need to study the Duhamel formulation of (3.3), where the additional contribution as compared to [44] is estimated via Lemma 2.3 as

$$\left\| J^s \int_0^t S(t-t') \mathbf{P}_{\text{LO}} \mathcal{Q}_\delta \partial_x v(t') dt' \right\|_{L_T^p L_x^q} \lesssim T\delta^{-1} \|v\|_{L_T^\infty L_x^2}. \quad (3.19)$$

This yields the second estimate (3.16) except for the case $(p, q) = (4, 4)$ and $\frac{1}{4} < s \leq \frac{1}{2}$.

It remains to consider the case $(p, q) = (4, 4)$ and $\frac{1}{4} < s \leq \frac{1}{2}$. In view of (3.14), we write $v = \mathbf{P}_+ v + \overline{\mathbf{P}_+ v}$ (recall that v is real-valued) and

$$\begin{aligned} \mathbf{P}_+ v &= \mathbf{P}_{+, \text{LO}} v - i\mathbf{P}_{+, \text{HI}}(e^{-iF} w) + \mathbf{P}_{+, \text{HI}}(\mathbf{P}_{+, \text{hi}} e^{-iF} \cdot \mathbf{P}_{\text{lo}}(e^{iF} v)) \\ &\quad - i\mathbf{P}_{+, \text{HI}}(\mathbf{P}_{+, \text{HI}} e^{-iF} \cdot \partial_x \mathbf{P}_{-, \text{hi}} e^{iF}) \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

Note that the estimates (3-18) (see also (3.19) above) and (3-14) in [44] on A_1 and A_3 , respectively, also hold for $\frac{1}{4} < s \leq \frac{1}{2}$. As for A_2 , by applying Lemmas 3.3 and 2.2, we have

$$\begin{aligned} \|J^s A_2\|_{L_{T,x}^4} &= \|J^s \mathbf{P}_{+, \text{HI}}(e^{-iF} w)\|_{L_{T,x}^4} \lesssim (1 + \|v\|_{L_T^\infty L_x^2}) \|J^s w\|_{L_{T,x}^4} \\ &\lesssim (1 + \|v\|_{L_T^\infty L_x^2}) \|w\|_{Y^{s, \frac{1}{2}}(T)} \end{aligned} \quad (3.20)$$

extending [44, (3-13)] to $\frac{1}{4} < s \leq \frac{1}{2}$.

Next, we treat A_4 . In view of the frequency localizations, we have

$$A_4 = -i \sum_{j, j_1=3}^{\infty} \sum_{j_2=1}^{\infty} \mathbf{1}_{j_1 \geq j, j_2} \cdot \mathbf{Q}_j \mathbf{P}_+ (\mathbf{Q}_{j_1} \mathbf{P}_+ e^{-iF} \cdot \mathbf{Q}_{j_2} \partial_x \mathbf{P}_- e^{iF}). \quad (3.21)$$

In the following, we work for fixed time t . For simplicity of notations, we suppress t -dependence. By the Littlewood-Paley theorem and Minkowski's integral inequality, we have

$$\begin{aligned} \|J^s A_4\|_{L_x^4} &\sim \left\| \left(\sum_{j=3}^{\infty} |\mathbf{Q}_j J^s \mathbf{P}_+(\mathbf{P}_{+, \text{HI}} e^{-iF} \cdot \partial_x \mathbf{P}_{-, \text{HI}} e^{iF})|^2 \right)^{\frac{1}{2}} \right\|_{L_x^4} \\ &\lesssim \left(\sum_{j=3}^{\infty} 2^{2(s-\frac{1}{4})j} \|\mathbf{Q}_j J^{\frac{1}{4}} \mathbf{P}_+(\mathbf{P}_{+, \text{HI}} e^{-iF} \cdot \partial_x \mathbf{P}_{-, \text{HI}} e^{iF})\|_{L_x^4}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.22)$$

We separately estimate the contributions to (3.22) by considering the following two cases: $2^{j_1} \sim 2^j$ and $2^{j_1} \sim 2^{j_2} \gg 2^j$, where j_1 and j_2 are as in (3.21).

• **Case 1:** $2^{j_1} \sim 2^j$.

In this case, we have $|j_1 - j| \leq 2$. Then, proceeding as in [44, (3-15)] with Lemma 3.2, Sobolev's inequality, and (3.5), we have

$$\begin{aligned} (3.22) &\lesssim \sum_{k=-2}^2 \left(\sum_{j_1=3}^{\infty} 2^{2(s-\frac{1}{4})j_1} \|\mathbf{Q}_{j_1+k} D^{\frac{1}{4}} \mathbf{P}_+(\mathbf{Q}_{j_1} \mathbf{P}_+ e^{-iF} \cdot \partial_x \mathbf{P}_{-, \text{HI}} e^{iF})\|_{L_x^4}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{j_1=3}^{\infty} 2^{2(s-\frac{1}{4})j_1} \|D^{\frac{5}{8}} \mathbf{Q}_{j_1} \mathbf{P}_+ e^{-iF}\|_{L_x^8}^2 \right)^{\frac{1}{2}} \|D^{\frac{5}{8}} \mathbf{P}_{-, \text{HI}} e^{iF}\|_{L_x^8} \\ &\lesssim \left(\sum_{j_1=3}^{\infty} 2^{2(s-\frac{1}{4})j_1} \|\mathbf{Q}_{j_1} \partial_x \mathbf{P}_+ e^{-iF}\|_{L_x^2}^2 \right)^{\frac{1}{2}} \|\partial_x \mathbf{P}_{-, \text{HI}} e^{iF}\|_{L_x^2} \\ &\lesssim \left(\sum_{j_1=3}^{\infty} 2^{2(s-\frac{1}{4})j_1} \|\mathbf{Q}_{j_1} \mathbf{P}_+(e^{-iF} v)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \|v\|_{L_x^2}. \end{aligned} \quad (3.23)$$

By Lemma 3.3 (recall that $\frac{1}{4} < s \leq \frac{1}{2}$), we have

$$\begin{aligned} &\left(\sum_{j_1=3}^{\infty} 2^{2(s-\frac{1}{4})j_1} \|\mathbf{Q}_{j_1} \mathbf{P}_+(e^{-iF} v)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|e^{-iF} v\|_{H_x^{s-\frac{1}{4}}} \lesssim (1 + \|v\|_{L_x^2}) \|v\|_{H_x^{s-\frac{1}{4}}}. \end{aligned} \quad (3.24)$$

Hence, from (3.22), (3.23), and (3.24), we conclude that the contribution from this case (after taking the L_T^4 -norm) is bounded by the right-hand side of (3.16).

- **Case 2:** $2^{j_1} \sim 2^{j_2} \gg 2^j$.

In this case, we have $|j_1 - j_2| \leq 2$ and $j \leq j_1 - 3$. Proceeding as in (3.23), we have

$$\begin{aligned}
(3.22) &\lesssim \sum_{k=-2}^2 \sum_{j_1=3}^{\infty} \left(\sum_{j=3}^{j_1-3} 2^{2(s-\frac{1}{4})j} \right. \\
&\quad \left. \times \|D^{\frac{1}{4}} \mathbf{P}_{+, \text{HI}}(\mathbf{Q}_{j_1} \mathbf{P}_+ e^{-iF} \cdot \mathbf{Q}_{j_1+k} \partial_x \mathbf{P}_{-, \text{hi}} e^{iF})\|_{L_x^4}^2 \right)^{\frac{1}{2}} \\
&\lesssim \sum_{k=-2}^2 \sum_{j_1=3}^{\infty} 2^{(s-\frac{1}{4})j_1} \|D^{\frac{5}{8}} \mathbf{Q}_{j_1} \mathbf{P}_+ e^{-iF}\|_{L_x^8} \|\mathbf{Q}_{j_1+k} D^{\frac{5}{8}} \mathbf{P}_{-, \text{hi}} e^{iF}\|_{L_x^8} \\
&\lesssim \sum_{k=-2}^2 \sum_{j_1=3}^{\infty} 2^{(s-\frac{1}{4})j_1} \|\mathbf{Q}_{j_1} \partial_x \mathbf{P}_+ e^{-iF}\|_{L_x^2} \|\mathbf{Q}_{j_1+k} \partial_x \mathbf{P}_{-, \text{hi}} e^{iF}\|_{L_x^2} \\
&\lesssim \left(\sum_{j_1=3}^{\infty} 2^{2(s-\frac{1}{4})j_1} \|\mathbf{Q}_{j_1} \mathbf{P}_+(e^{-iF} v)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \|v\|_{L_x^2},
\end{aligned} \tag{3.25}$$

where we used Cauchy-Schwarz's inequality (in j_1) in the last step. Hence, from (3.22), (3.25), and (3.24), we conclude that the contribution from this case (after taking the L_T^4 -norm) is bounded by the right-hand side of (3.16).

Putting everything together, we conclude (3.16) when $(p, q) = (4, 4)$ and $\frac{1}{4} < s \leq \frac{1}{2}$. \square

The following proposition allows us to estimate the gauged function w in terms of v .

Proposition 3.5. *Let $0 \leq s \leq \frac{1}{2}$. Then, there exists $\theta > 0$ such that*

$$\begin{aligned}
&\|\mathbf{1}_{[0, T]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s, -\frac{1}{2}}} \\
&\lesssim \|v\|_{L_{T,x}^4}^2 + \left(T^\theta \|v\|_{L_T^\infty L_x^2} + \|v\|_{L_{T,x}^4} + \|v\|_{X^{-1,1}(T)} \right) \|w\|_{X^{s, \frac{1}{2}}(T)} \\
&\quad + T^\theta \|v\|_{L_T^\infty L_x^2}^2 \|w\|_{Y^{s, \frac{1}{2}}(T)} + T^\theta \delta^{-1} (1 + \delta^{-1}) (1 + \|v\|_{L_T^\infty L_x^2})^2 \|v\|_{L_T^\infty H_x^s}
\end{aligned} \tag{3.26}$$

for any $0 < \delta \leq \infty$, $0 < T \leq 1$, and any smooth solution v to (3.2) the time interval on $[0, T]$, where $w = w(v)$ is as in (3.12) and $\mathcal{N}_\delta(w, v)$ is as in (3.13).

Note that the last term on the right-hand side of (3.26) contains a linear term $\delta^{-1}(1 + \delta^{-1})\|v\|_{L_T^\infty H_x^s}$. In order to implement a bootstrap argument, we need additional smallness. This is the primary reason that we keep a small power of T for this term. The factor T^θ on the other terms are used in (4.4). See Section 4.

We now present the proof of Proposition 3.5.

Proof of Proposition 3.5. By arguing as in [44, Proposition 3.5 and Lemma 3.7], we can bound the contribution from the first two terms on the right-hand side of (3.13) by the right-hand side of (3.26).

Let us first treat the fourth term on the right-hand side of (3.13). By (2.3), we have

$$\|\mathbf{1}_{[0, T]}(t) \cdot \mathbf{P}_0(v^2)w\|_{L_t^2 H_x^s} \lesssim T^\theta \|v\|_{L_T^\infty L_x^2}^2 \|w\|_{L_T^\infty H_x^s} \lesssim T^\theta \|v\|_{L_T^\infty L_x^2}^2 \|w\|_{Y^{s, \frac{1}{2}}(T)}. \tag{3.27}$$

We now turn our attention to the third term in (3.13). Proceeding as in [44, (3-47)] with $\partial_x F = v$ and Lemma 3.3, we have

$$\begin{aligned} & \|\mathbf{1}_{[0,T]}(t) \cdot \partial_x \mathbf{P}_{+,hi}(e^{iF} \mathcal{Q}_\delta v)\|_{Y^{s,-\frac{1}{2}}} \lesssim \|\partial_x \mathbf{P}_{+,hi}(e^{iF} \mathcal{Q}_\delta v)\|_{L_T^{1+\kappa} H_x^s} \\ & \leq \|\mathbf{P}_{+,hi}(e^{iF} v \mathcal{Q}_\delta v)\|_{L_T^{1+\kappa} H_x^s} + \|\mathbf{P}_{+,hi}(e^{iF} \mathcal{Q}_\delta \partial_x v)\|_{L_T^{1+\kappa} H_x^s} \\ & \lesssim T^\theta (1 + \|v\|_{L_T^\infty L_x^2}) \left(\|v \mathcal{Q}_\delta v\|_{L_T^\infty H_x^s} + \|\mathcal{Q}_\delta \partial_x v\|_{L_T^\infty H_x^s} \right) \end{aligned} \quad (3.28)$$

for any $\kappa > 0$. From the fractional Leibniz rule (Lemma 2.4), Sobolev's embedding, and Lemma 2.3, we have

$$\begin{aligned} \|v \mathcal{Q}_\delta v\|_{L_T^\infty H_x^s} & \lesssim \|v\|_{L_T^\infty H_x^s} \|\mathcal{Q}_\delta v\|_{L_{T,x}^\infty} + \|v\|_{L_T^\infty L_x^2} \|\mathcal{Q}_\delta v\|_{L_T^\infty W_x^{s,\infty}} \\ & \lesssim \|v\|_{L_T^\infty H_x^s} \|\mathcal{Q}_\delta v\|_{L_T^\infty H_x^1} + \|v\|_{L_T^\infty L_x^2} \|\mathcal{Q}_\delta v\|_{L_T^\infty H_x^{s+1}} \\ & \lesssim \delta^{-1} (1 + \delta^{-1}) \|v\|_{L_T^\infty H_x^s} \|v\|_{L_T^\infty L_x^2}. \end{aligned} \quad (3.29)$$

By Lemma 2.3, we also have

$$\|\mathcal{Q}_\delta \partial_x v\|_{L_T^\infty H_x^s} \lesssim \delta^{-1} (1 + \delta^{-1}) \|v\|_{L_T^\infty H_x^s}. \quad (3.30)$$

Hence, from (3.28), (3.29), and (3.30), we obtain

$$\|\mathbf{1}_{[0,T]}(t) \cdot \partial_x \mathbf{P}_{+,hi}(e^{iF} \mathcal{Q}_\delta v)\|_{Y^{s,-\frac{1}{2}}} \lesssim T^\theta \delta^{-1} (1 + \delta^{-1}) (1 + \|v\|_{L_T^\infty L_x^2})^2 \|v\|_{L_T^\infty H_x^s}. \quad (3.31)$$

Therefore, from (3.27) and (3.31), we conclude (3.26). \square

Remark 3.6. In the proof of Proposition 3.5, we handled the first two terms on the right-hand side of (3.13) by simply invoking the bilinear estimates in [44, Proposition 3.5 and Lemma 3.7]. We point out that these bilinear estimates (which hold on both the real line and the circle) are the key ingredients for the simplification and unification of the L^2 well-posedness argument for the BO equation (1.5) presented in [44].

Remark 3.7. We point out that the map $T \mapsto \|\mathbf{1}_{[0,T]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s,-\frac{1}{2}}}$ is continuous for smooth w and v . Indeed, given $h > 0$, it follows from the triangle inequality that

$$\begin{aligned} & \left| \|\mathbf{1}_{[0,T+h]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s,-\frac{1}{2}}} - \|\mathbf{1}_{[0,T]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s,-\frac{1}{2}}} \right| \\ & \leq \|\mathbf{1}_{[T,T+h]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s,-\frac{1}{2}}} \\ & \lesssim \|\mathbf{1}_{[T,T+h]} \cdot \mathcal{N}_\delta(w, v)\|_{X^{s,0}} = \|\mathcal{N}_\delta(w, v)\|_{L^2([T,T+h]; H^s)} \\ & \rightarrow 0, \end{aligned}$$

as $h \rightarrow 0$. A similar computation holds for $h < 0$, tending to 0. See [4, Lemma A.8], [9, Lemma 4.4], and [49, Remark 3.3] for the continuity property (in T) of the $X^{s,b}$ -norm and its variants under the multiplication by $\mathbf{1}_{[0,T]}$. See also [29, Lemma 6.3] and [19, Lemma 8.1] for similar statements in the context of the more complicated short-time Fourier restriction norm method.

4. WELL-POSEDNESS OF ILW IN L^2

In this section, we present the proof of Theorem 1.1. With Lemma 3.4 and Proposition 3.5 in hand, it essentially follows from the discussion in Section 4 of [44], and thus we keep our presentation brief. Our main goal is to prove local well-posedness of the renormalized

ILW (3.2) in $L^2(\mathbb{T})$. Once we establish local well-posedness, global well-posedness follows from the L^2 -conservation.

Fix $0 < \delta < \infty$ and $0 \leq s \leq \frac{1}{2}$. We only discuss the case where initial data has sufficiently small L^2 -norm. More precisely, in the remaining part of this section, we assume that all the initial data in the following discussion have the L^2 -norm less than some sufficiently small $\varepsilon_0 = \varepsilon_0(\min(\delta, 1)) > 0$ (to be chosen later). See Remark 4.2 for the general case.

4.1. A priori estimates for smooth solutions. Let us first derive a priori estimates for smooth solutions to (3.2). Let v be a smooth global solution to (3.2) and $w = w(v)$ be the gauged function defined in (3.12). Given $T > 0$, set

$$N_T^s(v) = \max \left(\|v\|_{L_T^\infty H_x^s}, \|v\|_{L_T^4 W_x^{s,4}}, \|w(0)\|_{H^s}, \|\mathbf{1}_{[0,T]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s, -\frac{1}{2}}} \right), \quad (4.1)$$

where $\mathcal{N}_\delta(w, v)$ is as in (3.13). We first note that from the Duhamel formulation of (3.13) with (2.2) (for $Y^{s, \frac{1}{2}}(T)$), Lemma 2.1, and (4.1), we have

$$\|w\|_{Y^{s, \frac{1}{2}}(T)} \lesssim \|w(0)\|_{H^s} + \|\mathbf{1}_{[0,T]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s, -\frac{1}{2}}} \lesssim N_T^s(v). \quad (4.2)$$

Thanks to the smoothness of v (and w), we see that $N_T^s(v)$ is continuous in T ; see Remark 3.7. Moreover, from the definition, $N_T^s(v)$ is non-decreasing in T . From (3.12) and Lemma 3.3, we have

$$\|w(0)\|_{H^s} \lesssim (1 + \|v(0)\|_{L^2}) \|v(0)\|_{H^s}. \quad (4.3)$$

See also [44, (3-49)]. Hence, from (4.1) and (4.3), we have

$$\lim_{T \rightarrow 0^+} N_T^s(v) \lesssim (1 + \|v(0)\|_{L^2}) \|v(0)\|_{H^s}. \quad (4.4)$$

In (4.4), the control on $\|\mathbf{1}_{[0,T]} \cdot \mathcal{N}_\delta(w, v)\|_{Y^{s, -\frac{1}{2}}}$ follows from (3.26) and (3.15). On the other hand, from Lemma 3.4 and Proposition 3.5 with (4.2) and (4.3), we have⁴

$$\begin{aligned} N_T^s(v) &\leq C_1(1 + \|v(0)\|_{L^2}) \|v(0)\|_{H^s} \\ &\quad + C_2(1 + \delta^{-1})^2 N_T^0(v)(1 + N_T^0(v)) N_T^s(v) \\ &\quad + C_3 T^\theta \delta^{-1} (1 + \delta^{-1}) N_T^s(v) \end{aligned} \quad (4.5)$$

for any $0 < T \leq 1$. We point out that there are linear terms on the right-hand sides of (3.16) and (3.26): $\|w\|_{Y^{s, \frac{1}{2}}(T)}$, $T\delta^{-1}\|v\|_{L_T^\infty L_x^2}$, and $T^\theta\delta^{-1}(1 + \delta^{-1})\|v\|_{L_T^\infty H_x^s}$, where the last two terms are bounded by the last term on the right-hand side of (4.5). As for the linear term $\|w\|_{Y^{s, \frac{1}{2}}(T)}$ on the right-hand side of (3.16), we can use (4.2) and (3.26) to replace it by $T^\theta\delta^{-1}(1 + \delta^{-1})\|v\|_{L_T^\infty H_x^s}$ (along with the other terms on the right-hand side of (3.26)), which is in turn bounded by the last term on the right-hand side of (4.5).

By choosing sufficiently small $T_* = T_*(\min(\delta, 1)) > 0$ such that $C_3 T_*^\theta \delta^{-1} (1 + \delta^{-1}) \leq \frac{1}{2}$, we then have

$$\begin{aligned} N_T^s(v) &\leq 2C_1(1 + \|v(0)\|_{L^2}) \|v(0)\|_{H^s} \\ &\quad + 2C_2(1 + \delta^{-1})^2 N_T^0(v)(1 + N_T^0(v)) N_T^s(v) \end{aligned} \quad (4.6)$$

⁴Note that the constants in (4.5) depend on s . We, however, use (4.5) only at two regularities (s and 0) and thus we can simply take the worse constants in (4.5).

for any $0 < T \leq T_*$. By a standard continuity argument, we have

$$N_T^0(v) \lesssim \varepsilon \quad (4.7)$$

for any $0 < T \leq T_*$, provided that $\|v(0)\|_{L^2} = \varepsilon \leq \varepsilon_0$ for some sufficiently small $\varepsilon_0 = \varepsilon_0(\min(\delta, 1))$. Then, by applying a continuity argument with (4.6) and (4.7), we obtain

$$N_T^s(v) \lesssim \|v(0)\|_{H^s} \quad (4.8)$$

for any $0 < T \leq T_*$.

Remark 4.1. From (4.5), we can choose $T_* = T_*(\min(\delta, 1)) \gtrsim 1$ and small $\varepsilon_0 = \varepsilon_0(\min(\delta, 1)) \geq \varepsilon_1 > 0$, uniformly in $\delta \geq 1$. The same comment applies to the discussion below.

4.2. Difference estimates. Let v_j , $j = 1, 2$, be smooth global solutions to (3.2) with

$$\|v_j(0)\|_{L^2} \leq \varepsilon \leq \varepsilon_0 = \varepsilon_0(\min(\delta, 1)), \quad (4.9)$$

and let F_j be the mean-zero primitive of v_j as defined in (3.4). Then, as in (3.10) and (3.12), define the associated gauged functions W_j and w_j , $j = 1, 2$, by setting

$$W_j = \mathbf{P}_{+, \text{hi}}(e^{iF_j}) \quad \text{and} \quad w_j = \partial_x W_j.$$

Then, from the discussion in the previous subsection (see (4.7) and (4.8) with (4.2)), we have

$$\begin{aligned} \max_{j=1,2} \|w\|_{Y^{0, \frac{1}{2}}(T)} &\lesssim N_T^0(v_j) \lesssim \varepsilon \leq \varepsilon_0, \\ \max_{j=1,2} \|w\|_{Y^{s, \frac{1}{2}}(T)} &\lesssim N_T^s(v_j) \lesssim \max_{j=1,2} \|v_j(0)\|_{H^s} =: M \end{aligned} \quad (4.10)$$

for any $0 < T \leq T_* = T_*(\min(\delta, 1))$, $j = 1, 2$.

Let $\tilde{v} = v_1 - v_2$, $\tilde{W} = W_1 - W_2$, and $\tilde{w} = w_1 - w_2$. By the mean value theorem with (3.7) and (3.8), we have

$$\begin{aligned} \|e^{\pm iF_1(0)} - e^{\pm iF_2(0)}\|_{L^\infty} &\leq \|F_1(0) - F_2(0)\|_{L^\infty} \lesssim \|\tilde{v}(0)\|_{L^2}, \\ \|e^{\pm iF_1} - e^{\pm iF_2}\|_{L_{T,x}^\infty} &\leq \|F_1 - F_2\|_{L_{T,x}^\infty} \lesssim \|\tilde{v}\|_{L_T^\infty L_x^2}. \end{aligned} \quad (4.11)$$

Then, from (3.12), Lemma 3.3, and (4.11) with (4.9), we have

$$\begin{aligned} \|\tilde{w}(0)\|_{H^s} &\lesssim (1 + \|v_1(0)\|_{L^2}) \|\tilde{v}(0)\|_{H^s} \\ &\quad + \left(\|\tilde{v}(0)\|_{L^2} + \|e^{iF_1(0)} - e^{iF_2(0)}\|_{L^\infty} (1 + \|v_1(0)\|_{L^2}) \right) \|v_2(0)\|_{H^s} \\ &\lesssim \|\tilde{v}(0)\|_{H^s} + \|v_2(0)\|_{H^s} \|\tilde{v}(0)\|_{L^2}. \end{aligned} \quad (4.12)$$

We now estimate \tilde{w} . From (3.13), we have

$$\begin{aligned} &\partial_t \tilde{w} - \mathcal{H} \partial_x^2 \tilde{w} \\ &= -2\partial_x \mathbf{P}_{+, \text{hi}}(W_1 \mathbf{P}_- \partial_x \tilde{v}) - 2\partial_x \mathbf{P}_{+, \text{hi}}(\tilde{W} \mathbf{P}_- \partial_x v_2) \\ &\quad - 2\partial_x \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{\text{lo}} e^{iF_1} \mathbf{P}_- \partial_x \tilde{v}) - 2\partial_x \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{\text{lo}}(e^{iF_1} - e^{iF_2}) \mathbf{P}_- \partial_x v_2) \\ &\quad + i\partial_x \mathbf{P}_{+, \text{hi}}(e^{iF_1} \mathcal{Q}_\delta \tilde{v}) + i\partial_x \mathbf{P}_{+, \text{hi}}((e^{iF_1} - e^{iF_2}) \mathcal{Q}_\delta v_2) \\ &\quad - i\mathbf{P}_0(v_1^2) \tilde{w} - i\mathbf{P}_0(v_1^2 - v_2^2) w_2 \\ &=: B_1 + \cdots + B_8. \end{aligned} \quad (4.13)$$

Note that the first four terms on the right-hand side are already treated in Section 4 of [44], which, in view of (4.10) and (4.11), are bounded by the first two terms on the right-hand side of (4.20) below.⁵ By a slight modification of (3.27) with (4.10), we have

$$\begin{aligned} \|B_7 + B_8\|_{Y^{s, -\frac{1}{2}}(T)} &\lesssim \|v_1\|_{L_T^\infty L_x^2}^2 \|\tilde{w}\|_{Y^{s, \frac{1}{2}}(T)} \\ &\quad + \left(\|v_1\|_{L_T^\infty L_x^2} + \|v_2\|_{L_T^\infty L_x^2} \right) \|\tilde{v}\|_{L_T^\infty L_x^2} \|w_2\|_{Y^{s, \frac{1}{2}}(T)} \\ &\lesssim \varepsilon^2 \|\tilde{w}\|_{Y^{s, \frac{1}{2}}(T)} + \varepsilon \|w_2\|_{Y^{s, \frac{1}{2}}(T)} \|\tilde{v}\|_{L_T^\infty L_x^2} \end{aligned} \quad (4.14)$$

for any $0 < T \leq T_*$. Proceeding as in the proof of Proposition 3.5 with (4.10), given an absolute small constant $c_0 > 0$ (to be chosen later), by possibly making $T_* = T_*(\min(\delta, 1)) > 0$ smaller, we can show that there exists $\theta > 0$ such that

$$\begin{aligned} \|B_5\|_{Y^{s, -\frac{1}{2}}(T)} &\leq CT^\theta \delta^{-1} (1 + \delta^{-1}) (1 + \|v_1\|_{L_T^\infty L_x^2}) \\ &\quad \times \left((1 + \|v_1\|_{L_T^\infty L_x^2}) \|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_1\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2} \right) \\ &\leq c_0 \left(\|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_1\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2} \right) \end{aligned} \quad (4.15)$$

for any $0 < T \leq T_*$.

Let us now estimate B_6 . By Cauchy-Schwarz's inequality, Sobolev's inequality, and Minkowski's integral inequality (see [44, (3-47)]), we have

$$\begin{aligned} \|B_6\|_{Y^{s, -\frac{1}{2}}(T)} &\lesssim \left\| \partial_x \mathbf{P}_{+, \text{hi}} \left((e^{iF_1} - e^{iF_2}) \mathcal{Q}_\delta v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \\ &\leq \left\| \mathbf{P}_{+, \text{hi}} \left(\partial_x (e^{iF_1} - e^{iF_2}) \mathcal{Q}_\delta v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \\ &\quad + \left\| \mathbf{P}_{+, \text{hi}} \left((e^{iF_1} - e^{iF_2}) \mathcal{Q}_\delta \partial_x v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \\ &\leq \left\| \mathbf{P}_{+, \text{hi}} \left(e^{iF_1} (v_1 - v_2) \mathcal{Q}_\delta v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \\ &\quad + \left\| \mathbf{P}_{+, \text{hi}} \left((e^{iF_1} - e^{iF_2}) v_2 \mathcal{Q}_\delta v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \\ &\quad + \left\| \mathbf{P}_{+, \text{hi}} \left((e^{iF_1} - e^{iF_2}) \mathcal{Q}_\delta \partial_x v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \end{aligned} \quad (4.16)$$

for any $\kappa > 0$. By applying Lemma 3.3 and proceeding as in (3.29) with (4.10), we have

$$\begin{aligned} &\left\| \mathbf{P}_{+, \text{hi}} \left(e^{iF_1} (v_1 - v_2) \mathcal{Q}_\delta v_2 \right) \right\|_{L_T^{1+\kappa} H_x^s} \\ &\lesssim \delta^{-1} (1 + \delta^{-1}) (1 + \|v_1\|_{L_T^\infty L_x^2}) \left(\|v_2\|_{L_T^\infty L_x^2} \|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_2\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2} \right) \\ &\lesssim \delta^{-1} (1 + \delta^{-1}) \left(\varepsilon \|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_2\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2} \right). \end{aligned} \quad (4.17)$$

⁵In fact, there is another term $\varepsilon \|\tilde{w}\|_{X^{s, \frac{1}{2}}(T)}$, controlling the second term on the right-hand side of (4.13); see [44, Proposition 3.5]. In (4.20), this term is already hidden on the left-hand side, which explains the reason for the factor 2 in front of the last term in (4.20).

By Lemma 3.3, (4.11), and (3.29) with (4.10), we have

$$\begin{aligned}
& \|\mathbf{P}_{+, \text{hi}}((e^{iF_1} - e^{iF_2})v_2 \mathcal{Q}_\delta v_2)\|_{L_T^{1+\kappa} H_x^s} \\
& \lesssim \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|e^{iF_1} - e^{iF_2}\|_{L_{T,x}^\infty} (1 + \|v_1\|_{L_T^\infty L_x^2}) \right) \|v_2 \mathcal{Q}_\delta v_2\|_{L_T^\infty H_x^s} \\
& \lesssim \delta^{-1} (1 + \delta^{-1}) (1 + \|v_1\|_{L_T^\infty L_x^2}) \|\tilde{v}\|_{L_T^\infty L_x^2} \|v_2\|_{L_T^\infty H_x^s} \|v_2\|_{L_T^\infty L_x^2} \\
& \lesssim \varepsilon \delta^{-1} (1 + \delta^{-1}) \|v_2\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{4.18}$$

Similarly, by Lemma 3.3, (4.11), and Lemma 2.3 with (4.10), we have

$$\begin{aligned}
& \|\mathbf{P}_{+, \text{hi}}((e^{iF_1} - e^{iF_2}) \mathcal{Q}_\delta \partial_x v_2)\|_{L_T^{1+\kappa} H_x^s} \\
& \lesssim \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|e^{iF_1} - e^{iF_2}\|_{L_{T,x}^\infty} (1 + \|v_1\|_{L_T^\infty L_x^2}) \right) \|\mathcal{Q}_\delta v_2\|_{L_T^\infty H_x^{s+1}} \\
& \lesssim \delta^{-1} (1 + \delta^{-1}) \|v_2\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{4.19}$$

Therefore, from the Duhamel formulation of (4.13), Lemma 2.1, (4.14), (4.15), (4.16), (4.17), (4.18), and (4.19) together with [44, the math display after (4-4)], (4.11), and (4.10), we obtain

$$\begin{aligned}
\|\tilde{w}\|_{Y^{s, \frac{1}{2}}(T)} & \leq C \|\tilde{w}(0)\|_{H^s} \\
& + C(\varepsilon + \|w_1\|_{X^{s, \frac{1}{2}}(T)}) \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|\tilde{v}\|_{L_{T,x}^4} + \|\tilde{v}\|_{X^{-1,1}(T)} \right) \\
& + C\varepsilon \|w_2\|_{Y^{s, \frac{1}{2}}(T)} \|\tilde{v}\|_{L_T^\infty L_x^2} \\
& + C\delta^{-1} (1 + \delta^{-1}) \left(\varepsilon \|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_2\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2} \right) \\
& + 2c_0 \left(\|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_1\|_{L_T^\infty H_x^s} \|\tilde{v}\|_{L_T^\infty L_x^2} \right)
\end{aligned} \tag{4.20}$$

for any $0 < T \leq T_*$, where c_0 is as in (4.15).

From (3.3), we have

$$\partial_t \tilde{v} - \mathcal{H} \partial_x^2 \tilde{v} = \mathcal{Q}_\delta \partial_x \tilde{v} + \partial_x ((v_1 + v_2) \tilde{v}). \tag{4.21}$$

Then, by a slight modification of the proof of (3.15) in Lemma 3.4 with (4.10), we have

$$\|\tilde{v}\|_{X^{-1,1}(T)} \lesssim T^\theta (1 + \delta^{-1}) \|\tilde{v}\|_{L_T^\infty L_x^2} + \varepsilon \|\tilde{v}\|_{L_{T,x}^4} \tag{4.22}$$

for any $0 < T \leq T_*$.

In the following, we estimate the $L_T^p W_x^{s,q}$ -norm of $\tilde{v} = \mathbf{P}_+ \tilde{v} + \overline{\mathbf{P}_+ \tilde{v}}$. From (3.14) with $e^{iF} v = -i \partial_x e^{iF}$, we have

$$\begin{aligned}
\mathbf{P}_+ \tilde{v} & = \mathbf{P}_{+, \text{LO}} \tilde{v} + \mathbf{P}_{+, \text{HI}} \tilde{v} \\
& = \mathbf{P}_{+, \text{LO}} \tilde{v} - i \mathbf{P}_{+, \text{HI}} (e^{-iF_1} \tilde{w}) - i \mathbf{P}_{+, \text{HI}} ((e^{-iF_1} - e^{-iF_2}) w_2) \\
& \quad - i \mathbf{P}_{+, \text{HI}} (\mathbf{P}_{+, \text{hi}} e^{-iF_1} \cdot \partial_x \mathbf{P}_{\text{lo}} (e^{iF_1} - e^{iF_2})) \\
& \quad - i \mathbf{P}_{+, \text{HI}} (\mathbf{P}_{+, \text{hi}} (e^{-iF_1} - e^{-iF_2}) \cdot \partial_x \mathbf{P}_{\text{lo}} e^{iF_2}) \\
& \quad - i \mathbf{P}_{+, \text{HI}} (\mathbf{P}_{+, \text{HI}} e^{-iF_1} \cdot \partial_x \mathbf{P}_{-, \text{hi}} (e^{iF_1} - e^{iF_2})) \\
& \quad - i \mathbf{P}_{+, \text{HI}} (\mathbf{P}_{+, \text{HI}} (e^{-iF_1} - e^{-iF_2}) \cdot \partial_x \mathbf{P}_{-, \text{hi}} e^{iF_2}) \\
& =: E_1 + \cdots + E_7.
\end{aligned} \tag{4.23}$$

As pointed out in the proof of Lemma 3.4, (4.23) agrees with the corresponding expression in [44, p.385] (modulo constants). In particular, when (i) $(p, q) = (\infty, 2)$ and $0 \leq s \leq \frac{1}{2}$ or (ii) $(p, q) = (4, 4)$ and $0 \leq s \leq \frac{1}{4}$, the $L_T^p W_x^{s,q}$ -norm of $\tilde{v} = \mathbf{P}_+ \tilde{v} + \overline{\mathbf{P}_+ \tilde{v}}$ is already estimated in [44, (4-8)], except for the additional term in the low-frequency contribution which is estimated as in (3.19):

$$\left\| J^s \int_0^t S(t-t') \mathbf{P}_{+, \text{LO}} \mathcal{Q}_\delta \partial_x \tilde{v}(t') dt' \right\|_{L_T^p L_x^q} \lesssim T \delta^{-1} \|\tilde{v}\|_{L_T^\infty L_x^2}. \quad (4.24)$$

Hence, we only need to consider the case $(p, q) = (4, 4)$ and $\frac{1}{4} < s \leq \frac{1}{2}$.

In the following, we work with $T \in (0, T_*]$ such that (4.10) holds. By applying $\mathbf{P}_{+, \text{LO}}$ to the Duhamel formulation of (4.21) and proceeding as in [44, (3-17) and (3-18)] with (4.10) and (4.24), given an absolute small constant $c_1 > 0$ (to be chosen later), we have

$$\begin{aligned} \|J^s E_1\|_{L_{T,x}^4} &\leq \|\tilde{v}(0)\|_{L^2} + C(\varepsilon + T(1 + \delta^{-1})) \|\tilde{v}\|_{L_T^\infty L_x^2} \\ &\leq \|\tilde{v}(0)\|_{L^2} + (C\varepsilon + c_1) \|\tilde{v}\|_{L_T^\infty L_x^2}, \end{aligned} \quad (4.25)$$

where the second step holds by possibly making $T_* = T_*(\min(\delta, 1)) > 0$ smaller. Proceeding as in (3.20) with (4.10), we have

$$\|J^s E_2\|_{L_{T,x}^4} \lesssim (1 + \|v_1\|_{L_T^\infty L_x^2}) \|\tilde{w}\|_{Y^{s, \frac{1}{2}}(T)} \lesssim \|\tilde{w}\|_{Y^{s, \frac{1}{2}}(T)}. \quad (4.26)$$

By Lemma 3.3, (4.11), Lemma 2.2 with (4.10), we have

$$\begin{aligned} \|J^s E_3\|_{L_{T,x}^4} &\lesssim \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|e^{-iF_1} - e^{-iF_2}\|_{L_{T,x}^\infty} (1 + \|v_1\|_{L_T^\infty L_x^2}) \right) \|J^s w_2\|_{L_{T,x}^4} \\ &\lesssim \|w_2\|_{Y^{s, \frac{1}{2}}(T)} \|\tilde{v}\|_{L_T^\infty L_x^2}. \end{aligned} \quad (4.27)$$

Proceeding as in [44, (3-14)] with (4.11) and (4.10), we have

$$\begin{aligned} \|J^s E_4\|_{L_{T,x}^4} &\lesssim \|v_1\|_{L_T^\infty L_x^2} \|e^{iF_1} v_1 - e^{iF_2} v_2\|_{L_T^\infty L_x^2} \\ &\lesssim \|v_1\|_{L_T^\infty L_x^2} \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|e^{iF_1} - e^{iF_2}\|_{L_{T,x}^\infty} \|v_2\|_{L_T^\infty L_x^2} \right) \\ &\lesssim \varepsilon \|\tilde{v}\|_{L_T^\infty L_x^2}. \end{aligned} \quad (4.28)$$

Similarly, proceeding as in [44, (3-14)] with (3.4), (4.11), and (4.10), we have

$$\begin{aligned} \|J^s E_5\|_{L_{T,x}^4} &\lesssim \|\partial_x \mathbf{P}_{+, \text{hi}}(e^{-iF_1} - e^{-iF_2})\|_{L_T^\infty L_x^2} \|v_2\|_{L_T^\infty L_x^2} \\ &\lesssim \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|e^{-iF_1} - e^{-iF_2}\|_{L_{T,x}^\infty} \|v_2\|_{L_T^\infty L_x^2} \right) \|v_2\|_{L_T^\infty L_x^2} \\ &\lesssim \varepsilon \|\tilde{v}\|_{L_T^\infty L_x^2}. \end{aligned} \quad (4.29)$$

Next, we treat E_6 and E_7 . Proceeding as in (4.29), we have

$$\|\partial_x \mathbf{P}_{-, \text{hi}}(e^{iF_1} - e^{iF_2})\|_{L_T^\infty L_x^2} \lesssim \|\tilde{v}\|_{L_T^\infty L_x^2}. \quad (4.30)$$

By Lemma 3.3 (recall that $\frac{1}{4} < s \leq \frac{1}{2}$) with (4.11) and (4.10), we have

$$\begin{aligned}
& \|e^{-iF_1}v_1 - e^{-iF_2}v_2\|_{L_T^\infty H_x^{s-\frac{1}{4}}} \leq \|e^{-iF_1}\tilde{v}\|_{L_T^\infty H_x^{s-\frac{1}{4}}} + \|(e^{-iF_1} - e^{-iF_2})v_2\|_{L_T^\infty H_x^{s-\frac{1}{4}}} \\
& \lesssim (1 + \|v_1\|_{L_T^\infty L_x^2})\|\tilde{v}\|_{L_T^\infty H_x^{s-\frac{1}{4}}} \\
& \quad + \left(\|\tilde{v}\|_{L_T^\infty L_x^2} + \|e^{-iF_1} - e^{-iF_2}\|_{L_{T,x}^\infty} (1 + \|v_1\|_{L_T^\infty L_x^2})\right)\|v_2\|_{L_T^\infty H_x^{s-\frac{1}{4}}} \\
& \lesssim \|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_2\|_{L_T^\infty H_x^s}\|\tilde{v}\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{4.31}$$

Thus, proceeding as in (3.22)-(3.25) with (4.30), (4.31), and (4.10), we obtain

$$\begin{aligned}
& \|J^s E_6\|_{L_{T,x}^4} + \|J^s E_7\|_{L_{T,x}^4} \\
& \lesssim (1 + \|v_1\|_{L_T^\infty L_x^2})\|v_1\|_{L_T^\infty H_x^s}\|\tilde{v}\|_{L_T^\infty L_x^2} \\
& \quad + \left(\|\tilde{v}\|_{L_T^\infty H_x^s} + \|v_2\|_{L_T^\infty H_x^s}\|\tilde{v}\|_{L_T^\infty L_x^2}\right)\|v_2\|_{L_T^\infty L_x^2} \\
& \lesssim \|v_1\|_{L_T^\infty H_x^s}\|\tilde{v}\|_{L_T^\infty L_x^2} + \varepsilon\|\tilde{v}\|_{L_T^\infty H_x^s} + \varepsilon\|v_2\|_{L_T^\infty H_x^s}\|\tilde{v}\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{4.32}$$

Therefore, putting (4.23), (4.25), (4.26), (4.27), (4.28), (4.29), and (4.32) together with [44, (4-8)] and (4.11), we obtain

$$\begin{aligned}
& \|J^s \tilde{v}\|_{L_T^p L_x^q} \lesssim \|\tilde{v}(0)\|_{L^2} + \left(\|w_2\|_{Y^{s,\frac{1}{2}}(T)} + \|v_1\|_{L_T^\infty H_x^s} + \varepsilon\|v_2\|_{L_T^\infty H_x^s} + c_1\right)\|\tilde{v}\|_{L_T^\infty L_x^2} \\
& \quad + \varepsilon\|\tilde{v}\|_{L_T^\infty H_x^s} + \|\tilde{w}\|_{Y^{s,\frac{1}{2}}(T)}
\end{aligned} \tag{4.33}$$

for $0 \leq s \leq \frac{1}{2}$, $(p, q) = (\infty, 2)$ or $(4, 4)$, and any $0 < T \leq T_*$, where c_1 is as in (4.25).

By collecting (4.12), (4.20), (4.22), and (4.33) for $s = 0$ with (4.10), we first obtain

$$\|\tilde{v}\|_{L_T^\infty L_x^2} + \|\tilde{v}\|_{L_{T,x}^4} + \|\tilde{w}\|_{Y^{0,\frac{1}{2}}(T)} + \|\tilde{v}\|_{X^{-1,1}(T)} \lesssim \|\tilde{v}(0)\|_{L^2} \tag{4.34}$$

for any $0 < T \leq T_*$, provided that we choose (i) $\varepsilon_0 = \varepsilon_0(\min(\delta, 1)) > 0$ in (4.10) sufficiently small such that

$$\varepsilon_0 \delta^{-1} (1 + \delta^{-1}) \ll 1 \tag{4.35}$$

and (ii) $c_0 > 0$ in (4.15) and (4.20) and $c_1 > 0$ in (4.25) sufficiently small, which can be guaranteed by choosing $T_* = T_*(\min(\delta, 1)) > 0$ sufficiently small. Finally, from (4.12), (4.20), (4.22), and (4.33) with (4.34), we conclude that, with M as in (4.10),

$$\|J^s \tilde{v}\|_{L_T^\infty L_x^2} + \|J^s \tilde{v}\|_{L_{T,x}^4} + \|\tilde{w}\|_{Y^{s,\frac{1}{2}}(T)} + \|\tilde{v}\|_{X^{-1,1}(T)} \leq C(M)\|\tilde{v}(0)\|_{H^s} \tag{4.36}$$

for any $0 < T \leq T_*$, provided that $\varepsilon_0, c_0 > 0$ are sufficiently small as above.

4.3. Well-posedness. Once we have (4.36), local well-posedness of the renormalized ILW (3.2) (and hence of the original ILW equation (1.1)) follows from a standard argument. Given $0 \leq s \leq \frac{1}{2}$, let $u_0 \in H_0^s(\mathbb{T})$ with $2\|u_0\|_{L^2} \leq \varepsilon_0 = \varepsilon_0(\min(\delta, 1))$ and consider the sequence $\{u_{0,k}\}_{k \in \mathbb{N}}$ of smooth mean-zero initial data $u_{0,k} = \Pi_{\leq k} u_0$, where $\Pi_{\leq k}$ is the frequency projector onto the (spatial) frequencies $\{n \in \mathbb{Z} : |n| \leq k\}$. Let v_k be the smooth global solution to (3.2) with $v_k|_{t=0} = u_{0,k}$. Then, it follows from (4.36) that v_k converges to a limit v in $C([0, T_*]; H^s(\mathbb{T})) \cap L^4([0, T_*]; W^{s,4}(\mathbb{T}))$ and that the associated gauged function $w_k = w_k(v_k)$ (defined as in (3.12)) converges to a limit w in $Y^{s,\frac{1}{2}}(T_*)$, where $T_* = T_*(\min(\delta, 1)) > 0$ is as in the previous subsections. It is easy to verify that the limit v satisfies the equation (3.2) with $v|_{t=0} = u_0$ and that the limits v and w satisfy (3.12). In the current periodic setting,

uniqueness of solutions and local Lipschitz continuity of the solution map for the renormalized ILW (3.2) follow from (4.36). Furthermore, by noting that the local existence time $T_* = T_*(\min(\delta, 1)) > 0$ depends only on $\delta > 0$ (as long as we choose ε_0 (= the upper bound on the L^2 -norm of initial data) sufficiently small such that (4.35) is satisfied), we conclude global well-posedness of (3.2) from the conservation of the L^2 -norm. Finally, by undoing the Galilean transform (3.1), we obtain global well-posedness of the original ILW equation (1.1). This concludes the proof of Theorem 1.1.

Remark 4.2. In this section, we proved local and global well-posedness of the renormalized ILW (3.2) (and then of ILW (1.1)) with the small L^2 -norm assumption. In the general case, we apply a scaling argument and reduce the situation to the small L^2 -norm case as in [13, 42, 43, 44]. Given $\lambda \geq 1$, let \mathcal{S}_λ be the $\dot{H}^{-\frac{1}{2}}$ -invariant scaling operator defined by

$$\mathcal{S}_\lambda(u)(t, x) = \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x). \quad (4.37)$$

Recall that the BO equation (1.5) on \mathbb{R} is invariant under the action of \mathcal{S}_λ . While the renormalized ILW (3.2) on \mathbb{R} does not enjoy any scaling symmetry, a direct computation shows that if v is a solution to (3.2) on \mathbb{R} with the depth parameter δ , then $v_\lambda = \mathcal{S}_\lambda(v)$ is a solution to (3.2) on \mathbb{R} with the depth parameter $\lambda\delta$:

$$\partial_t v_\lambda - \mathcal{T}_{\lambda\delta} \partial_x^2 v_\lambda = \partial_x(v_\lambda^2). \quad (4.38)$$

Namely, *the family of the renormalized ILW equations with depth parameters $0 < \delta < \infty$ remains invariant under the action of \mathcal{S}_λ .* In the periodic setting, if v satisfies (3.2) on \mathbb{T} then $v_\lambda = \mathcal{S}_\lambda(v)$ satisfies (4.38) on the dilated torus $\mathbb{T}_\lambda = \mathbb{R}/(2\pi\lambda\mathbb{Z})$. With $u_0 = v(0)$, the scaled initial data for v_λ satisfying (4.38) is given by $u_{0,\lambda}(x) = \lambda^{-1}u_0(\lambda^{-1}x)$. Noting that

$$\|u_{0,\lambda}\|_{L^2(\mathbb{T}_\lambda)} = \lambda^{-\frac{1}{2}}\|u_0\|_{L^2(\mathbb{T})},$$

we can choose $\lambda \gg 1$ such that $\|u_{0,\lambda}\|_{L^2(\mathbb{T}_\lambda)} \leq \varepsilon_0 = \varepsilon_0(\min(\delta, 1))$. We point out that we can choose $\varepsilon_0(\min(\delta, 1))$ independently of $\lambda \geq 1$ (since $\lambda\delta \geq \delta$). As pointed out in [44, Section 7], all the estimates hold true on \mathbb{T}_λ with constants independent of $\lambda \geq 1$. Here, the most important (and well-known) point is that the periodic L^4 -Strichartz estimate (Lemma 2.2) holds true on \mathbb{T}_λ with a constant independent of $\lambda \geq 1$. With this observation, we can repeat the argument presented in this section and prove global well-posedness of (4.38) in $H^s(\mathbb{T}_\lambda)$ with the small L^2 -norm assumption. Then, by undoing the scaling, we conclude global well-posedness of (3.2) (and hence of (1.1)) in $H^s(\mathbb{T})$ without the small L^2 -norm assumption.

Remark 4.3. The bound (4.11), coming from (3.7) and (3.8), played a crucial role in the argument on \mathbb{T} presented above, yielding local Lipschitz continuity of the solution map in the periodic case. As pointed out in Remark 3.1, however, (3.7) and (3.8) (and hence (4.11)) do not hold in the real line case. This is the primary reason for the failure of local uniform continuity of the solution map for ILW on \mathbb{R} constructed in Theorem 1.1. Nonetheless, under the extra assumption (3.9), one can still prove (3.7) and (3.8) on \mathbb{R} (but the argument uses the equation; see [44, Lemma 4.1] in the case of the BO equation), giving local Lipschitz continuity of the solution map restricted to the class (3.9) (say, for fixed $v_1(0)$) consisting of elements with the identical low-frequency parts. See Section 4 in [44] for a further discussion.

5. DEEP-WATER LIMIT OF ILW IN L^2

In this section, by slightly modifying the argument in Section 4, we present the proof of Theorem 1.2. As in the previous sections, we only consider the periodic case. Moreover, we only consider the case with sufficiently small L^2 -norms.

• **Part 1:** In this part, we consider the mean-zero case. Namely, given $0 \leq s \leq \frac{1}{2}$, fix a mean-zero function $u_0 \in H_0^s(\mathbb{T})$ and let $\{u_{0,\delta}\}_{1 \leq \delta < \infty}$ be a net in $H_0^s(\mathbb{T})$ (of mean-zero functions) such that $u_{0,\delta}$ converges to u_0 in $H^s(\mathbb{T})$ as $\delta \rightarrow \infty$.

We first establish convergence of solutions to the renormalized ILW (3.2) to the associated solution to the BO equation (1.5). Let u be the global solution to the BO equation (1.5) with $u|_{t=0} = u_0$ constructed in [44], and let v_δ , $1 \leq \delta < \infty$, be the global solution to (3.2) with $v_\delta|_{t=0} = u_{0,\delta}$ constructed in Theorem 1.1. Recalling that $\mathcal{Q}_\infty = 0$, the renormalized ILW (3.2) formally reduces to the BO equation (1.5) when $\delta = \infty$. In view of this, we set $v_\infty = u$ and $u_{0,\infty} = u_0$. For each $1 \leq \delta \leq \infty$, let F_δ be the mean-zero primitive of v_δ as defined in (3.4), and let W_δ and w_δ be the associated gauged functions as in (3.10) and (3.12).

Let N_T^s be as in (4.1). Then, the a priori bound (4.6) holds uniformly in $1 \leq \delta \leq \infty$ (including $\delta = \infty$):

$$N_T^s(v_\delta) \lesssim (1 + \|u_{0,\delta}\|_{L^2}) \|u_{0,\delta}\|_{H^s} + N_T^0(v_\delta)(1 + N_T^0(v_\delta))N_T^s(v_\delta) \quad (5.1)$$

for any $0 < T \leq T_*$, where $T_* > 0$ is now independent of $1 \leq \delta \leq \infty$. By assuming that $\|u_{0,\infty}\|_{L^2} = \|u_0\|_{L^2} \leq \varepsilon$ for sufficiently small $\varepsilon > 0$, it follows from the convergence of $u_{0,\delta}$ to $u_{0,\infty}$ in $H^s(\mathbb{T})$ that there exists $\delta_0 = \delta_0(\varepsilon) \geq 1$ such that $\|u_{0,\delta}\|_{L^2} \lesssim \varepsilon$ for any $\delta \geq \delta_0$. Then, from (5.1), we have

$$N_{T_*}^0(v_\delta) \lesssim \varepsilon \quad \text{and} \quad N_{T_*}^s(v_\delta) \lesssim \|u_{0,\delta}\|_{H^s} \leq M \quad (5.2)$$

for some $M > 0$, uniformly in $\delta_0 \leq \delta \leq \infty$.

Let $\tilde{v}_\delta = v_\delta - v_\infty$, $\tilde{W}_\delta = W_\delta - W_\infty$, and $\tilde{w}_\delta = w_\delta - w_\infty$. The difference \tilde{w}_δ satisfies

$$\begin{aligned} & \partial_t \tilde{w}_\delta - \mathcal{H} \partial_x^2 \tilde{w}_\delta \\ &= -2\partial_x \mathbf{P}_{+, \text{hi}}(W_\delta \mathbf{P} - \partial_x \tilde{v}_\delta) - 2\partial_x \mathbf{P}_{+, \text{hi}}(\tilde{W}_\delta \mathbf{P} - \partial_x v_\infty) \\ & \quad - 2\partial_x \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{\text{lo}} e^{iF_\delta} \mathbf{P} - \partial_x \tilde{v}_\delta) - 2\partial_x \mathbf{P}_{+, \text{hi}}(\mathbf{P}_{\text{lo}}(e^{iF_\delta} - e^{iF_\infty}) \mathbf{P} - \partial_x v_\infty) \\ & \quad + i\partial_x \mathbf{P}_{+, \text{hi}}(e^{iF_\delta} \mathcal{Q}_\delta v_\delta) + 0 \\ & \quad - i\mathbf{P}_0(v_\delta^2) \tilde{w}_\delta - i\mathbf{P}_0(v_\delta^2 - v_\infty^2) w_\infty. \end{aligned} \quad (5.3)$$

Compare this with (4.13), where the difference appears only in the fifth term and the sixth term (which is 0 in (5.3)). In estimating the fifth term on the right-hand side of (5.3), we instead use (3.31). Hence, from the discussion in Subsection 4.2 leading to (4.20) with (5.2), we have

$$\begin{aligned} \|\tilde{w}_\delta\|_{Y^{s, \frac{1}{2}}(T_*)} &\leq C \|\tilde{w}_\delta(0)\|_{H^s} \\ & \quad + C(\varepsilon + \|w_\delta\|_{X^{s, \frac{1}{2}}(T_*)}) \left(\|\tilde{v}_\delta\|_{L_{T_*}^\infty L_x^2} + \|\tilde{v}_\delta\|_{L_{T_*, x}^4} + \|\tilde{v}_\delta\|_{X^{-1,1}(T_*)} \right) \\ & \quad + C\varepsilon \|w_\infty\|_{Y^{s, \frac{1}{2}}(T_*)} \|\tilde{v}_\delta\|_{L_{T_*}^\infty L_x^2} + T_*^\theta M \delta^{-1}, \end{aligned} \quad (5.4)$$

uniformly in $\delta_0 \leq \delta \leq \infty$. From (1.5) and (3.3), we have

$$\partial_t \tilde{v}_\delta - \mathcal{H} \partial_x^2 \tilde{v}_\delta = \mathcal{Q}_\delta \partial_x v_\delta + \partial_x((v_\delta + v_\infty) \tilde{v}_\delta). \quad (5.5)$$

Then, by a slight modification of the proof of (3.15) in Lemma 3.4 with (5.2), we have

$$\begin{aligned} \|\tilde{v}_\delta\|_{X^{-1,1}(T_*)} &\lesssim \|\tilde{v}_\delta\|_{L_{T_*}^\infty L_x^2} + \varepsilon \|\tilde{v}_\delta\|_{L_{T_*,x}^4} + \delta^{-1} \|v_\delta\|_{L_{T_*}^\infty L_x^2} \\ &\lesssim \|\tilde{v}_\delta\|_{L_{T_*}^\infty L_x^2} + \varepsilon \|\tilde{v}_\delta\|_{L_{T_*,x}^4} + \varepsilon \delta^{-1}, \end{aligned} \quad (5.6)$$

uniformly in $\delta_0 \leq \delta \leq \infty$.

Lastly, note that (5.5) is different from (4.21) only on the first term on the right-hand side of (5.5) (under the identification of v_1 and v_2 with v_δ and v_∞ , respectively). Namely, the bound (4.33) holds except for $c_1 \|\tilde{v}\|_{L_T^\infty L_x^2}$ in (4.33), coming from (4.24) and (4.25). In this case, we instead apply (3.19) with (5.2) to estimate the contribution from the first term on the right-hand side of (5.5). This yields

$$\begin{aligned} \|J^s \tilde{v}_\delta\|_{L_{T_*}^p L_x^q} &\lesssim \|\tilde{v}_\delta(0)\|_{L^2} + \left(\|w_\infty\|_{Y^{s,\frac{1}{2}}(T_*)} + \|v_\delta\|_{L_{T_*}^\infty H_x^s} + \varepsilon \|v_\infty\|_{L_{T_*}^\infty H_x^s} \right) \|\tilde{v}_\delta\|_{L_{T_*}^\infty L_x^2} \\ &\quad + \varepsilon \|\tilde{v}_\delta\|_{L_{T_*}^\infty H_x^s} + \|\tilde{w}_\delta\|_{Y^{s,\frac{1}{2}}(T_*)} + \varepsilon \delta^{-1} \end{aligned} \quad (5.7)$$

for $0 \leq s \leq \frac{1}{2}$ and $(p, q) = (\infty, 2)$ or $(4, 4)$, uniformly in $\delta_0 \leq \delta \leq \infty$, where the last term represents the contribution from the first term on the right-hand side of (5.5).

Therefore, putting (5.4), (5.6), and (5.7) together with (4.12) (for $\tilde{w}_\delta(0)$ and $\tilde{v}_\delta(0)$) and proceeding as in Subsection 4.2 (first for $s = 0$ as in (4.34) and then for s as in (4.36)), we obtain

$$\begin{aligned} &\|J^s \tilde{v}_\delta\|_{L_{T_*}^\infty L_x^2} + \|J^s \tilde{v}\|_{L_{T_*,x}^4} + \|\tilde{w}_\delta\|_{Y^{s,\frac{1}{2}}(T_*)} + \|\tilde{v}_\delta\|_{X^{-1,1}(T_*)} \\ &\leq C(M) \|\tilde{v}_\delta(0)\|_{H^s} + C(M) \delta^{-1}, \end{aligned} \quad (5.8)$$

uniformly in $\delta_0 \leq \delta \leq \infty$, provided that $T_* > 0$ sufficiently small. (Recall that $T_* > 0$ is independent of $1 \leq \delta \leq \infty$.) Here, the last term on the right-hand side of (5.8) comes from the last term in (5.6). Since the right-hand side of (5.8) tends to 0 as $\delta \rightarrow \infty$, we conclude that, as $\delta \rightarrow \infty$, the solution v_δ to (3.2) converges to the solution $u = v_\infty$ to (1.5) in $C([0, T_*]; H^s(\mathbb{T})) \cap L^4([0, T_*]; W^{s,4}(\mathbb{T}))$. Given any $T_0 \gg 1$, we can iterate this local-in-time convergence argument and conclude convergence on the entire interval $[0, T_0]$.

Next, we transfer this convergence back to the original ILW equation (1.1). Let u_δ be the global solution to (1.1) with $u_\delta|_{t=0} = u_{0,\delta}$. Then, from (3.1), we have $u_\delta = \tau_{-\delta^{-1}} v_\delta = \tau(-\delta^{-1}, v_\delta)$. Letting u be the global solution to the BO equation (1.5) with $u|_{t=0} = u_0$ as above, it follows from the triangle inequality that

$$\begin{aligned} \|u - u_\delta\|_{C_{T_0} H_x^s} &\leq \|u - \tau(-\delta^{-1}, u)\|_{C_{T_0} H_x^s} + \|\tau(-\delta^{-1}, u) - u_\delta\|_{C_{T_0} H_x^s} \\ &\leq \|u - \tau(-\delta^{-1}, u)\|_{C_{T_0} H_x^s} + \|u - v_\delta\|_{C_{T_0} H_x^s}. \end{aligned} \quad (5.9)$$

By noting that $u = \tau(0, u)$, it follows from Lemma 2.5 that the first term on the right-hand side of (5.9) tends to 0 as $\delta \rightarrow \infty$, while the second term tends to 0 as $\delta \rightarrow \infty$ in view of the discussion above. This proves convergence of u_δ to u in $C([0, T_0]; H^s(\mathbb{T}))$ for any $T_0 > 0$.

• **Part 2:** In this step, we consider the general case without the mean-zero assumption. Given $u_0 \in H^s(\mathbb{T})$, let $\{u_{0,\delta}\}_{1 \leq \delta < \infty}$ be a net in $H^s(\mathbb{T})$ such that $u_{0,\delta}$ converges to u_0 in $H^s(\mathbb{T})$ as $\delta \rightarrow \infty$. Note that the spatial mean $\mu(u_{0,\delta})$ of $u_{0,\delta}$ converges to the spatial mean $\mu(u_0)$ of u_0 as $\delta \rightarrow \infty$.

Let u be the global solution to the BO equation (1.5) with $u|_{t=0} = u_0$ constructed in [44], and let u_δ , $1 \leq \delta < \infty$, be the global solution to (1.1) with $u_\delta|_{t=0} = u_{0,\delta}$ constructed in

Theorem 1.1. Then, by setting $\tilde{u} = \Gamma_{u_0}(u)$ and $\tilde{u}_\delta = \Gamma_{u_{0,\delta}}(u_\delta)$, where Γ is the Galilean transform defined in (1.7), we see that \tilde{u} (and \tilde{u}_δ , respectively) is the global solution to (1.5) with the mean-zero initial data $\tilde{u}|_{t=0} = \tilde{u}_0 = u_0 - \mu(u_0)$ (and to (1.1) with the mean-zero initial data $\tilde{u}_\delta|_{t=0} = \tilde{u}_{0,\delta} = u_{0,\delta} - \mu(u_{0,\delta})$, respectively). Hence, from Part 1, we see that, given any $T_0 > 0$, \tilde{u}_δ converges to \tilde{u} in $C([0, T_0]; H^s(\mathbb{T}))$ as $\delta \rightarrow \infty$.

With τ_h as in (2.9), we have

$$u = \tau(2\mu(u_0), \tilde{u}) + \mu(u_0) \quad \text{and} \quad u_\delta = \tau(2\mu(u_{0,\delta}), \tilde{u}_\delta) + \mu(u_{0,\delta}).$$

Therefore, in view of the convergence of $\mu(u_{0,\delta})$ and \tilde{u}_δ to $\mu(u_0)$ and \tilde{u} , respectively, we conclude from the joint continuity of τ (Lemma 2.5) that u_δ converges to u in $C([0, T_0]; H^s(\mathbb{T}))$ as $\delta \rightarrow \infty$. This concludes the proof of Theorem 1.2.

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REFERENCES

- [1] L. Abdelouhab, J.L. Bona, M. Felland, J.-C. Saut, *Nonlocal models for nonlinear, dispersive waves*, Phys. D 40 (1989), no. 3, 360–392.
- [2] T. Alazard, M. Ifrim, D. Tataru, *A Morawetz inequality for water waves*, Amer. J. Math. 144 (2022), no. 3, 607–699.
- [3] . Benyi, T. Oh, *The Sobolev inequality on the torus revisited*, Publ. Math. Debrecen 83 (2013), no. 3, 359–374.
- [4] . Benyi, T. Oh, O. Pocovnicu, *On the probabilistic Cauchy theory of the cubic nonlinear Schrodinger equation on \mathbb{R}^d , $d \geq 3$* , Trans. Amer. Math. Soc. Ser. B 2 (2015), 1–50.
- [5] . Benyi, T. Oh, T. Zhao, *Fractional Leibniz rule on the torus*, to appear in Proc. Amer. Math. Soc.
- [6] J. Bona, D. Lannes, J.-C. Saut, *Asymptotic models for internal waves*, J. Math. Pures Appl. 89 (2008), no. 6, 538–566.
- [7] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrodinger equations*, Geom. Funct. Anal. 3 (1993), no. 2, 107–156.
- [8] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal. 3 (1993), no. 3, 209–262.
- [9] B. Bringmann, *Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity II: Dynamics*, J. Eur. Math. Soc. (JEMS) 26 (2024), no. 6, 1933–2089.
- [10] N. Burq, F. Planchon, *On well-posedness for the Benjamin-Ono equation*, Math. Ann. 340 (2008), no. 3, 497–542.
- [11] A. Chapouto, J. Forlano, G. Li, T. Oh, D. Pilod, *Low regularity a priori bounds for the intermediate long wave equation*, to appear in Proc. Amer. Math. Soc. Ser. B.
- [12] A. Chapouto, G. Li, T. Oh, *Deep-water and shallow-water limits of statistical equilibria for the intermediate long wave equation*, in preparation.
- [13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp global well-posedness for KdV and modified KdV on \mathbb{R} and \mathbb{T}* , J. Amer. Math. Soc. 16 (2003), no. 3, 705–749.
- [14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Symplectic nonsqueezing of the Korteweg-de Vries flow*, Acta Math. 195 (2005), 197–252.
- [15] Y. Deng, N. Tzvetkov, N. Visciglia, *Invariant measures and long time behaviour for the Benjamin-Ono equation III*, Comm. Math. Phys. 339 (2015), no. 3, 815–857.
- [16] R. Freire, F. Linares, C. Munoz, G. Ponce, *On local energy decay for solution of the Benjamin-Ono equation*, Ann. Inst. H. Poincare Anal. Non Linaire 41 (2024), no. 3, 725–747.

- [17] P. Gérard, T. Kappeler, P. Topalov, *Sharp well-posedness results of the Benjamin-Ono equation in $H^s(\mathbb{T}, \mathbb{R})$ and qualitative properties of its solutions*, Acta Math. 231 (2023), no. 1, 31–88.
- [18] M. Gubinelli, H. Koch, T. Oh, *Renormalization of the two-dimensional stochastic nonlinear wave equations*, Trans. Amer. Math. Soc. 370 (2018), no. 10, 7335–7359.
- [19] Z. Guo, T. Oh, *Non-existence of solutions for the periodic cubic nonlinear Schrödinger equation below L^2* , Int. Math. Res. Not. 2018, no. 6, 1656–1729.
- [20] B. Harrop-Griffiths, M. Ifrim, D. Tataru, *Finite depth gravity water waves in holomorphic coordinates*, Ann. PDE 3 (2017), no. 1, Paper No. 4, 102 pp.
- [21] S. Herr, *Well-posedness results for dispersive equations with derivative nonlinearities*, Thesis (Ph.D.)–Universität Dortmund. 2006. viii+116 pp.
- [22] M. Ifrim, J.-C. Saut, *The lifespan of small data solutions for Intermediate Long Wave equation (ILW)* arXiv:2305.05102 [math.AP].
- [23] M. Ifrim, D. Tataru, *Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation*, Ann. Sci. Éc. Norm. Supér. 52 (2019), no. 2, 297–335.
- [24] A. Ionescu, C. Kenig, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces*, J. Amer. Math. Soc. 20 (2007), no. 3, 753–798.
- [25] A. Ionescu, C. Kenig, *Local and global wellposedness of periodic KP-I equations*, Mathematical aspects of nonlinear dispersive equations, 181–211, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.
- [26] R.J. Iório, Jr., *On the Cauchy problem for the Benjamin-Ono equation*, Comm. Partial Differential Equations 11 (1986), no. 10, 1031–1081.
- [27] R.I. Joseph, *Solitary waves in a finite depth fluid*, J. Phys. A 10 (1977), no. 12, 225–227.
- [28] C. Kenig, K. Koehn, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Res. Lett. 10 (2003), no. 5-6, 879–895.
- [29] C. Kenig, D. Pilod, *Well-posedness for the fifth-order KdV equation in the energy space*, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2551–2612.
- [30] C. Kenig, G. Ponce, L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. 71 (1993), no. 1, 1–21.
- [31] C. Kenig, G. Ponce, L. Vega, *Uniqueness properties of solutions to the Benjamin-Ono equation and related models*, J. Funct. Anal. 278 (2020), no. 5, 108396, 14 pp.
- [32] R. Killip, T. Laurens, M. Vişan, *Sharp well-posedness for the Benjamin-Ono equation*, Invent. Math. 236 (2024), no. 3, 999–1054.
- [33] C. Klein, J.-C. Saut, *Nonlinear dispersive equations–inverse scattering and PDE methods*, Applied Mathematical Sciences, 209. Springer, Cham, [2021], ©2021. xx+580 pp.
- [34] H. Koch, N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$* , Int. Math. Res. Not. 2003, no. 26, 1449–1464.
- [35] H. Koch, N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation*, Int. Math. Res. Not. 2005, no. 30, 1833–1847.
- [36] T. Kubota, D.R.S. Ko, L.D. Dobbs, *Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth*, J. Hydronautics 12 (1978), no. 4, 157–165.
- [37] G. Li, *Deep-water and shallow-water limits of the generalised intermediate long wave equation*, Nonlinearity 37 (2024), no. 7, Paper No. 075001, 44 pp.
- [38] G. Li, *Deep-water and shallow-water limits of the intermediate long wave equation: from deterministic and statistical viewpoints*, Thesis (Ph.D.)–The University of Edinburgh. 2022.
- [39] G. Li, T. Oh, G. Zheng, *On the deep-water and shallow-water limits of the intermediate long wave equation from a statistical viewpoint*, arXiv:2211.03243 [math.AP].
- [40] F. Linares, D. Pilod, J.-C. Saut, *Dispersive perturbations of Burgers and hyperbolic equations I: Local theory*, SIAM J. Math. Anal. 46 (2014), no. 2, 1505–1537.
- [41] F. Linares, G. Ponce, *On decay and asymptotic properties of solutions to the intermediate long wave equation*, arXiv:2406.19111 [math.AP].
- [42] L. Molinet, *Global well-posedness in the energy space for the Benjamin-Ono equation on the circle*, Math. Ann. 337 (2007), no. 2, 353–383.
- [43] L. Molinet, *Global well-posedness in L^2 for the periodic Benjamin-Ono equation*, Amer. J. Math. 130 (2008), no. 3, 635–683.
- [44] L. Molinet, D. Pilod, *The Cauchy problem for the Benjamin-Ono equation in L^2 revisited*, Anal. PDE 5 (2012), no. 2, 365–395.

- [45] L. Molinet, D. Pilod, S. Vento, *On well-posedness for some dispersive perturbations of Burgers' equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 35 (2018), no. 7, 1719–1756.
- [46] L. Molinet, J.-C. Saut, N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal. 33 (2001), no. 4, 982–988.
- [47] L. Molinet, S. Vento, *Improvement of the energy method for strongly nonresonant dispersive equations and applications*, Anal. PDE 8 (2015), no. 6, 1455–1495.
- [48] C. Muñoz, G. Ponce, J.-C. Saut, *On the long time behavior of solutions to the intermediate long wave equation*, SIAM J. Math. Anal. 53 (2021), no. 1, 1029–1048.
- [49] T. Oh, J. Quastel, P. Sosoe, *Global dynamics for the stochastic KdV equation with white noise as initial data*, Trans. Amer. Math. Soc. Ser. B 11 (2024), 420–460.
- [50] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Differential Integral Equations 4 (1991), no. 3, 527–542.
- [51] J.-C. Saut, *Benjamin-Ono and intermediate long wave equations: modeling, IST and PDE. Nonlinear dispersive partial differential equations and inverse scattering*, 95–160, Fields Inst. Commun., 83, Springer, New York, [2019], ©2019.
- [52] T. Tao, *Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbf{R})$* , J. Hyperbolic Differ. Equ. 1 (2004), no. 1, 27–49.
- [53] T. Tao, *Nonlinear dispersive equations. Local and global analysis*. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.
- [54] N. Tzvetkov, N. Visciglia, *Gaussian measures associated to the higher order conservation laws of the Benjamin-Ono equation*, Ann. Sci. Éc. Norm. Supér. 46 (2013), no. 2, 249–299.
- [55] N. Tzvetkov, N. Visciglia, *Invariant measures and long-time behavior for the Benjamin-Ono equation*, Int. Math. Res. Not. IMRN 2014, no. 17, 4679–4714.
- [56] N. Tzvetkov, N. Visciglia, *Invariant measures and long time behaviour for the Benjamin-Ono equation II*, J. Math. Pures Appl. 103 (2015), no. 1, 102–141.

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