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Feedback stabilization of a wave equation coupled with non-expansive systems with input constraints

Ling Ma** Vincent Andrieu* Daniele Astolfi*
Mathieu Bajodek* Xuyang Lou**

* Univ. Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007,
F-69100 Villeurbanne, France (e-mail: name.surname@univ-lyon1.fr)

** School of IoT Engineering, Jiangnan University, Wuxi 214122, China

Abstract: We study the stabilization of a boundary controlled wave equation coupled with a non-expansive ordinary differential equation in the presence of actuator nonlinearities. The open-loop dynamics of the wave equation is conservative, while the ODE is marginally stable. The proposed feedback control law is derived through the forwarding-Lyapunov method, utilizing a weak Lyapunov functional and a Sylvester equation. Well-posedness and global asymptotic stability of the closed-loop system are established.

Keywords: Stabilization, wave equation, input constraints, semigroup theory.

1. INTRODUCTION

Currently, there is considerable research focus on addressing the stabilization challenges of interconnected systems. These systems involve a combination of partial differential equations (PDEs) and ordinary differential equations (ODEs), as highlighted in works such as (Bekiaris-Liberis and Krstic, 2014; Natarajan, 2021; Mathieu et al., 2022). Additionally, attention is given to systems where PDEs are interconnected with other PDEs, as discussed in works like (Auriol and Pietri, 2022). The preceding research primarily relies on backstepping methods. However, when confronted with nonlinearity in the PDE, identifying a suitable target system for stability analysis remains a complex task.

In this article, we focus on the feedback stabilization of a wave equation, particularly in the presence of input constraints, cascaded with a neutrally stable linear system described by an ODE. This type of system, for instance, occurs in oil drilling systems, as illustrated in (Bekiaris-Liberis and Krstic, 2014). In such systems, the drillstring is modeled as a wave PDE, while an ODE describes the dynamics of the bottom drill bit. Specifically, in our case, the first subsystem is a wave equation characterized by conservative properties, and the control action is located at the boundary and subjected to actuator nonlinearities, similar to those discussed in (Vanspranghe et al., 2021; Prieur et al., 2016). Such constraints are modeled by a nonlinear monotonic function which may include for instance a saturation effect. The second subsystem is an ODE, marginally stable and driven by the output of the wave equation. The investigation of a PDE system coupled with integral action has, for instance, been explored in (Terrand-Jeanne et al., 2019; Marx et al., 2021; Ma et al.,

2023; Vanspranghe and Brivadis, 2023). The case where integral action is employed in conjunction with an actuator nonlinearity has also been explored in (Logemann et al., 1999; Marx et al., 2020; Ma et al., 2024).

In our work, inspired by internal model methods (Paunonen and Pohjolainen, 2014; Paunonen, 2019; Guo and Meng, 2020), we examine the case in which the added dynamics is an oscillatory system. Similarly to (Ma et al., 2024), we assume that both systems are independently neutrally stable. This sets us apart from a series of works in which the PDE system is supposed to be open-loop exponentially stable, e.g. (Gauvrit et al., 2023; Lhachemi and Prieur, 2023). The implication is that we cannot rely on properties of robustness and interconnection for cascaded system control. In turns, the control needs to incorporate stabilizing effects on both systems. For this reason, and considering the actuator nonlinearity, we propose to rely on weak Lyapunov functions. Notably, the theoretical analysis includes an approximate observability condition and employs LaSalle's invariance principle. Numerical simulation results are presented to illustrate the effectiveness of the proposed methodology.

The paper is organized as follows. In Section 2, we introduce the problem to be solved in this work, along with properties of the wave equation in open-loop. Section 3 describes the design of a boundary controller with internal dynamics. Sections 4 and 5 present the main results concerning the well-posedness and global asymptotic stability of the closed-loop system. Section 6 provides the numerical simulation results. Finally, conclusions are drawn in Section 7.

Notation. The set $\mathbb{R}_+ = [0, \infty)$ and \mathbb{R} represents the real numbers, \mathbb{N} stands for $\mathbb{N} = \{0, 1, 2, \dots\}$, \mathbb{Z} is the set $\{\dots, -1, 0, 1, \dots\}$, and I_n denotes the identity matrix in \mathbb{R}^n . Let $H^1(0, 1) \subset L^2(0, 1)$ be the Hilbert space of absolutely continuous functions over the interval $(0, 1)$

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with a first-order square-integrable derivative. $H_0^1(0, 1)$ is the closure in $L^2(0, 1)$ of the set of smooth functions that vanish at $x = 0$. $|x|$ represents the Euclidean norm of x , where x in \mathbb{R}^n . For a matrix A , A^\top denotes the transpose.

2. PROBLEM STATEMENT

2.1 System description

In this paper, we consider the stabilization problem of the following interconnected system composed of a wave equation and an ODE:

$$\begin{aligned} \omega_{tt}(x, t) &= \omega_{xx}(x, t), & (t, x) \in \mathbb{R}_+ \times (0, 1), \\ \omega(0, t) &= 0, & t \in \mathbb{R}_+, \\ \omega_x(1, t) &= \psi(u(t)), & t \in \mathbb{R}_+, \\ \dot{z}(t) &= Sz(t) + G\omega(1, t), & t \in \mathbb{R}_+, \end{aligned} \quad (1)$$

where $\omega : \mathbb{R}_+ \times (0, 1) \rightarrow \mathbb{R}$, $z : \mathbb{R}_+ \rightarrow \mathbb{R}^2$, the matrices S, G are defined as

$$S = \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where θ is a positive constant value, $u : \mathbb{R} \rightarrow \mathbb{R}$ is the control input, and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a function which satisfies the following definition 1.

Definition 1. We say that $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a monotonic function if it is continuous and satisfies the following properties:

- $\psi(s) = 0$, if and only if $s = 0$;
- the function $\psi(\cdot)$ is nondecreasing, i.e., for all $(s_1, s_2) \in \mathbb{R}^2$, $(\psi(s_1) - \psi(s_2))(s_1 - s_2) \geq 0$.

The control input is constrained as specified in Definition 1, where these constraints may take the form of saturations or other localized nonlinearities. The dynamics of variable z can be interpreted as an internal model, see, e.g. (Paunonen and Pohjolainen, 2014).

Our goal is to design a controller stabilizing the system's origin (1). It is important to note that the z -component, governed by the skew-symmetric matrix S , is only marginally stable, and (S, G) is a controllable pair. We remark that the class of systems under consideration in this work is very similar to (Gauvrit et al., 2023). However, we cannot use directly their results for two reasons: first, we allow generic monotonic nonlinearities in the input (while in (Gauvrit et al., 2023) they only consider saturation functions); and second, we consider the marginal case in which S is not asymptotically stable.

In the rest of the paper, we will show that the stabilization problem of the system (1) can be solved by a proposed control law by using a (weak) Lyapunov method for some particular values of θ .

2.2 The dissipative property of the wave equation

To obtain the stabilizing control law, we propose to follow the method introduced in (Ma et al., 2024). In the first step, we demonstrate the dissipativity of the infinite-dimensional part of the considered system. Consider the energy function of the following form:

$$E(t) = \frac{1}{2} \int_0^1 [\omega_t^2(x, t) + \omega_x^2(x, t)] dx. \quad (2)$$

The norm is defined by the energy function $E(\cdot)$, which is equivalent to the usual norm on a Hilbert space $\mathcal{H} := H_0^1(0, 1) \times L^2(0, 1)$ defined as

$$\|\xi(t)\|_{\mathcal{H}} = \|\omega(\cdot, t)\|_{H_0^1(0, 1)} + \|\omega_t(\cdot, t)\|_{L^2(0, 1)}, \quad (3)$$

implying that there exist two positive constants c_1 and c_2 such that the following inequality holds:

$$c_1 \|\xi(t)\|_{\mathcal{H}}^2 \leq E(t) \leq c_2 \|\xi(t)\|_{\mathcal{H}}^2.$$

Taking the time derivative of (2), by using the boundary conditions of the wave equation in (1) and integration by parts, one obtains that

$$\dot{E}(t) = [\omega_t(x, t)\omega_x(x, t)]_0^1 = \omega_t(1, t)\psi(u(t)).$$

Hence, since when $u(t) = 0$ we obtain $\dot{E}(t) = 0$, we can conclude that the energy is conserved and the system of the wave equation part is marginally stable.

3. FEEDBACK DESIGN

Following (Ma et al., 2024), to address the stabilization of system (1), it is crucial to notice that the wave equation has the conservative property in the open loop sense as stated in Section 2 and the z -component is marginally stable. Our control method is based on an operator \mathcal{M} , which is a unique solution to a Sylvester equation. In the second step, we formulate a (weak) Lyapunov functional to derive the feedback law.

Let \mathcal{H} be the state space of system (1). We define an operator \mathcal{A} in open loop, and the domain $D(\mathcal{A}) \subset \mathcal{H}$ is specified as:

$$\frac{d}{dt}\xi = \mathcal{A}\xi := \begin{pmatrix} \omega_t \\ \omega_{xx} \end{pmatrix}, \quad \xi = \begin{pmatrix} \omega \\ \omega_t \end{pmatrix}, \quad (4)$$

$$D(\mathcal{A}) := \{\xi \in H^2(0, 1) \times H^1(0, 1) : \omega(0) = \omega_x(1) = 0\}.$$

To derive the controller, we would like to construct the operator $\mathcal{M} : \mathcal{H} \rightarrow \mathbb{R}^{2 \times 1}$, which is the solution to the following Sylvester equation:

$$\mathcal{M}\mathcal{A}\xi = S\mathcal{M}\xi + G\mathcal{C}\xi, \quad \forall \xi \in D(\mathcal{A}), \quad (5)$$

where $\mathcal{C} : \xi \in D(\mathcal{A}) \mapsto \omega(1) \in \mathbb{R}$.

Remark 1. According to the standard theory (Phóng, 1991, Lemma 22), the existence and uniqueness of the Sylvester equation (in this paper like (5)) cannot be conclusively established at now, as it necessitates that the spectrum $\sigma(\mathcal{A})$ and $\sigma(S)$ are disjoint nonempty sets. However, in the subsequent lemma, we provide an explicit solution to equation (5).

Lemma 1. For any $\theta > 0$ satisfying

$$\theta \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{N},$$

the equation (5) admits a unique solution, given by

$$\mathcal{M}\xi = \int_0^1 M(x)\xi(x) dx, \quad M(x) = \begin{bmatrix} -\frac{\sin(\theta x)}{\cos \theta} & 0 \\ 0 & -\frac{\sin(\theta x)}{\theta \cos \theta} \end{bmatrix}. \quad (6)$$

Based on the previous operator \mathcal{M} defined in (5), we select now the Lyapunov-like functional as follows:

$$V(t) = E(t) + \frac{\mu}{2} (z(t) - \mathcal{M}\xi(t))^\top (z(t) - \mathcal{M}\xi(t)), \quad (7)$$

where μ is a positive scalar. According to (6) and (1), one gets that

$$\mathcal{M}\mathcal{A}\xi(t) = \dot{z}(t) - \frac{\sin \theta}{\theta \cos \theta} G\psi(u(t)). \quad (8)$$

Assuming that V is C^1 and differentiating (7) with respect to time along the trajectory of (1), by utilizing equation (8), we obtain

$$\dot{V}(t) = \left(\omega_t(1, t) + \mu \frac{\sin \theta}{\theta \cos \theta} (z(t) - \mathcal{M}\xi(t))^\top G \right) \psi(u(t)).$$

Consequently, we design the control law as follows:

$$u(t) = -\omega_t(1, t) - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (z(t) - \mathcal{M}\xi(t)). \quad (9)$$

With the expression of $M(x)$ in (6), we obtain the explicit form of the controller $u(t)$ in (9) as follows:

$$u(t) = -\omega_t(1, t) - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top z(t) - \mu \frac{\sin \theta}{\theta^2 \cos^2 \theta} \int_0^1 \sin(\theta x) \omega_t(x, t) dx, \quad (10)$$

which is well defined for any $\theta \notin \{\frac{\pi}{2} + k\pi, k \in \mathbb{N}\}$.

Consequently, with the controller $u(t)$ as defined in (10), we can further derive the time derivative of $V(t)$ (if it exists) as follows:

$$\dot{V}(t) = -\psi(u(t))u(t) \leq 0. \quad (11)$$

In the subsequent results, the main objective is to show that the origin of the closed-loop system with the controller (10) is globally asymptotically stable. To this end, we first define another Hilbert space $\mathcal{H}_e = \mathcal{H} \times \mathbb{R}^2$. With the proposed controller given in (10), the closed-loop system can be expressed in an abstract form as:

$$\dot{\chi}(t) = \mathcal{A}_\psi \chi(t), \quad (12)$$

where the nonlinear operator $\mathcal{A}_\psi : D(\mathcal{A}_\psi) \subset \mathcal{H}_e \rightarrow \mathcal{H}_e$ which is defined as

$$\mathcal{A}_\psi \begin{pmatrix} p \\ v \\ r \end{pmatrix} = \begin{pmatrix} v \\ p'' \\ Sr + Gp(1) \end{pmatrix}, \quad \chi = \begin{pmatrix} p \\ v \\ r \end{pmatrix},$$

and

$$D(\mathcal{A}_\psi) = \left\{ H^2(0, 1) \times H^1(0, 1) \times \mathbb{R}^2 : p(0) = 0, p'(1) = \psi(-v(1) - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top r - \mu \frac{\sin \theta}{\theta^2 \cos^2 \theta} \int_0^1 \sin(\theta x) v(x) dx) \right\}. \quad (13)$$

4. WELL-POSEDNESS AND LYAPUNOV STABILITY RESULT

In this section, we show that there exists a unique solution to (12). A crucial step in proving the following theorem is demonstrating that the operator \mathcal{A}_ψ is an m -dissipative operator (see (Miyadera, 1992, Lemma 2.13)) in Hilbert space \mathcal{H}_e . To this end, we first define the following set

$$\mathcal{S} := \left\{ s \in \mathbb{R} \mid s > 0, (1 - s \cos(s)) \tan(s) > 0, \tan(s) \sin(s) > 0 \right\} \setminus \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{N} \right\}. \quad (14)$$

Theorem 1. Suppose that $\psi(\cdot)$ is a monotonic function satisfying Definition 1 and moreover one of the following conditions:

- $\theta \in \mathcal{S}$ with \mathcal{S} defined as in (14);
- $\theta \notin \{\frac{\pi}{2} + k\pi, k \in \mathbb{N}\}$ and there exist $\bar{\psi} > \underline{\psi} > 0$ such that $\underline{\psi} < \psi(s) < \bar{\psi}$ for any $s \in \mathbb{R}$;
- $\theta \notin \{\frac{\pi}{2} + k\pi, k \in \mathbb{N}\}$ and $\psi(s) = s$ for any $s \in \mathbb{R}$.

Then, for every initial condition $\chi_0 \in \mathcal{H}_e$ (resp. $\chi_0 \in D(\mathcal{A}_\psi)$), there exists a mild solution (resp. classical) $\chi \in$

$C^0(\mathbb{R}_+; \mathcal{H}_e)$ (resp. $C^1(\mathbb{R}_+; \mathcal{H}_e) \cap C^0(\mathbb{R}_+; D(\mathcal{A}_\psi))$) to (12). Moreover, there exists $c_0 > 0$ such that

$$\|\chi(t)\|_{\mathcal{H}_e} \leq c_0 \|\chi_0\|_{\mathcal{H}_e}, \quad \forall \chi_0 \in \mathcal{H}_e, \forall t \geq 0; \quad \|\mathcal{A}_\psi \chi(t)\|_{\mathcal{H}_e} \leq c_0 \|\mathcal{A}_\psi \chi_0\|_{\mathcal{H}_e}, \quad \forall \chi_0 \in D(\mathcal{A}_\psi), \forall t \geq 0. \quad (15)$$

Proof. According to (Miyadera, 1992, Lemma 2.13), establishing the m -dissipativity of the operator \mathcal{A}_ψ involves verifying two crucial properties. Firstly, it requires showing the dissipativity of \mathcal{A}_ψ . Secondly, we need to prove the range condition, specifically, ensuring that $\text{Ran}(\mathcal{A}_\psi - \lambda \mathcal{I}_{\mathcal{H}_e}) = \mathcal{H}_e$ for some $\lambda > 0$, with Ran as the range operator and $\mathcal{I}_{\mathcal{H}_e}$ as the identity operator in the Hilbert space \mathcal{H}_e .

Dissipativity of the the operator \mathcal{A}_ψ . Given $\chi_i = (\xi_i, r_i)^\top$, $\xi_i = (p_i, v_i)^\top$ with $\chi_i \in D(\mathcal{A}_\psi)$ for $i = 1, 2$, we define the following inner product

$$\langle \chi_1, \chi_2 \rangle_V = \int_0^1 (p'_1(x)p'_2(x) + v_1(x)v_2(x)) dx + \mu \left(r_1 - \int_0^1 M(x)\xi_1 dx \right)^\top \left(r_2 - \int_0^1 M(x)\xi_2 dx \right). \quad (16)$$

Next, we define $\tilde{p} = p_1 - p_2$, $\tilde{v} = v_1 - v_2$, $\tilde{r} = r_1 - r_2$ and $u_i(t) = -(v_i(1) + \mu \frac{\sin \theta}{\theta \cos \theta} G^\top r_i + \mu \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \frac{\sin(\theta x)}{\theta \cos \theta} v_i(x) dx)$, with $i = 1, 2$.

By denoting $\tilde{\chi} = \chi_1 - \chi_2$ and $\delta \mathcal{A}_\psi(\tilde{\chi}) = \mathcal{A}_\psi(\chi_1) - \mathcal{A}_\psi(\chi_2)$, we compute

$$\begin{aligned} & \langle \tilde{\chi}, \delta \mathcal{A}_\psi(\tilde{\chi}) \rangle_V \\ &= \int_0^1 (\tilde{p}'(x)\tilde{v}'(x) + \tilde{v}(x)\tilde{p}''(x)) dx + \\ & \mu \left[\tilde{r} - \int_0^1 M(x)\tilde{\xi} dx \right]^\top \left[S\tilde{r} + G\tilde{p}(1) - \int_0^1 M(x) \begin{pmatrix} \tilde{v} \\ \tilde{p}'' \end{pmatrix} dx \right]. \end{aligned}$$

Noting that

$$S\tilde{r} + G\tilde{p}(1) - \int_0^1 M(x) \begin{pmatrix} \tilde{v} \\ \tilde{p}'' \end{pmatrix} dx = G \frac{\sin \theta}{\theta \cos \theta} \tilde{p}'(1),$$

we further obtain

$$\begin{aligned} \langle \tilde{\chi}, \delta \mathcal{A}_\psi(\tilde{\chi}) \rangle_V &= \tilde{v}(1)\tilde{p}'(1) + \mu \frac{\sin \theta}{\theta \cos \theta} \tilde{p}'(1) G^\top \tilde{r} \\ & \quad + \mu \frac{\sin \theta}{\theta \cos \theta} \tilde{p}'(1) \int_0^1 \frac{\sin(\theta x)}{\theta \cos \theta} \tilde{v}(x) dx \\ &= \tilde{p}'(1) \left(\tilde{v}(1) + \mu \frac{\sin \theta}{\theta \cos \theta} G^\top \tilde{r} + \mu \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \frac{\sin(\theta x)}{\theta \cos \theta} \tilde{v}(x) dx \right) \\ &= -\tilde{p}'(1) (u_1(t) - u_2(t)) \\ &= -(\psi(u_1(t)) - \psi(u_2(t))) (u_1(t) - u_2(t)) \leq 0. \end{aligned}$$

We can conclude that the operator \mathcal{A}_ψ is dissipative.

Range condition. We need to prove that, for some constant $\lambda > 0$, for any $(\bar{p}, \bar{v}, \bar{r})$ in $\mathcal{H}_e = \mathcal{H} \times \mathbb{R}^2$, there exists (p, v, r) in $D(\mathcal{A}_\psi)$ such that the following equation holds:

$$\mathcal{A}_\psi \begin{pmatrix} p \\ v \\ r \end{pmatrix} - \lambda \begin{pmatrix} p \\ v \\ r \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{v} \\ \bar{r} \end{pmatrix}. \quad (17)$$

A more precise representation of the former equation is expressed as follows:

$$\begin{cases} v - \lambda p = \bar{p}, \\ p'' - \lambda v = \bar{v}, \\ Sr + Gp(1) - \lambda r = \bar{r}, \\ p(0) = 0, v(0) = 0, p'(1) = \psi(U), \end{cases} \quad (18)$$

where

$$U = -v(1) - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top r - \mu \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \frac{\sin(\theta x)}{\theta \cos \theta} v(x) dx. \quad (19)$$

From the equation (18), we obtain the following relations:

$$\begin{aligned} v &= \lambda p + \bar{p}, \\ r &= (S - \lambda I_2)^{-1}(\bar{r} - Gp(1)). \end{aligned} \quad (20)$$

From the first and second line of (18), we get

$$p'' - \lambda^2 p = \lambda \bar{p} + \bar{v}. \quad (21)$$

Now, substituting (20) into (19), we obtain

$$\begin{aligned} U &= -(\lambda - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (S - \lambda I_2)^{-1} G) p(1) - \bar{p}(1) \\ &\quad - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (S - \lambda I_2)^{-1} \bar{r} \\ &\quad - \mu \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \sin(\theta x) [\lambda p(x) + \bar{p}(x)] dx. \end{aligned} \quad (22)$$

To get the solution of the above equation (21), by using integration by parts and with the boundary condition $p(0) = 0$ in (18), we get

$$\begin{aligned} \begin{bmatrix} p(x) \\ p'(x) \end{bmatrix} &= \exp(Jx) \begin{bmatrix} 0 \\ p'(0) \end{bmatrix} \\ &\quad + \int_0^x \exp(J(x-s)) \begin{bmatrix} 0 \\ \bar{v}(s) + \lambda \bar{p}(s) \end{bmatrix} ds, \end{aligned} \quad (23)$$

where

$$J = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix}, \quad \exp(Jx) = \begin{bmatrix} \cosh(\lambda x) & \frac{1}{\lambda} \sinh(-\lambda x) \\ \lambda \sinh(\lambda x) & \cosh(\lambda x) \end{bmatrix},$$

with \sinh (resp. \cosh) being the hyperbolic sinus (resp. cosine). Now, we need to verify the condition $p'(1) = \psi(U)$ as stated in (18). For the U term in (22), we observe that, given $\bar{p}(1)$, \bar{r} , and \bar{p} , the expression of U can be written as follows:

$$U = \alpha_1 p'(0) + \beta_1, \quad (24)$$

where

$$\begin{aligned} \beta_1 &= -\bar{p}(1) - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (S - \lambda I_2)^{-1} \bar{r} \\ &\quad - \mu \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \sin(\theta x) \bar{p}(x) dx \end{aligned} \quad (25)$$

and

$$\begin{aligned} \alpha_1 p'(0) &= -(\lambda - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (S - \lambda I_2)^{-1} G) p(1) \\ &\quad - \mu \lambda \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \sin(\theta x) p(x) dx. \end{aligned} \quad (26)$$

For the first part in (26), we get the expression that:

$$\lambda - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (S - \lambda I_2)^{-1} G = \lambda + \frac{\lambda \mu \sin \theta}{(\lambda^2 + \theta^2) \theta \cos \theta}. \quad (27)$$

For the second part in (26) we use integration by parts and get

$$\begin{aligned} &\mu \lambda \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \sin(\theta x) \exp(Jx) dx \\ &= \frac{\theta^2}{\theta^2 + \lambda^2} \left(-\mu \lambda \frac{\sin \theta}{\theta^2} \exp(J) \right. \\ &\quad \left. - \mu \lambda \frac{\sin \theta}{\theta^2 \cos \theta} I_2 + \mu \lambda \frac{J \sin^2 \theta}{\theta^3 \cos \theta} \exp(J) \right). \end{aligned} \quad (28)$$

According to the above equation, and recalling the notation $C = (1, 0)$, one gets that

$$\begin{aligned} \alpha_1 &= C \left[-\left(\lambda + \frac{\lambda \mu \sin \theta}{(\lambda^2 + \theta^2) \theta \cos \theta} \right) \exp(J) \right. \\ &\quad \left. - \mu \lambda \frac{\sin \theta}{\theta \cos \theta} \int_0^1 \sin(\theta x) \exp(Jx) dx \right] G \\ &= C \left(-A \exp(J) - \frac{\mu \lambda \sin \theta}{(\lambda^2 + \theta^2) \cos \theta} I_2 \right) G, \end{aligned} \quad (29)$$

where

$$A = \left(\lambda I_2 + \frac{\lambda \mu \sin \theta}{(\lambda^2 + \theta^2) \theta \cos \theta} I_2 - \frac{\mu \lambda \sin \theta}{\lambda^2 + \theta^2} I_2 + \frac{\mu \lambda J \sin^2 \theta}{(\lambda^2 + \theta^2) \theta \cos \theta} \right)$$

and $\exp(J)$ is obtained from (23). Using the definition of J and A , the expression of α_1 can be further simplified as

$$\alpha_1 = -\left(1 + \frac{\mu \tan \theta (1 - \theta \cos \theta)}{(\lambda^2 + \theta^2) \theta} \right) \sinh(\lambda) - \frac{\mu \lambda \tan \theta \sin \theta}{(\lambda^2 + \theta^2) \theta} \cosh(\lambda). \quad (30)$$

From equation (23), we can get

$$p'(1) = \alpha_2 p'(0) + \beta_2, \quad (31)$$

with (23) into account, we find

$$\alpha_2 = G^\top \exp(J) G = \cosh(\lambda) > 0, \quad (32)$$

and

$$\beta_2 = G^\top \int_0^1 \exp(J(1-s)) \begin{bmatrix} 0 \\ \bar{v}(s) + \lambda \bar{p}(s) \end{bmatrix} ds. \quad (33)$$

We consider the three assumptions given in Theorem 1:

a) θ belongs to \mathcal{S} .

Note that $\tan \theta (1 - \theta \cos \theta) > 0$ and $\tan \theta \sin \theta > 0$. Hence, for $\mu > 0$ and $\lambda > 0$, referring to (30), we can conclude that $\alpha_1 < 0$. Considering a function $f(p') = \psi(\alpha_1 p' + \beta_1) - (\alpha_2 p' + \beta_2)$, where $f(p')$ is continuous and given $\alpha_1 < 0$ and $\alpha_2 > 0$ we observe that since $f(p')$ is continuous, there exists a $p'(0)$, such that $f(p'(0)) = 0$, and this implies $\psi(\alpha_1 p'(0) + \beta_1) = (\alpha_2 p'(0) + \beta_2)$.

b) $\psi(\cdot)$ is upper and lower bounded.

Given the expression for α_2 in (32) with $\lambda > 0$, it is evident that $\alpha_2 \neq 0$. As $\psi(\cdot)$ is a saturation function, there exist upper and lower bounds $\bar{\psi} > \underline{\psi} > 0$, such that $\bar{\psi} > \psi(\cdot) > \underline{\psi}$. Given $\alpha_2 \neq 0$, there always exists $p'(0)$ that satisfies this equality $\psi(\alpha_1 p'(0) + \beta_1) = \alpha_2 p'(0) + \beta_2$.

c) $\psi(\cdot)$ is linear.

In this case, given $\alpha_2 \neq 0$, there exists $p'(0)$ such that the equation $\alpha_1 p'(0) + \beta_1 = \alpha_2 p'(0) + \beta_2$ holds.

Therefore, the solution p obtained from (23), v and r from (20), hence we can conclude that there exists (p, v, r) in $D(\mathcal{A}_\psi)$ such that the equation (17) is satisfied. In other words, the range condition holds. Now we can conclude that the operator \mathcal{A}_ψ is an m -dissipative operator.

With the m -dissipativity of the operator \mathcal{A}_ψ in hand, we apply (Brezis, 1973, Theorem 3.1, p. 54). This theorem ensures the existence of nonlinear contraction semigroup $t \mapsto e^{\mathcal{A}_\psi t}$ such that, for each $\chi(t)$ in $D(\mathcal{A}_\psi)$, $e^{\mathcal{A}_\psi t} \chi$ is a solution to (12). Furthermore, the solution belongs to $C^1(\mathbb{R}_+; \mathcal{H}_e)$ and satisfies the inequalities

$$\|e^{\mathcal{A}_\psi t} \chi\|_V \leq \|\chi\|_V, \quad \|\mathcal{A}_\psi e^{\mathcal{A}_\psi t} \chi\|_V \leq \|\mathcal{A}_\psi \chi\|_V. \quad (34)$$

Note that, the norm $\|\chi(t)\|_V$ is equivalent to the norm $\|\chi(t)\|_{\mathcal{H}_e}$, meaning that

$$\underline{c} \|\chi(t)\|_{\mathcal{H}_e} \leq \|\chi(t)\|_V \leq \bar{c} \|\chi(t)\|_{\mathcal{H}_e}.$$

This proves Theorem 1. \square

5. GLOBAL ASYMPTOTIC STABILITY ANALYSIS

We establish now the global asymptotic stability result for the closed-loop system (12). To this end, we will rely on the (weak) Lyapunov functional (7) and LaSalle-like arguments.

Theorem 2. Suppose the conditions of Theorem 1 are satisfied and, in addition, suppose that $\theta \notin \{k\pi, k \in \mathbb{N}\}$.

Then, the origin of the closed-loop system (12) is globally asymptotically stable, that is, for every initial condition $\chi_0 \in D(\mathcal{A}_\psi)$, there exists a unique solution $\chi(t)$ such that

$$\lim_{t \rightarrow +\infty} \|e^{\mathcal{A}_\psi t} \chi_0\|_{\mathcal{H}_e} = 0. \quad (35)$$

Proof. As shown in (11), we get along C^1 solutions $\dot{V} \leq -u(t)\psi(u(t))$. In the following, we will show that $\chi = 0$ is the only solution to the system (12) which satisfies for all t ,

$$u(t) = -(\omega_t(1, t) + \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (z(t) - \mathcal{M}\xi)) = 0. \quad (36)$$

To this end, we note that the time derivative of the above equation (36), based on the equation (8), satisfies

$$\begin{aligned} -\omega_{tt}(1, t) - \mu \frac{\sin \theta}{\theta \cos \theta} G^\top (\dot{z}(t) - \mathcal{M}\mathcal{A}\xi) \\ = -\omega_{tt}(1, t) - \mu \frac{\sin^2 \theta}{\theta^2 \cos^2 \theta} \psi(u(t)) = 0. \end{aligned} \quad (37)$$

Since $\psi(u(t)) = 0$ (by using the Point 1 in Definition 1), this implies $\omega_{tt}(1, t) = 0$.

Furthermore, we consider the solution to the following wave equation:

$$\begin{cases} \omega_{tt}(x, t) = \omega_{xx}(x, t), & (t, x) \in \mathbb{R}_+ \times (0, 1), \\ \omega(0, t) = 0, & t \in \mathbb{R}_+, \\ \omega_x(1, t) = 0, & t \in \mathbb{R}_+, \\ \omega_{tt}(1, t) = 0, & t \in \mathbb{R}_+. \end{cases} \quad (38)$$

Forgetting the last boundary condition, solutions are in the form

$$\begin{aligned} \omega(x, t) = \sum_{k \geq 0} \sin((k\pi + \frac{\pi}{2})x) \\ \times (a_k \sin((k\pi + \frac{\pi}{2})t) + b_k \cos((k\pi + \frac{\pi}{2})t)), \end{aligned} \quad (39)$$

where a_k, b_k depend on the initial conditions.

The last condition gives, for all $t \in \mathbb{R}_+$

$$\begin{aligned} 0 = \omega_{tt}(1, t) = \sum_{k \geq 0} (k\pi + \frac{\pi}{2})^2 (-1)^{k+1} \\ \times (a_k \sin((k\pi + \frac{\pi}{2})t) + b_k \cos((k\pi + \frac{\pi}{2})t)). \end{aligned} \quad (40)$$

Since, $e_k = \{t \mapsto (\sin(k\pi + \frac{\pi}{2})t), \cos(k\pi + \frac{\pi}{2})t)\}$ forms an orthogonal basis of $L^2(0, 2)$, it implies that $a_k = b_k = 0$ for all k .

Therefore the only solution to (38) is $\omega(x, t) = 0$ for all (x, t) . From the equation (36), it yields the explicit relationship:

$$\begin{aligned} -\omega_t(1, t) - \mu \frac{\sin \theta}{\theta \cos \theta} \left(z_2(t) + \frac{1}{\theta \cos \theta} \int_0^1 \sin(\theta x) \omega_t(x, t) dx \right) \\ = 0. \end{aligned}$$

Considering $\omega_t(x, t) = 0$, and since $\theta \notin \{k\pi, k \in \mathbb{Z}\}$, the former equation implies that

$$G^\top z(t) + \frac{1}{\theta \cos \theta} \int_0^1 \sin(\theta x) \omega_t(x, t) dx = 0,$$

with $\theta \neq 0$, we can conclude that $z(t) = 0$ for all $t \geq 0$.

From Theorem 1 and Equation (15), it implies that the exists $c > 0$ such that

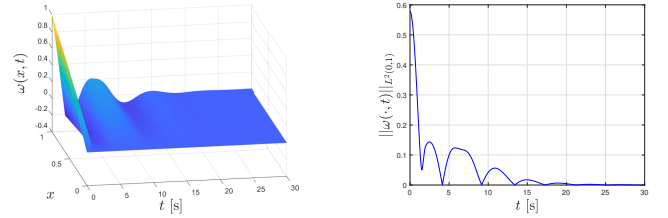
$$\|\omega_t(\cdot, t)\|_{H_0^1(0,1)} + \|\omega_{xx}(\cdot, t)\|_{L^2(0,1)} + |S z(t) + G\omega(1, t)| < c.$$

The above inequality means the precompactness of the trajectory and we can use Theorem 7.1 in (Slemrod, 1989) to conclude the proof. \square

6. NUMERICAL SIMULATION

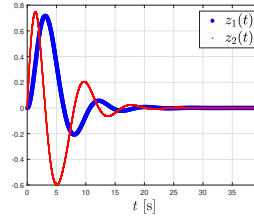
In the simulation, we use the finite difference method to solve the closed-loop system (12) numerically with a time and space step of 0.002. The initial condition are set as $\omega_0(x) = x, \omega_{t0}(x) = 0$ and $z_0 = 0$ in (1). Additionally, we adopt the saturation function with a level $L = 0.2$ as the nonlinear function $\psi(\cdot)$, which satisfies the Definition 1. Let the controller parameter $\mu = 0.5$ and $\theta = 0.5$.

Figs.1(a) and 1(b) illustrate the time evolution of $\omega(x, t)$ and the corresponding norm. It is evident from the figures that both $\omega(x, t)$ and the norm of it converge to zero over time. Furthermore, it is evident from Fig.1(c) that the z -component decreases to zero as time increases. In Fig.1(d), we observe that the controller $u(t)$ undergoes saturation twice, once around 1-2.5 seconds and again around 4-6.6 seconds. The simulation results give evidence of the results we provided in Theorem 1 and 2.

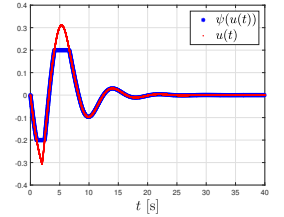


(a) Evolution of $\omega(x, t)$.

(b) $\|\omega(\cdot, t)\|_{L^2(0,1)}$.



(c) Evolution of $z(t)$.



(d) $u(t)$ and $\psi(u(t))$.

Fig. 1. The transient dynamics of system (1) with controller (10).

7. CONCLUSION

In this paper, we have solved the stabilization problem of a wave equation coupled with a non-expansive ODE. Our focus has been on the control part imposed on the wave equation, subject to some actuator nonlinearities described by monotonic functions. Our proposed solution is based on the forwarding-Lyapunov method. With the observability property, we demonstrate that global asymptotic stability of the closed-loop system. As a future work we would like to describe a general procedure in the context of abstract systems in which each subsystem is an infinite-dimensional system, such as a wave equation coupled with a transport equation.

Appendix A. PROOF OF LEMMA 1.

According to Riesz representation theorem in (Brezis, 2011, Theorem 4.11), the operator \mathcal{M} is defined as:

$$\begin{aligned} \mathcal{M}\xi &= \int_0^1 M(x)\xi(x) dx \\ &= \int_0^1 [M_1(x)\omega(x) + M_2(x)\omega_t(x)] dx, \end{aligned} \quad (\text{A.1})$$

where $M(x) = [M_1(x) \ M_2(x)] : [0, 1] \mapsto \mathbb{R}^{2 \times 2}$ is a continuous function with $M_1(x) \in \mathbb{R}^{2 \times 1}$ and $M_2(x) \in \mathbb{R}^{2 \times 1}$. To obtain the function $M(x)$, we apply the definition of the operator \mathcal{M} as given in equation (A.1). Referring to the abstract system description in (4), one gets that

$$\mathcal{M}A\xi = \int_0^1 [M_1(x)\omega_t(x) + M_2(x)\omega_{tt}(x)] dx.$$

Based on (4) and by using the integration by parts, we get the explicit form:

$$\begin{aligned} \mathcal{M}A\xi &= \int_0^1 [M_1(x)\omega_t(x) + M_2(x)\omega_{xx}(x)] dx \\ &= \int_0^1 M_1(x)\omega_t(x) dx - M_2(0)\omega_x(0) \\ &\quad - M_2'(1)\omega(1) + \int_0^1 M_2''(x)\omega(x) dx, \end{aligned} \quad (\text{A.2})$$

and

$$SM\xi = \int_0^1 S [M_1(x)\omega(x) + M_2(x)\omega_t(x)] dx. \quad (\text{A.3})$$

Based on equations (5), (A.2), (A.3), and recalling $\mathcal{C}\xi = \omega(1)$, we get the following boundary value problem:

$$\begin{cases} M_1(x) = SM_2(x), \\ SM_1(x) = M_2''(x), \\ -M_2'(1) = G, \quad M_2(0) = 0. \end{cases} \quad (\text{A.4})$$

From the first line of the equation (A.4), we get that

$$M_2''(x) = -\theta^2 M_2(x), \quad (\text{A.5})$$

where we used $S^2 = -\theta^2 I$. Letting $M_2(x) = (b_1(x), b_2(x))^T$, we obtain the general solution: $b_1(x) = \nu_1 \cos(\theta x) + \nu_2 \sin(\theta x)$ and $b_2(x) = \nu_3 \cos(\theta x) + \nu_4 \sin(\theta x)$. Applying the boundary conditions from (A.4), we find $b_1(0) = \nu_1 = 0$ and $b_2(0) = \nu_3 = 0$. Furthermore, we have $b_1'(1) = \nu_2 \theta \cos \theta = 0$ and $b_2'(1) = \nu_4 \theta \cos \theta = -1$, which leads to $\nu_2 = 0$ and $\nu_4 = -\frac{1}{\theta \cos \theta}$ which requires $\theta \neq \frac{\pi}{2} + k\pi$ and $k \in \mathbb{N}$. Consequently, by denoting $C = (1, 0)$, the solution is given by $M_1(x) = -\frac{\sin(\theta x)}{\cos \theta} C^T$, $M_2(x) = -\frac{\sin(\theta x)}{\theta \cos \theta} G$.

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