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The distribution of minimal fractions on the Stern-Brocot tree

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Abstract. The Stern-Brocot (SB) tree enumerates all irreducible rationals, uniquely positioned with ties to continued fractions and Christoffel words. In this work, we study a quotiented SB tree, focusing on fractions at levels multiple of 3 and their presence on minimal paths defined by the second-order balancedness of Christoffel words. We describe the distribution of all fractions belonging to a minimal path at these levels, detailing their positions, and provide a comprehensive study of the gaps between consecutive fractions, revealing structural patterns tied to the quotiented tree. In prior work, fractions $\frac{F_n}{F_{n+1}}$ with F_n from the Fibonacci sequence were found to be on a minimal path (zigzag path). Analogously, we study four other minimal paths that define the boundary of the set of all minimal paths.

Keywords: Stern-Brocot Tree · Christoffel words · minimal paths · balancedness · Continued fractions.

1 Introduction

Balancedness appears to be a fundamental concept in the emerging field of combinatorics on words. One prominent class of binary balanced words is the Christoffel words, which are of significant interest due to their deep ties to geometry, algebra, and number theory (see [1] for definitions, initial properties, and an extensive bibliography). Each Christoffel word represents the discretization of a line segment with a rational slope and can be uniquely factored into two distinct Christoffel words. This standard factorization enables the organization of all Christoffel words in a binary tree structure, known as the Christoffel tree, with the root being the word $w = 01 = (0, 1)$.

The concept of balancedness allows us to define, for any binary word w , a balance matrix B_w , [10, 11] where the maximal element indicates its order of balancedness, denoted $\delta(w)$. This study focuses specifically on Christoffel words that are balanced, which means that their order of balancedness is 1. For such words, each row of the balance matrix is itself a binary word, allowing us to compute its order of balancedness again. This process yields a second-order balance matrix U_w , [10, 11] whose maximal element is defined as the second-order balance value, $\delta^2(w)$.

The matrix U_w exhibits properties that enable its recursive construction based on three $U_{w'}$ matrices, corresponding to specific ancestors w' of w in the Christoffel tree [11].

Additionally, the correspondence between a Christoffel word and the continued fraction representation of its rational slope allows us to extend the notion of balancedness to the Stern-Brocot tree (hereafter SB tree). The SB tree is a binary tree in which irreducible rational numbers are arranged using the Farey sum operator, defined independently in [3] and [9]. This operator closely mirrors the standard decomposition of Christoffel words. In [5], the authors establish a link between the second-order balance value of a Christoffel word and the structure of the continued fraction representing the slope of its associated discrete segment. Building on this, they characterized the minimal paths (with respect to the second-order balance value) as being the paths in the SB tree which minimize the growth of the value of δ^2 . By combining algebraic and arithmetic techniques applied to continued fraction representations, they showed that any fraction appearing on a minimal path, called minimal fractions, at a level multiple of 3 in SB tree are those rationals whose continued fraction expansions begin with $[0; 1]$, end with 2, and feature middle terms composed of blocks $(1, 1, 1)$, $(2, 1)$, $(1, 2)$, and (3) .

In this paper, extending these previous works, we explore the Stern-Brocot tree quotiented by three, $SB/3$, focusing on fractions appearing at levels multiples of 3. We provide a clear distribution of all minimal fractions at these levels, detailing their positions and conducting a comprehensive study of the gaps between consecutive fractions, revealing structural patterns tied to the tree's quotiented form. Furthermore, we examine five distinct branches of this quotiented tree, each converging to known irrational limits, such as the golden ratio, depending on the branch's recursive construction. Our results illuminate the interplay between level periodicity, fraction distribution, and balancedness properties, while the irrational limits highlight the tree's asymptotic behavior under this novel quotienting framework.

2 Notations

In this section, we introduce some basic notation used in the literature [4].

2.1 Words and Christoffel words

A word w defined over an alphabet A is the concatenation of letters of A , and we denote by A^* the set of all words formed over an alphabet A . We let $|w|$ and $|w|_a$ denote respectively the length of the word w and the number of occurrences of the letter a in w . A word $w \in A^*$ is a palindrome if it is also read the same from right to left, for example, $w = 10101$. Conventionally, let $w^0 = \varepsilon$ be the empty word, and we say w is a *primitive* word if it is not the power of a nonempty word. The notation $w[i \dots j]$ is used to represent the factor of the word w from position i to position j , and $w^\omega = ww \dots w \dots$ is the circular word associated with w . We

are particularly interested in primitive Christoffel words, which encode a specific path below the line segment from $(0,0)$ and (b, a) . Crucially, there are no integer points between this path and the segment, except at the endpoints, where a and b are relatively prime. The encoding takes 0 for an horizontal step and 1 for the vertical one, and give the word $C\left(\frac{a}{b}\right)$. As an example, we take $(a, b) = (7, 2)$, and its Christoffel word $C\left(\frac{2}{7}\right)$ is the binary word 000010001, see Figure 1.

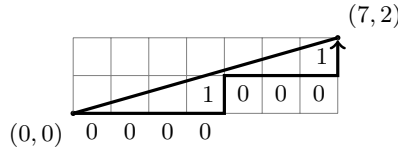


Fig. 1. The Christoffel path goes from $(0,0)$ to $(7,2)$ and $C\left(\frac{2}{7}\right) = 000010001$.

Any Christoffel word w can be factorized in a unique way into two Christoffel words $w = (u, v)$ using the standard factorization [1]. In the literature, we have two morphisms, G and \tilde{D} , defined from $A^* \times A^* \mapsto A^* \times A^*$ as follows $G(u, v) = (u, uv)$ and $\tilde{D}(u, v) = (uv, v)$, where (u, v) is the standard factorization of any Christoffel word w . With these two morphisms, we can generate all the Christoffel words and construct the Christoffel tree starting from the Christoffel word $w = 01 = (0, 1)$. Note that the Christoffel words appear once in the Christoffel tree.

2.2 Stern-Brocot tree and continued fractions

In this section, we introduce the Stern-Brocot tree, which is the tree containing all irreducible rational numbers. This tree distributes these fractions using the Farey sum defined as follows: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. In [6], we have all the details related to the construction of the tree. Figure 2 shows the first levels of SB tree. Every

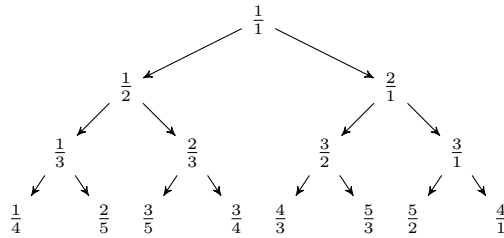


Fig. 2. The first levels of the Stern-Brocot tree.

irreducible rational number $\frac{a}{b}$ on SB tree has its continuous fraction defined as

follows: $\frac{a}{b} = [a_0, \dots, a_z]$, with $a_0 \geq 0$; $a_i \geq 1$ for $1 \leq i \leq z$ and if $z \geq 2$ then $a_z \geq 2$, and we write:

$$\frac{a}{b} = a_0 + \frac{1}{a_1 + \frac{1}{\dots + a_z}}$$

If $a_z = 1$, we define $[a_0, \dots, a_z] = [a_0, \dots, a_{z-1} + 1]$. The position and level of every rational number in the SB tree are determined by its continued fraction. We can have several references to better understand the continued fraction distribution and properties, example [7]. In particular, the fraction $\frac{a}{b} = [a_0, \dots, a_z]$ belongs to the level k , where $k = (\sum_{i=0}^z a_i) - 1$. We request the reader to refer to [10, 11] for the details of steps required to arrive at a/b from $1/1$. The key ingredient is the directive sequence defined below.

Definition 1 ([11]). *The directive sequence $\Delta(\frac{a}{b}) = R^{a_0} L^{a_1} R^{a_2} \dots p^{a_z-1}$, where $p \in \{R, L\}$ depending on the parity of z , and the letters L, R represents the movement to the left and right respectively on SB tree.*

We conclude this section with some remarks.

Remark 1. The two trees, Christoffel and Stern-Brocot, are isomorphic by identifying the morphism G to the letter L , and the morphism \tilde{D} to R . There are plenty of properties for all the fractions in the SB tree. For the full study, check [1, 8].

Remark 2. The two children of the fraction $\frac{2}{3}$ are exactly $\frac{3}{5}$ and $\frac{3}{4}$. We notice that the numerators of these two nodes form a palindrome. Using the Farey sum, and the construction of the SB tree, we note that the numerators of fractions descending from $\frac{2}{3}$ will form a palindrome at any level. We choose to highlight this property which will be used later in our work.

2.3 First and second order Balance matrices

A binary word w is said to be k -balanced if, and only if, for all factors u, v of w , we have: $|u| = |v| \implies ||u|_1 - |v|_1| \leq k$. If $k = 1$, w is said to be balanced. Christoffel words have a variety of properties, in particular, they are balanced, see [2]. In [11], the authors introduced the balanced matrix (B_w) , which allows us to obtain the balance order called $(\delta = \max(B_w))$ for any binary word w . The elements $B_w[i, j]$ are obtained by taking a window of length i and counting the number of occurrences of the letter 1 starting from position j over the circular word w^ω , and then normalizing the answer. See Figure 3.

In particular, when the binary word w is a Christoffel word, all the rows of B_w are binary words. For that, the authors of [10, 11] took a step further by defining the second order of balance, (δ^2) , induced from the first order. This value is obtained by first constructing the second-order balance matrix defined as follows:

Definition 2 ([11]). Let $w = C\left(\frac{a}{b}\right)$ be a Christoffel word and B_w its binary matrix. The elements of the second order balance matrix U_w are given by: $U_w[i, j] = \max(B_{(B_w[i])[j]})$, where $1 \leq i, j \leq a + b - 1$.

To obtain (δ^2) , we search for the maximum value of the matrix U_w :

Definition 3 ([11]). Let $w = C\left(\frac{a}{b}\right)$ be a Christoffel word and B_w its binary matrix. The second order of balance is the maximum value of U_w ; $\delta^2(w) = \max(U_w)$.

To simplify notations, we use $B_{\frac{a}{b}}, U_{\frac{a}{b}}$ to represent respectively $B_{C(\frac{a}{b})}$, and $U_{C(\frac{a}{b})}$. See Fig. 3 for an example.

$$B_{\frac{2}{7}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_{\frac{2}{7}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Fig. 3. The associated matrices B_w , and U_w for $w = C\left(\frac{2}{7}\right) = 000010001$.

In [11], we have a detailed recursive construction of U_w , based on some smaller Christoffel words and the continued fractions of their slopes. This construction leads to some results related to the value of δ^2 . For example, we know that in Theorem 2 of [5], for each level k in SB tree, we have $\delta_k^2 \geq \lceil \frac{k}{3} \rceil$, where δ_k^2 is the lower bound of the second order balance value of any element at level k in SB tree. This recursive construction allows us to determine the value of any rational number on SB tree as we can see in Theorem 1.

Theorem 1 ([5]). Let $\frac{a}{b} = [a_0, a_1, \dots, a_z]$. Then we have the following.

$$\delta^2\left(C\left(\frac{a}{b}\right)\right) = \begin{cases} \delta^2\left(C\left(\frac{u}{v}\right)\right) & \text{if } a_{z-1} \geq 2 \text{ and } a_z = 2, \\ \delta^2\left(C\left(\frac{\rho}{\theta}\right)\right) + 1 & \text{elsewhere.} \end{cases}$$

$$\text{where: } \frac{u}{v} = [a_0, a_1, \dots, a_{z-1} + 1]; \text{ and } \frac{\rho}{\theta} = \begin{cases} [a_0, a_1, \dots, a_z - 2] & \text{if } a_z \geq 4 \\ [a_0, a_1, \dots, a_{z-1} + 1] & \text{if } a_z = 3 \\ [a_0, a_1, \dots, a_{z-2}] & \text{if } a_z = 2. \end{cases}$$

With this theorem, we can construct another tree, called second order δ^2 tree. In this tree, we consider only the left half side, SB_L of the SB tree, where the slopes are less or equal than 1, and at every rational number position we associate its corresponding δ^2 value. See Figure 4.

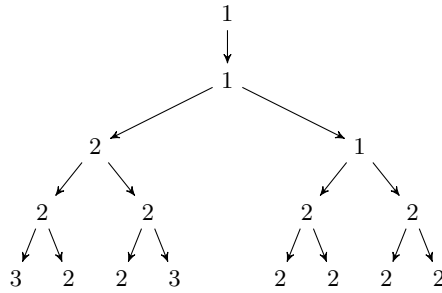


Fig. 4. The first levels of δ^2 values in SB_L tree.

2.4 Minimal paths in SB tree

After the construction of the second order δ^2 tree, the authors of [5] studied some particular branches, called paths. We define the path $P(\frac{a}{b})$ as the ordered set of fractions lying on SB tree from its root $\frac{1}{1}$ till $\frac{a}{b}$. For example, $P(\frac{8}{13}) = [\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}]$. Analogously, we define the δ_2 path $P_{\delta_2}(\frac{a}{b})$ as the ordered set of δ_2 values of fractions in $P(\frac{a}{b})$. For example, $P_{\delta_2}(\frac{8}{13}) = [1, 1, 1, 2, 2, 2]$, and $P_{\delta_2}(\frac{1}{5}) = [1, 1, 2, 2, 3]$. We notice that the first elements of $P_{\delta_2}(\frac{a}{b})$ starts always with two consecutive ones due to Theorem 1. In [5], the minimal path of SB_L is defined as follows.

Definition 4. [5] *The minimal path MP_k at level k in SB tree is defined as the sequence $[1, 1, 1, 2, 2, \dots, \lfloor \frac{k}{3} \rfloor]$.*

We say a fraction at level k is *minimal* if its δ_2 path is MP_k . For example, $\frac{8}{13}$ is a minimal fraction, while $\frac{1}{5}$ is not. The first minimal path was introduced in [5], and is called the zigzag path.

Definition 5. *The zigzag path is the path defined by the fractions in SB tree having $[0, 1, 1, \dots]$ as continued fraction.*

In fact, the zigzag path, in which we follow the fractions by descending alternatively first left and then right in SB tree, contains exactly the ratio of terms in Fibonacci sequence $\frac{F_n}{F_{n+1}}$.

Lemma 1. [5] *The zigzag path converges to the reciprocal of the golden ratio φ and minimizes the growth of δ^2 on SB tree.*

The result below helps us to categorize the fractions $\frac{a}{b}$ on SB tree, and belonging to a level k , multiple of 3, as minimal fractions or not. This characteristic is based on the values for the elements of the continued fraction of $\frac{a}{b}$.

Theorem 2. [5, Theorem 4] *Let $\frac{a}{b}$ be a minimal fraction at level $k = 3n$ with $n \geq 1$. Then the continued fraction representation of $\frac{a}{b}$ is of the form $[0, 1, P(\Omega), 2]$ where $P(\Omega)$ is a concatenation of elements of $\Omega = \{\omega_0 = (1, 2), \omega_1 = (1, 1, 1), \omega_2 = (2, 1), \omega_3 = (3)\}$ and conversely.*

For example, the continued fraction of the minimal fraction $\frac{8}{13}$ is $[0, 1, 1, 1, 1, 2]$ and satisfies the conditions. A small enumeration allows us to determine for any level k on SB tree the number of minimal fractions. In particular, when the level k is multiple of 3, we have by [5, Corollary 1], that the number of minimal fractions is $4^{\frac{k}{3}-1}$. Henceforth, we consider the subtree of SB tree with children descending from the parent $\frac{2}{3}$. We shall, by abusing notation, refer to it as SB tree.

3 Construction of $SB_{/3}$ tree from SB tree

In this section, we study the octree $SB_{/3}$ defined below and we introduce some new results.

1. At level one, we put the root $\frac{2}{3}$.
2. For level two, we choose 8 descendants of $\frac{2}{3}$ coming from level 6 of SB_L .
3. For level k , we choose 8^{k-1} descendants coming from level $3k$ of SB_L tree.

In other words, $SB_{/3}$ tree is nothing but SB_L tree where we consider only the levels that are non-zero multiples of 3. The number $|SB_{/3}[k]|$ of nodes at the level k in $SB_{/3}$ is 8^{k-1} which is precisely half of $|SB_L[3k]|$. See Figure 5 for partial $SB_{/3}$ up to level $k = 3$.

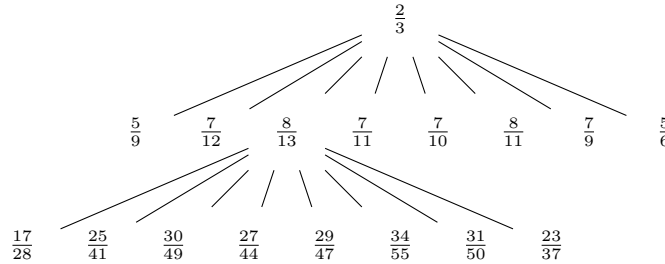


Fig. 5. Partial $SB_{/3}$ up to level 3

Let us introduce some notations. We fix a level $k \geq 1$ in $SB_{/3}$.

1. Let $SB_{/3}[k]$ be the ordered set of fractions in $SB_{/3}$ at level k and let $i^{(a/b)}$ denote the position (index) of the fraction a/b in $SB_{/3}[k]$.
2. Let f_k^ℓ and f_k^r denote the leftmost and the rightmost minimal fraction respectively appearing at level k with respective positions i_k^ℓ and i_k^r .
3. Let us denote the gaps $g_k^\ell = i_k^\ell - 1$ and $g_k^r = 8^{k-1} - i_k^r$. Let g_k denote the maximum gap between two consecutive minimal fractions at level k . Let $f_k^{c\ell}$ and f_k^{cr} denote the minimal fractions with respective positions $i_k^{c\ell}$ and i_k^{cr} with gap g_k with $f_k^{c\ell}$ being on the left side of f_k^{cr} . Clearly $g_k = i_k^{cr} - i_k^{c\ell} - 1$.

Example 1. 1. Let $k = 2$. We have $f_k^\ell = 7/12$, $f_k^r = 7/9$, $i_k^\ell = 2$, $i_k^r = 7$ and $g_k = 2$. Furthermore, $f_k^{c\ell} = 8/13$ and $f_k^{cr} = 8/11$ with $i_k^{c\ell} = 3$ and $i_k^{cr} = 6$.
 2. Let $k = 3$. We have $f_k^\ell = 26/45$, $f_k^r = 26/33$, $i_k^\ell = 10$, $i_k^r = 55$ and $g_k = 18$.
 Furthermore, $f_k^{c\ell} = 31/50$ and $f_k^{cr} = 31/43$ with $i_k^{c\ell} = 23$ and $i_k^{cr} = 42$.

Proposition 1. *Let f be a minimal fraction in $SB_{/3}[k]$ and let c_1, \dots, c_8 be its ordered children at level $k + 1$. Then c_2, c_3, c_6 and c_7 are minimal.*

Proof. The only fractions that satisfy the conditions of Theorem 2 are precisely c_2, c_3, c_6, c_7 . This suffices.

The fractions c_2, c_3, c_6 , and c_7 have specific continued fractions.

Proposition 2. *Let $a/b = [0, 1, P(\Omega), 2]$ be a minimal fraction at level k of $SB_{/3}$. The minimal fractions descendants from a/b at level $k + 1$ of $SB_{/3}$ are c_i ; $i \in \{2, 3, 6, 7\}$. These minimal fractions have one of the following continued fractions: $[0, 1, P(\Omega), \omega_0, 2]$, $[0, 1, P(\Omega), \omega_1, 2]$, $[0, 1, P(\Omega), \omega_2, 2]$, or $[0, 1, P(\Omega), \omega_3, 2]$.*

Proof. Let $a/b = [0, 1, P(\Omega), 2]$ be a minimal fraction at level k of $SB_{/3}$. The 8 possible paths to reach the level $k + 1$ of $SB_{/3}$ are: $LLL, LLR, LRL, LRR, RLL, RLR, RRL$ and RRR . They lead respectively to all the children c_i , $1 \leq i \leq 8$. By Proposition 1, only c_2, c_3, c_6 , and c_7 are minimal fractions, which arise from the paths LLR, LRL, RLR , and RRL respectively. With simple manipulation of the continued fraction of a/b using the directive sequence defined in Definition 1, we get the required result.

Figure 6 shows all the possible paths to pass from level 1 to level 6 in SB_L , and highlights the minimal fractions of $SB_{/3}$.

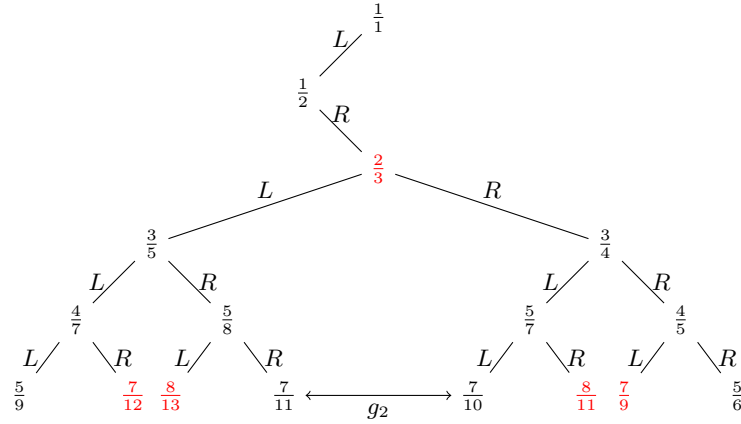


Fig. 6. Partial SB_L containing $2/3$ up to level 6 along with minimal fractions highlighted in red with the left and right sense of descent. The arrow g_2 represents the gap $g_2 = 2$.

3.1 Recursive nature of g_k^ℓ and g_k

The gaps g_k^ℓ and g_k progress recursively. We have $g_2^\ell = 1$ and $f_2^\ell = 7/12$. It is not difficult to see that f_3^ℓ must be the leftmost minimal child of f_2^ℓ . By Proposition 1, we know that it must be the second child of f_2^ℓ . As $g_2^\ell = 1$, 8 children coming from $5/9$ cannot be minimal. Furthermore, the first child of f_2^ℓ is non-minimal. All in all, we have $g_3^\ell = 8 + 1 = 9$. Similarly $g_4^\ell = 8 \times 9 + 1 = 73$. Generally, we have $g_k^\ell = 8g_{k-1}^\ell + 1$. Similarly we have $g_k^r = 8g_{k-1}^r + 1$. As $g_2^\ell = g_2^r = 1$, we have $g_k^\ell = g_k^r$ for all $k > 1$.

Let us now deal with g_k . As $g_2 = 2$, there are two non-minimal fractions between $f_2^{c\ell}$ and f_2^{cr} . Clearly their 16 children are also non-minimal. Furthermore, by Proposition 1, the last (resp. first) child of $f_2^{c\ell}$ (resp. f_2^{cr}) is non-minimal. We thus have $g_3 = 8 \times 2 + 2 = 18$. Generally, we have $g_k = 8g_{k-1} + 2$. Finally, by induction, we have $g_k = 2g_k^\ell$. We summarize it in the following result.

Theorem 3. *Let $g_1^\ell = g_1^r = g_1 = 0$. Then, for all $k \geq 1$, we have*

1. $g_k^\ell = 8g_{k-1}^\ell + 1$ and $g_k^r = 8g_{k-1}^r + 1$ with $g_k^\ell = g_k^r$
2. $g_k = 8g_{k-1} + 2$ and $g_k = 2g_k^\ell$

We shall see in Section 3.2 that the minimal fractions are distributed symmetrically on $SB_{/3}[k]$. Thus, the gap g_k is induced by g_k^ℓ and g_k^r appears exactly at the center of $SB_{/3}[k]$.

Using generating series for the recursive sequences g_k^ℓ , g_k^r and g_k , we have the following exact formulae for them.

Theorem 4. *Let $k \geq 1$. Then $g_k^\ell = g_k^r = \frac{8^{k-1}-1}{7}$.*

3.2 Distribution of minimal fractions at a given level

We now consider the distribution of minimal fractions at level k . We shall show that the distribution of minimal fractions at a given level is symmetric. From Theorem 2, we know that the continued fraction representation of minimal fractions is of form $[0, 1, P(\Omega), 2]$ where $P(\Omega)$ is a concatenation of elements of $\Omega = \{\omega_0 = (1, 2), \omega_1 = (1, 1, 1), \omega_2 = (2, 1), \omega_3 = (3)\}$.

Lemma 2. *Let $W_i = [0, 1, \omega_i, b_4, \dots, b_m]$ for $0 \leq i \leq 3$ be continued fractions. Let N_i and D_i denote respectively the numerator and the denominator of W_i for $0 \leq i \leq 3$. Then,*

1. $N_0 = N_3$ and $D_0 + D_3 = 3N_0$. In particular $\frac{N_0}{D_0} \oplus \frac{N_3}{D_3} = \frac{2}{3}$.
2. $N_1 = N_2$ and $D_1 + D_2 = 3N_1$. In particular $\frac{N_1}{D_1} \oplus \frac{N_2}{D_2} = \frac{2}{3}$.

Lemma 2 can be easily proved using induction and Remark 2. Using it, we have the following result about the symmetric distribution of minimal fractions.

Theorem 5. *If a/b is a minimal fraction at level k then there exists a minimal fraction a'/b' at level k with $a/b \oplus a'/b' = 2/3$. Furthermore, a/b and a'/b' are equidistant from the extremities of $SB_{/3}[k]$ i.e. $i(a/b) + i(a'/b') = |SB_{/3}[k]| + 1$.*

Proof. We use the same notations as in Lemma 2. By Theorem 2, the continued fraction representation of a/b is of form $W_i = [0, 1, P_i(\Omega), 2]$ where $P_i(\Omega)$ is a concatenation of elements of Ω . Then by Lemma 2, there exists W_j such that $W_i \oplus W_j = 2/3$. Furthermore, at level 2 of $SB_{/3}$, W_0, W_3 are equidistant from the extremities of $SB_{/3}[2]$. Idem for W_1 and W_2 . Thus, by induction, a/b and a/b' are equidistant.

3.3 Continuous fractions of boundary paths

In Section 2.4, we described the zigzag path, which is a minimal path. In this section, we consider 4 special minimal paths that define the boundary of the set of minimal fractions in $SB_{/3}$.

Definition 6 (Boundary paths). *The minimal path \mathcal{P}^ℓ (resp. $\mathcal{P}^{cl}, \mathcal{P}^{cr}, \mathcal{P}^r$) is defined as the sequence of fractions $(f_k^\ell)_{k \geq 1}$ (resp. $(f_k^{cl})_{k \geq 1}, (f_k^{cr})_{k \geq 1}, (f_k^r)_{k \geq 1}$)*

In Proposition 2, we described the continued fraction representations of minimal children of a minimal fraction a/b . Furthermore, if a/b is also on a boundary path then we the following.

Proposition 3. *Let $P(\Omega)$ be a concatenation of elements of Ω and let $a/b = [0, 1, P(\Omega), 2] \in SB_{/3}$ be a minimal fraction appearing at level k on some boundary path \mathcal{P} . Then the fraction with continued fraction $[0, 1, P(\Omega), \omega_0, 2]$ is minimal and appears on \mathcal{P} at level $k + 1$.*

Proof. By Proposition 2, $[0, 1, P(\Omega), \omega_0, 2]$ is minimal. Due to the recursive construction of SB tree, it suffices to prove the result for level $k = 2$ in $SB_{/3}$ tree i.e. at the sixth level of SB tree. We apply Definition 1 and the proof of Theorem 1 in [5], we see that the continued fractions of fractions on boundary paths at level $k + 1 = 3$ end with $(\omega_0, 2)$.

In the next theorem, we use Proposition 3 to describe the continued fraction representations of four boundary paths and compute their irrational limits.

Theorem 6. *For all $k \geq 2$, the continued fraction representations of $f_k^\ell, f_k^{cl}, f_k^{cr}$ and f_k^r are, respectively, $[0, 1, \omega_0, \omega_0^{k-2}, 2]$, $[0, 1, \omega_1, \omega_0^{k-2}, 2]$, $[0, 1, \omega_2, \omega_0^{k-2}, 2]$ and $[0, 1, \omega_3, \omega_0^{k-2}, 2]$. Furthermore, $\mathcal{P}^\ell, \mathcal{P}^{cl}, \mathcal{P}^{cr}$ and \mathcal{P}^r converge respectively to $\frac{1}{\sqrt{3}}, \frac{16-\sqrt{3}}{23}, \frac{17+\sqrt{3}}{26}$ and $\frac{3+\sqrt{3}}{6}$.*

Proof. We shall only treat the case of \mathcal{P}^ℓ . The remaining cases are similar. We apply Proposition 3 to \mathcal{P}^ℓ to obtain $f_k^\ell = [0, 1, \omega_0, \omega_0^{k-2}, 2]$. Let us show that \mathcal{P}^ℓ converges to $\frac{1}{\sqrt{3}}$. Suppose f_k^ℓ converges to α . Clearly $\alpha = \frac{1}{1+\frac{1}{\beta}}$ with $\beta = [1, 2, 1, 2, \dots] = [1, 2, \beta]$. Straightforward computations yield the positive value of $\beta = \frac{1}{2}(1 + \sqrt{3})$. Then the limit of $\mathcal{P}^\ell = [0, 1, \omega_0, \omega_0^{k-2}, 2]$ is $\frac{1}{1+\frac{1}{\beta}} = \frac{1}{\sqrt{3}}$.

4 Link with Christoffel tree

We described the isomorphism between the SB tree and the Christoffel tree. Hence, we can rephrase the previous results in the context of Christoffel words. We consider a particular subtree T' of the Christoffel tree which is made of 4 levels of the SB tree. It starts with the standard factorization of $w = (x, y)$, where w is the Christoffel word of slope a/b which is a minimal fraction in $SB_{/3}$. Proposition 1 shows that the following Christoffel words in T' have a rational slope which is minimal, and we denote them: $w_2 = (x^3y, x^2y)$, $w_3 = (x^2y, x^2yxy)$, $w_6 = (xyxy^2, xy^2)$, and $w_7 = (xy^2, xy^3)$, see Figure 7.

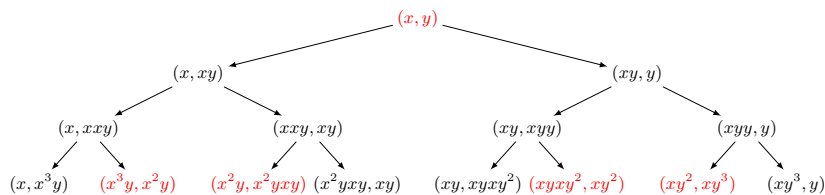


Fig. 7. The subtree, T' , made of 4 levels of the Christoffel tree using the two morphisms, G and \tilde{D} mentioned earlier. The words in red are Christoffel words with a minimal rational slope.

In fact, the two morphisms allow us to generate more Christoffel words by concatenating on the left or the right side of the initial word $w = (x, y)$ some copies of x or of y , depending on the side we choose in the tree. We define $\text{Prop}(w') := (|w'|_x, |w'|_y)$ as the spread of copies of x and y in any word w' descendant of the word $w = (x, y)$. In other words, we count how many times the factor x or y appears in w' . In particular, $\text{Prop}(w) = (1, 1)$. The spread is symmetric at each level, in the sense that the number of occurrences of the factor x on the left side is equal to the number of occurrences of the factor y on the right side, and vice versa. For that, it is enough to consider only the left-hand side of T' . The spread at level 2 of T' is the couple $(2, 1)$. At level 3, we have the couples $(3, 1)$, and $(3, 2)$. All these words are associated with minimal rational slopes. At level 4, we need to decide which fractions are minimal and not. By checking the spread of all the words at level 4 of T' , we obtain the result displayed in Table 1.

words	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
$\text{Prop}(w_i)$	$(4, 1)$	$(5, 2)$	$(5, 3)$	$(4, 3)$	$(3, 4)$	$(3, 5)$	$(2, 5)$	$(1, 4)$

Table 1. The spread of the copies of x and y for the words at level 4 of T' .

We note that the words w_2, w_3, w_6 , and w_7 have the highest spread on the copies of x or y (depending on whether we are on the left or right side of the tree). This aligns with the definition of δ^2 , being the study of the spread of the letter 1 in a Christoffel word. Recall that a minimal fraction belongs to a path that minimizes the growth of δ^2 . In other words, for a Christoffel word, the smaller the value of δ^2 , the better balanced the word is, with the letters 1 being more evenly spread and distributed within the Christoffel word.

5 Conclusion and future works

This study has provided a thorough examination of the quotient tree $SB/3$, revealing significant insights into its structure, minimal fraction distribution, and balance. By focusing on the levels indexed by multiples of three of the SB tree, and analyzing distinct branches converging to known irrational limits, we have illuminated the intricate relationships between these mathematical entities. The findings contribute to the broader field of combinatorics on words, offering a foundation for future research and exploration. There are various ways in which one can proceed to further analyze the minimal fractions. Let us list a few.

1. We study the class of Christoffel words with minimal rational slopes and characterize some properties related to this class.
2. For a fixed natural number n , we study its co-prime partitions (a, b) with $a < b$, such that a/b is a minimal fraction. If such a partition exists, we say n is *minimal*. As not every natural number is minimal, it is worth investigating minimal natural numbers.
3. We observed that the numerators and the denominators of minimal fractions on boundary paths are defined recursively. This deserves some investigation along with three Fibonacci type paths discussed previously.

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