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Interplay between discretization and controllability of linear delay systems: an algebraic viewpoint

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Abstract

In this paper, we give an in depth study of linear delay systems controllability preservation/alteration through discretization. We make use of a module theoretic framework acting as a unifying one for most of the existing delay system controllability notions. We propose a formal generic definition of a discretization scheme and illustrate through examples that controllability properties may be lost through discretization. Then, we introduce the notion of preservation (that is, a measure of quantifying the ability of the discretizer to preserve controllability properties) and prove that for a given discretizer, we can always find a delay system for which even the torsion-free controllability (which is the weakest controllability notion) is not preserved. Finally, we reverse the situation, and show that for any given delay system, preserving discretizers exist.

Keywords: Delay systems, controllability, discretization, module theoretic setting.

2000 MSC: 34K35, 93B05, 93B25.

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1. Introduction

Delay systems are at the same time an important class arising in practice (due to, e.g., sensor and/or actuator delays, communication or computation delays) and one of the simplest class of infinite-dimensional systems (e.g., a boundary controlled wave equation without damping is equivalent to a delay system). Since delay systems naturally mix a differential and a difference nature, many bridging notions between continuous delay systems and sampled data systems have been proposed in the literature [26, 27, 40]. In this context, a natural question is: what is the influence of discretization on the class of delay systems? A crucial property that one aims to preserve under the discretization operation is controllability. The goal of this paper is to provide an extensive study of the controllability of linear delay systems preservation or alteration through discretization using an algebraic framework.

Although linear delay systems may be considered as the simplest class of infinite-dimensional systems, the presence of time-delays leads to more complex analysis than in the finite-dimensional case (i.e., without delay). This is particularly true for the controllability problem, central to automatic control. Many different notions of controllability (for instance, torsion-free controllability, projective controllability, free controllability, spectral controllability, etc.) have been developed in

the delay systems literature, either from a functional analytic approach (see, e.g., [1, 2, 4, 42, 67]) or from a more algebraic viewpoint (see, e.g., [6, 17, 46, 56], where an analogous formalism to that considered in this paper is used; [8, 11, 39, 57], where systems over rings are considered; [7, 35, 61, 66], where distributional rings are employed; and also [28, 30, 50], where dual trajectory modules over an operator ring are used). In the linear finite-dimensional case, all these various controllability properties are equivalent and reduce to Kalman's controllability, but in the case of delay systems, when introduced, they seem, for most of them, apparently unrelated. Therefore an interesting problem, studied and solved in [25], is the possible existence of a unifying framework under which all these notions could be interpreted. In this context, the first contribution of the paper is to *recall and systematize, within an algebraic framework, the interconnections among the different existing controllability notions in a more pedagogical manner* than the current literature, aiming to make them more accessible for readers less familiar with algebraic concepts.

Then, another natural question is *whether the different controllability notions for delay systems are preserved through discretization*. The answer to this question is not always obvious and all situations seem possible, as illustrated by the following two examples (which are generalized and discussed in detail in Section 3.3.3). Consider the following one-dimensional delay system represented by:

$$\dot{x}(t) = \frac{1}{L}x(t) + \left(1 - h\frac{1}{L}\right)u(t) - u(t - L),$$

with $L > 0$ and $h > 0$ such that $h \neq L(1 - e^{-1})$. Such a system is spectrally controllable (a delay system is said to be spectrally controllable if all its spectral modes are controllable, see Subsection 2.2.8 for a precise definition). On the other hand, its associated discretized system, obtained by a classical backward difference discretization scheme with discretization period h , is never controllable. Consider now the following dynamical system represented by:

$$\dot{x}(t) = s_0x(t) + p\left(e^{-L\frac{d}{dt}}\right)(u)(t),$$

where s_0 and $L > 0$ are real numbers and the polynomial p is such that $p(s_0) = 0$. It can be shown that this system is not spectrally controllable, while its discretization, obtained via a discretizer defined with the help of the polynomial p and another polynomial q (such that their greatest common divisor is 1), is controllable.

The present paper deals with these questions and gives, using an algebraic framework, an in depth study of controllability of linear delay systems preservation or alteration through discretization. The benefit of the algebraic framework is that it allows us to gain a much deeper understanding of controllability and its relations with discretization. More precisely, one may be tempted to discretize a delay system (or a PDE, in space) and obtain a finite-dimensional system. If the discretization period is sufficiently small, then the discretized system will be a good enough approximation of the infinite-dimensional system, and one may be tempted to believe that studies on delay and PDE systems are useless. The results obtained in this paper show that this is not at all the case, and that controllability properties may be lost through discretization, whatever the discretization period may be.

We give a complete analysis of the interplay between different notions of controllability for linear delay systems (which are defined as finitely generated $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -modules, where \mathbb{R} is the field of real numbers and δ_L is the delay operator of delay L) and discretization, in the commensurate delay case. First, we provide a generic formal definition of a discretizer $\mathbb{D}_{p,q,h}$, based on a rational identity approximation, with discretization period h , and associated to two $\mathbb{R}[\delta_h]$ -polynomials p and q , where $\mathbb{R}[\delta_h]$ is the ring of shift operators δ_h of length h . At first sight, from a controllability

viewpoint, to work with the distributed extension (which is defined over the ring of entire functions, contains distributed delay, and is used for spectral controllability) of the delay system in a continuous setting or to work with its discretized version (obtained by applying $\mathbb{D}_{p,q,h}$ in a discrete setting seems equivalent (since for both the distributed extension and discretized system associated to the original delay system, the different notions of controllability are equivalent)). We show in the paper that, in general, this is not true. To that end, we first analyze how the main obstructions to torsion-free controllability and free controllability (that is, common factors and common zeros of the Fitting ideal's generators of the delay system) are modified by the applied discretization scheme. We show that common factors (which correspond to the presence of torsion elements) are always transmitted through the proposed discretization schemes, thus if the original delay system is not torsion-free controllable, then there is no discretizer $\mathbb{D}_{p,q,h}$ leading to a controllable discrete counterpart of the system. On the other hand, we prove that for the common zeros which do not yield common factors (describing the case when the system is torsion-free controllable, but not free controllable) to be preserved for the discretized version of the system, we need an additional condition, meaning that a torsion-free controllable delay system, which is not free controllable, can lead via discretization to a (free) controllable discrete system. Given these observations, we then introduce the notions of ideal discretizer and of preservation (a measure of quantifying the ability of the discretizer to preserve controllability properties) and show first that the only universally preserving discretizer (i.e., that preserves specific controllability properties for any delay system) is the ideal discretizer; and second, that for a fixed discretizer $\mathbb{D}_{p,q,h}$, we can always find a delay system for which even the torsion-free controllability (which is the weakest controllability notion) is not preserved. Then we reverse the situation, and prove in Theorems 3.1 and 3.2, that for any given delay system, preserving discretizers always exist. To sum up the interest of the paper is threefold:

- first, to review, using the algebraic framework, the relations between the various scattered, and which originally seemed unrelated, controllability notions in the literature;
- second, to give generic formal definitions of discretization schemes;
- third, to establish the existence of discretization schemes which preserve important controllability notions.

The paper is organized as follows. In Section 2, general algebraic definitions, relations between the different types of controllability and various controllability criteria are recalled. In Section 3 we give our main results: we first introduce a generic formal definition of discretization schemes for delay systems in Subsection 3.2, and then, discretizers and the interplay between controllability and discretization are studied in Subsection 3.3. The introduced notions are illustrated via simple examples throughout the paper. A recall on some algebraic notions is given in Appendix A. The reader familiar with of freeness, torsion-freeness, projectivity, Fitting ideal and those who want to quickly delve into the main findings of the article can simply take a look at Section 2 (Definitions 2.1, to 2.6 and Proposition 2.2), before reading Section 3 containing the new results.

Notations list

General notations: \mathbb{R} , the field of real numbers; \mathbb{Q} , the field of rational numbers; \mathbb{C} , the field of complex numbers; for a matrix P with entries in a ring R , $\text{rk}_R P$, the rank of P over R .

R-systems: R , ring; Λ , R -module; \mathcal{O} , ring of entire functions; P_Λ , presentation matrix of Λ ; $\mathfrak{F}(\Lambda)$, Fitting ideal of Λ .

Useful rings for finite-dimensional systems: $\mathbb{R}[\frac{d}{dt}]$, ring of differential operators; $\mathbb{R}[\delta_h]$, ring of shift operators δ_h of length h .

Delay systems: L , commensurate delay; δ_L , delay operator of delay L ; Λ , linear delay system; Λ_e , exponential version of Λ ; Λ_d , distributed extension of Λ ;

Useful rings for delay systems and discretization: $\mathbb{R}[\delta_L]$ the ring of polynomials in the delay operator δ_L ; $\mathbb{R}[\frac{d}{dt}, \delta_L]$, the polynomial ring (on which delay systems are defined); $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$, the ring of exponential polynomials; $\mathbb{R}[\frac{d}{dt}][e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$, the distributed delay ring; $\mathbb{R}[\delta_h, \ln(\delta_h)]$, a ring isomorphic to $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$; $\mathbb{R}[s, z]$, the ring of polynomials in $s \in \mathbb{C}$ and $z \in \mathbb{C}$ associated to Laplace/Z transforms; $\mathbb{R}[[\frac{d}{dt}]]$, the power series ring in $\frac{d}{dt}$; $\mathbb{R}[[\delta_h]]$, the power series ring in δ_h ; $\mathbb{R}(\delta_L)$, the field of fractions in the delay operator δ_L ; $\mathbb{R}(\delta_L)[\frac{d}{dt}]$, the ring of polynomials in $\frac{d}{dt}$ with coefficients in the ring of fractions $\mathbb{R}(\delta_L)$; $\mathbb{R}[[\frac{d}{dt}]][\delta_L]$, the ring of polynomials in the delay operator δ_L and series in $\frac{d}{dt}$; $\mathbb{R}[\delta_h, \pi^{-1}]$, the ring of polynomials in δ_h and π^{-1} , where $\pi \in \mathbb{R}[\delta_h]$.

System discretizers: $\mathbb{D}_{p,q,h}$, discretizer with discretization period h , associated to the $\mathbb{R}[\delta_h]$ -polynomials p, q ; $\mathbb{D}_{\infty,h}$, ideal discretizer with discretization period h ; $(\mathbb{D}_{p_k, q_k, h})_{k \in \mathbb{N}}$, discretizer sequence with discretization period h , associated to the sequences of $\mathbb{R}[\delta_h]$ -polynomials $(p_k)_{k \in \mathbb{N}}, (q_k)_{k \in \mathbb{N}}$;

Operator discretizers: $\mathfrak{d}_{p,q,h}$, discretizer of an operator, and $\mathfrak{d}_{p,q,h}^c$ compensated discretizer of an operator, with discretization period h , associated to $\mathbb{R}[\delta_h]$ -polynomials p and q ; $\mathfrak{d}_{\infty,h}$ ideal operator discretizer with discretization period h .

2. Review on controllability notions and their relations in a unifying algebraic framework

Our motivation of using the present algebraic framework comes from:

- The *intrinsic character* of the various notions and definitions (where, for instance, properties like controllability and observability of control systems, which are seen as modules, correspond to algebraic properties of the module).
- The *important flexibility* in defining the structural notions, through base change (i.e., change of the ring of coefficients) and various module theoretic properties (e.g., torsion-freeness, projectivity, freeness).
- The ability to *unify most of* the existing *structural notions* (in particular, controllability and observability) of the delay system's literature using tools as those introduced in [25].
- The *constructive character* of the structural notions, as demonstrated by the numerous computer algebra implementations (see, e.g., [12, 13, 51]).
- The *independence* of the obtained structural results *from* often tricky *existence and uniqueness* prerequisites, as is usual for infinite dimensional systems thanks to functional analytic techniques.
- The ability for this *same framework* to be used both *for finite and infinite-dimensional linear systems*, the main difference being the operator ring.

Most of the commutative algebra concepts, that we will use, may be found in standard textbooks, such as [16, 38, 54]. In particular, the basic definitions of module, ideal, free module, etc., can be found for example in pages 11 to 17 of [16]. We recall next the main notions used in the paper and illustrate them via several examples. All the rings and algebras we consider in the sequel are commutative, with unity, and without zero divisors.

2.1. Delay systems and dynamics

2.1.1. Module, torsion and freeness: definitions

A *module* shares the same axioms as a vector space, the only difference being that its scalars are taken in a ring instead of being taken in a field. A *free module* is one that has a basis (contrary to vector spaces, a module does not always have a basis), i.e., a linearly independent and generating family. The reader is also referred to Appendix A, p. 47, where a formal definition is given.

A *torsion element* z of a module Λ over a ring R (commutative with no zero divisor) is such that there exists $r \in R$, $r \neq 0$ such that $rz = 0$. In other words, the family $\{z\}$ is linearly dependent over R . A module is said to be *torsion* if all its elements are torsion. In Sections 2.1.2, 2.2.1, and 2.2.2, definitions of these notions over the ring of difference-differential operators are recalled.

2.1.2. Delay system

A linear delay system corresponds to a finite set of variables satisfying a finite number of difference-differential linear equations. Its typical form considered in the time-delay systems literature is as follows:

$$E(\delta_L)\dot{\mathbf{x}} = A(\delta_L)\mathbf{x} + B(\delta_L)\mathbf{u} \quad (1)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, $\dot{\mathbf{x}}$ denotes the differentiation with respect to time (i.e., $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$), and the matrices A , B , and E are such that $A \in \mathbb{R}[\delta_L]^{n \times n}$, $B \in \mathbb{R}[\delta_L]^{n \times m}$, $E \in \mathbb{R}[\delta_L]^{n \times n}$, with E of full rank, and where $\mathbb{R}[\delta_L]$ is the ring of polynomials in the delay operator δ_L of delay L . Hereafter, we explain how we can interpret this form in the algebraic framework. For the sake of simplicity, we shall *restrict* ourselves to *commensurate delays*.

A linear delay system can be modelled as a linear algebraic structure over the ring of difference-differential operators $\mathbb{R}[\frac{d}{dt}, \delta_L]$, that of a module. The ring $\mathbb{R}[\frac{d}{dt}, \delta_L]$ is the polynomial ring in two indeterminates d/dt (the differentiation with respect to the time operator) and δ_L (the delay operator of delay L), with coefficients in \mathbb{R} , i.e., an element of $\mathbb{R}[\frac{d}{dt}, \delta_L]$ consists of finite sums of the form:

$$r(\frac{d}{dt}, \delta_L) = \sum_{i,j} c_{ij} \frac{d^i}{dt^i} \delta_L^j, \quad \text{where } c_{ij} \in \mathbb{R}.$$

An $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ is said to be *generated* by the family $\{w_i\}_{i \in I}$, $I \subset \mathbb{N}$, ($w_i \in \Lambda$), if any element of Λ is expressed as a finite linear combination of the w_i 's with coefficients in $\mathbb{R}[\frac{d}{dt}, \delta_L]$. In other words:

$$\forall w \in \Lambda, \exists J \subset I, J \text{ finite}, \quad \exists (a_i)_{i \in J} \in \mathbb{R}[\frac{d}{dt}, \delta_L]^{\#J}, \quad w = \sum_{i \in J} a_i(\frac{d}{dt}, \delta_L) w_i,$$

where $\#J$ denotes the cardinal of J . In the case where I is finite, for example equal to $\{1, \dots, \alpha\}$, the module Λ is said to be *finitely generated*. It is denoted by $\Lambda = [w_1, \dots, w_\alpha]$ and w_1, \dots, w_α are called *generators* of Λ . In addition, a family $\{z_1, \dots, z_\mu\}$ of elements in Λ is said to be $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -*linearly independent* if there exists no non-trivial linear relation with coefficients in $\mathbb{R}[\frac{d}{dt}, \delta_L]$ between its elements. In other words:

$$\forall c_1, \dots, c_\mu \in \mathbb{R}[\frac{d}{dt}, \delta_L], \quad \left(\sum_{i=1}^{\mu} c_i(\frac{d}{dt}, \delta_L) z_i = 0 \right) \Rightarrow (\forall i = 1, \dots, \mu, c_i = 0).$$

The family is said to be $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -*linearly dependant* otherwise. The cardinal of an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -linearly

independent family of Λ , maximal with respect to inclusion, is called the *rank* of Λ and denoted $\text{rk}_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$.

The definition of a linear delay system is directly related to the system's equations as they are given in an analysis framework. A *finitely presented* module Λ over $\mathbb{R}[\frac{d}{dt}, \delta_L]$ is entirely characterized by the datum of finitely generated generators and relations verified by the latter. The module Λ is then said to be given by generators and relations. More precisely, this module is defined through:

- a (free and finitely generated) generators module: $\mathcal{W} = [W_1, \dots, W_\alpha]_{\mathbb{R}[\frac{d}{dt}, \delta_L]} = [\mathbf{W}]_{\mathbb{R}[\frac{d}{dt}, \delta_L]}$, corresponding to the various variables of the system (see Example 2.1 below);
- a (free and finitely generated) relations module: $\mathcal{E} = [P_\Lambda(\frac{d}{dt}, \delta_L) \mathbf{W}]_{\mathbb{R}[\frac{d}{dt}, \delta_L]}$, where the relations $P_\Lambda(\frac{d}{dt}, \delta_L) \mathbf{W} = 0$ are the difference-differential equations satisfied by the W_i 's, and where P_Λ is a $(\beta \times \alpha)$ -matrix with elements in $\mathbb{R}[\frac{d}{dt}, \delta_L]$ and $\text{rk}_{\mathbb{R}[\frac{d}{dt}, \delta_L]} P_\Lambda = \beta$ (see Example 2.1).

Here P_Λ is called a presentation matrix. Since both \mathcal{W} and \mathcal{E} are freely generated, then the module Λ is isomorphic to \mathcal{W}/\mathcal{E} which is an alternative characterization of a finitely presented module. We can now give the concise definition of a system:

Definition 2.1. A linear delay system Λ is a finitely presented $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module.

The above definition states that an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ , or, simply phrased, a delay system, is an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module with finite presentation (see, e.g., [16, p.17], [54, p. 90]):

$$\mathcal{E} \longrightarrow \mathcal{W} \longrightarrow \Lambda \longrightarrow 0, \quad (2)$$

where \mathcal{E} and \mathcal{W} are finitely generated free modules of generators and relations. The system variables (i.e., states, inputs, and outputs) are generators of \mathcal{W}/\mathcal{E} and correspond to the equivalence classes of this quotient. In the paper, we consider the case of linear delay systems on $\mathbb{R}[\frac{d}{dt}, \delta_L]$ only, so often the terms “linear” and “delay” or the specific ring $\mathbb{R}[\frac{d}{dt}, \delta_L]$ will be omitted. For an element w in Λ (i.e., a system variable), we shall denote indistinctly \dot{w} or $\frac{d}{dt}w$ for the time-derivative of w , and $w^{(i)}$ or $\frac{d^i}{dt^i}w$, $i > 1$, for the higher order derivatives.

Remark 2.1. Note that for the definition of a linear delay system, instead of $\mathbb{R}[\frac{d}{dt}, \delta_L]$, we could equally well have chosen $k[\frac{d}{dt}, \delta_L]$ with another base field k , such as \mathbb{Q} , the field of rational numbers. For example, the choice $k = \mathbb{Q}$, would be appropriate in the case when implementable solutions are sought (since this is not the goal of the paper, we continue to work with \mathbb{R} , the field of real numbers, but all the subsequent definitions and theorems also hold with rings such as $k = \mathbb{Q}$).

Example 2.1. Consider the delay system given by (1) which can also be written as:

$$E(\delta_L) \frac{d}{dt} \mathbf{x} = A(\delta_L) \mathbf{x} + B(\delta_L) \mathbf{u}.$$

The freely generated modules \mathcal{W} and \mathcal{E} associated to (1) are given by:

$$\begin{aligned} \mathcal{W} &= [\mathbf{X}, \mathbf{U}]_{\mathbb{R}[\frac{d}{dt}, \delta_L]} = \{p_1(\frac{d}{dt}, \delta_L) \mathbf{X} + p_2(\frac{d}{dt}, \delta_L) \mathbf{U} \mid p_1, p_2 \in \mathbb{R}[\frac{d}{dt}, \delta_L]\} \\ \mathcal{E} &= [E(\delta_L) \frac{d}{dt} \mathbf{X} - A(\delta_L) \mathbf{X} - B(\delta_L) \mathbf{U}]_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \end{aligned}$$

$$= \left\{ Q\left(\frac{d}{dt}, \delta_L\right) \left(E(\delta_L) \frac{d}{dt} \mathbf{X} - A(\delta_L) \mathbf{X} - B(\delta_L) \mathbf{U} \right) \mid Q \in \mathbb{R}\left[\frac{d}{dt}, \delta_L\right]^{n \times n} \right\},$$

with the family $\{\mathbf{X}, \mathbf{U}\}$ generating the module \mathcal{W} , and the $n \times (n + m)$ presentation matrix $P_\Lambda\left(\frac{d}{dt}, \delta_L\right) = \begin{pmatrix} E(\delta_L) \frac{d}{dt} - A(\delta_L) & B(\delta_L) \end{pmatrix}$ defining the relations module \mathcal{E} . Observe that \mathbf{X} and \mathbf{U} can be seen as internal variables giving rise to the final system variables \mathbf{x} and \mathbf{u} , defined as equivalence classes. More precisely, the above system can be seen as an $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$ -module Λ isomorphic to the quotient module \mathcal{W}/\mathcal{E} . We call \mathbf{x} and \mathbf{u} the generators of this quotient, i.e., we have $\mathcal{W}/\mathcal{E} = [\mathbf{x}, \mathbf{u}]$, and the equivalence classes \mathbf{x} and \mathbf{u} satisfy the equation

$$E(\delta_L) \dot{\mathbf{x}} - A(\delta_L) \mathbf{x} - B(\delta_L) \mathbf{u} = 0,$$

since \mathcal{E} is identically zero in the quotient.

2.1.3. Delay dynamics with input

An input to a linear system with delay is such that any variable in this system satisfies an equation in which the input is involved (possibly trivially). The notions of input and dynamics are then defined as follows.

Definition 2.2. An $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$ -linear dynamics, or an $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$ -dynamics, or a (linear) dynamics over $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$, is an $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$ -system Λ equipped with an input $\mathbf{u} = (u_1, \dots, u_m)$, i.e., a finite subset of Λ , such that the quotient module $\Lambda/[\mathbf{u}]$ is torsion. The input \mathbf{u} is said to be independent if $[\mathbf{u}]$ is a free $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$ -module of rank m .

Thus any variable w in the dynamics is such that there are p, q_1, \dots, q_m , polynomials in $\frac{d}{dt}$ and δ_L , with coefficients in \mathbb{R} , with $p \neq 0$, such that

$$p\left(\frac{d}{dt}, \delta_L\right) w = \sum_{i=1}^m q_i\left(\frac{d}{dt}, \delta_L\right) u_i$$

where $u_i, 1 \leq i \leq m$, denote the input variables. Then, *modulo* the input, in the quotient $\Lambda/[\mathbf{u}]$ (i.e., formally setting $\bar{u}_i = \bar{0}$, where by $\bar{\cdot}$ we denote the equivalence classes in $\Lambda/[\mathbf{u}]$), we have

$$p\left(\frac{d}{dt}, \delta_L\right) \bar{w} = \bar{0},$$

and \bar{w} is a torsion element.

Remark 2.2. The notion of state, in the present framework, can be defined as a finite, independent family, maximal in the sense of inclusion, of $[\Lambda/[\mathbf{u}]]_{\mathbb{R}[\delta_L]}$. Indeed, $\Lambda/[\mathbf{u}]$ being torsion, $[\Lambda/[\mathbf{u}]]_{\mathbb{R}[\delta_L]}$ is of finite rank and, taking $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$ a finite sequence whose underlying set is a maximally independent family of the latter, we get :

$$\forall i \in \{1, \dots, n\} \text{ the family } \{\dot{\bar{x}}_i, \bar{\mathbf{x}}\} \text{ is linearly } \mathbb{R}[\delta_L]\text{-dependant.}$$

In the module Λ , this becomes:

$$E(\delta_L) \dot{\mathbf{x}} = A(\delta_L) \mathbf{x} + \sum_{j=0}^{\mu} B_j(\delta_L) \mathbf{u}^{(j)}, \quad (3)$$

where $A(\delta_L) \in \mathbb{R}[\delta_L]^{n \times n}$, $B_i(\delta_L) \in \mathbb{R}[\delta_L]^{n \times m}$, for $0 \leq j \leq \mu$, $E(\delta_L) \in \mathbb{R}[\delta_L]^{n \times n}$ is diagonal and of full rank over $\mathbb{R}[\delta_L]$. Observe that the form of equation (3) is more general than that of the typical linear delay system (1) which is usually considered in the literature; it is however this general form that appears naturally from an algebraic viewpoint. Equation (3) actually corresponds to a particular type of neutral or advanced form, since $E(\delta_L)$ is diagonal and of full rank.

From the above definition and remarks, it follows that the rank of the module Λ corresponds to the number of linearly independent input variables and, in general, the rank of the presentation matrix P_Λ is larger than the rank of the module.

We would like to highlight the fact that when defining the different notions of controllability using the algebraic framework, one does not need to distinguish between state and input variables (since all controllability concepts are defined as properties of the module). Therefore in the following subsections, we state all results for delay systems but illustrate them by delay (controlled) dynamics.

2.1.4. Finite-dimensional systems

Two other common instances of systems are given when the operators ring is the ring of differential operators $\mathbb{R}[\frac{d}{dt}]$, which yields continuous-time linear finite-dimensional systems [19], and when the ring is $\mathbb{R}[\delta_h]$, with δ_h the shift operator of length h , which yields linear discrete-time finite-dimensional systems [21]. Hence, the following system:

$$\dot{\mathbf{x}} = \frac{d}{dt}\mathbf{x} = A\mathbf{x} + B\mathbf{u}, \quad (4)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, and the constant matrices A and B are of appropriate dimensions, can be defined (in the same way as for delay linear systems) as an $\mathbb{R}[\frac{d}{dt}]$ -module. Likewise system

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{u}[k], \quad (5)$$

with $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, A and B constant matrices of appropriate dimensions, and $\mathbf{x}[k] = \mathbf{x}(kh)$, where kh , for $k \in \mathbb{N}$ and fixed $h > 0$, are the discrete-time instances, can be rewritten as $\mathbf{x}(kh) = A\mathbf{x}(kh-h) + B\mathbf{u}(kh-h)$, or, equivalently, as

$$\mathbf{x} = A\delta_h\mathbf{x} + B\delta_h\mathbf{u}, \quad (6)$$

which can be seen as an $\mathbb{R}[\delta_h]$ -module. (common examples of principal ideal rings), all notions of torsion-freeness, projectivity and freeness (that we recall in the next subsections) coincide, resulting in the well known Kalman controllability notion, see [19, 20, 22, 23, 24] for several fruitful consequences of this framework. Therefore, for linear finite-dimensional systems all controllability notions, recalled in Subsection 2.2 below, are equivalent and we no longer distinguish them. In particular, we will not distinguish the different controllability properties for the discretized version of a delay system, since the corresponding discrete system is a finite-dimensional system on $\mathbb{R}[\delta_h]$.

2.2. Controllability notions and criteria

Before defining the controllability notions considered in the paper, we need first to recall several algebraic notions, together with their relations to usual control theoretic concepts, for the convenience of the reader. Then, general definitions for different controllability notions will be given, followed by some examples.

2.2.1. Torsion free delay systems

An $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ is called *torsion-free* if its torsion submodule $T(\Lambda)$ (defined as the largest torsion module included in Λ) is such that $T(\Lambda) = \{0\}$. If Λ is an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -dynamics and torsion elements exist, then they satisfy a differential-difference equation not influenced by any of the input variables.

Remark 2.3. For finite-dimensional system, that is, on $\mathbb{R}[\frac{d}{dt}]$ or $\mathbb{R}[\delta_h]$, the torsion submodule corresponds to the uncontrollable Kalman subspace.

Example 2.2. Consider the following finite-dimensional (i.e., without delay) system Λ :

$$\dot{x}_1 = x_2 + u \quad (7a)$$

$$\dot{x}_2 = x_1 + u. \quad (7b)$$

By setting $z_1 = x_2 - x_1$ and $z_2 = x_2 + x_1$, system (7) transforms into:

$$\dot{z}_1 = z_1 \quad (8a)$$

$$\dot{z}_2 = z_2 + 2u, \quad (8b)$$

from which it is clear that the uncontrollable Kalman space corresponds to the $\mathbb{R}[\frac{d}{dt}]$ -module generated by z_1 . More precisely, the system without delay $[z_1, z_2, u]_{\mathbb{R}[\frac{d}{dt}]}$ admits the following decomposition:

$$\Lambda = [z_1, z_2, u]_{\mathbb{R}[\frac{d}{dt}]} = [z_1]_{\mathbb{R}[\frac{d}{dt}]} \oplus [z_2, u]_{\mathbb{R}[\frac{d}{dt}]}$$

where $[z_1]_{\mathbb{R}[\frac{d}{dt}]}$ is the uncontrollable part and $[z_2, u]_{\mathbb{R}[\frac{d}{dt}]}$ the controllable (equivalently, torsion-free) part. Hence, the torsion submodule is $T(\Lambda) = [z_1]_{\mathbb{R}[\frac{d}{dt}]}$.

2.2.2. Free delay systems

A system Λ of rank m is said to be *free* if there exists a finite family of m elements in Λ which is both $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -linearly independent and generator. This family is called a *basis* of Λ .

Remark 2.4. For finite-dimensional systems (i.e., without delay), the existence of a basis corresponds to the existence of a controllable canonical form (that is, to Kalman controllability). To simplify, consider the single input system $\dot{\mathbf{x}} = A\mathbf{x} + bu$, which is controllable if, and only if, there exists an invertible state transformation $\mathbf{z} = F\mathbf{x}$ such that $\dot{\mathbf{x}} = A\mathbf{x} + bu$ is transformed into:

$$\dot{z}_i = z_{i+1}, \quad 1 \leq i \leq n-1, \quad (9a)$$

$$\dot{z}_n = \alpha_1 z_1 + \alpha_2 z_2 + \dots + \alpha_n z_n + u. \quad (9b)$$

System (9) corresponds to a $\mathbb{R}[\frac{d}{dt}]$ -free module, with basis z_1 . Indeed, z_1 is a generator, as (9) proves, and $\{z_1\}$ is linearly independent (otherwise z_1 would be a torsion element, and the module $[z_1]_{\mathbb{R}[\frac{d}{dt}]}$ would be a part of the uncontrollable Kalman space, which is impossible since, in this case, the system being controllable, the uncontrollable Kalman space is simply $\{0\}$).

2.2.3. Projective delay systems

An intermediate property between torsion-freeness and freeness is that of projectivity. A module Λ on a commutative ring is said to be *projective*, if there exists a module Σ on the same ring such that the sum $\Lambda \oplus \Sigma$ is free (see, e.g., [16, A3.2 p. 615]); Σ is then also projective for the same reason.

A characterisation of projectivity is the following: a module Λ (over a commutative ring R), is projective if the presentation matrix $P_\Lambda \in R^{\beta \times \alpha}$ (with $\text{rk}_R P_\Lambda = \beta$) is right-invertible (that is, if there exists $Q \in R^{\alpha \times \beta}$ such that $P_\Lambda Q = I_\beta$).

Remark 2.5. *Note that this characterisation, for delay systems, is directly linked to the existence of Bezout matrix identities. If we consider a dynamics Λ with equations:*

$$\dot{\mathbf{x}} = A(\delta_L)\mathbf{x} + B(\delta_L)\mathbf{u}$$

with $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, and A and B matrices with coefficients in $\mathbb{R}[\delta_L]$ of appropriate dimensions, the projectivity of Λ amounts to the existence of matrices \tilde{A} and \tilde{B} with coefficients in $\mathbb{R}[\frac{d}{dt}, \delta_L]$ such that

$$\left(\frac{d}{dt} I_n - A(\delta_L) \right) \tilde{A}(\frac{d}{dt}, \delta_L) + B(\delta_L) \tilde{B}(\frac{d}{dt}, \delta_L) = I_n,$$

where I_n denotes the $n \times n$ -identity matrix. This type of equation can be used for stabilisation purposes by dynamic state feedback, in the spirit of what has been done by Vidyasagar (cf. [62]).

For a general ring of scalars, projectivity is an intermediate notion between the torsion-free character and freeness. For rings of polynomial type, we will see in Subsection 2.2.10, that there is in fact no difference, as projectivity and freeness coincide. On the other hand, for more elaborate rings the distinction is justified.

Remark 2.6. *For finite-dimensional systems (that is, on $\mathbb{R}[\frac{d}{dt}]$ or $\mathbb{R}[\delta_h]$), the three notions introduced above (torsion-freeness, freeness and projectivity) are equivalent.*

2.2.4. Extending or restricting the base ring and inverting elements

In general, the base change operation, a classic operation in algebra (see, e.g., [16]) allows changing the perspective on an algebraic object. Several examples will be mentioned here: the extension (respectively, restriction) of the ring of operators to a larger (respectively, smaller) ring, and the inversion of certain elements of the former ring.

Extension of scalars. Consider an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -algebra A containing $\mathbb{R}[\frac{d}{dt}, \delta_L]$ and Λ an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module. An A -module is formed by formally multiplying elements of Λ by scalars of A . This formal multiplication is a special case of a composition operation called the tensor product (see [16, p. 63] for a precise definition). The A -module thus obtained is denoted as $A \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ and is said to be obtained from Λ by *extension of scalars*.

Let us take, for example, Λ a delay system and $A = \mathbb{R}(\delta_L)[\frac{d}{dt}]$, the ring of partial fractions in δ_L . The operators acting on the module $A \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ are polynomials in $\frac{d}{dt}$ with coefficients in $\mathbb{R}(\delta_L)$. By setting $K = \mathbb{R}(\delta_L)$, we find ourselves with $K[\frac{d}{dt}]$, and in this sense, a system with delays can be considered as a finite-dimensional system with coefficients in the abstract field $\mathbb{R}(\delta_L)$.

Restriction of scalars. Let us also consider the inverse operation; let R be a ring contained in $\mathbb{R}[\frac{d}{dt}, \delta_L]$, Λ be an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ module and $f : R \rightarrow \mathbb{R}[\frac{d}{dt}, \delta_L]$ a ring morphism. We can now define an R -module structure on Λ by setting $rz = f(r)z$, for $r \in R$ and $z \in \Lambda$. We say that we have obtained this R -module by *restriction of scalars* from Λ and denote it by $f^*\Lambda$.

Inverting elements: localisation. The technique for inverting a certain subset of elements in the ring of operators is known as localisation. Let \mathcal{S} be a multiplicatively closed subset of $\mathbb{R}[\frac{d}{dt}, \delta_L]$, i.e., for all $\pi_1, \pi_2 \in \mathcal{S}$, $\pi_1\pi_2 \in \mathcal{S}$ and $1 \in \mathcal{S}$. The *localized in \mathcal{S} of $\mathbb{R}[\frac{d}{dt}, \delta_L]$* is the ring, denoted $\mathcal{S}^{-1}\mathbb{R}[\frac{d}{dt}, \delta_L]$, made up of elements of the form p/σ (meaning $p\sigma^{-1}$) where $p \in \mathbb{R}[\frac{d}{dt}, \delta_L]$ and $\sigma \in \mathcal{S}$ (see [16, p. 59] for a precise definition). All elements of a multiplicative set are thus inverted. Consider, for example, in $\mathbb{R}[\frac{d}{dt}, \delta_L]$ the set $\mathcal{S} = \{\delta_L^i \mid i \in \mathbb{N}\}$. The localisation procedure leads to the ring $\mathcal{S}^{-1}\mathbb{R}[\frac{d}{dt}, \delta_L] = \mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$, which is called the ring of partial Laurent *polynomials in δ_L* . Moreover, $\mathbb{R}(\delta_L)[\frac{d}{dt}]$ can be found by taking $\mathcal{S} = \mathbb{R}[\delta_L]$. Note that the ring $\mathbb{R}(\delta_L)[\frac{d}{dt}]$ strictly contains $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$: indeed, $1/(1 + \delta_L)$ belongs to the first but not to the second.

2.2.5. General controllability definitions

For an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system (and in particular, for an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -dynamics), we recall next the definition of several notions of controllability. Varying the rings under consideration by extending the base ring (as explained before, through suitable tensor products, in the spirit of what is done along Grothendieck's ideas in modern algebraic geometry, see, e.g., [29], [31]), yields a virtually infinite number of controllability notions, as the following definition exemplifies.

Definition 2.3. *Let R be a ring such that $\mathbb{R}[\frac{d}{dt}, \delta_L] \subseteq R$. The $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ is said to be R -torsion-free controllable (resp. R -projective controllable, R -free controllable) if the R -module $R \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ is torsion-free (resp. projective, free).*

By abuse of language, often we will not distinguish between the controllability notion of the $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ and the corresponding property of the R -module $R \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ (for instance, when we say that the $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ is R -free, we mean that Λ is R -free controllable or, equivalently, that the R -module $R \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ is free). In addition to base change, the three distinct algebraic notions (torsion-freeness, projectivity and freeness) allow for an even greater number of controllability notions.

2.2.6. Illustrative examples

Consider some simple examples of the previous definitions.

Example 2.3. *Consider:*

$$\frac{d}{dt}x_1 = \delta_L x_2 + u \tag{10a}$$

$$\frac{d}{dt}x_2 = x_2, \tag{10b}$$

whose associated $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ , given by (10), is not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free, since x_2 is a torsion element: indeed, we have $(\frac{d}{dt} - 1)x_2 = 0$.

Example 2.4. Consider:

$$\frac{d}{dt}x_1 = x_2 \quad (11a)$$

$$\frac{d}{dt}x_2 = \delta_L x_1 + u, \quad (11b)$$

whose associated $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ , given by (11), is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -projective. Indeed, the associated presentation matrix P_Λ , given by:

$$P_\Lambda = \begin{pmatrix} \frac{d}{dt} & -1 & 0 \\ -\delta_L & \frac{d}{dt} & -1 \end{pmatrix}$$

admits a right inverse in $\mathbb{R}[\frac{d}{dt}, \delta_L]$:

$$\begin{pmatrix} \frac{d}{dt} & -1 & 0 \\ -\delta_L & \frac{d}{dt} & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ -\frac{d}{dt} & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The above $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ is in fact $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free, with basis x_1 . The generator character follows from:

$$x_2 = \frac{d}{dt}x_1 \quad (12a)$$

$$u = \dot{x}_2 - \delta_L x_1 = \frac{d^2}{dt^2}x_1 - \delta_L x_1, \quad (12b)$$

while the independent character comes from the fact that a relation like $r(\frac{d}{dt}, \delta_L)x_1 = 0$, for $r \in \mathbb{R}[\frac{d}{dt}, \delta_L]$, cannot be deduced from equations (12).

Example 2.5. Consider the system:

$$\frac{d}{dt}x_1 = x_2 + \delta_L u \quad (13a)$$

$$\frac{d}{dt}x_2 = x_1. \quad (13b)$$

The associated $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ , given by (13), is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free, but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free (we will show this in Subsection 2.2.10, when discussing criteria for controllability, see Example 2.13). Equations (13) defined not over $\mathbb{R}[\frac{d}{dt}, \delta_L]$ but over $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$ yield a free module. In other words, $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ is a free $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$ -module, with basis x_2 . Indeed, we have:

$$\begin{aligned} x_1 &= \frac{d}{dt}x_2 \\ u &= \delta_L^{-1} \left(\frac{d}{dt}x_1 - x_2 \right). \end{aligned}$$

2.2.7. Controllability implications and π -freeness

Elementary homological algebra [54] yields:

Proposition 2.1. Let R be any ring such that $\mathbb{R}[\frac{d}{dt}, \delta_L] \subset R$. Then R -free controllability implies R -projective controllability, which in turn implies R -torsion-free controllability.

We have seen in Example 2.5, that there are systems that are $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable, and in order to render them free, it is, in fact, sufficient to invert a single polynomial in δ_L (in the case of Example 2.5, δ_L itself). Therefore it seems natural to seek, by inversion of certain elements of the ring of operators, to free up a module that is not initially free. The following result, adapted from [55, Proposition 2.12.17, p. 233], shows how starting from a system admitting a not very restrictive controllability notion, one can get a stronger property (recall that a principal ideal ring is a ring in which every ideal is principal, that is, generated by a single element).

Proposition 2.2. *Let Λ be an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable delay system and S a multiplicative part of $\mathbb{R}[\frac{d}{dt}, \delta_L]$ such that $S^{-1}\mathbb{R}[\frac{d}{dt}, \delta_L]$ is a principal ideal ring. Then, there exists $\pi \in S$ such that Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L, \pi^{-1}]$ -free controllable. The system Λ is called π -free, while the polynomial $\pi \in S$ is called a freeing polynomial of Λ .*

For single input delay systems, the proof of the above proposition is constructive and an algorithm that explicitly constructs $\pi \in S$ is given in [45].

Example 2.6. *Consider the system Λ with equation:*

$$\dot{x} = x + (1 + \delta_L)u \quad (14)$$

This system is trivially $(1 + \delta_L)$ -free with basis x since $u = (1 + \delta_L)^{-1}(\dot{x} - x)$.

2.2.8. Relations with spectral controllability

One of the most widely used notions in a functional analysis setting is a modal type of controllability known as spectral controllability (see, for example, [49], [5]). Its popularity is partly due to the fact that a spectrally controllable dynamics can be stabilized by static state feedback including distributed delays (see, for example, [41]). For dynamic equations of the form

$$\dot{\mathbf{x}} = A(\delta_L)\mathbf{x} + B(\delta_L)\mathbf{u} \quad (15)$$

with $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, $A \in \mathbb{R}[\delta_L]^{n \times n}$, $B \in \mathbb{R}[\delta_L]^{n \times m}$, it is expressed by

$$\forall s \in \mathbb{C}, \quad \text{rk}_{\mathbb{C}}[sI_n - A(e^{-Ls}) \quad B(e^{-Ls})] = n.$$

Let us give an interpretation of spectral controllability in our framework. We introduce two preliminary Definitions, 2.4 and 2.5, the interpretation being given in Definition 2.6 and Proposition 2.3. First recall that a Taylor type series of an entire function f is given by:

$$f(t - L) = \sum_{i=0}^{\infty} \frac{(-L)^i}{i!} f^{(i)}(t), \quad (16)$$

which can actually be rewritten as:

$$\begin{aligned} f(t - L) &= \delta_L f(t) = \sum_{i=0}^{\infty} \frac{(-L)^i}{i!} f^{(i)}(t) = \sum_{i=0}^{\infty} \frac{(-L)^i}{i!} \frac{d^i f}{dt^i}(t) = \sum_{i=0}^{\infty} \frac{(-L)^i}{i!} \frac{d^i}{dt^i} \cdot f(t) \\ &= \left(\sum_{i=0}^{\infty} \frac{(-L)^i}{i!} \frac{d^i}{dt^i} \right) \cdot f(t) = e^{-L \frac{d}{dt}} \cdot f(t) \end{aligned} \quad (17)$$

implying that the delay operator can be seen as an infinite order differential operator of exponential type:

$$\delta_L = e^{-L \frac{d}{dt}}. \quad (18)$$

Hence, one can state the following natural embedding from $\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}]$ to $\mathbb{R}[\frac{d}{dt}]$, where \mathbf{Mod}_R denotes the category of R -modules, for a ring R , see Appendix A for the definitions of categories and functors.

Definition 2.4. *Let Λ be a linear delay system. We shall call the exponential version of Λ , denoted by Λ_e , the $\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}]$ -system obtained by the following functor from $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, \delta_L]}$ to $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}]}$:*

$$\begin{aligned} \tilde{\Lambda} &= \mathbb{R}[\frac{d}{dt}][\delta_L] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda && \text{extension of scalars} \\ \Lambda_e &= \psi_e^* \tilde{\Lambda} && \text{restriction of scalars} \end{aligned}$$

with ψ_e the ring homomorphism

$$\begin{aligned} \psi_e : \mathbb{R}[\frac{d}{dt}][\delta_L] &\longrightarrow \mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}] \\ \delta_L &\longmapsto e^{-L \frac{d}{dt}}, \end{aligned}$$

where $\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}]$ is viewed as a subring of $\mathbb{R}[\frac{d}{dt}]$, and ψ_e^* is the restriction of scalars functor from $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, \delta_L]}$ to $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}]}$ associated to ψ_e .

Definition 2.5. *Let Λ be a linear delay system. The extension of scalars*

$$\mathbb{R}(\frac{d}{dt})[e^{-L \frac{d}{dt}}, e^{L \frac{d}{dt}}] \cap \mathcal{O} \otimes_{\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}]} \Lambda_e$$

is called the distributed extension or the $\mathbb{R}(\frac{d}{dt})[e^{-L \frac{d}{dt}}, e^{L \frac{d}{dt}}] \cap \mathcal{O}$ -extension of Λ , and is denoted by Λ_d .

Recall that \mathcal{O} , used in the above definition, is the ring of entire functions with pointwise defined multiplication. We give next the property of *spectral controllability* (see, e.g., [1], [4], [47], [53]), which is particularly useful in the infinite-dimensional case, recall its algebraic characterisation (see, e.g., [65]), and discuss the relations with the algebraic notions of controllability.

Definition 2.6. *Denote by \mathcal{L} the Laplace embedding $\mathbb{R}[\frac{d}{dt}, e^{-L \frac{d}{dt}}] \rightarrow \mathcal{O}$. Consider a delay system Λ with presentation matrix P_Λ . Let Λ_e and P_{Λ_e} be the exponential version of Λ and the presentation matrix of Λ_e , respectively. The system Λ is said to be spectrally controllable if the \mathcal{O} -matrix $\hat{P}_{\Lambda_e} = \mathcal{L}(P_{\Lambda_e})$ satisfies $\exists k \in \mathbb{N} : \forall s \in \mathbb{C} : \text{rk}_{\mathbb{C}} \hat{P}_{\Lambda_e}(s) = k$.*

Proposition 2.3. *Consider a delay system Λ . The following conditions are equivalent:*

- (i) *The system Λ is spectrally controllable.*
- (ii) *The $\mathbb{R}(\frac{d}{dt})[e^{-L \frac{d}{dt}}, e^{L \frac{d}{dt}}] \cap \mathcal{O}$ -module Λ_d is torsion-free.*

(iii) The $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -module Λ_d is free.

Let us highlight that since the ring $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$, by means of which Λ_d is defined, has the property of being a Bézout ring (i.e., an integral domain in which the sum of two principal ideals is again a principal ideal), the torsion-freeness and freeness properties are equivalent. Notice that condition (i) simply says that the rank of the presentation matrix of Λ_e , in the Laplace domain, is constant. This can be interpreted as the generalisation of Popov-Belevitch-Hautus criterion for controllability of linear finite-dimensional control systems (according to which the system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{u} = (u_1, \dots, u_m)$, is controllable if, and only if, $\text{rk}(sI_n - A \quad B) = n$, for all $s \in \mathbb{C}$).

We consider now two simple examples for spectral controllability.

Example 2.7. Consider the $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ :

$$\dot{x} = (1 + \delta_L)u. \quad (19)$$

The presentation matrix, in the Laplace domain, $\hat{P}_{\Lambda_e}(s) = [s \quad 1 + e^{-Ls}]$ of its exponential version is such that $\text{rk}_{\mathbb{C}} \hat{P}_{\Lambda_e}(s) = 1, \forall s \in \mathbb{C}$. Thus, the distributed extension Λ_d of Λ

$$\Lambda_d = \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O} \otimes_{\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]} \Lambda_e$$

is torsion-free. Since $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ is a Bézout ring, torsion-freeness is equivalent to freeness, and Λ_d is free. Indeed, equation (19) is equivalent to

$$\dot{x} + (-1 - \delta_L)u = \dot{x} + (1 - \delta_L)u - 2u = 0.$$

Hence $u = \frac{1}{2}(\dot{x} + (1 - \delta_L)u) = \frac{1}{2}(\dot{x} + \frac{d}{dt}\Delta_L u) = \frac{1}{2}\frac{d}{dt}(x + \Delta_L u)$, with $\Delta_L = (1 - \delta_L)/\frac{d}{dt}$ implying that Λ_d is free with basis $x + \Delta_L u$, and finally that Λ is spectrally controllable.

Example 2.8. Consider the following $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ :

$$\dot{x} = (1 - \delta_L)u. \quad (20)$$

The presentation matrix of the exponential version Λ_e , in the Laplace domain, $\hat{P}_{\Lambda_e}(s) = [s \quad 1 - e^{-Ls}]$ is such that for $s = 0$, we have $\text{rk}_{\mathbb{C}} \hat{P}_{\Lambda_e}(0) = 0$. Thus, the distributed extension Λ_d of Λ

$$\Lambda_d = \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O} \otimes_{\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]} \Lambda_e \text{ is not torsion-free.}$$

Indeed, the element $x - \Delta_L u$ satisfies $\frac{d}{dt}(x - \Delta_L u) = 0$, with $\Delta_L = (1 - \delta_L)/\frac{d}{dt}$, and is a torsion element in Λ_d . Finally, we deduce that Λ is not spectrally controllable.

2.2.9. Fitting ideal

In this subsection we recall some tools useful to decide whether a system is torsion, torsion-free, or projective. We introduce an algebraic object associated with a system Λ which is built with the minors of the presentation matrix P_Λ and which, unlike the latter (which depends on the choices of the generators and the relations modules of Λ), is unique and intrinsically associated to the module Λ .

An *ideal* \mathfrak{a} of $\mathbb{R}[\frac{d}{dt}, \delta_L]$ is an additive subgroup of $\mathbb{R}[\frac{d}{dt}, \delta_L]$ such that for any $\alpha \in \mathfrak{a}$ and $r \in \mathbb{R}[\frac{d}{dt}, \delta_L]$, $r\alpha \in \mathfrak{a}$. It is therefore a subset of $\mathbb{R}[\frac{d}{dt}, \delta_L]$ which is also an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module (it follows that definitions like “finitely generated ideal” can be formulated in a similar way as those recalled in Subsection 2.1.2). We will use the standard notation $\mathfrak{a} = ((\alpha_i)_{i \in I})$ to denote the ideal \mathfrak{a} generated by the family $(\alpha_i)_{i \in I}$.

For an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ given by generators and relations as in Subsection 2.1.2, and such that $\text{rk}_{\mathbb{R}[\frac{d}{dt}, \delta_L]} P_\Lambda = \beta$, the Fitting ideal of Λ , denoted by $\mathfrak{F}(\Lambda)$, is defined as the ideal of $\mathbb{R}[\frac{d}{dt}, \delta_L]$ generated by the determinants of all $\beta \times \beta$ submatrices, that is, the minors of order β , of P_Λ (see [16, definition 20.4 p. 493]). The Fitting ideal is an intrinsic algebraic object associated to a module, yielding a projectivity criterion for various rings (see, e.g., [16]):

Proposition 2.4. *An R -module Λ , where R is a commutative ring, is projective if, and only if, $\mathfrak{F}(\Lambda) = R$.*

The following examples show how the Fitting ideal can be computed and applied as a tool for different controllability properties.

Example 2.9. *Consider again the system Λ of Example 2.7 given by:*

$$\dot{x} = (1 + \delta_L)u$$

as an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module. The associated Fitting ideal is $\mathfrak{F}(\Lambda) = (\frac{d}{dt}, 1 + \delta_L)$, and we see that $\mathfrak{F}(\Lambda) \neq \mathbb{R}[\frac{d}{dt}, \delta_L]$; hence, Λ is not projective.

On the other hand, if Λ is envisioned as an $\mathbb{R}[\frac{d}{dt}, \delta_L, (1 + \delta_L)^{-1}]$ -system, we have:

$$\mathfrak{F}((1 + \delta_L)^{-1}\Lambda) = (\frac{d}{dt}(1 + \delta_L)^{-1}, 1) = (1 + \delta_L)^{-1}\mathbb{R}[\frac{d}{dt}, \delta_L, (1 + \delta_L)^{-1}],$$

and it follows from Proposition 2.4 that the $\mathbb{R}[\frac{d}{dt}, \delta_L, (1 + \delta_L)^{-1}]$ -system Λ is projective.

Example 2.10. *Consider again the system Λ of Example 2.7, and its distributed extension Λ_d , for which we have:*

$$\mathfrak{F}(\Lambda_d) = (\frac{d}{dt}, \frac{d}{dt}\Delta_L - 2) = \left\{ \frac{d}{dt}p + \frac{d}{dt}\Delta_L q - 2q, \forall p, q \in \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O} \right\},$$

where $\Delta_L = \frac{1 - \delta_L}{d/dt}$. Then, setting, with no loss of generality:

$$\begin{aligned} p &= -\Delta_L q + \bar{p}, & \text{with } \bar{p} &\in \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O} \\ q &= -\frac{1}{2}\bar{q}, & \text{with } \bar{q} &\in \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O} \end{aligned}$$

we get:

$$\begin{aligned} \mathfrak{F}(\Lambda_d) &= \left\{ \frac{d}{dt}\bar{p} + \bar{q}, \forall \bar{p}, \bar{q} \in \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O} \right\} = (\frac{d}{dt}, 1) \\ &= \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}, \end{aligned}$$

and deduce that the $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -dynamics Λ_d is projective.

2.2.10. Controllability criteria for delay systems

We recall next two criteria for $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free and $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllability (see [25]).

On a polynomial ring, freeness and projectivity are equivalent. This is the celebrated Quillen-Suslin theorem (see [52], [58]) answering to Serre's conjecture (see [37]). Quillen-Suslin's theorem may here be stated as:

Proposition 2.5. *An $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -delay system is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable if, and only if, it is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -projective controllable.*

This implies the following criterion for free controllability, where $\hat{P}_\Lambda = \mathcal{L}\mathcal{L}(P_\Lambda)$, with $\mathcal{L}\mathcal{L} : \mathbb{R}[\frac{d}{dt}, \delta_L] \rightarrow \mathcal{O}^2$ is the Laplace/Z transform embedding. Similarly, considering Λ as defined by generators and relations, i.e., $\Lambda = \mathcal{W}/\mathcal{E}$, we define $\hat{\Lambda} = \hat{\mathcal{W}}/\hat{\mathcal{E}}$ where:

- $\hat{\mathcal{W}} = [\hat{W}_1, \dots, \hat{W}_\alpha]_{\mathbb{R}[s, z]} = [\hat{\mathbf{W}}]_{\mathbb{R}[s, z]}$;
- $\hat{\mathcal{E}} = [\hat{P}_\Lambda(s, z) \hat{\mathbf{W}}]_{\mathbb{R}[s, z]}$,

where $\hat{W}_i = \mathcal{L}\mathcal{L}(W_i)$, $1 \leq i \leq \alpha$. Then, $\hat{\Lambda}_e$ is defined along the same lines. In this paper, we will always use the hat symbol for the Laplace/Z transform.

Proposition 2.6. *A delay system Λ with presentation matrix P_Λ such that $\text{rk}_{\mathbb{R}[\frac{d}{dt}, \delta_L]} P_\Lambda = \beta$, with β the number of rows of P_Λ , is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable if, and only if,*

$$\forall (s, z) \in \mathbb{C}^2, \quad \text{rk}_{\mathbb{C}} \hat{P}_\Lambda(s, z) = \beta,$$

which is equivalent to the minors of \hat{P}_Λ of order β (equivalently, the generators of $\mathfrak{F}(\hat{\Lambda})$) having no common zeros in \mathbb{C}^2 .

According to the above proposition, for free controllability, the minors of \hat{P}_Λ of order β never annihilate simultaneously. This property can be seen as the generalisation of the Popov-Belevitch-Hautus criterion (see, e.g., [34], [25]).

A criterion for $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllability is the following:

Proposition 2.7. *An $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -delay system Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable if, and only if, the greatest common divisor (GCD) of the $\beta \times \beta$ minors of P_Λ belongs to \mathbb{R} , if, and only if, the minors of P_Λ of order β (equivalently, the generators of $\mathfrak{F}(\Lambda)$) have no common factors.*

Note that both criteria are given in terms of the presentation matrix P_Λ , but that for freeness we work with \hat{P}_Λ in the Laplace/Z domain since we are investigating the common zeros, while for torsion-freeness, we need to consider the matrix P_Λ (so with elements in $\mathbb{R}[\frac{d}{dt}, \delta_L]$) since we are looking for the common factors of its minors of order β .

Remark 2.7. *Consider the continuous-time finite-dimensional system Λ , which is an $\mathbb{R}[\frac{d}{dt}]$ -module, and $\hat{\Lambda}$ on $\mathbb{R}[s]$. Having no common factor between the generators of $\mathfrak{F}(\Lambda)$ is equivalent to having no common zeros between the generators of $\mathfrak{F}(\hat{\Lambda})$; indeed, torsion-freeness is equivalent to freeness for $\mathbb{R}[\frac{d}{dt}]$ -modules (since $\mathbb{R}[\frac{d}{dt}]$ is a principal ideal ring). The same property is true for discrete-time finite-dimensional systems (i.e., modules over $\mathbb{R}[\delta_h]$) and for the distributed extension of a delay system since the distributed extension is a module over $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ which is a Bézout ring. Recall also that torsion-freeness is equivalent to freeness for Bézout rings (see, e.g., Definition 2.6).*

Let us now illustrate the previous controllability criteria on some examples.

Example 2.11. Consider the following system, which we denote by Λ^α , depending on the parameter α :

$$\frac{d}{dt}x_1 = x_2 + \alpha\delta_L u \quad (21a)$$

$$\frac{d}{dt}x_2 = x_1 + \delta_L u \quad (21b)$$

with $\alpha \in \{0, 1\}$. Its presentation matrix is given by:

$$P_{\Lambda^\alpha} = \begin{pmatrix} \frac{d}{dt} & -1 & -\alpha\delta_L \\ -1 & \frac{d}{dt} & -\delta_L \end{pmatrix}. \quad (22)$$

The corresponding minors are, where we denote by m_i , $1 \leq i \leq 3$, the determinant of the 2×2 -matrix containing all columns of P_Λ except the i th one (we will use this notation throughout the paper):

$$m_1\left(\frac{d}{dt}, \delta_L\right) = \delta_L(1 + \alpha\frac{d}{dt}), \quad m_2\left(\frac{d}{dt}, \delta_L\right) = -\delta_L\left(\frac{d}{dt} + \alpha\right), \quad m_3\left(\frac{d}{dt}, \delta_L\right) = \frac{d^2}{dt^2} - 1, \quad (23)$$

and their Laplace/Z transform are:

$$\hat{m}_1(s, z) = z(1 + \alpha s), \quad \hat{m}_2(s, z) = -z(s + \alpha), \quad \hat{m}_3(s, z) = s^2 - 1. \quad (24)$$

When $\alpha = 1$, it is clear from (23), that $\frac{d}{dt} + 1$ is a common factor between the m_i 's. This corresponds to $x_2 - x_1$ being a torsion element. Indeed, subtracting (21b) and (21a), one gets: $\frac{d}{dt}(x_2 - x_1) = -(x_2 - x_1)$. Thus $\Lambda^{\alpha=1}$ is not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable.

Then, when $\alpha = 0$, there is no common factor between the m_i 's. On the other hand, from (24), it follows immediately that $(s, z) = (\pm 1, 0)$ are common zeros of the \hat{m}_i 's. Thus $\Lambda^{\alpha=0}$ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion free controllable but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable.

Example 2.12. Consider again the module Λ of Example 2.4, with equations:

$$\begin{aligned} \frac{d}{dt}x_1 &= x_2 \\ \frac{d}{dt}x_2 &= \delta_L x_1 + u. \end{aligned}$$

and recall that we showed (by finding a basis) that Λ is free controllable. We will apply next the criterion of Proposition 2.6 to reach the same conclusion. The system Λ admits the presentation matrix P_Λ :

$$P_\Lambda = \begin{pmatrix} \frac{d}{dt} & -1 & 0 \\ -\delta_L & \frac{d}{dt} & -1 \end{pmatrix}.$$

The 2×2 minors of its Laplace/Z transform \hat{P}_Λ are:

$$\hat{m}_1(s, z) = \begin{vmatrix} -1 & 0 \\ s & -1 \end{vmatrix} = 1, \quad \hat{m}_2(s, z) = \begin{vmatrix} s & 0 \\ -z & -1 \end{vmatrix} = -s, \quad \hat{m}_3(s, z) = \begin{vmatrix} s & -1 \\ -z & s \end{vmatrix} = s^2 - z.$$

They have no common zeros (as polynomials in s, z elements of $\mathbb{R}[s, z]$), showing that the system is free controllable.

Example 2.13. Consider again the system Λ of Example 2.5:

$$\frac{d}{dt}x_1 = x_2 + \delta_L u \quad (25a)$$

$$\frac{d}{dt}x_2 = x_1 \quad (25b)$$

We stated in Example 2.5 that Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free, but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free. To see this, we form the associated presentation matrix P_Λ , given by:

$$P_\Lambda = \begin{pmatrix} \frac{d}{dt} & -1 & -\delta_L \\ -1 & \frac{d}{dt} & 0 \end{pmatrix}.$$

Its associated 2×2 minors are:

$$m_1(\frac{d}{dt}, \delta_L) = \frac{d}{dt}\delta_L, \quad m_2(\frac{d}{dt}, \delta_L) = -\delta_L, \quad m_3(\frac{d}{dt}, \delta_L) = \left(\frac{d}{dt}\right)^2 - 1.$$

and their Laplace/Z transforms are:

$$\hat{m}_1(s, z) = sz, \quad \hat{m}_2(s, z) = -z, \quad \hat{m}_3(s, z) = s^2 - 1.$$

The minors have their greatest common divisor (GCD) equal to 1 in $\mathbb{R}[\frac{d}{dt}, \delta_L]$ and their Laplace/Z transform have $(s, z) = (1, 0)$ as a common zero. Hence the associated module Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free (see Propositions 2.7 and 2.6, p. 17 for torsion-freeness and freeness criteria over $\mathbb{R}[\frac{d}{dt}, \delta_L]$).

Example 2.14. Consider again the system Λ of Example 2.7 given by:

$$\dot{x} = (1 + \delta_L)u. \quad (26)$$

This system can be seen in various ways, according to the ring over which the module is envisioned.

1. The $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ is a torsion-free but not free $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system. Its presentation matrix P_Λ and its Laplace/Z transform \hat{P}_Λ (over $\mathbb{R}[s, z]$) are given by:

$$P_\Lambda = \begin{bmatrix} \frac{d}{dt} & -1 - \delta_L \end{bmatrix}, \quad \hat{P}_\Lambda = [s \quad -1 - z], \quad (27)$$

and one readily sees that $\frac{d}{dt}$ and $1 + \delta_L$ have no common factor but that s and $1 + z$ have $(s, z) = (0, -1)$ as a common zero. Hence Λ is torsion-free but not free.

2. Equation (26) also corresponds to a torsion-free but not free $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -system. The torsion-freeness and absence of freeness characters of Λ translate to the exponential version of this latter module, in virtue of the isomorphism of the rings $\mathbb{R}[\frac{d}{dt}, \delta_L]$ and $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ and the fact that the freeness and torsion-freeness are purely algebraic properties.
3. Equation (26) corresponds to a free $\mathbb{R}[\frac{d}{dt}, \delta_L, (1+\delta_L)^{-1}]$ -system. Indeed, on the ring $\mathbb{R}[\frac{d}{dt}, \delta_L, (1+\delta_L)^{-1}]$, we eliminate the common zero $(s, z) = (0, -1)$. Thus, \hat{P}_Λ has rank one for all (s, z) , $s \in \mathbb{C}$, $z \in \mathbb{C}, z \neq -1$, and the system $\mathbb{R}[\frac{d}{dt}, \delta_L, (1+\delta_L)^{-1}] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ is free.
4. Finally, equation (26) corresponds to a free $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -system (as we have seen in Example 2.7 by finding a basis of the distributed extension Λ_d of Λ). Indeed, in virtue of Definition 2.6, one has: $\forall s \in \mathbb{C}$, $\text{rk}_{\mathbb{C}}[s \quad -(1+e^{-Ls})] = 1$, see also Subsection 2.2.8 above.

Example 2.15. Consider the system Λ with equation:

$$\dot{x}(t) + \alpha \dot{x}(t-L) = u(t) + \beta u(t-L) \quad (28)$$

where $\alpha, \beta \in \{-1, 1\}$. Let $P_\Lambda = [\frac{d}{dt}(1+\alpha\delta_L) \quad -(1+\beta\delta_L)] = [m_2(\frac{d}{dt}, \delta_L) \quad m_1(\frac{d}{dt}, \delta_L)]$ be the presentation matrix of the $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ . In the Laplace/ Z domain, we have: $\hat{P}_\Lambda = [s(1+\alpha z) \quad -(1+\beta z)] = [\hat{m}_2(s, z) \quad \hat{m}_1(s, z)]$. We shall consider three cases:

1. Case $\alpha = \beta$, considering the delay system Λ (i.e., an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module).

In this case, m_1 and m_2 have a common factor, and a torsion element exists (recall Proposition 2.7, p. 17). Indeed, one has: $(1+\alpha\delta_L)(\frac{d}{dt}x - u) = 0$, and $\frac{d}{dt}x - u$ is a torsion element.

2. Case $\alpha \neq \beta$, considering the delay system Λ (i.e., an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module).

Then, there are no common factor between m_1 and m_2 in $\mathbb{R}[\frac{d}{dt}, \delta_L]$. Then, the pair $(s, z) = (0, -1/\beta)$ is a common zero between \hat{m}_1 and \hat{m}_2 which does not induce a common factor between m_1 and m_2 .

Hence, by virtue of Propositions 2.6, 2.7, system Λ is torsion-free but not free.

3. Case $\beta = -1, \alpha = 1$, considering the exponential delay system Λ_e (i.e., an $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -module).

The presentation matrix of Λ_e is such that $\hat{P}_{\Lambda_e} = [s(1+e^{-Ls}) \quad -(1-e^{-Ls})]$. In this case, $s = 0$ is a common zero between $\hat{m}_{1,e}(s) = -(1-e^{-Ls})$ and $\hat{m}_{2,e}(s) = s(1+e^{-Ls})$ which does not induce a common factor between the corresponding $m_{1,e}$ and $m_{2,e}$.

4. Case $\beta = -1, \alpha = 1$, considering the distributed extension Λ_d (i.e., an $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -module).

On the ring $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$, we get the following torsion element for Λ_d :

$$\frac{d}{dt}(1+e^{-L\frac{d}{dt}})x - (1-e^{-L\frac{d}{dt}})u = 0.$$

Let $\Delta_L \in \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ be such that $\Delta_L = (1-e^{-L\frac{d}{dt}})/\frac{d}{dt}$. Then the above relation becomes: $\frac{d}{dt}(1+e^{-L\frac{d}{dt}})x - \frac{d}{dt}\Delta_L u = \frac{d}{dt}((1+e^{-L\frac{d}{dt}})x - \Delta_L u) = 0$ and the element $(1+e^{-L\frac{d}{dt}})x - \Delta_L u$ is a torsion element of Λ_d .

2.2.11. Simple basis construction

Let us now show how to determine a basis for an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable delay system in the simple case where the number of independent equations is equal to the number of generators minus one, see [45]. Note that this corresponds to the case of a single input. Moreover, the presentation matrix P_Λ associated to the system Λ is a $\beta \times (\beta + 1)$ -matrix, and below we denote by $m_i \in \mathbb{R}[\frac{d}{dt}, \delta_L]$, for $1 \leq i \leq \beta$, the determinant of the $\beta \times \beta$ -matrix containing all columns of P_Λ except the i th one.

Proposition 2.8. *Consider a delay system Λ given by generators and relations, with:*

$$\Lambda = [w_1, \dots, w_{\beta+1}] = [\mathbf{w}], \quad P_\Lambda(\frac{d}{dt}, \delta_L)\mathbf{w} = 0,$$

where $P_\Lambda \in \mathbb{R}[\frac{d}{dt}, \delta_L]^{\beta \times (\beta+1)}$ and $\text{rk}_{\mathbb{R}[\frac{d}{dt}, \delta_L]} P_\Lambda = \beta$, and suppose that Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable. Then there exists $r_1, \dots, r_{\beta+1} \in \mathbb{R}[\frac{d}{dt}, \delta_L]$ such that:

$$r_1 m_1 + \dots + r_{\beta+1} m_{\beta+1} = 1,$$

and ω defined by:

$$\omega = \sum_{i=1}^{\beta+1} (-1)^{(i-1)} r_i(\frac{d}{dt}, \delta_L) w_i$$

is a basis of Λ .

The proof of the above proposition is constructive and was given in [45]. However, for the sake of completeness, we briefly recall it here. It will also be used in the example of Section 4 to construct a basis for the system considered there.

PROOF. Recall first that $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllability is equivalent to $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -projective controllability, for which we have the following criteria in terms of the presentation matrix P_Λ . If Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -projective controllable, then P_Λ is right-invertible:

$$\exists Q \in \mathbb{R}[\frac{d}{dt}, \delta_L]^{(\beta+1) \times \beta} \quad P_\Lambda Q = I_\beta.$$

Let $\mathfrak{F}(\Lambda) = (m_1, \dots, m_{\beta+1})$ be the Fitting ideal associated with the minors of order β of Λ . Since Λ is projective, we have $\mathfrak{F}(\Lambda) = \mathbb{R}[\frac{d}{dt}, \delta_L]$ and therefore $1 \in \mathfrak{F}_\Lambda$, implying that

$$\exists r_1, \dots, r_{\beta+1} \in \mathbb{R}[\frac{d}{dt}, \delta_L] : \quad r_1 m_1 + \dots + r_{\beta+1} m_{\beta+1} = 1.$$

The r_i 's can be formally computed using basis Gröbner techniques (see, for example, [14]). Let $\mathbf{r} = (r_1, -r_2, \dots, (-1)^\beta r_{\beta+1})$ and $M = \begin{bmatrix} \mathbf{r} \\ P_\Lambda \end{bmatrix}$. Then M is a unimodular $(\beta + 1) \times (\beta + 1)$ -matrix by construction, since its determinant, being equal to $r_1 m_1 + \dots + r_{\beta+1} m_{\beta+1}$ is 1. Then M admits a (unique) right inverse, and setting $\mathbf{m} = (m_1, -m_2, \dots, (-1)^\beta m_{\beta+1})$, the $(\beta + 1) \times (\beta + 1)$ -matrix $U = \begin{bmatrix} \mathbf{m}^T & Q \end{bmatrix}$ is the right inverse of M , that is,

$$MU = \begin{bmatrix} \mathbf{r} \\ P_\Lambda \end{bmatrix} \begin{bmatrix} \mathbf{m}^T & Q \end{bmatrix} = I_{\beta+1}$$

implying

$$P_\Lambda U = P_\Lambda [\mathbf{m}^T \quad Q] = [0 \quad I_\beta].$$

Now let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{\beta+1})$ be such that

$$U\boldsymbol{\omega} = [\mathbf{m}^T \quad Q]\boldsymbol{\omega} = \mathbf{w}. \quad (29)$$

It follows that

$$P_\Lambda U\boldsymbol{\omega} = P_\Lambda \mathbf{w} = 0 = [0 \quad I_\beta]\boldsymbol{\omega}.$$

Therefore, all ω_i except ω_1 are zero and from $\mathbf{w} = U\boldsymbol{\omega}$, we have :

$$w_i = (-1)^{(i-1)} m_i\left(\frac{d}{dt}, \delta_L\right) \omega_1.$$

Hence $\omega = \omega_1$ is a basis of Λ . Finally, multiplying (29) by M , we get:

$$\omega = \omega_1 = \sum_{i=1}^{\beta+1} (-1)^{(i-1)} r_i\left(\frac{d}{dt}, \delta_L\right) w_i.$$

□

Remark 2.8. *Note that the preceding proposition is in fact valid over any polynomial ring, and more generally over any ring R for which projectivity is equivalent to freeness for R -modules (for instance Bézout rings). In particular, it is valid over $\mathbb{R}\left[\frac{d}{dt}\right]$ and $\mathbb{R}[\delta_h]$.*

2.2.12. Relations with associated literature notions

The following result of [25] gives some relations between the previous algebraic notions and some related controllability notions of the literature, namely, weak controllability (see, e.g., [44]) and reachability (see, e.g., [8, 57]):

Proposition 2.9. *Let Λ be a delay system. Λ is weakly controllable if, and only if, Λ is $\mathbb{R}(\delta_L)\left[\frac{d}{dt}\right]$ -free controllable. Λ is reachable if, and only if, Λ is $\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]$ -free controllable.*

An interpretation of approximate controllability (see, e.g., [1, 47, 67]) for delay systems of the form (30) below, in terms of the previous algebraic notions, is the following (see [25]):

Proposition 2.10. *Let Λ be the delay system associated to the following equation*

$$\dot{\mathbf{x}} + E(\delta_L)\dot{\mathbf{x}} = A(\delta_L)\mathbf{x} + B(\delta_L)\mathbf{u} \quad (30)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, $A, E \in \mathbb{R}[\delta_L]^{n \times n}$, and $B \in \mathbb{R}[\delta_L]^{n \times m}$. Let ν denote the largest order of the delay operator involved in $B(\delta_L)$. The system Λ is approximately controllable if, and only if, it is spectrally controllable and the $\mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}\right]$ -module $e^{L\nu\frac{d}{dt}}\Lambda/e^{L\frac{d}{dt}}\Lambda$ is $\mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}\right]$ -torsion-free controllable.

3. Main results: discretizers and interplay between controllability and discretization

In this section we present our main results. First we provide, in Subsection 3.2, a formal definition of a discretizer, and then we give a complete analysis of the interplay between controllability and discretization in Subsection 3.3. We introduce the notion of preservation (that is, a measure

of quantifying the ability of the discretizer to preserve controllability properties) and show that for a fixed discretizer, we can always find a delay system for which even the torsion-free controllability (which is the weakest controllability notion) is not preserved (Proposition 3.5). We then reverse the situation, and prove in Theorems 3.1 and 3.2 that for any given delay system, preserving discretizers exist.

3.1. Informal definition of a discretization scheme

Before giving a formal definition of a generic discretization scheme in the algebraic framework, recall that at the beginning of Subsection 2.2.8, we showed that the delay operator can be seen as an infinite order differential operator of exponential type. The same property holds for the "discretization" shift operator:

$$\delta_h = e^{-h \frac{d}{dt}}, \quad (31)$$

and then the differential operator is also given as an infinite order "discretization" shift operator of logarithmic type:

$$\frac{d}{dt} = -\frac{1}{h} \ln(\delta_h). \quad (32)$$

From this, we deduce that a discretizer should be built using approximants of $-\ln(\delta_h)/h$.

Example 3.1. *Some examples of classical approximants of $-\ln(\delta_h)/h$ are:*

- *Polynomial Taylor approximants. They include first-order, second-order (until n th-order) forward discretisation schemes.*
- *Chebyshev polynomial approximants (see [59, 63]).*
- *Padé approximants (see, e.g., [10, 15, 64]). An example of a first order Padé approximant of $\ln(1 + \delta_h)$ is the Tustin discretization scheme:*

$$\ln(1 + \delta_h) \approx \frac{\delta_h}{1 + \frac{\delta_h}{2}} \quad (33)$$

implying:

$$\ln(\delta_h) \approx 2 \frac{\delta_h - 1}{\delta_h + 1}, \text{ and, finally, } \frac{1}{h} \ln(\delta_h) \approx 2 \frac{\delta_h - 1}{h(\delta_h + 1)}, \quad (34)$$

which is a Tustin type discretization scheme.

- *Continuous fraction approximants [9], a superclass of the previous Padé approximants.*
- *Hermite-Padé algebraic approximants [33], which is another superclass of Padé approximants.*

Observe from the above examples of classical approximants that a discretization scheme translates the differentiation operator $\frac{d}{dt}$ into a fraction $p(\delta_h)/q(\delta_h)$, where $p, q \in \mathbb{R}[\delta_h]$. In order to grasp the notion of discretizer which will be formally defined in Subsection 3.2 below, let us now give the following generic example.

Example 3.2. Recall Example 2.1 with equations:

$$E(\delta_L) \frac{d}{dt} \mathbf{x} = A(\delta_L) \mathbf{x} + B(\delta_L) \mathbf{u} \quad (35)$$

with $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_m)$, $E, A \in \mathbb{R}[\delta_L]^{n \times n}$, $B \in \mathbb{R}[\delta_L]^{n \times m}$, E being of rank n over $\mathbb{R}[\delta_L]$. We have seen that the $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -module Λ is isomorphic to the following quotient (which is a module given by generators and relations, recall the discussion of Subsection 2.1.2):

$$\Lambda \approx [\mathbf{X}, \mathbf{U}] / [(\frac{d}{dt} E(\delta_L) - A(\delta_L)) \mathbf{X} - B(\delta_L) \mathbf{U}] \quad (36)$$

The associated presentation matrix is given, in the Laplace domain, by:

$$\hat{P}_\Lambda = [sE(z) - A(z) \quad -B(z)]. \quad (37)$$

Since by discretization the differentiation operator $\frac{d}{dt}$ is transformed into a fraction $p(\delta_h)/q(\delta_h)$, where h is the discretization period and p and q are $\mathbb{R}[\delta_h]$ -polynomials, when discretizing, equation (35) becomes:

$$\frac{p(\delta_h)}{q(\delta_h)} E(\delta_h^M) \mathbf{x} = A(\delta_h^M) \mathbf{x} + B(\delta_h^M) \mathbf{u}, \quad (38)$$

where we assumed $L = Mh$, with $M \in \mathbb{N}$. Multiplying both sides by $q(\delta_h)$ yields:

$$p(\delta_h) E(\delta_h^M) \mathbf{x} = q(\delta_h) A(\delta_h^M) \mathbf{x} + q(\delta_h) B(\delta_h^M) \mathbf{u} \quad (39)$$

which can be seen as the following discretized module $\mathbb{D}_{p,q,h}(\Lambda)$:

$$\mathbb{D}_{p,q,h}(\Lambda) \approx [\mathbf{X}, \mathbf{U}] / [(p(\delta_h) E(\delta_h^M) - q(\delta_h) A(\delta_h^M)) \mathbf{X} - q(\delta_h) B(\delta_h^M) \mathbf{U}], \quad (40)$$

whose associated, presentation matrix $P_{\mathbb{D}_{p,q,h}(\Lambda)}$ and its Z transform are given, respectively, by:

$$P_{\mathbb{D}_{p,q,h}(\Lambda)} = [p(\delta_h) E(\delta_h^M) - q(\delta_h) A(\delta_h^M) \quad -q(\delta_h) B(\delta_h^M)], \quad (41)$$

$$\hat{P}_{\mathbb{D}_{p,q,h}(\Lambda)} = [p(z) E(z^M) - q(z) A(z^M) \quad -q(z) B(z^M)]. \quad (42)$$

Thus, a discretizer $\mathbb{D}_{p,q,h}$ can be viewed as a mapping which transforms systems of the form Λ , given by (36), into systems of the form $\mathbb{D}_{p,q,h}(\Lambda)$, given by (40). This is formalized, in Definition 3.3, through the notion of functor between module categories.

3.2. Discretization scheme: formal definition

Based on the discussion and examples of Subsection 3.1, we introduce next the notion of (rational) identity approximation associated to two fixed polynomials $p, q \in \mathbb{R}[\delta_h]$ with no common factor, and then we propose a formal definition for a discretizer. In the following definition, by abuse of notation, we continue to denote by p and q the corresponding polynomials on $\mathbb{R}[e^{-h} \frac{d}{dt}]$.

Definition 3.1 (Identity approximation). Let h be a real number dividing L (i.e., such that $L = Mh$, with $M \in \mathbb{N}$), and $p, q \in \mathbb{R}[\delta_h]$ be two fixed polynomials with no common factor. The ring

morphism $\phi_{\mathbb{D}_{p,q,h}}$ defined by

$$\begin{aligned}\phi_{\mathbb{D}_{p,q,h}} : \mathbb{R}\left[\frac{d}{dt}, \delta_L\right] &\rightarrow (q)^{-1}\mathbb{R}[\delta_h] \\ \left(\frac{d}{dt}, \delta_L\right) &\mapsto \left(\frac{p(\delta_h)}{q(\delta_h)}, \delta_h^M\right)\end{aligned}$$

is a (rational) identity approximation if its associated morphism $\phi_{\mathbb{D}_{p,q,h}}^e$ defined by

$$\begin{aligned}\phi_{\mathbb{D}_{p,q,h}}^e : \mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}\right] &\rightarrow (q)^{-1}\mathbb{R}\left[e^{-h\frac{d}{dt}}\right] \\ \left(\frac{d}{dt}, e^{-L\frac{d}{dt}}\right) &\mapsto \left(\frac{p(e^{-h\frac{d}{dt}})}{q(e^{-h\frac{d}{dt}})}, e^{-Mh\frac{d}{dt}}\right),\end{aligned}$$

with the same p and q as for $\phi_{\mathbb{D}_{p,q,h}}$ (but seen as polynomials on $\mathbb{R}[e^{-h\frac{d}{dt}}]$), satisfies

$$\lim_{h \rightarrow 0} \phi_{\mathbb{D}_{p,q,h}}^e \left(\frac{d}{dt}\right) = -\lim_{h \rightarrow 0} \frac{\ln(e^{-h\frac{d}{dt}})}{h} = \frac{d}{dt}.$$

Remark 3.1. Note that the limit $\lim_{h \rightarrow 0} \phi_{\mathbb{D}_{p,q,h}}^e \left(\frac{d}{dt}\right) = \frac{d}{dt}$ does not belong to $(q)^{-1}\mathbb{R}[e^{-h\frac{d}{dt}}]$, and the results comes from the fact that $\delta_h = e^{-h\frac{d}{dt}}$; hence

$$-\frac{\ln(\delta_h)}{h} = -\frac{\ln(e^{-h\frac{d}{dt}})}{h} = -\frac{-h\frac{d}{dt}}{h} = \frac{d}{dt},$$

which is in $\mathbb{R}[e^{-h\frac{d}{dt}}]$.

Definition 3.2 (Polynomial embedding). Let h be a real number dividing L (i.e., such that $L = Mh$, with $M \in \mathbb{N}$), and $q \in \mathbb{R}[\delta_h]$ be a fixed polynomial. The ring morphism $\psi_{\mathbb{D}_{q,h}}$ associated to h and q is defined by

$$\begin{aligned}\psi_{\mathbb{D}_{q,h}} : (q)^{-1}\mathbb{R}[\delta_h] &\rightarrow \mathbb{R}[\delta_h] \\ r(\delta_h) &\mapsto q^{\lambda_r}(\delta_h)r(\delta_h).\end{aligned}$$

where λ_r is the highest degree in q^{-1} of r .

In the above definition, r is seen as a polynomial with indeterminates δ_h and q^{-1} . To clarify what we mean by “the highest degree in q^{-1} of r ”, consider the following example:

Example 3.3. Consider $q(\delta_h) = 1 + \delta_h$, and $r(\delta_h) = \delta_h(1 + \delta_h)^{-1} - \delta_h^2(1 + \delta_h)^{-2}$. The highest degree in q^{-1} in r is $\lambda_r = 2$. Then:

$$\psi_{\mathbb{D}_{q,h}}(r) = (1 + \delta_h)^2 r(\delta_h) = \delta_h(1 + \delta_h) - \delta_h^2.$$

We formalize next the definition of a discretizer using the notion of functor between categories. Recall that a category consists of two sorts of items (the “objects” and the “morphisms” or

“arrows”), and that we use \mathbf{Mod}_R to denote the category of R -modules, for a ring R , see the beginning of Subsection 2.2.8 and Appendix A. Notice that the objects of $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, \delta_L]}$ are simply delay systems, while the objects of $\mathbf{Mod}_{\mathbb{R}[\delta_h]}$ are discrete systems.

Definition 3.3 (Discretizer). *Let h be a real number dividing L (i.e., $L = Mh$, $M \in \mathbb{N}$), and $p, q \in \mathbb{R}[\delta_h]$ be two fixed polynomials with no common factor, such that $\phi_{\mathbb{D}_{p,q,h}}$ is an identity approximation and $\psi_{\mathbb{D}_{q,h}}$ is a polynomial embedding. A (system) discretizer is a functor $\mathbb{D}_{p,q,h}: \mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \rightarrow \mathbf{Mod}_{\mathbb{R}[\delta_h]}$ defined by: for any delay system Λ (that is, object of $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, \delta_L]}$), we have*

$$\begin{aligned} \tilde{\Lambda} &= \mathbb{R}[\frac{d}{dt}, \delta_h, q(\delta_h)^{-1}] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda && \text{(extn)} \\ \bar{\Lambda} &= \phi_{\mathbb{D}_{p,q,h}}^* \tilde{\Lambda} && \text{(rest)} \\ \mathbb{D}_{p,q,h}(\Lambda) &= \psi_{\mathbb{D}_{q,h}} \bar{\Lambda} && \text{(mult)}. \end{aligned}$$

For any $r \in \mathbb{R}[\frac{d}{dt}, \delta_L]$, the operator discretizer $\mathbb{d}_{p,q,h}$ is defined by $\mathbb{d}_{p,q,h}(r) = \phi_{\mathbb{D}_{p,q,h}}(r)$, while the compensated operator discretizer $\mathbb{d}_{p,q,h}^c$ is defined by $\mathbb{d}_{p,q,h}^c(r) = \psi_{\mathbb{D}_{q,h}} \circ \phi_{\mathbb{D}_{p,q,h}}(r)$.

Remark 3.2. *An exponential (system) discretizer can be defined, mutatis mutandis, as a functor from $\mathbf{Mod}_{\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]}$ to $\mathbf{Mod}_{\mathbb{R}[e^{-h\frac{d}{dt}}]}$, by replacing δ_L (resp., δ_h) by $e^{-L\frac{d}{dt}}$ (resp., $e^{-h\frac{d}{dt}}$), and $\phi_{\mathbb{D}_{p,q,h}}$ by $\phi_{\mathbb{D}_{p,q,h}}^e$. Similarly, for any $r \in \mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$, one can define the exponential operator discretizer and the exponential compensated operator discretizer. By abuse of notation, we shall use the same notation for discretizers and exponential discretizers. These notions shall be instrumental for the Fitting ideal study made in Propositions 3.2 and 3.3. More precisely it will be used to see how to transfer common factors between the Fitting ideal’s generators from the original delay system to the discretized system.*

Note that in the above definition, (extn) and (rest) correspond, respectively, to an extension and a restriction of scalars (with $\phi_{\mathbb{D}_{p,q,h}}^*$ the restriction of scalars functor associated to $\phi_{\mathbb{D}_{p,q,h}}$), while (mult) is the multiplication by q . When the context is clear enough a system discretizer will be simply called a discretizer.

Note also that we could have defined an operator discretizer by means of a functor between ring categories. We have not done so, to simplify the exposition of the paper.

The denominator torsion case should be avoided. Observe that an intermediary step in defining the discretized system is $\bar{\Lambda}$ whose scalars are in $(q)^{-1}\mathbb{R}[\delta_h]$; it is thus meaningful to discretize a system Λ only when

$$q \notin \mathfrak{F}(\mathbb{T}(\Lambda)),$$

where $\mathbb{T}(\Lambda)$ is the torsion module of Λ , as illustrated by the following example.

Example 3.4. *Consider the following delay system Λ :*

$$q(\delta_L)\dot{x}(t) = q(\delta_L)u(t)$$

with the Fitting ideal:

$$\mathfrak{F}(\Lambda) = \left(\frac{d}{dt}q(\delta_L), q(\delta_L) \right), \quad (43)$$

and for which $\dot{x} - u$ is clearly a torsion element since $q(\delta_L)(\dot{x} - u) = 0$. Setting $z_1 = \dot{x} - u$ and $z_2 = \dot{x} + u$, one has:

$$\Lambda = [z_1]_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \oplus [z_2]_{\mathbb{R}[\frac{d}{dt}, \delta_L]} = \mathbb{T}(\Lambda) \oplus \Lambda/\mathbb{T}(\Lambda),$$

and it follows that Λ has torsion (i.e., $\mathbb{T}(\Lambda) \neq 0$). Now consider the discretized system $\mathbb{D}_{p,q,h}(\Lambda)$, where $h = L$, q is the $\mathbb{R}[\delta_h]$ -polynomial associated to that appearing in (43), and $p \in \mathbb{R}[\delta_h]$ is any polynomial such that p and q have no common factor. Observe that $\mathbb{D}_{p,q,h}(\Lambda)$ has the following Fitting ideal

$$\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda)) = \left(\frac{p(\delta_h)}{q(\delta_h)}q(\delta_h), q(\delta_h) \right) = (p, q),$$

from which we deduce that $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda)) = \mathbb{R}[\delta_h]$. Indeed, since $\text{GCD}(p, q) = 1$, by Bézout's theorem, there exists r_p and r_q in $\mathbb{R}[\delta_h]$ such that $r_p p + r_q q = 1$. Thus the ideal (p, q) generated by p and q contains 1, and $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda)) = (p, q) = \mathbb{R}[\delta_h]$. It follows that $\mathbb{D}_{p,q,h}(\Lambda)$ is controllable.

To sum up, by discretizing, we transformed a delay system that was not even torsion-free into a controllable discret system.

The only way to avoid the case of the above example (allowing to transform, by discretizing, uncontrollable systems into controllable ones, which from a physical viewpoint, does not have any sense) is to impose $q \notin \mathfrak{F}(\mathbb{T}(\Lambda))$. This will be supposed in the sequel. From now on, we make the following assumptions:

Assumption 3.1 (Discretization period divides L). For any delay system Λ and discretizer with discretization period h , acting on Λ , the real number h is supposed to divide L , i.e., $L = Mh$, with $M \in \mathbb{N}$.

Assumption 3.2 (Denominator with no torsion). For any fixed delay system Λ and discretizer $\mathbb{D}_{p,q,h}$, the polynomial q is supposed to be such that $q \notin \mathfrak{F}(\mathbb{T}(\Lambda))$. In that case, we say that Λ and $\mathbb{D}_{p,q,h}$ satisfy the denominator with no torsion assumption.

Example 3.5 (Classical discretization schemes). We next discuss some classical discretization schemes from the above definition viewpoint.

- *Backward difference discretization.* In this case, the morphism $\phi_{\mathbb{D}_{p,q,h}}$ is

$$\phi_{\mathbb{D}_{p,q,h}} \left(\frac{d}{dt} \right) = \frac{1 - \delta_h}{h},$$

i.e., $p = (1 - \delta_h)/h$ and $q = 1$ (recall that p and q do not have common factors, so they are actually unique up a multiplicative constant). Since we will use this discretization scheme to illustrate several comments or results, we simply denote it by $\mathbb{D}_{B,h}$ in the rest of the paper.

○ *Forward difference discretization.* In this case, the morphism $\phi_{\mathbb{D}_{p,q,h}}$ is

$$\phi_{\mathbb{D}_{p,q,h}}\left(\frac{d}{dt}\right) = \frac{\delta_h^{-1} - 1}{h} = \frac{1 - \delta_h}{h\delta_h},$$

i.e., $p = (1 - \delta_h)$ and $q = h\delta_h$.

○ *Tustin discretization.* In this case, the morphism $\phi_{\mathbb{D}_{p,q,h}}$ is

$$\phi_{\mathbb{D}_{p,q,h}}\left(\frac{d}{dt}\right) = \frac{2(1 - \delta_h)}{h(1 + \delta_h)},$$

i.e., $p = 2(1 - \delta_h)$ and $q = h(1 + \delta_h)$.

Remark 3.3. Any p and $q \in \mathbb{R}[e^{-h\frac{d}{dt}}]$ of Definition 3.1, p. 24, such that

$$\lim_{h \rightarrow 0} \frac{p(e^{-h\frac{d}{dt}})}{q(e^{-h\frac{d}{dt}})} = \frac{d}{dt}$$

yield a discretizer. In other words, setting $e^{-h\frac{d}{dt}} = \delta_h$, any \tilde{p} and $\tilde{q} \in \mathbb{R}[\delta_h]$ such that

$$\lim_{h \rightarrow 0} \frac{\tilde{p}(\delta_h)}{\tilde{q}(\delta_h)} = -\frac{\ln \delta_h}{h}$$

yield a discretizer.

Remark 3.4. One could imagine more general discretizers through:

- (1) a non polynomial ring morphism $\phi_{\mathbb{D}_h} : \mathbb{R}[\frac{d}{dt}, \delta_L] \rightarrow \mathbb{R}[\delta_h]$ corresponding to an identity approximation, *i.e.*, $\phi_{\mathbb{D}_h}^e : \mathbb{R}[\frac{d}{dt}, \delta_L] \rightarrow \mathbb{R}[e^{-h\frac{d}{dt}}]$ being such that

$$\lim_{h \rightarrow 0} \phi_{\mathbb{D}_h}^e\left(\frac{d}{dt}\right) = -\frac{d}{dt},$$

(for instance, $\phi_{\mathbb{D}_h}(z)$ could be, *e.g.*, an analytic spline or a wavelet approximation of $-\ln z/h$);

- (2) an embedding $\psi_{\mathbb{D}_h} : \mathbb{R}[\delta_h] \rightarrow \mathbb{R}[\delta_h]$, whose prototype is the truncation of the series.

Then, using the same construction for the functor as in Definition 3.3, one gets a discretizer. More precisely the (extn), (rest), (mult) are adapted as:

$$\begin{aligned} \tilde{\Lambda} &= \mathbb{R}[\frac{d}{dt}][[\delta_h]] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda && \text{(extn)} \\ \bar{\Lambda} &= \phi_{\mathbb{D}_h}^* \tilde{\Lambda} && \text{(rest)} \\ \mathbb{D}_h(\Lambda) &= \psi_{\mathbb{D}_h} \bar{\Lambda} && \text{(mult)}. \end{aligned}$$

3.3. Interplay between controllability and discretization

3.3.1. Ring and dimension level analysis

On a gross level, the results of Subsection 2.2 can be summed up by the following proposition:

Proposition 3.1. *Let Λ be a delay system, and $h \in \mathbb{R}$, $p, q \in \mathbb{R}[\delta_h]$ satisfying Assumptions 3.1 and 3.2. We have the following:*

- (1) *For Λ : $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free $\Leftrightarrow \mathbb{R}[\frac{d}{dt}, \delta_L]$ -projective $\not\Rightarrow \mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free.*
- (2) *For Λ_e : $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -free $\Leftrightarrow \mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -projective $\not\Rightarrow \mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -torsion-free.*
- (3) *For Λ_d : $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free $\Leftrightarrow \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -projective $\Leftrightarrow \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -torsion-free.*
- (4) *For $\mathbb{D}_{p,q,h}(\Lambda)$: $\mathbb{R}[\delta_h]$ -free $\Leftrightarrow \mathbb{R}[\delta_h]$ -projective $\Leftrightarrow \mathbb{R}[\delta_h]$ -torsion-free.*

PROOF. The proof follows directly from the results of Section 2.2.5 and Appendix A. \square

From Proposition 3.1 (in particular, from the equivalences in items (3) and (4)), one may naively be tempted to conclude that from a controllability viewpoint, to work with Λ_d in a continuous setting or to work on $\mathbb{D}_{p,q,h}(\Lambda)$ in a discrete setting is equivalent. Indeed, in both cases of items (3) and (4), the various controllability notions coincide since for $\mathbb{D}_{p,q,h}(\Lambda)$, the corresponding ring $\mathbb{R}[\delta_h]$ is a principal ideal ring, while for Λ_d , the ring $R_d = \mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ is a Bézout ring (see the comment following Definition 2.6, p. 14). Thus, one could think that a spectrally controllable system Λ (or equivalently, the corresponding controllable system Λ_d on $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$) is necessarily transformed by discretization into a controllable system on $\mathbb{R}[\delta_h]$. The following example shows that, in general this is not true.

Example 3.6. *Consider the following system:*

$$\frac{d}{dt}x = (1 + \epsilon\delta_L)u, \quad (44)$$

with $\epsilon \in \{-1, 0, 1\}$, that we denote by Λ , and whose presentation matrix is, in the Laplace/Z transform domain: $\hat{P}_\Lambda = [s \quad 1 + \epsilon z]$. The presentation matrix of Λ_d is, in the Laplace domain: $\hat{P}_{\Lambda_d} = [s \quad 1 + \epsilon e^{-Ls}]$.

We have the following controllability properties:

- *If $\epsilon = 0$, then Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free (no common zeros between the minors of \hat{P}_Λ).*
- *If $\epsilon \in \{-1, 1\}$, then Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free, but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free (no common factor between the minors of P_Λ , but with $(s, z) = (0, -\epsilon)$ as a common zero between the minors of \hat{P}_Λ).*
- *If $\epsilon = 1$, then Λ is spectrally controllable, equivalently, Λ_d is $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free (no common zeros between the minors of \hat{P}_{Λ_d}).*
- *If $\epsilon = -1$, then Λ is not spectrally controllable, equivalently, Λ_d has a torsion element (a common zero between the minors of \hat{P}_{Λ_d} , or a common factor in $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ between the minors of P_Λ).*

Following the same line as in Example 3.2, the discretized system $\mathbb{D}_{p,q,h}(\Lambda)$ is given by the following equation:

$$p(\delta_h)x = q(\delta_h)(1 + \epsilon\delta_h^M)u \quad (45)$$

and the associated presentation matrix is, in the Z transform domain:

$$\hat{P}_{\mathbb{D}_{p,q,h}(\Lambda)} = [\hat{p}(z) \quad -\hat{q}(z)(1 + \epsilon z^M)]. \quad (46)$$

The discretized system $\mathbb{D}_{p,q,h}(\Lambda)$ is controllable (recall that all notions are equivalent for discrete systems), if, and only if, $\hat{p}(z)$ and $\hat{q}(z)(1 + \epsilon z^M)$ have no common zeros in \mathbb{C} . The case when \hat{p} and \hat{q} have common zeros is excluded by the definition of a discretizer (recall that, see Remark 2.7, p, q having no common factor is equivalent to \hat{p}, \hat{q} having no common zeros), hence $\mathbb{D}_{p,q,h}(\Lambda)$ is controllable if, and only if, p and $(1 + \epsilon z^M)$ have no common zero in \mathbb{C} (in other words, if, and only if, $p(\sqrt[M]{-1/\epsilon}) \neq 0$).

The above example shows that in the case $\epsilon = 0$, the discrete system is always controllable. However, when $\epsilon \in \{-1, 1\}$, there always exist discretization methods that fail to preserve even the torsion-free controllability inherent in the original delay system. We will further look into this by taking into consideration the different controllability criteria recalled in Subsection 2.2.10 and analyzing how the common factors and common zeros of the Fitting ideal's generators (i.e., the obstructions to torsion-freeness and freeness) are modified by the applied discretization scheme.

Recall that since for $\mathbb{D}_{p,q,h}(\Lambda)$, all controllabilities notions are equivalent, we no longer distinguish them when referring to the discrete version of the delay system Λ . More precisely, if $\mathbb{D}_{p,q,h}(\Lambda)$ is $\mathbb{R}[\delta_h]$ -free (equivalently $\mathbb{R}[\delta_h]$ -projective, equivalently $\mathbb{R}[\delta_h]$ -torsion-free), then we will simply say that $\mathbb{D}_{p,q,h}(\Lambda)$ is controllable.

3.3.2. Fitting ideal level analysis

Common factors and common zeros through discretization. The next proposition studies how the common factors and common zeros of the Fitting ideal's generators of a delay system are modified under discretization. Let Λ be a delay system, and $m_{1,e}(\frac{d}{dt}, e^{-L\frac{d}{dt}}), \dots, m_{\rho,e}(\frac{d}{dt}, e^{-L\frac{d}{dt}})$ be the generators of $\mathfrak{F}(\Lambda_e)$, where Λ_e is the exponential version of Λ . The Laplace transform of these generators are exponential polynomials in s and can be expressed as:

$$\hat{m}_{i,e}(s) = \sum_{k=0}^{\mu_i} \zeta_{ik}(s) e^{-kLs}, \quad (47)$$

where $\zeta_{ik} \in \mathbb{R}[s]$. For fixed $1 \leq i \leq \rho$, let $\zeta_i^g(s)$ be the GCD of the ζ_{ik} , $k = 0, \dots, \mu_i$. Let also $\mathcal{Z}(\mathfrak{F}(\hat{\Lambda}_e))$ denote the set of common zeros of the $\hat{m}_{i,e}$'s.

The so-called Shapiro conjecture (concerning the common zeros of two exponential polynomials), solved in the commensurate delay case in [60], can be stated as follows: *two exponential polynomials have an infinite number of common zeros if, and only if, they have a common factor which is an exponential polynomial.* We shall thus distinguish two possibilities in the following proposition: either $\mathcal{Z}(\mathfrak{F}(\hat{\Lambda}_e))$ is infinite (equivalently, there is a common factor between the $\hat{m}_{i,e}$'s), or $\mathcal{Z}(\mathfrak{F}(\hat{\Lambda}_e))$ is finite. When $\mathcal{Z}(\mathfrak{F}(\hat{\Lambda}_e))$ is finite, i.e., there are a finite number of common zeros between the $\hat{m}_{i,e}$'s which do not induce common factors, we shall consider a typical case, corresponding to the following assumption.

Assumption 3.3 (Polynomial and exponential sum common zero). *We suppose that $\mathcal{Z}(\mathfrak{F}(\hat{\Lambda}_e))$ is finite, equal to $\{s_1, \dots, s_\eta\}$, and that the $\hat{m}_{i,e}$'s are partitioned as follows:*

- firstly, $\hat{m}_{i,e}$, for $1 \leq i \leq \nu$, with $1 \leq \nu < \rho$ are such that $\prod_{j=0}^{\eta} (s - s_j)$ divide all ζ_i^g , for $1 \leq i \leq \nu$, (i.e., s_0, \dots, s_η are common zeros of $\hat{m}_{1,e}, \dots, \hat{m}_{\nu,e}$);

◦ *secondly, $\hat{m}_{i,e}$, for $\nu + 1 \leq i \leq \rho$, have $\prod_{j=0}^{\eta} (e^{-Ls} - e^{-Ls_j})$ as a common factor.*

Notice that in the above assumption, we could also have taken the two extreme cases $\nu = 0$ and $\nu = \rho$ (i.e., all common zeros come from terms of the form $\prod_{j=0}^{\eta} (e^{-Ls} - e^{-Ls_j})$, respectively, of the form $\prod_{j=0}^{\eta} (s - s_j)$), but this would correspond to the case of common zeros inducing common factors, which is excluded by condition (2) of Proposition 3.2 below. The case of Assumption 3.3 can thus be seen as the union of the two extreme possibilities for the common zeros between some of the $\hat{m}_{i,e}$'s: either a polynomial in s or an exponential sum (i.e., an exponential polynomial like in (47), with $\zeta_{ik} \in \mathbb{R}$). Note that this case, because of its simplicity, allows us to state a precise result in the next proposition for the controllability of the discretized system. The general case of common zeros not being common factors is indeed more involved (see, e.g., [3]).

Proposition 3.2. *Let Λ be a delay system, and $\mathbb{D}_{p,q,h}$ a discretizer satisfying Assumptions 3.1 and 3.2. Then:*

- (1) *Let r be a common factor of the $\mathfrak{F}(\Lambda_e)$'s generators. Then the compensated operator discretizer $\mathfrak{d}_{p,q,h}^c(r)$ of r is a common factor of $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda))$'s generators;*
- (2) *Let $\bar{\mathcal{Z}}(\mathfrak{F}(\hat{\Lambda}_e))$ denote the set of common zeros which do not induce common factors between the generators of $\mathfrak{F}(\Lambda_e)$. Suppose that $\bar{\mathcal{Z}}(\mathfrak{F}(\hat{\Lambda}_e))$ is not empty, and that $\mathfrak{F}(\hat{\Lambda}_e)$'s generators satisfy Assumption 3.3. Then, for any $s_j \in \bar{\mathcal{Z}}(\mathfrak{F}(\hat{\Lambda}_e))$, let $z_{s_j,k} = \sqrt[M]{e^{-Ls_j} e^{\frac{2ik\pi}{M}}}$, for $k \in \mathbb{Z}$. If $z_{s_j,k}$, for any fixed $k \in \mathbb{Z}$, satisfies*

$$\hat{p}(z_{s_j,k}) - s_j \hat{q}(z_{s_j,k}) = 0,$$

then $z_{s_j,k}$ is a common zero of the $\widehat{\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda))}$'s generators.

PROOF. The first part of Proposition 3.2 is obvious. Let us detail the situation of common zeros of $\mathfrak{F}(\hat{\Lambda}_e)$'s generators. Suppose that there are common zeros which do not induce common factors between the generators of $\mathfrak{F}(\Lambda_e)$.

Due to Assumption 3.3, the generators of $\mathfrak{F}(\hat{\Lambda}_e)$ are of the form:

$$\begin{aligned} \hat{m}_{i,e}(s) &= \prod_{j=0}^{\eta} (s - s_j) \cdot \sigma_i(s, e^{-Ls}), & 1 \leq i \leq \nu, \\ \hat{m}_{i,e}(s) &= \prod_{j=0}^{\eta} (e^{-Ls} - e^{-Ls_j}) \cdot \sigma_i(s, e^{-Ls}), & \nu + 1 \leq i \leq \rho, \end{aligned} \tag{48}$$

where σ_i , for $1 \leq i \leq \rho$, are such that there is no common zeros between all of them. Notice that we use the notation $\sigma_i(s, e^{-Ls})$ to indicate that σ_i 's are exponential polynomials in s . Explicitly including e^{-Ls} in the arguments of σ_i will facilitate the application of discretizers and Z transform. The application of the identity approximation yields:

$$\begin{aligned} \widehat{\phi_{\mathbb{D}_{p,q,h}}^e(m_{i,e})} &= \prod_{j=0}^{\eta} \left(\frac{\hat{p}}{\hat{q}} - s_j \right) \cdot \sigma_i \left(\frac{\hat{p}}{\hat{q}}, z^M \right), & 1 \leq i \leq \nu, \\ \widehat{\phi_{\mathbb{D}_{p,q,h}}^e(m_{i,e})} &= \prod_{j=0}^{\eta} (z^M - e^{-Ls_j}) \cdot \sigma_i \left(\frac{\hat{p}}{\hat{q}}, z^M \right), & \nu + 1 \leq i \leq \rho. \end{aligned}$$

Then the application of the compensated operator discretizer leads to:

$$\begin{aligned}\widehat{\mathfrak{d}_{p,q,h}^c(m_{i,e})} &= \prod_{j=0}^{\eta} (\hat{p} - s_j \hat{q}) \cdot \tilde{\sigma}_i(\hat{p}, \hat{q}, z^M), & 1 \leq i \leq \nu, \\ \widehat{\mathfrak{d}_{p,q,h}^c(m_{i,e})} &= \prod_{j=0}^{\eta} (z^M - e^{-Ls_j}) \cdot \tilde{\sigma}_i(\hat{p}, \hat{q}, z^M), & \nu + 1 \leq i \leq \rho,\end{aligned}$$

where $\tilde{\sigma}_i$ corresponds to σ_i multiplied by $\hat{q}^{\lambda_{m_{i,e}}}$, where $\lambda_{m_{i,e}}$ is the highest degree in \hat{q}^{-1} of $\hat{m}_{i,e}$ (recall Definition 3.2).

Then, any of the s_j will induce a common zero of the $\widehat{\mathfrak{d}_{p,q,h}^c(m_{i,e})}$'s if the following two polynomials:

$$\hat{p}(z) - s_j \hat{q}(z) \quad \text{and} \quad z^M - e^{-Ls_j},$$

have a common zero. The zeros of $z^M - e^{-Ls_j}$ are the following:

$$z_{s_j,k} = \sqrt[M]{e^{-Ls_j} e^{\frac{2ik\pi}{M}}}, \quad k \in \mathbb{Z}.$$

Then, $z_{s_j,k}$ is a common zero of the $\widehat{\mathfrak{d}_{p,q,h}^c(m_{i,e})}$'s if one has:

$$\hat{p}(z_{s_j,k}) - s_j \hat{q}(z_{s_j,k}) = 0$$

for some $k \in \mathbb{Z}$, wherefrom the result. □

Remark 3.5. *Proposition 3.2 discusses the two major obstructions to controllability:*

- (1) *The presence of common factors between the generators of the Fitting ideal $\mathfrak{F}(\Lambda_e)$ corresponds to the presence of torsion elements. Note that, from a purely algebraic standpoint, the same common factors as those of $\mathfrak{F}(\Lambda_e)$ appear when studying $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda))$, so common factors are always transmitted through discretization and if the original delay system Λ is not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable, then there is no discretizer $\mathbb{D}_{p,q,h}$ leading to a controllable discrete counterpart $\mathbb{D}_{p,q,h}(\Lambda)$.*
- (2) *The second part of the proposition deals with the case of common zeros between generators of the Fitting ideal $\mathfrak{F}(\hat{\Lambda}_e)$ which do not yield common factors. This corresponds to the case of Λ_e being $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -torsion-free but not $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ -free.*

Both cases of the proposition (a common factor of $\mathfrak{F}(\Lambda_e)$'s generators and a common zero of $\mathfrak{F}(\hat{\Lambda}_e)$'s generators which does not induce a common factor) translate into a common factor when considering the distributed extension Λ_d (which is an $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -module), thus into Λ not being spectrally controllable. This actually allows, under Assumption 3.3 and the hypothesis of Proposition 3.2(2), to have a common zero between $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda))$'s generators (implying non controllability of $\mathbb{D}_{p,q,h}(\Lambda)$).

3.3.3. A disturbing situation

On the ring and dimension level analysis, things appear to be easy to handle. But these levels are quite coarse, and when one looks deeper into a finer, Fitting ideal analysis, the situation appears to be more tricky, as shown by Proposition 3.2(2) and highlighted by the following examples.

Example 3.7. Consider the following example

$$\dot{x}(t) = s_0x(t) + (1 - hs_0)^M u(t) - u(t - L), \quad (49)$$

whose associated module is denoted by Λ , and where $L > 0$, $h > 0$ and $M \in \mathbb{N}$ are related by $L = hM$ and, $s_0 \in \mathbb{R}$ is such that $1 - hs_0 > 0$ and:

$$\ln(1 - hs_0) \neq -\frac{L}{M} s_0. \quad (50)$$

The Fitting ideal of the exponential version Λ_e is

$$\mathfrak{F}(\Lambda_e) = \left(\frac{d}{dt} - s_0, (1 - hs_0)^M - e^{-L\frac{d}{dt}} \right).$$

The system is spectrally controllable, since (50) is equivalent to: $e^{-Ls_0} \neq (1 - hs_0)^M$, and there is no common zeros between $s - s_0$ and $(1 - hs_0)^M - e^{-Ls}$. Through a backward difference discretization of period h (the same h that appears in (49)), we get (recall the notation $\mathbb{D}_{B,h}$, see Example 3.5, p.27)

$$\mathfrak{F}(\mathbb{D}_{B,h}(\Lambda)) = \left(\frac{1 - \delta_h}{h} - s_0, (1 - hs_0)^M - \delta_h^M \right) = (1 - hs_0 - \delta_h) \left(\frac{1}{h}, r(\delta_h) \right).$$

The generators of $\mathfrak{F}(\mathbb{D}_{B,h}(\Lambda))$ have a common factor, implying that $\mathbb{D}_{B,h}(\Lambda)$ is non controllable, whatever the value of s_0 satisfying condition (50) is.

Example 3.8. Let the $\mathbb{R}[e^{-L\frac{d}{dt}}]$ -polynomial p and $s_0 \in \mathbb{R}$ be such that $\hat{p}(e^{-Ls_0}) = 0$. Consider a delay system Λ with exponential version Λ_e given by:

$$\dot{x}(t) = s_0x(t) + p(e^{-L\frac{d}{dt}})u(t). \quad (51)$$

The Fitting ideal of $\hat{\Lambda}_e$ is:

$$\mathfrak{F}(\hat{\Lambda}_e) = (s - s_0, p(e^{-Ls})),$$

and it is clear that the generators of $\mathfrak{F}(\hat{\Lambda}_e)$ have the common zero $s = s_0$, implying that the system is not spectrally controllable. The discretization of Λ via $\mathbb{D}_{p,q,h}$, where the $\mathbb{R}[\delta_h]$ -polynomial p is the same as that appearing in (51), has the following Fitting ideal

$$\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda)) = (p(\delta_h) - s_0q(\delta_h), p(\delta_h^M)q(\delta_h))$$

whose generators have no common factor, since $\text{GCD}(p, q) = 1$. Hence, the discretized system is controllable.

3.3.4. Ideal discretizer and ideal discretizer sequences

In the light of the above examples, one can easily deduce that the lack of controllability transmission through discretization is due to the approximate nature of the discretizers (note, for instance, that the backward difference discretization mapping $\mathbb{D}_{B,h}$ comes from a truncated Taylor expansion). We define next the concepts of ideal discretizer and ideal discretizer sequence. An ideal discretizer is not defined as a usual discretizer but rather as a limit of rational discretizers.

As it has been previously noted, any discretizer can be seen (through the identity approximation) as an approximation of $-\ln(\delta_h)/h$. The ideal discretizer is precisely this $-\ln \delta_h/h$.

Definition 3.4. Let $h \in \mathbb{R}$ such that Assumption 3.1 is verified. Consider the ring isomorphism $\phi_{\mathbb{D}_{\infty,h}}$ defined by

$$\begin{aligned} \phi_{\mathbb{D}_{\infty,h}} : \mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}\right] &\longrightarrow \mathbb{R}[\delta_h, \ln(\delta_h)] \\ \left(\frac{d}{dt}, e^{-L\frac{d}{dt}}\right) &\longmapsto \left(-\frac{\ln(\delta_h)}{h}, \delta_h^M\right). \end{aligned} \quad (52)$$

The ideal discretizer with discretization period h is the functor $\mathbb{D}_{\infty,h}: \mathbf{Mod}_{\mathbb{R}\left[\frac{d}{dt}, \delta_L\right]} \rightarrow \mathbf{Mod}_{\mathbb{R}[\delta_h, \ln(\delta_h)]}$ associated to $\phi_{\mathbb{D}_{\infty,h}}$, which can be characterized through Fitting ideals by:

$$\mathfrak{F}(\mathbb{D}_{\infty,h}(\Lambda)) = \phi_{\mathbb{D}_{\infty,h}}(\mathfrak{F}(\Lambda_e)),$$

where $\phi_{\mathbb{D}_{\infty,h}}(\mathfrak{F}(\Lambda_e))$ is the ideal whose generators are the transformation through $\phi_{\mathbb{D}_{\infty,h}}$ of $\mathfrak{F}(\Lambda_e)$'s generators. For any polynomial $r \in \mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}\right]$, we define the ideal operator discretization of r by $\mathfrak{d}_{\infty,h}(r) = \phi_{\mathbb{D}_{\infty,h}}(r)$.

Remark 3.6. Note that $\mathbb{D}_{\infty,h}(\Lambda)$ is an $\mathbb{R}[\delta_h, \ln(\delta_h)]$ -module and no longer an $\mathbb{R}[\delta_h]$ -module; hence torsion-freeness and freeness of $\mathbb{D}_{\infty,h}(\Lambda)$ are no longer equivalent as they were for $\mathbb{D}_{p,q,h}(\Lambda)$.

Remark 3.7. Note that there is no need to define the exponential version of the ideal discretizer, since it would refer to exactly the same object, in virtue of the isomorphism $\left(\frac{d}{dt}, e^{-L\frac{d}{dt}}\right) \mapsto \left(-\frac{\ln \delta_h}{h}, \delta_h\right)$ between $\mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}\right]$ and $\mathbb{R}[\delta_h, \ln(\delta_h)]$.

The isomorphism $\left(\frac{d}{dt}, e^{-L\frac{d}{dt}}\right) \mapsto \left(-\frac{\ln \delta_h}{h}, \delta_h\right)$ preserves the common factors and common zeros between $\mathbb{R}\left[\frac{d}{dt}, e^{-L\frac{d}{dt}}\right]$ and $\mathbb{R}[\delta_h, \ln(\delta_h)]$. Indeed, Proposition 3.2 becomes:

Proposition 3.3. Let Λ be a delay system and $h \in \mathbb{R}$ such that Assumption 3.1 is verified. Then the following statements hold:

- (1) Let r be a common factor of $\mathfrak{F}(\Lambda_e)$'s generators. Then $\mathfrak{d}_{\infty,h}(r)$ is a common factor of $\mathfrak{F}(\mathbb{D}_{\infty,h}(\Lambda))$'s generators.
- (2) Let $\bar{\mathfrak{Z}}(\mathfrak{F}(\hat{\Lambda}_e))$ denote the set of common zeros which do not induce common factors between the generators of $\mathfrak{F}(\hat{\Lambda}_e)$. Suppose that $\bar{\mathfrak{Z}}(\mathfrak{F}(\hat{\Lambda}_e))$ is not empty, and that $\mathfrak{F}(\hat{\Lambda}_e)$'s generators satisfy Assumption 3.3. Then, for any $s_i \in \bar{\mathfrak{Z}}(\mathfrak{F}(\hat{\Lambda}_e))$, e^{-hs_i} is a common zero of $\mathfrak{F}(\widehat{\mathbb{D}_{\infty,h}(\Lambda)})$'s generators.

PROOF. The first part of the proposition is obvious. Let us detail the situation of a common zero of $\mathfrak{F}(\hat{\Lambda}_e)$'s generators. Suppose that there is a common zero which is not a common factor between the generators of $\mathfrak{F}(\hat{\Lambda}_e)$. Due to Assumption 3.3, the generators of $\mathfrak{F}(\hat{\Lambda}_e)$ are of the form:

$$\begin{aligned} \hat{m}_{i,e}(s) &= \prod_{j=0}^{\eta} (s - s_j) \cdot \sigma_i(s, e^{-Ls}), & 1 \leq i \leq \nu, \\ \hat{m}_{i,e}(s) &= \prod_{j=0}^{\eta} (e^{-Ls} - e^{-Ls_j}) \cdot \sigma_i(s, e^{-Ls}), & \nu + 1 \leq i \leq \rho, \end{aligned} \quad (53)$$

where σ_i , for $1 \leq i \leq \rho$, are such that there is no common zero between all of them, and where we use the same notation $\sigma_i(s, e^{-Ls})$ for exponential polynomials in s as in the proof of Proposition 3.2. The application of $\phi_{\mathbb{D}_{\infty,h}}$ yields:

$$\begin{aligned}\widehat{\phi_{\mathbb{D}_{\infty,h}}(m_{i,e})} &= \prod_{j=0}^{\eta} \left(\frac{\ln z}{h} - s_j \right) \sigma_i \left(\frac{\ln z}{h}, z^M \right), \quad 1 \leq i \leq \nu, \\ \widehat{\phi_{\mathbb{D}_{\infty,h}}(m_{i,e})} &= \prod_{j=0}^{\eta} (z^M - e^{-Ls_j}) \sigma_i \left(\frac{\ln z}{h}, z^M \right), \quad \nu + 1 \leq i \leq \rho,\end{aligned}$$

Then, s_j will induce a common zero of the $\hat{\mathbb{d}}_{\infty,h}(\hat{m}_i)$'s if, and only if, the following two functions:

$$-\frac{\ln z}{h} - s_j \quad \text{and} \quad z^M - e^{-Ls_j},$$

have a common zero. The zeros of both functions are the same:

$$z_{s_j} = e^{-hs_j}$$

wherefrom the result. □

Definition 3.5. A sequence $(\mathbb{D}_{p_k, q_k, h})_{k \in \mathbb{N}}$ of discretizers whose limit, when k tends to infinity, is the ideal discretizer, is called an ideal discretizer sequence.

Example 3.9. The simplest form of an ideal discretizer sequence is $(\mathbb{D}_{p_k, 1, h})_{k \in \mathbb{N}}$, where for fixed $k \in \mathbb{N}$, we have

$$p_k = \frac{1}{h} \sum_{i=0}^k \frac{(1 - \delta_h)^i}{i},$$

corresponding to a k^{th} order Taylor approximation of $\ln \delta_h$. Note that these discretization maps may induce additional torsion elements in the delay module. Indeed, consider the basic discretizer $\mathbb{D}_{p_2, 1, h}$, given by

$$\mathbb{D}_{p_2, 1, h} \left(\frac{d}{dt} \right) = \frac{1 - \delta_h}{h} + \frac{(1 - \delta_h)^2}{2h} = \frac{1 - \delta_h}{h} \left(1 + \frac{1 - \delta_h}{2} \right) = \frac{(1 - \delta_h)(3 - \delta_h)}{2h}$$

and the following example

$$\dot{x}(t) = 3^M u(t) - u(t - L),$$

with $M \in \mathbb{N}$ such that $L = hM$, whose Fitting ideal is

$$\left(\frac{d}{dt}, 3^M - e^{-L \frac{d}{dt}} \right).$$

The corresponding system is clearly spectrally controllable. On the other hand, its discretization

through $\mathbb{D}_{p2,1,h}$ has the following Fitting ideal

$$\left(\frac{1}{2h} (1 - \delta_h)(3 - \delta_h), 3^M - \delta_h^M \right) = (3 - \delta_h) \left(\frac{1}{2h} (1 - \delta_h), \sum_{k=1}^M 3^{M-k} \delta_h^{k-1} \right),$$

hence the discretized system is non controllable.

3.3.5. Preserving discretizers

In order to study discretizers, one needs a property quantifying the optimality of a discretizer. This is the notion of preservation, as defined in Definition 3.6 below, where the asterisk * symbol stands for one of the following torsion-free, projective or free character (i.e., $* \in \{\text{torsion-free, projective, free}\}$).

Definition 3.6. *Let R be a ring containing $\mathbb{R}[\frac{d}{dt}, \delta_L]$. A discretizer $\mathbb{D}_{p,q,h}$ (resp. $\mathbb{D}_{\infty,h}$) is said to be:*

- R -* unconditionally preserving for an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ if whenever Λ is R -* controllable, then $\mathbb{D}_{p,q,h}(\Lambda)$ (resp. $\mathbb{D}_{\infty,h}(\Lambda)$) is $\mathbb{R}[\delta_h]$ -free controllable.
- universally R -* unconditionally preserving if $\mathbb{D}_{p,q,h}(\Lambda)$ (resp. $\mathbb{D}_{\infty,h}(\Lambda)$) is R -* unconditionally preserving for any $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ .

3.3.6. Discretizer centric study

When discretizing a system, one should choose a discretizer with the best properties for the desired control objective. For instance, a reasonable goal is to choose the polynomials p and q and the discretization period h such that the corresponding discretizer $\mathbb{D}_{p,q,h}$ preserves the most frequently encountered controllability properties, that is, $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-freeness and spectral controllability (which amounts to $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -freeness). Thus appropriate properties required for the applied discretizers seem to be $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-freeness and $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -freeness unconditional preservation (see Definition 3.6 just above). We will discuss these properties in Propositions 3.4-3.5 below.

According to the following result, the ideal discretizer is $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free universally preserving (that is, it is an optimal discretizer from a spectral controllability viewpoint):

Proposition 3.4. *The ideal discretizer $\mathbb{D}_{\infty,h}$ is universally $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free unconditionally preserving.*

PROOF. Consider an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ . When Λ is spectrally controllable (i.e., $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free), the generators of $\mathfrak{F}(\Lambda_e)$ have no common zeros. Then, there are no common zeros between the generators of $\mathfrak{F}(\mathbb{D}_{\infty,h}(\Lambda))$ (since $\mathbb{R}[\frac{d}{dt}, e^{-L\frac{d}{dt}}]$ and $\mathbb{R}[\delta_h, \ln(\delta_h)]$ are isomorphic with preservation of analytic properties between the ring indeterminates, see Proposition 3.3). Thus the discretized system $\mathbb{D}_{\infty,h}(\Lambda)$ is controllable. \square

The examples of Section 3.3.3, p. 32, indicate that, in general, apart from the ideal discretizer, a discretizer cannot be universally preserving (i.e., cannot preserve specific controllability properties for *any* delay system). Recall that the weakest controllability notion for delay systems is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllability. We consider next a generic example to investigate the existence of universally $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free preserving discretizer.

Example 3.10. Let Λ be the delay system associated to the following equations:

$$\dot{x}_1 = a_1(\delta_L)x_1 + (\delta_L + \zeta_1)x_2 \quad (54a)$$

$$\dot{x}_2 = a_2(\delta_L)x_1 + (\delta_L + \zeta_2)u, \quad (54b)$$

with $a_1, a_2 \in \mathbb{R}[\delta_L]$, $\zeta_1, \zeta_2 \in \mathbb{R}$. This system is easily seen to be π -free with $\pi = (\delta_L + \zeta_1)(\delta_L + \zeta_2)$. Indeed, x_1 is a basis of $\pi^{-1}\mathbb{R}[\frac{d}{dt}, \delta_L] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$ since:

$$\begin{aligned} x_2 &= (\delta_L + \zeta_1)^{-1}(\dot{x}_1 - a_1x_1) = \pi^{-1}(\delta_L + \zeta_2)(\dot{x}_1 - a_1x_1) \\ u &= (\delta_L + \zeta_2)^{-1}(\dot{x}_2 - a_2x_1) = (\delta_L + \zeta_2)^{-1}((\delta_L + \zeta_1)^{-1}(\ddot{x}_1 - a_1\dot{x}_1) - a_2x_1) \\ &= \pi^{-1}((\ddot{x}_1 - a_1\dot{x}_1) - (\delta_L + \zeta_1)a_2x_1). \end{aligned}$$

In the Laplace/Z-transform domain, the presentation matrix associated to Λ is given by:

$$\hat{P}_\Lambda = \begin{pmatrix} s - \hat{a}_1 & -z - \zeta_1 & 0 \\ -\hat{a}_2 & s & -z - \zeta_2 \end{pmatrix},$$

and the corresponding minors are:

$$\hat{m}_1(s, z) = (z + \zeta_1)(z + \zeta_2), \quad \hat{m}_2(s, z) = -(s - \hat{a}_1(z))(z + \zeta_2), \quad \hat{m}_3(s, z) = s(s - \hat{a}_1(z)) - \hat{a}_2(z)(z + \zeta_1).$$

Consider a discretizer $\mathbb{D}_{p,q,h}$ with $h = L$ and $p, q \in \mathbb{R}[\delta_h]$ satisfying Assumption 3.2. In the Laplace/Z-transform domain, the presentation matrix of the discretized system $\mathbb{D}_{p,q,h}(\Lambda)$ is

$$\hat{P}_{\mathbb{D}_{p,q,h}(\Lambda)} = \begin{pmatrix} \hat{p} - \hat{a}_1\hat{q} & -(z + \zeta_1)\hat{q} & 0 \\ -\hat{a}_2\hat{q} & \hat{p} & -(z + \zeta_2)\hat{q} \end{pmatrix},$$

and the corresponding 2×2 -minors (that we denote using the index \mathbb{D}) are:

$$\hat{m}_{1\mathbb{D}}(z) = (z + \zeta_1)(z + \zeta_2)\hat{q}^2, \quad \hat{m}_{2\mathbb{D}}(z) = -(\hat{p} - \hat{a}_1\hat{q})(z + \zeta_2)\hat{q}, \quad \hat{m}_{3\mathbb{D}}(z) = \hat{p}(\hat{p} - \hat{a}_1\hat{q}) - \hat{a}_2(z + \zeta_1)\hat{q}^2.$$

The associated zero sets are then:

$$\begin{aligned} \mathcal{Z}(\hat{m}_{1\mathbb{D}}) &= \{-\zeta_1\} \cup \{-\zeta_2\} \cup \mathcal{Z}(\hat{q}), \quad \mathcal{Z}(\hat{m}_{2\mathbb{D}}) = \{-\zeta_2\} \cup \mathcal{Z}(\hat{q}) \cup \mathcal{Z}(\hat{p} - \hat{a}_1\hat{q}), \\ \mathcal{Z}(\hat{m}_{3\mathbb{D}}) &= \{z \in \mathbb{C} \mid \hat{p}(p - \hat{a}_1\hat{q}) = \hat{a}_2(z + \zeta_1)\hat{q}^2\}. \end{aligned}$$

Let us consider the case when $\hat{m}_{1\mathbb{D}}(z_0) = 0$, for $z_0 \in \mathbb{C}$. Firstly, notice that the existence of z_0 such that $\hat{q}(z_0) = 0$ cannot lead to a common zero between the above three minors. Indeed, $\hat{q}(z_0) = 0$ implies $\hat{m}_{1\mathbb{D}}(z_0) = 0$ and $\hat{m}_{2\mathbb{D}}(z_0) = 0$. Then $\hat{m}_{3\mathbb{D}}(z_0) = \hat{p}^2(z_0)$, which cannot be zero since $\text{GCD}(\hat{p}, \hat{q}) = 1$. Thus, we are left with two cases to ensure $\hat{m}_{1\mathbb{D}}(z_0) = 0$: first, when $z_0 = -\zeta_1$ and second, when $z_0 = -\zeta_2$.

Case 1). Suppose that $z_0 = -\zeta_1$. Then the remaining two minors are:

$$\hat{m}_{2\mathbb{D}}(-\zeta_1) = -((\hat{p} - \hat{a}_1\hat{q})(z + \zeta_2)\hat{q})(-\zeta_1), \quad \hat{m}_{3\mathbb{D}}(-\zeta_1) = (\hat{p}(\hat{p} - \hat{a}_1\hat{q}))(-\zeta_1),$$

and, observe that considering $a_1(z)$ begin any polynomial such that

$$\hat{p}(-\zeta_1) - \hat{a}_1(-\zeta_1)\hat{q}(-\zeta_1) = 0$$

yields a common factor between $\hat{m}_{2\mathbb{D}}$ and $\hat{m}_{3\mathbb{D}}$, and thus a torsion element.

Case 2). Suppose that $z_0 = -\zeta_2$. Then $\hat{m}_{2\mathbb{D}}(s_0) = 0$, and to have $\hat{m}_{3\mathbb{D}}(s_0) = 0$, we can take $a_2(z)$ any polynomial such that:

$$\hat{a}_2(-\zeta_2)(\zeta_1 - \zeta_2)\hat{q}^2(-\zeta_2) = \hat{p}(-\zeta_2)(\hat{p}(-\zeta_2) - \hat{a}_1(-\zeta_2)\hat{q}(-\zeta_2)). \quad (55)$$

Thus, for any given discretizer $\mathbb{D}_{p,q,h}$, we can find a delay system Λ of form (54), i.e., a π -free delay system (where $\pi = (\delta_h + \zeta_1)(\delta_h + \zeta_2)$) such that $\mathbb{D}_{p,q,h}(\Lambda)$ is non controllable (i.e., not $\mathbb{R}[\delta_h]$ -free).

As suggested by the previous example, the following result shows that there is no universally $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free preserving discretizer (in other words, there is no discretizer $\mathbb{D}_{p,q,h}$ satisfying the most basic universal preservation property).

Proposition 3.5. *Let $\mathbb{D}_{p,q,h}$, with $p, q \in \mathbb{R}[\delta_h]$, be a discretizer. Then, there exists an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ such that Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion free controllable but $\mathbb{D}_{p,q,h}(\Lambda)$ is not controllable.*

PROOF. Consider a discretizer $\mathbb{D}_{p,q,h}$. Examples 3.10 shows that, we can always construct a torsion-free controllable delay system Λ of the form (54), with $L = h$, and such that Assumption 3.2 is satisfied for q and Λ , and leading to a noncontrollable discrete system $\mathbb{D}_{p,q,h}(\Lambda)$. \square

3.3.7. System centric study

We have just seen that, except for $\mathbb{D}_{\infty,h}$, there is no universally preserving discretizer. A natural question is whether we can reverse the situation. More precisely, for a given delay system Λ , we would like to decide whether preserving discretizers (at least for the most frequently encountered controllability properties) exist.

Theorem 3.1. *Let Λ be a $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free controllable delay system (equivalently, a spectrally controllable delay system). There exists an unconditionally $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free preserving discretizer $\mathbb{D}_{p,q,h}$ for Λ .*

PROOF. Let the function $\sigma : \mathbb{C} \rightarrow \mathbb{R}$ defined by:

$$\sigma(z) = \sum_{i=1}^{\ell} \left| \hat{m}_{i,e} \left(-\frac{1}{h} \ln z, z \right) \right|^2.$$

be the sum of squares associated with the generators $\hat{m}_{i,e}$, $1 \leq i \leq \ell$, of $\mathfrak{F}(\hat{\Lambda}_e)$. The assumption of Λ being $\mathbb{R}(\frac{d}{dt})[e^{-L\frac{d}{dt}}, e^{L\frac{d}{dt}}] \cap \mathcal{O}$ -free controllable amounts to σ having no zero. Consider an ideal discretizer sequence $(\mathbb{D}_{p_k, q_k, h})_{k \in \mathbb{N}}$. Let now, for any $k > 0$, σ_k be the sum of squares associated with the minors of $\mathfrak{F}(\mathbb{D}_{p_k, q_k, h}(\Lambda))$:

$$\sigma_k(z) = \sum_{i=1}^{\ell} \left| \hat{m}_{i\mathbb{D}}(z) \right|^2,$$

where $\hat{m}_{i\mathbb{D}}(z)$ is simply the Z transform of $m_{i\mathbb{D}}\left(\phi_{\mathbb{D}_{p_k, q_k, h}}(\delta_h), \delta_h\right)$. The claim we want to prove is equivalent to the existence of a k such that σ_k has no zero. Since $(\mathbb{D}_{p_k, q_k, h})_{k \in \mathbb{N}}$ is an ideal discretizer sequence, we have $\lim_{k \rightarrow \infty} \sigma_k = \sigma$. Then, by continuity of all σ_k 's, we deduce that there exists a k such that σ_k has no zero. \square

Another natural property of the system Λ that one should preserve is torsion-freeness. Recall that any $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable system Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L, \pi^{-1}]$ -free controllable, for a suitable freeing polynomial $\pi \in \mathbb{R}[\delta_L]$ (cf. Proposition 2.2, p. 13). Motivated by this property, we formulate the following definition:

Definition 3.7. *A discretizer $\mathbb{D}_{p, q, h}$ is said to be: $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free preserving for an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -system Λ , conditionally to $\mathbb{R}[\delta_h, \pi^{-1}(\delta_h)]$, where $\pi \in \mathbb{R}[\delta_L]$ is a freeing polynomial for Λ , if whenever Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free, then $\mathbb{D}_{p, q, h}(\Lambda)$ is $\mathbb{R}[\delta_h, \pi^{-1}(\delta_h)]$ -free controllable.*

We have the following result concerning the discretization:

Theorem 3.2. *Let Λ be an $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable system. There exists a discretizer $\mathbb{D}_{p, q, h}$ which is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free preserving for Λ , conditionally to $\mathbb{R}[\delta_h, \pi^{-1}]$, where π is a freeing polynomial of Λ .*

PROOF. Let $\sigma : \mathbb{C} \rightarrow \mathbb{R}$ defined by:

$$\sigma(z) = \sum_{i=1}^{\ell} |\hat{m}_{i, e}(-\frac{1}{h} \ln z, z)|^2,$$

be the sum of squares associated with the generators $\hat{m}_{i, e}$ of $\mathfrak{F}(\hat{\Lambda}_e)$. The assumption of Λ being $\mathbb{R}[\frac{d}{dt}, \delta_L, \pi^{-1}]$ -free controllable amounts to:

$$\mathcal{Z}(\sigma) \subseteq \mathcal{Z}(\hat{\pi}).$$

Consider now an ideal discretizer sequence $(\mathbb{D}_{p_k, q_k, h})_{k \in \mathbb{N}}$. Using the same arguments and notations as those of the proof of Theorem 3.1, we conclude that σ_k satisfies $\mathcal{Z}(\sigma_k) \subseteq \mathcal{Z}(\hat{\pi})$. Hence there are no other zeros except those associated to $\hat{\pi}$, thus $\mathbb{D}_{p_k, q_k, h}(\Lambda)$ is $\mathbb{R}[\delta_h, \pi^{-1}]$ -controllable. \square

To illustrate Theorems 3.1 and 3.2, let us revisit the situation of system (49), see Example 3.7 in Subsection 3.3.3.

Example 3.11. *Consider again:*

$$\dot{x}(t) = s_0 x(t) + (1 - h s_0)^M u(t) - u(t - L)$$

with s_0, L, h and M satisfying the conditions of Example 3.7 (so the above system is spectrally controllable, equivalently Λ_d is free controllable). Consider also its discretization by $\mathbb{D}_{p_2, 1, h}$ (see relation (51) for the definition of the polynomial p_2). Then $\mathfrak{F}(\mathbb{D}_{p_2, 1, h}(\Lambda))$ is generated by

$$\left(\frac{(1 - \delta_h)(3 - \delta_h)}{2h} - s_0, (1 - h s_0)^M - \delta_h^M \right) = \left(\frac{\delta_h^2 - 4\delta_h + 3 - 2h s_0}{2h}, (1 - h s_0)^M - \delta_h^M \right)$$

and since the generators of $\mathfrak{F}(\mathbb{D}_{p_2, 1, h}(\Lambda))$ have no common factor, it follows that $\mathbb{D}_{p_2, 1, h}(\Lambda)$ is controllable.

Example 3.12. *Let us consider again:*

$$\dot{x}(t) = s_0 x(t) + (1 - hs_0)^M u(t) - u(t - L)$$

but with s_0 , L , h and M not satisfying the conditions of Example 3.7, i.e., there is a common zero between $s - s_0$ and $(1 - hs_0)^M - e^{-Ls}$. Thus, the above system is not spectrally controllable; but it is $\mathbb{R}[\frac{d}{dt}, \delta_L, ((1 - hs_0)^M - \delta_L)^{-1}]$ -free controllable. Then, considering the discretization by $\mathbb{D}_{p_2,1,h}$, the Fitting ideal $\mathfrak{F}(\mathbb{D}_{p_2,1,h}(\Lambda))$ is generated by

$$\left(\frac{(1 - \delta_h)(3 - \delta_h)}{2h} - s_0, (1 - hs_0)^M - \delta_h^M \right) = \left(\frac{\delta_h^2 - 4\delta_h + 3 - 2hs_0}{2h}, (1 - hs_0)^M - \delta_h^M \right)$$

and the generators of $\mathfrak{F}(\mathbb{D}_{p_2,1,h}(\Lambda))$ have no common factor when we exclude the factors of $(1 - hs_0)^M - \delta_h^M$, it follows that $\mathbb{D}_{p_2,1,h}(\Lambda)$ is $\mathbb{R}[\delta_h, ((1 - hs_0)^M - \delta_h^M)^{-1}]$ -controllable.

4. Illustrative example: a single population Wilson-Cowan system

We give next a simple example from the field of neurosciences, illustrating some of the previous analysis.

The so-called Wilson-Cowan model is a macroscopic neural model whose equations involve the proportion of cells in a population that are active per unit time. The effective behavior of the population relies on interactions between excitatory and inhibitory cells, where $\nu_e(t)$ and $\nu_i(t)$ are the proportion of excitatory and inhibitory cells firing per unit time at instant t . Thus, $\nu_{i,e}(t) = 0$ corresponds to a low-activity resting state. Excitatory (resp., inhibitory) neurons make their neighbors more (resp., less) likely to become active, and activation is a nonlinear, typically sigmoidal function S_e (resp., S_i) of the presently active proportion of cells. These assumptions yield the following (nonlinear) model:

$$\tau_e \dot{\nu}_e(t) = -\nu_e(t) + S_e(w_{ee}\nu_e(t) - w_{ie}\nu_i(t - L) + u(t)) \quad (56a)$$

$$\tau_i \dot{\nu}_i(t) = -\nu_i(t) + S_i(w_{ei}\nu_e(t - L) - w_{ii}\nu_i(t)) \quad (56b)$$

where $w_{kl} \leq 0$, for $k, l = e, i$, describe the strength of connection from the cell type k to l , τ_k is the timescale of population k dynamics, L is the propagation delay of the electric signal from excitatory (resp., inhibitory) to inhibitory (resp., excitatory) population, and u is an external excitation signal (such as, e.g., coming from another neural population). The above nonlinear time-delay Wilson-Cowan model can be simplified by taking, for instance, its linear approximation. We consider next the following simplified linear version, adapted from [48]:

$$\tau_e \dot{\nu}_e(t) = -\nu_e(t) + w_{ee}\nu_e(t) - w_{ie}\nu_i(t - L) + u(t) \quad (57a)$$

$$\tau_i \dot{\nu}_i(t) = -\nu_i(t) + w_{ei}\nu_e(t - L) \quad (57b)$$

Firstly, notice that if $w_{ei} = 0$, then the system is not torsion-free controllable since (57b) becomes $\tau_i \dot{\nu}_i(t) = -\nu_i(t)$ and the evolution of ν_i is uncontrolled, i.e., ν_i is a torsion element.

Secondly, if $w_{ei} \neq 0$, we have:

$$\nu_e(t) = \frac{1}{w_{ei}} (\tau_i \dot{\nu}_i(t + L) + \nu_i(t + L)), \quad u(t) = \tau_e \dot{\nu}_e(t) + (1 - w_{ee})\nu_e(t) + w_{ie}\nu_i(t - L)$$

equivalently:

$$\nu_e = \frac{1}{w_{ei}} \left(\tau_i \frac{d}{dt} \delta_L^{-1} + \delta_L^{-1} \right) \nu_i, \quad u = \left(\frac{\tau_e \tau_i}{w_{ei}} \frac{d^2}{dt^2} \delta_L^{-1} + \frac{\tau_e + (1 - w_{ee}) \tau_i}{w_{ei}} \frac{d}{dt} \delta_L^{-1} + \frac{1 - w_{ee}}{w_{ei}} \delta_L^{-1} + w_{ie} \delta_L \right) \nu_i$$

Thus the delay system Λ with equations (57) is $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$ -free controllable with basis ν_i . Another view is that system (57) admits the following presentation matrix:

$$P_\Lambda = \begin{pmatrix} \tau_e \frac{d}{dt} + 1 - w_{ee} & w_{ie} \delta_L & -1 \\ -w_{ei} \delta_L & \tau_i \frac{d}{dt} + 1 & 0 \end{pmatrix}.$$

The associated 2×2 -minors (which are the generators of $\mathfrak{F}(\Lambda)$) are given by:

$$m_1 = \tau_i \frac{d}{dt} + 1, \quad m_2 = -w_{ei} \delta_L, \quad m_3 = \left(\tau_e \frac{d}{dt} + 1 - w_{ee} \right) \left(\tau_i \frac{d}{dt} + 1 \right) + w_{ei} w_{ie} \delta_L^2,$$

In the case $w_{ei} = 0$, we see that the generators of $\mathfrak{F}(\Lambda)$ have $\tau_i \frac{d}{dt} + 1$ as a common factor. Hence, Λ is not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable, which corresponds to ν_i being a torsion element satisfying $(\tau_i \frac{d}{dt} + 1) \nu_i = 0$.

If $w_{ei} \neq 0$, it is clear that the above generators of $\mathfrak{F}(\Lambda)$ have no common factor; indeed, $\tau_i \frac{d}{dt} + 1$ and δ_L are co-prime in $\mathbb{R}[\frac{d}{dt}, \delta_L]$, hence $\text{GCD}(m_2, m_3) = 1$. From the above expressions, by taking the Laplace/Z transform, we deduce that \hat{m}_i 's have $(s, z) = (-1/\tau_i, 0)$ as a common zero, which proves that Λ is indeed $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable but not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable. On the other hand, eliminating $z = 0$ (i.e., considering $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}] \otimes_{\mathbb{R}[\frac{d}{dt}, \delta_L]} \Lambda$), the \hat{m}_i 's have no common zero.

Let us now investigate the controllability of the discretized system. Assuming that $L = Mh$, $M \in \mathbb{N}$, the equations associated to $\mathbb{D}_{p,q,h}(\Lambda)$, where $\mathbb{D}_{p,q,h}$ is a given discretizer, are:

$$(\tau_e p + q - w_{ee} q) \nu_e = -w_{ie} \delta_h^M q \nu_i + q u \tag{58a}$$

$$(\tau_i p + q) \nu_i = w_{ei} \delta_h^M q \nu_e \tag{58b}$$

The associated presentation matrix is of the form:

$$P_{\mathbb{D}_{p,q,h}(\Lambda)} = \begin{pmatrix} \tau_e p + q - w_{ee} q & w_{ie} \delta_h^M q & -q \\ -w_{ei} \delta_h^M q & \tau_i p + q & 0 \end{pmatrix},$$

and the (2×2) -minors (which are the generators of $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda))$) are:

$$m_{1\mathbb{D}} = (\tau_i p + q) q, \quad m_{2\mathbb{D}} = -w_{ei} \delta_h^M q^2, \quad m_{3\mathbb{D}} = (\tau_e p + q - w_{ee} q) (\tau_i p + q) + w_{ei} w_{ie} \delta_h^{2M} q^2.$$

Similarly to system Λ , given by (57), we distinguish again the following two cases:

1) *Case $w_{ei} = 0$ (Λ not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable).* We can see that, whatever the discretization scheme we choose (i.e., whatever p and q may be in $\mathbb{R}[\delta_h]$), the generators of $\mathfrak{F}(\mathbb{D}_{p,q,h}(\Lambda))$ have $\tau_i p + q$ as a common factor. Hence, $\mathbb{D}_{p,q,h}(\Lambda)$ is not controllable, as Proposition 3.2(1) states.

2) Case $w_{ei} \neq 0$ (Λ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free controllable, not $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -free controllable, but $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$ -free). Through the discretizer $\mathbb{D}_{p,q,h}$ we obtain that:

$$\begin{aligned}\mathcal{Z}(\hat{m}_{1\mathbb{D}}) &= \mathcal{Z}(\tau_i \hat{p} + \hat{q}) \cup \mathcal{Z}(\hat{q}), & \mathcal{Z}(\hat{m}_{2\mathbb{D}}) &= \{0\} \cup \mathcal{Z}(\hat{q}), \\ \mathcal{Z}(\hat{m}_{3\mathbb{D}}) &= \mathcal{Z}((\tau_e \hat{p} + \hat{q} - w_{ee} \hat{q})(\tau_i \hat{p} + \hat{q}) + w_{ei} w_{ie} z^{2M} \hat{q}^2).\end{aligned}$$

It follows that if $z \in \mathcal{Z}(\hat{q})$, then $z \in \mathcal{Z}(\hat{m}_{1\mathbb{D}})$ and $z \in \mathcal{Z}(\hat{m}_{2\mathbb{D}})$, but $\tau_i \tau_e \hat{p}^2(z) \neq 0$ since, by hypothesis, \hat{p} and \hat{q} have no common zero, thus $z \notin \mathcal{Z}(\hat{m}_{3\mathbb{D}})$. Therefore, $z \in \mathcal{Z}(\hat{q})$ cannot be a common zero of the $\hat{m}_{i\mathbb{D}}$'s and the only possibility for $\hat{m}_{1\mathbb{D}}$ and $\hat{m}_{2\mathbb{D}}$ to have a common zero is that $\tau_i \hat{p}(0) + \hat{q}(0) = 0$ (the common zero being $z = 0$). Then, we also have $\hat{m}_{3\mathbb{D}}(0) = 0$.

Summarizing, any discretizer $\mathbb{D}_{p,q,h}$ such that $\tau_i \hat{p}(0) + \hat{q}(0) \neq 0$ will yield a controllable discretized system $\mathbb{D}_{p,q,h}(\Lambda)$ and any discretizer $\mathbb{D}_{p,q,h}$ such that $\tau_i \hat{p}(0) + \hat{q}(0) = 0$ will yield a non controllable discretized system $\mathbb{D}_{p,q,h}(\Lambda)$. Therefore, when $\tau_i \hat{p}(0) + \hat{q}(0) \neq 0$, the discretizer $\mathbb{D}_{p,q,h}$ is $\mathbb{R}[\frac{d}{dt}, \delta_L, \delta_L^{-1}]$ -free unconditionally preserving for Λ . When $\tau_i \hat{p}(0) + \hat{q}(0) = 0$, this is no longer the case. Note however that, from (58b), $\mathbb{D}_{p,q,h}(\Lambda)$ is $\delta_h q$ -free, in the sense of Proposition 2.2, with basis ν_i . Thus, in the sense of Definition 3.7, $\mathbb{D}_{p,q,h}$ is $\mathbb{R}[\frac{d}{dt}, \delta_L]$ -torsion-free preserving for Λ , conditionally to $\mathbb{R}[\delta_h, (\delta_h q)^{-1}]$.

Let us now consider the case $\tau_i \hat{p}(0) + \hat{q}(0) \neq 0$ and investigate what the basis of $\mathbb{D}_{p,q,h}(\Lambda)$ can be, using the constructive procedure of Proposition 2.8. Suppose again that $w_{ei} \neq 0$ and let, as in Proposition 2.8, r_1, r_2, r_3 in $\mathbb{R}[\delta_h]$ be such that: $r_1 m_{1\mathbb{D}} + r_2 m_{2\mathbb{D}} + r_3 m_{3\mathbb{D}} = 1$. Then, ω given by $\omega = \sum_{i=1}^3 (-1)^{(i-1)} r_i(\delta_h) w_i$, with $w_1 = \nu_e, w_2 = \nu_i, w_3 = u$, is a basis of $\mathbb{D}_{p,q,h}(\Lambda)$. Although the procedure for obtaining this basis is constructive (through the extended Euclidean algorithm yielding the r_i 's), the explicit form of the basis is not given. Let us now consider a very simple discretizer scheme, to exhibit an explicit form of such a basis. Consider the discretisation using a usual backward difference discretization scheme, i.e., approximating $\frac{d}{dt}$ by $\frac{1-\delta_h}{h}$ (with h the discretization step) and thus defining a discretizer for which $p = 1 - \delta_h$ and $q = h$. The associated discrete system is given by:

$$\begin{aligned}(\tau_e(1 - \delta_h) + h(1 - w_{ee})) \nu_e + w_{ie} h \delta_h^M \nu_i - hu &= 0 \\ -w_{ei} h \delta_h^M \nu_e + (h + \tau_i(1 - \delta_h)) \nu_i &= 0,\end{aligned}$$

and the preceding minors become:

$$m_{1\mathbb{D}} = \tau_i \frac{1 - \delta_h}{h} + 1, \quad m_{2\mathbb{D}} = -w_{ei} \delta_h^M, \quad m_{3\mathbb{D}} = \left(\tau_e \frac{1 - \delta_h}{h} + 1 - w_{ee} \right) \left(\tau_i \frac{1 - \delta_h}{h} + 1 \right) + w_{ei} w_{ie} \delta_h^{2M}.$$

It is clear that the corresponding $\hat{m}_{i\mathbb{D}}$'s have no common zeros in \mathbb{C} ; thus $\mathbb{D}_{p,q,h}(\Lambda)$ is controllable.

In this specific case, one can set $\omega = r_1 \nu_e + r_2 \nu_i$, $r_1, r_2 \in \mathbb{R}[\delta_h]$ such that in

$$\begin{pmatrix} -w_{ei} h \delta_h^M & h + \tau_i(1 - \delta_h) \\ r_1 & r_2 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_i \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \end{pmatrix} \quad (59)$$

the determinant $-w_{ei} h \delta_h^M r_1 + (h + \tau_i(1 - \delta_h)) r_2$ is equal to 1. For instance:

$$\text{if } M = 2, \text{ then } \quad r_1 = \frac{\tau_i^2}{\tau_i^2 h w_{ei} + 2\tau_i h^2 w_{ei} + h^3 w_{ei}}, \quad r_2 = \frac{\tau_i}{\tau_i^2 + 2\tau_i h + h^2} \delta_h + \frac{1}{\tau_i + h},$$

$$\begin{aligned} \text{while if } M = 3, \text{ then } \quad r_1 &= \frac{\tau_i^3}{\tau_i^3 h w_{ei} + 3\tau_i^2 h^2 w_{ei} + 3\tau_i h^3 w_{ei} + h^4 w_{ei}} \delta_h, \\ r_2 &= \frac{\tau_i^2}{\tau_i^3 + 3\tau_i^2 h + 3\tau_i h^2 + h^3} \delta_h^2 + \frac{\tau_i}{\tau_i^2 + 2\tau_i h + h^2} \delta_h + \frac{1}{\tau_i + h}. \end{aligned}$$

Indeed, one then has, from (59):

$$\begin{pmatrix} \nu_e \\ \nu_i \end{pmatrix} = \begin{pmatrix} r_2 & -h - \tau_i(1 - \delta_h) \\ -r_1 & -w_{ei} h \delta_h^M \end{pmatrix} \begin{pmatrix} 0 \\ \omega \end{pmatrix} \quad (60)$$

yielding the generating character of ω .

5. Conclusion

In this paper, we proposed a formal definition of discretizers for delay systems within an algebraic setting in which notions are intrinsic. We studied the interplay between controllability, in particular, spectral controllability, and discretization. We saw that it is possible to have a controllable (e.g., spectrally, torsion-free controllable) continuous-time delay system yielding a non controllable discretized system and to have an uncontrollable continuous-time delay system leading to a controllable discretized system. We introduced the notion of preserving discretizer yielding, for a spectrally controllable system, a controllable discretized one.

We saw that, no fixed discretizer is universally preserving in the sense that, for a fixed discretizer $\mathbb{D}_{p,q,h}$, we can always construct a torsion-free delay system Λ such that the discretized system $\mathbb{D}_{p,q,h}(\Lambda)$ is not controllable. On the contrary, we showed that, for a given torsion-free delay system, it is always possible to find a preserving discretizer. In the paper, we did not study the effect of sampled data control on a continuous system, as in, e.g., [36, 43] (see also [18, 32] for recent surveys on the subject). This can be the subject of future research.

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Appendix A. Brief recall of algebraic notions

Appendix A.1. Modules and their properties

A module has the same axioms as vector spaces, the difference lying in the fact that scalars are taken in a ring instead of a field. More specifically:

Definition Appendix A.1. *Given a ring R , an R -module M is an abelian group with an action of R , that is, a map $R \times M \rightarrow M$, written $(r, m) \mapsto rm$, satisfying for all $r, s \in R$ and $m, n \in M$:*

$$\begin{aligned}
 r(sm) &= (rs)m && \text{(associativity)} \\
 r(m+n) &= rm + rn \\
 (r+s)m &= rm + sm && \text{(distributivity)} \\
 1m &= m
 \end{aligned}$$

The fact that scalars are taken in a ring instead of in a field leads to more complex situations than for vector spaces. For instance, and contrarily to vector spaces, modules do not necessarily have bases; when a module admits a basis, then it is called *free*.

Definition Appendix A.2. *A free R -module is a module that is isomorphic to a direct sum of copies of R . In other words, a free module has a so called basis, i.e., a minimally generating and maximally independent family.*

Appendix A.2. Categories and functors

Categories are the context for discussing general properties of systems such as groups, rings, modules, sets, or topological spaces, in tandem with their respective transformations: homomorphisms, functions, or continuous maps.

Definition Appendix A.3. *A category \mathbf{C} consists of three ingredients: a class $\text{obj}(\mathbf{C})$ of objects, a set of morphisms $\text{Hom}(A, B)$ for every ordered pair (A, B) of objects, and composition $\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$, denoted by $(f, g) \mapsto fg$, where $f \in \text{Hom}(B, C)$ and $g \in \text{Hom}(A, B)$, for every ordered triple A, B, C of objects. These ingredients are subject to the following axioms:*

- (1) *the Hom sets are pairwise disjoint; that is, each $f \in \text{Hom}(A, B)$ has a unique domain A and a unique target B ;*
- (2) *for each object A , there is an identity morphism $1_A \in \text{Hom}(A, A)$ such that $f1_A = f$ and $1_B f = f$ for all $f : A \rightarrow B$;*
- (3) *composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h(gf) = (hg)f$.*

Functors are homomorphisms of categories:

Definition Appendix A.4. *If \mathbf{C} and \mathbf{D} are categories, then a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is a function such that*

- (1) *if $A \in \text{obj}(\mathbf{C})$, then $F(A) \in \text{obj}(\mathbf{D})$,*
- (2) *if $f : A \rightarrow A$ in \mathbf{C} , then $F(f) : F(A) \rightarrow F(A)$ in \mathbf{D} ,*
- (3) *if $A \xrightarrow{f} A' \xrightarrow{g} A''$ in \mathbf{C} , then*

$$F(A) \xrightarrow{F(f)} F(A') \xrightarrow{F(g)} F(A'') \quad \text{in } \mathbf{D} \text{ and}$$

$$F(gf) = F(g)F(f)$$

- (4) *$F(1_A) = 1_{F(A)}$ for every $A \in \text{obj}(\mathbf{C})$.*