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Stability of non-conservative cross diffusion model and approximation by stochastic particle systems

Vincent Bansaye ^{*} Alexandre Bertolino [†] Ayman Moussa [‡]

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Abstract

We study the stability of non-conservative deterministic cross diffusion models and prove that they are approximated by stochastic population models when the populations become locally large. In this model, the individuals of two species move, reproduce and die with rates sensitive to the local densities of the two species. Quantitative estimates are obtained and convergence is proved soon as the population per site and the number of sites go to infinity. The proofs rely on the extension of stability estimates via duality approach initiated in [4] under a smallness condition and the development of large deviation estimates for structured population models, which are of independent interest. The proofs also involve martingale estimates in H^{-1} and improve the approximation results in the conservative case as well.

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1 Introduction

In [31], Shigesada, Kawasaki and Teramoto proposed a system of equations describing the spatial distribution of interacting individuals. This system, named after their initials SKT, focuses on the population densities u and v of two species and include two type of interactions: competition for resources and mutual agitation. These phenomena translate in the model as nonlinearities. Adding a reproduction rate for each species and considering only interspecific agitation, the system takes the general form

$$\begin{cases} \partial_t u - \Delta((d_1 + a_1 v)u) = u(\rho_1 - s_{11}u - s_{12}v), \\ \partial_t v - \Delta((d_2 + a_2 u)v) = v(\rho_2 - s_{21}u - s_{22}v), \\ (u, v)(0, \cdot) = (u_0, v_0). \end{cases} \quad (1)$$

The coefficients ρ_1 and ρ_2 are the reproduction rates, while the s_{ij} coefficients are the interspecific and intraspecific competition terms for resources. In comparison with standard reaction-diffusion equations, the originality of this system stems from the cross-diffusion terms a_k which model the interspecific agitation. In the previous model all the coefficients are assumed non-negative and $d_1 d_2 \neq 0$. This last nondegeneracy condition amounts for the existence, for each species, of intrinsic agitation. This system was originally introduced in [31] because the cross-diffusion terms induce a repulsive behavior at the macroscopic level for each species, leading to the existence of (stable) segregated equilibria. The study of these asymptotic states is historically the first mathematical facet that has been explored on the SKT system, and it is still an active domain of research, see for instance [7] and the reference therein. Surprisingly enough, the Cauchy problem for this system only started with [25] and has been puzzling the mathematical community since then. While construction of global weak solutions is now well-understood ([8, 15, 27, 17]) for SKT systems and its variants, their uniqueness and link with more regular solutions for which only local statements are available [2, 10, 11, 21, 14] offer challenging questions. In comparison, the derivation of SKT-type models, that is their realization as rigorous limit of microscopic systems in an appropriate scaling limit, is by far a younger area of research as the first rigorous attempt of derivation appeared in 2015 with the program of Fontbona and Méléard [20]. We discard here the fast-reaction limit program proposed in [23] and followed for instance rigorously in [16] because this asymptotic may not be understood as a derivation from a simpler (microscopic) model. The approximation of the system by a collection of finite brownian trajectories led to several developments [9, 29] and ultimately to [17]. In an another direction, Daus, Desvillettes and Dietert proposed in [12] another derivation program which relies on a spatial discretization. In this description, followed in dimension 1, individuals move (in pairs) on a discrete grid with jumps on neighbor sites which are both random in direction and time. The frequency at which those jumps occur is directly linked to the number of agents present on the site. In [12], the link with the stochastic model was yet formal, while the motion by pair allowed reversibility properties but was not justified in terms of modeling. An hydrodynamic limit has been implemented in [4] leading to a complete derivation of the conservative SKT model in the

one dimensional case. However, [4] invited several natural generalization and in particular the case of non-conservative cross-diffusion systems, including reaction terms like in (1) and the relaxation of the scaling required for convergence of the stochastic process.

This aim of this paper is to fill this gap. We explore the rigorous derivation of non-conservative SKT models of the following form

$$\begin{cases} \partial_t u - \Delta(\mu_1(v)u) = R_1(u, v), \\ \partial_t v - \Delta(\mu_2(u)v) = R_2(u, v) \\ (u, v)(0, \cdot) = (u_0, v_0). \end{cases} \quad (2)$$

Here, R_k and μ_k are generic nonlinearities, with μ_1 and μ_2 lower bounded by a positive constant. As in [4], this SKT system will be obtained as the limit of a collection of interacting random walks, now also branching and dying. The new difficulties come from the additional (local) non-linearity due to birth and deaths. The main novelty is the control of the number of birth and deaths to get the convergence of the stochastic system and quantify it. The proofs rely on time change and comparison arguments and provide large deviations estimates which are of independent interest. A second input of this work is the extension of the stability estimates of [4], under a smallness condition, which allows both to cover reaction terms (births and deaths) and more general motion rates. Last but not least, we improve here the convergence obtained in [4]. More precisely, the convergence of the stochastic process was proven (only) when the number of individuals per site N is large enough compared to the number of sites M , namely $N \gg M^2 \gg 1$. Indeed, the normalisation M^2/N appeared naturally in quadratic variations of martingales when accelerating time by factor M^2 and renormalizing population size by N . Using here estimates of the jumps of martingales in H^{-1} , we refine the bounds on martingales and only need that $N, M \rightarrow \infty$. This yields the minimal setting expected for our convergence and more generally for local approximation of the stochastic process and its deterministic limit. Inspiration here came from the works of Blount [5, 6] who considered one single species and linear term for the motion. Our result actually also complement the result of Blount [6] by relaxing scaling conditions on N, M . Let us finally mention that we refine the distance used to measure the gap between the stochastic process and its deterministic limit, establishing pathwise and uniform-in-time convergence in the spirit of the Skorokhod topology. This strengthens the approximation result of [4], even in the conservative setting. We believe that this approach can be extended to larger dimensions and more complex spatial structure, including random graphs. But this also raises new difficulties, that will be mentioned later, and we hope to achieve it in a forthcoming work.

In Subsection 1.1 below, we collect all the notations that will be used in this work and then state our main results in Subsection 1.2. In Section 2 we state and prove crucial (deterministic) stability estimate both in continuous and semi-discrete setting, which are then used in Section 3 to establish the convergence of a stochastic process defined on a discrete grid. In Section 3, we first construct and characterize the stochastic process. We then prove martingale estimates to control the fluctuations of the stochastic process. In Subsection 3.3, we control the number of birth and deaths and the associated cumulative rates along time, but using suitable time change on jump Markov process. We can then combine these ingredients to quantify the convergence of the sequence of stochastic processes to crossed diffusion.

1.1 Notation and definitions

We fix $d \in \mathbb{N}$ an integer. Throughout this article, we denote $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ the d -dimensional torus, on which for any function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ the usual L^p norm, mean and L^2 scalar product are denoted $\|u\|_{L^p} = \left(\int_{\mathbb{T}^d} |u|^p\right)^{1/p}$, $[u] = \int_{\mathbb{T}^d} u$ and $\langle u, v \rangle_{L^2}$. We denote $(H^s(\mathbb{T}^d), \|\cdot\|_{H^s})$ the Sobolev spaces and $\langle \cdot, \cdot \rangle_{H^s}$ the associated scalar product. Their definition and properties are written in Section 4.2 of the appendix.

For a time $T > 0$, $Q_T := [0, T] \times \mathbb{T}^d$ is the closed periodic cylinder and we define the following norm:

$$\|z\|_T^2 = \|z\|_{L^\infty([0, T], H^{-1}(\mathbb{T}^d))}^2 + \|z\|_{L^2(Q_T)}^2. \quad (3)$$

For a positive integer M , \mathbb{T}_M is the discretized 1-dimensional torus:

$$\mathbb{T}_M := \{x_j \mid j \in \mathbb{N} \cap [0, M-1]\}, \quad \text{with } x_j := \frac{j}{M}.$$

For a function $\mathbf{u} \in \mathbb{R}^{\mathbb{T}_M}$, we will often denote $u_j := \mathbf{u}(x_j)$. We equip \mathbb{T}_M with the uniform measure of total mass 1 and denote by $\ell^2(\mathbb{T}_M)$ the finite-dimensional vector space of real valued functions over \mathbb{T}_M . It will often be equipped with the ℓ^2 scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{2, M} = \frac{1}{M} \sum_{j=1}^M u_j v_j,$$

sometimes also simply denoted $\langle \mathbf{u}, \mathbf{v} \rangle$. Of course \mathbf{u} and \mathbf{v} can be understood as vectors but throughout all this work, we manipulate these objects as functions and this is why we write $\mathbf{u}\mathbf{v}$ for the element of $\ell^2(\mathbb{T}_M)$ whose j -th component is $u_j v_j$. The product of functions $\mathbf{u}, \mathbf{v} \in \ell^2(\mathbb{T}_M)$ corresponds to the coordinate by coordinate product and will simply be denoted $\mathbf{u}\mathbf{v}$.

For a time $T > 0$, $Q_{T, M} := [0, T] \times \mathbb{T}_M$ is the discrete analogue of Q_T , which we equip with the product measure. For $0 \leq i \leq M$, the vector $\mathbf{e}_i \in \ell^2(\mathbb{T}_M)$ is defined by $\mathbf{e}_M = \mathbf{e}_0$ if $i = M$ and

$$\mathbf{e}_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

We denote by $\mathbf{1} \in \ell^2(\mathbb{T}_M)$ the constant function on \mathbb{T}_M with value 1. For all $\mathbf{u} \in \ell^2(\mathbb{T}_M)$ we will also use the notation $u_j := \mathbf{u}(x_j)$ with the convention $u_M = u_0$. The operator $\Delta_M : \ell^2(\mathbb{T}_M) \rightarrow \ell^2(\mathbb{T}_M)$ is then defined by

$$(\Delta_M u)_j = M^2(u_{j+1} + u_{j-1} - 2u_j).$$

We also define the discrete analogues of L^p norms by setting

$$\|\mathbf{u}\|_{p, M} = \left(\frac{1}{M} \sum_{j=1}^M |u_j|^p \right)^{1/p}, \quad \|\mathbf{u}\|_\infty = \max_{1 \leq j \leq M} |u_j|.$$

We will also denote $[\mathbf{u}]_M$ the mean of \mathbf{u} :

$$[\mathbf{u}]_M = \frac{1}{M} \sum_{j=1}^M u_j.$$

Operator Δ_M being symmetric with real coefficients and having its kernel equal to constant functions, it induces an automorphism on its range that is the space of all mean free functions. We denote Δ_M^{-1} the reciprocal automorphism. By integration by parts or Abel transform, one can see that $-\Delta_M$ is non-negative so $-\Delta_M^{-1}$ is positive definite on mean free functions and one can define an analogue of the negative Sobolev norm H^{-1} and the associated scalar product:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{-1,M} = [\mathbf{u}]_M [\mathbf{v}]_M + \langle \mathbf{u} - [\mathbf{u}]_M, -\Delta_M^{-1}(\mathbf{v} - [\mathbf{v}]_M) \rangle,$$

$$\|\mathbf{u}\|_{-1,M} = \langle \mathbf{u}, \mathbf{u} \rangle_{-1,M}^{1/2}.$$

We refer to Subsection 2.2 for details. When nothing is specified on the index, for discrete functions $\langle \cdot, \cdot \rangle$ will denote the $\langle \cdot, \cdot \rangle_{2,M}$ scalar product. For more details about Sobolev spaces and in particular H^{-1} , see Appendix 4.2. Norm $\|\cdot\|_T$ also has its discrete analogue:

$$\|\mathbf{u}\|_{T,M}^2 = \sup_{t \leq T} \|\mathbf{u}(t)\|_{-1,M}^2 + \int_0^T \|\mathbf{u}(t)\|_{2,M}^2 dt.$$

We now present interpolations/restrictions allowing transitions between the continuous and discrete world. For $f : \mathbb{T} \rightarrow \mathbb{R}$ continuous, we denote by

$$\hat{f}^M = f|_{\mathbb{T}_M} : \mathbb{T}_M \rightarrow \mathbb{R}$$

the restriction of f to \mathbb{T}_M . For $\mathbf{u} : \mathbb{T}_M \rightarrow \mathbb{R}$, the $\mathbb{T} \rightarrow \mathbb{R}$ function $\pi_M(\mathbf{u}) \in L^2(\mathbb{T})$ is the affine interpolation of \mathbf{u} :

$$\pi_M(u)(x) = \sum_{k=1}^M \theta(M(x - x_k)) u(x_k), \quad \text{where } \theta(z) = (1 - |z|)^+.$$

To alleviate notations, when no confusion is possible we omit the M exponent. For example, \hat{u}^M becomes \hat{u} .

For two quantities A and B , we will denote $A \lesssim B$ if there exists a constant c depending only on the parameters of the problem, e.g. μ_i, R_i or K_0 and the solution (u, v) defined in Hypotheses 1 below, such that $A \leq cB$.

For any real number x , we denote by $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$ respectively its positive and negative parts.

For a matrix A , A^T is the transposed matrix and for a probability event E , E^c is the complementary event. For a finite-dimensional vector space X and a time $T > 0$, $\mathbb{D}([0, T], X)$ is the Skorokhod space, i.e. the space of right-continuous functions $f : [0, T] \rightarrow X$ which have left-hand limits.

Throughout this document, finite dimensional vectors will be noted with bold characters and random variables with capital letters. leading to the following conventions:

Notation	Object
\mathbf{u}	Function on \mathbb{T}_M
u	Function on \mathbb{T}
u_j	j -th coordinate of \mathbf{u}
\mathbf{U}	Random function on \mathbb{T}_M
U	Random real number

Time-dependent quantities will be indexed by the discretization parameters M and N . We choose to omit the corresponding indices in the proofs. For example, a function $\mathbf{u}^M : \mathbb{R}_+ \rightarrow \ell^2(\mathbb{T}_M)$ and a stochastic process $H^{M,N} : \mathbb{R}_+ \rightarrow \mathbb{R}$ will be respectively denoted \mathbf{u} and H in proofs.

We will also rely on the following setting for the parabolic equations composing the system (2).

Definition 1.1. For any $s \in \mathbb{R}$, any bounded function $\mu \in L^\infty(Q_T)$, $F \in L^1([0, T]; H^s(\mathbb{T}^d))$ and any $z_0 \in L^1(\mathbb{T}^d)$ we say that $z \in L^1(Q_T)$ is a solution of the Cauchy problem

$$\begin{aligned}\partial_t z - \Delta(\mu z) &= F, \\ z(0) &= z^0,\end{aligned}$$

if, for all $\varphi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{T}^d)$ there holds

$$-\int_{Q_T} z(\partial_t \varphi + \mu \Delta \varphi) = \int_{\mathbb{T}^d} z^0 \varphi(0) + \int_{Q_T} F \varphi.$$

1.2 Main results

This work contains three main results, ranging from the stability of the continuous SKT system to the convergence of the sequence of random processes.

1.2.1 Stability estimate and semi discrete approximation for the SKT system

The first result is an extension of the stability estimate given by Theorem 1 of [4]. This extension includes reaction terms and more general interspecific agitation (diffusion). Its proof in Section 2 will also enlighten the other proofs by focusing on a key argument: the use of a duality lemma. Such approach involves the following smallness conditions

$$\|u\|_{L^\infty(Q_T)} \|v\|_{L^\infty(Q_T)} < \frac{\alpha_1 \alpha_2}{\|\mu'_1\|_{L^\infty(\mathbb{R}_+)} \|\mu'_2\|_{L^\infty(\mathbb{R}_+)}} , \quad (4)$$

where $\|\mu'_i\|_{L^\infty(\mathbb{R}_+)}$ stands for the Lipschitz constant of μ_i .

Theorem 1. Assume that μ_i, R_i are globally Lipschitz and $\inf_{\mathbb{R}_+} \mu_i = \alpha_i > 0$ for $i \in \{1, 2\}$. Then for any $T > 0$ and all non-negative solutions $(u, v), (\bar{u}, \bar{v}) \in L^\infty(Q_T)$ of (2) with respective initial conditions (u_0, v_0) and (\bar{u}_0, \bar{v}_0) , and (u, v) verifying the smallness condition (4),

$$\| \|u - \bar{u}\|_T^2 + \| \|v - \bar{v}\|_T^2 \lesssim C e^{cT} \left(\|u_0 - \bar{u}_0\|_{H^{-1}}^2 + \|v_0 - \bar{v}_0\|_{H^{-1}}^2 \right),$$

where c and the constant behind \lesssim depend only on the L^∞ norm and Lipschitz constants of μ_i and R_i and on the difference between the two members of (4).

Remark. Condition (4) is naturally linked to the so-called Petrovskii condition for quasilinear parabolic systems which asks that all eigenvalues of the diffusion matrix have positive real part (we refer to [21] for more details on this condition). For the system at stake, namely (2), the diffusive matrix is given by

$$\begin{pmatrix} \mu_1(u_2) & \mu'_1(u_2)u_1 \\ \mu'_2(u_1)u_2 & \mu_2(u_1) \end{pmatrix}.$$

Since both μ_1 and μ_2 are positively lower-bounded, Petrovskii's condition amounts to ask that the determinant of the previous matrix remains positive that is

$$\mu_1(u_2)\mu_2(u_1) - \mu_1'(u_2)\mu_2'(u_1)u_1u_2 > 0.$$

Since $\alpha_i = \inf_{\mathbb{R}_+} \mu_i$, Petrovskii's condition is directly implied by (4). Note however that Hadamard's ellipticity condition – which asks that the diffusion matrix has a positive symmetric part and is known to be too restrictive for the study of quasilinear parabolic systems – is not implied by (4) (again, we refer to [21] for a discussion about this other ellipticity condition).

As a corollary, we get a uniqueness result for the solutions of (2):

Corollary 1.1. *If μ_i is uniformly Lipschitz for $i \in \{1, 2\}$ and if there exists $T > 0$ and a solution $(u, v) \in L^\infty(Q_T)$ of (2) satisfying the smallness condition (4), then (u, v) is the unique solution in $L^\infty(Q_T)$ of (2) with the initial condition (u_0, v_0) .*

We now turn to the semidiscrete system. The second result is about the convergence of space discretizations of (2) to a solution of (2). More precisely, let us introduce a semi-discrete model, first proposed in [13]. On the discrete torus of dimension 1 \mathbb{T}_M , consider the following Cauchy problem:

$$\begin{cases} \frac{d}{dt} \mathbf{u}^M = \Delta_M(\mu_1(\mathbf{v}^M)\mathbf{u}^M) + R_1(\mathbf{u}^M, \mathbf{v}^M) \\ \frac{d}{dt} \mathbf{v}^M = \Delta_M(\mu_2(\mathbf{u}^M)\mathbf{v}^M) + R_2(\mathbf{u}^M, \mathbf{v}^M) \\ (\mathbf{u}^M(0), \mathbf{v}^M(0)) = (\mathbf{u}_0^M, \mathbf{v}_0^M) \end{cases}, \quad (5)$$

where $\mathbf{u}^M, \mathbf{v}^M \in \mathcal{C}^1(\mathbb{R}_+, \ell^2(\mathbb{T}_M))$ and $\mathbf{u}_0^M, \mathbf{v}_0^M \in \ell^2(\mathbb{T}_M)$, which is finite dimensional. The study of the stability of this system and its perturbations will be a key step between the continuous limiting system (Theorem 1) and the convergence of the sequence of stochastic processes (Theorem 3). Throughout the semi-discrete part in Section 2, we will make the following assumptions:

Hypotheses 1. (i) *We work in dimension $d = 1$.*

(ii) *For $i \in \{1, 2\}$, functions μ_i are of class \mathcal{C}^3 and globally Lipschitz and lower-bounded by an $\alpha_i > 0$. We denote L_i the Lipschitz constant of μ_i .*

(iii) *For $i \in \{1, 2\}$, functions R_i have the structure $R_1(u, v) = u\lambda_1(u, v)$, $R_2(u, v) = v\lambda_2(u, v)$ with λ_1, λ_2 globally Lipschitz and there exists $\rho_0 \in \mathbb{R}$ such that*

$$\lambda_1(u, v) + \lambda_2(u, v) \leq \rho_0, \quad \lambda_1^-(u, v)^2 + \lambda_2^-(u, v)^2 \lesssim 1 + u(1 + \lambda_1^-(u, v)) + v(1 + \lambda_2^-(u, v)),$$

where λ_i^- is the negative part of λ_i .

Working in dimension $d = 1$ will be technically convenient and particularly useful when comparing the semi-discrete system and the limiting PDE. As commented in the next section, extensions are expected in higher dimension and even for more general space structures.

Hypothesis 2. *For some $T > 0$, there exists a solution $(u, v) \in \mathcal{C}^0([0, T]; H^2(\mathbb{T})) \cap L^2([0, T]; H^3(\mathbb{T}))$ of (2) satisfying the smallness condition*

$$\|u\|_{L^\infty(Q_T)} \|v\|_{L^\infty(Q_T)} < \frac{\alpha_1\alpha_2}{\|\mu_1'\|_{L^\infty(\mathbb{R}_+)} \|\mu_2'\|_{L^\infty(\mathbb{R}_+)}}. \quad (6)$$

We note that the solution in Hypothesis 2 is unique by Corollary 1.1 and that the smallness condition makes sense in the class of functions considered since (u, v) is actually bounded thanks to the Sobolev embedding $H^2(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Moreover this condition is non-empty and solutions indeed exist. More precisely, in dimension d for $s > d/2$ the Cauchy problem for the SKT is well-posed in the $H^s(\mathbb{T})$ with local existence (for non-negative initial data) of a non-negative $\mathcal{C}^0([0, T]; H^s(\mathbb{T}^d)) \cap L^2(0, T; H^{s+1}(\mathbb{T}^d))$ solution [21]. Under a smallness assumption on the initial data, this solution is global. We refer to [21, Theorems 1, 2, 3] for details.

We wish to prove that $(\mathbf{u}^M, \mathbf{v}^M)$ converges to (u, v) . Therefore, we compare u (ou v) to the piecewise affine interpolation $\pi_M(\mathbf{u}^M) \in L^2(\mathbb{T})$ of \mathbf{u}^M . The second main result quantifies this discretization error:

Theorem 2. *Under Hypotheses 1 and 2, we also assume that the initial data \mathbf{u}_0^M and \mathbf{v}_0^M are non negative and that*

$$\sup_{M \in \mathbb{N}} \left(\|\mathbf{u}_0^M\|_{2,M} + \|\mathbf{v}_0^M\|_{2,M} \right) < \infty.$$

Then, for any $M \in \mathbb{N}$,

$$\|\|\pi_M(\mathbf{u}^M) - u\|_T^2 + \|\|\pi_M(\mathbf{v}^M) - v\|_T^2 \lesssim e^{\exp(cT)} \left(\|\pi_M(\mathbf{u}_0^M) - u_0\|_{H^{-1}(\mathbb{T})}^2 + \|\pi_M(\mathbf{v}_0^M) - v_0\|_{H^{-1}(\mathbb{T})}^2 + \delta_M \right),$$

where $\delta_M \rightarrow 0$ as $M \rightarrow \infty$ and c is a constant. Moreover $(\delta_M)_M$, c and the constant behind \lesssim depend only on $\sup_{M \in \mathbb{N}} \left(\|\mathbf{u}_0^M\|_{2,M} + \|\mathbf{v}_0^M\|_{2,M} \right)$, the solution (u, v) of (2) and the parameters μ_i, R_i of the problem.

Corollary 1.2. *Under the assumptions of Theorem 2, if moreover*

$$\|\|\pi_M(\mathbf{u}_0^M) - u_0\|_{H^{-1}(\mathbb{T})}^2 + \|\|\pi_M(\mathbf{v}_0^M) - v_0\|_{H^{-1}(\mathbb{T})}^2 \xrightarrow{M \rightarrow \infty} 0,$$

then $(\pi_M(\mathbf{u}^M), \pi_M(\mathbf{v}^M)) \xrightarrow{M \rightarrow \infty} (u, v)$ for the $\|\cdot\|_T$ norm.

1.2.2 Convergence of the stochastic particle systems

First, let us define the stochastic process $(\mathbf{U}^{M,N}(t), \mathbf{V}^{M,N}(t))_{t \geq 0}$ measuring the population densities along time. The number of sites is still $M \in \mathbb{N}$. We also use a scaling of local population size by factor $N \in \mathbb{N}$. Thus, $NU_j^{M,N}(t)$ (resp. $NV_j^{M,N}(t)$) is the number of individuals of the first species (resp. second species) on site $x_j = j/M$ at time t .

Informally, each individual of species 1 may move from site x_j to its left neighbor $x_j - 1/M$ and to its right neighbor $x_j + 1/M$ with the same individual rate

$$M^2 \mu_1(v_j)$$

where $v_j \in \mathbb{R}_+$ is the density of individuals of the other species on the same site and μ_1 satisfies the regularity condition of Hypothesis 1 *i*). The random walk associated with each individual is thus centered and scaling M^2 is the classical diffusive scaling.

Each individual of species 1 in each site j may also give birth on the same site or die with respective individual birth and death rate

$$b_1(u_j, v_j), \quad d_1(u_j, v_j),$$

where $(u_j, v_j) \in \mathbb{R}_+^2$ are the local densities of each species. Each individual of species 2 behaves similarly with respective individual rates $M^2\mu_2(u_j)$, $b_2(u_j, v_j)$ and $d_2(u_j, v_j)$.

More precisely, the transition rates ν of the process $(\mathbf{U}^{M,N}(t), \mathbf{V}^{M,N}(t))_{t \geq 0}$ are defined as follows for $\epsilon \in \{-1, 1\}$ and $1 \leq j \leq M$:

$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) &\longrightarrow (\mathbf{u} + \frac{1}{N}\mathbf{e}_{j+\epsilon} - \frac{1}{N}\mathbf{e}_j, \mathbf{v}) && \text{at rate } \nu_1^{M,N}(\mathbf{u}, \mathbf{v}, \mathbf{e}_{j+\epsilon} - \mathbf{e}_j) := NM^2u_j\mu_1(v_j) \\
&(\mathbf{u}, \mathbf{v} + \frac{1}{N}\mathbf{e}_{j+\epsilon} - \frac{1}{N}\mathbf{e}_j) && \text{at rate } \nu_2^{M,N}(\mathbf{u}, \mathbf{v}, \mathbf{e}_{j+\epsilon} - \mathbf{e}_j) := NM^2v_j\mu_1(u_j) \\
&(\mathbf{u} + \frac{1}{N}\mathbf{e}_j, \mathbf{v}) && \text{at rate } \nu_1^{M,N}(\mathbf{u}, \mathbf{v}, \mathbf{e}_j) := Nu_jb_1(u_j, v_j) \\
&(\mathbf{u}, \mathbf{v} + \frac{1}{N}\mathbf{e}_j) && \text{at rate } \nu_2^{M,N}(\mathbf{u}, \mathbf{v}, \mathbf{e}_j) := Nv_jb_2(u_j, v_j) \\
&(\mathbf{u}, \mathbf{v} - \frac{1}{N}\mathbf{e}_j) && \text{at rate } \nu_2^{M,N}(\mathbf{u}, \mathbf{v}, -\mathbf{e}_j) := Nv_jd_2(u_j, v_j), \\
&(\mathbf{u} - \frac{1}{N}\mathbf{e}_j, \mathbf{v}) && \text{at rate } \nu_1^{M,N}(\mathbf{u}, \mathbf{v}, -\mathbf{e}_j) := Nu_jd_1(u_j, v_j).
\end{aligned}$$

We assume that functions μ_i and

$$\lambda_i(u, v) := b_i(u, v) - d_i(u, v)$$

satisfy Hypotheses 1 for $i \in \{1, 2\}$. We give now a trajectorial representation of the process using Poisson point processes. For convenience, we denote by

$$\mathcal{T} = \bigcup_{j=1}^M \{\mathbf{e}_j, -\mathbf{e}_j, \mathbf{e}_{j+1} - \mathbf{e}_j, \mathbf{e}_{j-1} - \mathbf{e}_j\}$$

the set of the different types of transition, respectively births, deaths, motions to the right and to the left. Let $\mathcal{N}_1, \mathcal{N}_2$ two independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{T}$ of intensity $dt \otimes d\rho \otimes \text{Card}$ where Card is the counting measure on \mathcal{T} . We consider finite initial population densities $(\mathbf{U}^{M,N}(0), \mathbf{V}^{M,N}(0))$. Then, the process $(\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ is the unique strong solution in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}_+^{2M})$ of the following SDE

$$\mathbf{U}^{M,N}(t) = \mathbf{U}_0^{M,N} + \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1^{M,N}(\mathbf{U}^{M,N}(s^-), \mathbf{V}^{M,N}(s^-), \theta)} \theta \mathcal{N}_1(ds, d\rho, d\theta) \quad (7)$$

$$\mathbf{V}^{M,N}(t) = \mathbf{V}_0^{M,N} + \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_2^{M,N}(\mathbf{U}^{M,N}(s^-), \mathbf{V}^{M,N}(s^-), \theta)} \theta \mathcal{N}_2(ds, d\rho, d\theta). \quad (8)$$

We construct rigorously $(\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ and justify this result in Subsection 3.1 using classical localisation arguments and moments estimates on the total population sizes. This result is obtained under Hypothesis 1 for coefficients and L^1 moment condition for the initial value. This covers our framework, see forthcoming (23). We will observe that construction and pathwise uniqueness, moment conditions can be relaxed by a truncation procedure. Actually, for the proof of the convergence we need higher moment assumptions than bounded first moments and will assume :

Hypotheses 3. *i) There exist $p_0 > 3$ and $K_0 > 0$ such that for all $K \geq K_0$ and $M, N \geq 1$,*

$$\mathbb{P} \left([\mathbf{U}_0^{M,N}]_M + [\mathbf{V}_0^{M,N}]_M \geq K \right) \lesssim \frac{1}{(MK)^{p_0}}.$$

ii) For $i \in \{1, 2\}$, functions b_i, d_i are globally Lipschitz and there exists $\rho_0 \geq 0$ and $\alpha < 1$ such that for any $(u, v) \in \mathbb{R}_+^2$,

$$b_i(u, v) \leq \rho_0 + \alpha d_i(u, v), \quad d_1(u, v)^2 + d_2(u, v)^2 \lesssim 1 + u(1 + d_1(u, v)) + v(1 + d_2(u, v)).$$

Hypothesis 3 *i)* concerns the tail distribution of initial conditions and implies that the p -order moments of $[\mathbf{U}_0^{M,N}]_M$ and $[\mathbf{V}_0^{M,N}]_M$ are bounded uniformly in M, N for all $p < p_0$. Let us give two simple and relevant examples, for which the hypothesis is satisfied for any $p_0 > 3$. This holds when the initial population densities $\mathbf{U}_0^{M,N}(x_j)$ and $\mathbf{V}_0^{M,N}(x_j)$ are deterministic and equal to $u_0^N(x_j)$ and $v_0^N(x_j)$, where u_0^N, v_0^N are bounded. The tail condition is also satisfied when the initial population sizes $N\mathbf{U}_0^{M,N}(x_j)$ and $N\mathbf{V}_0^{M,N}(x_j)$ have a Poisson distribution of respective parameters $Nu_0^N(x_j)$ and $Nv_0^N(x_j)$, where u_0^N, v_0^N are (still) bounded.

Hypothesis 3 *ii)* concerns birth and death rates. It is satisfied for classical reactions terms of the form

$$b_1(u, v) = b_1, \quad b_2(u, v) = b_2, \quad d_1(u, v) = d_1 + c_{11}u + c_{12}v, \quad d_2(u, v) = d_2 + c_{21}u + c_{22}v,$$

where $d_i, c_{i,j}$ ($i, j \in \{1, 2\}$) are non negative coefficients. This contains the competitive Lotka Volterra model (logistic reaction, when $c_{ij} > 0$) and branching processes (linear reaction term, when $c_{ij} = 0$) and the conservative case (no reaction term, when all coefficients are zero). It also includes the following family of birth and death rates : $d_1(u, v) \sim u^\alpha + v^\beta$, $d_2(u, v) \sim u^\gamma + v^\delta$ with $\alpha, \delta \geq 1$, $2\beta \leq \delta + 1$ and $2\gamma \leq \alpha + 1$.

Theorem 3. *Assume that Hypotheses 1, 2 and 3 are satisfied. Then, there exists a sequence $(\delta_M)_M$ going to 0 as $M \rightarrow \infty$ and a constant $c > 0$ such that for any $N, M \geq 1$,*

$$\mathbb{E} \left[\left\| \pi_M(\mathbf{U}^{M,N}) - u \right\|_T^2 + \left\| \pi_M(\mathbf{V}^{M,N}) - v \right\|_T^2 \right] \lesssim e^{\exp(cT)} \left(D_0^{M,N} + \delta_M + \frac{T}{\sqrt{N}} \right),$$

where

$$D_0^{M,N} = \mathbb{E} \left[\left\| \pi_M(\mathbf{U}_0^{M,N}) - u_0 \right\|_{H^{-1}(\mathbb{T})}^2 + \left\| \pi_M(\mathbf{V}_0^{M,N}) - v_0 \right\|_{H^{-1}(\mathbb{T})}^2 \right]. \quad (9)$$

This result is stated here for the second moment of our norm $\| \cdot \|$. It will be proved more generally for moments $p \geq 1$. The constant c and the constant behind \lesssim in this estimate do not depend on T and depend on the initial conditions $(\mathbf{U}_0^{M,N}, \mathbf{V}_0^{M,N})$ only through the values of K_0, γ_0 in Hypothesis (3) *i)*. We immediately derive from this estimate the following convergence of the stochastic model to the crossed diffusion by letting M and N go to infinity.

Corollary 1.3. *Assume that Hypotheses 1, 2 and 3 are satisfied and that $D_0^{M,N}$ defined in (9) goes to zero as (M, N) goes to (∞, ∞) . Then the following convergence holds:*

$$\mathbb{E} \left[\left\| \pi_M(\mathbf{U}^{M,N(M)}) - u \right\|_T^2 + \left\| \pi_M(\mathbf{V}^{M,N(M)}) - v \right\|_T^2 \right] \xrightarrow{M, N \rightarrow \infty} 0.$$

Let us explain the main ingredients for this convergence and a brief outline of the proof sections. Our approach is perturbative and relies on stability estimate of the limiting system. Roughly, the semi-martingale decomposition of the stochastic process allows to see it as a deterministic semi-discrete flow with a random source given by a martingale, which is an integral with respect to compensated Poisson measure. The stability result obtained through duality estimates in Section 2 (see Proposition 2.10) allows to prove strong estimates of the form

$$\|\pi_M(\mathbf{U}^{M,N}) - u\|_T^2 \lesssim e^{c \int_{Q_T, M} 1 + (\mathbf{U}^{M,N})^2 + (\mathbf{V}^{M,N})^2} \times \mathcal{R}^{M,N}(T) \quad \text{a.s.}$$

The process $\mathcal{R}^{M,N}$ goes to 0 and quantifies the approximation. It involves the control of martingales for a suitable H^{-1} norm. The term $\int 1 + (\mathbf{U}^{M,N})^2 + (\mathbf{V}^{M,N})^2$ is inherited from births and deaths via Grönwall inequality. We need fine estimates to prove that this term is not large and does not degrade the approximation and the convergence to 0 coming from $\mathcal{R}^{M,N}$. These estimates are obtained through large deviation type techniques for multi-type birth and death processes and are of independent interest.

2 Stability of the continuous and semi-discrete systems

2.1 Duality lemma and proof of Theorem 1

To study (2), it will be useful to look at the following Kolmogorov equation:

$$\begin{cases} \partial_t z - \Delta(\mu z) = \Delta f + r \\ z(0, \cdot) = z_0 \end{cases} \quad (10)$$

To bound the solution of this equation in our context, we rely on a duality lemma. This idea goes back to the works of [28, 30] on reaction diffusion systems. The lemma we propose to prove is to assimilate with the more recent duality lemma presented in [4]. This estimate allows for a precise treatment of L^2 source terms, which will be useful to bound the reaction part in (2).

Lemma 2.1. *If $\mu \in L^\infty(Q_T)$ is lower-bounded by $\alpha > 0$, then for all $f, r \in L^2(Q_T)$ and $z_0 \in H^{-1}(\mathbb{T}^d)$, there exists a unique weak solution z for (10). Moreover, $z \in \mathcal{C}([0, T], H^{-1}(\mathbb{T}^d))$ and satisfies*

$$\begin{aligned} & \|z(T)\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_{Q_T} \mu z^2 \\ & \leq \|z_0\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_0^T [\mu(s)][z(s)]^2 ds + 2 \int_0^T \|r(s)\|_{H^{-1}(\mathbb{T}^d)} \|z(s)\|_{H^{-1}(\mathbb{T}^d)} ds + \frac{1}{\alpha} \|f\|_{L^2(Q_T)}^2. \end{aligned}$$

Proof. When r is mean free, existence and uniqueness come from Lemma 1 of [4], because r can be integrated to Δf thanks to Proposition 4.2 of the appendix. If r has non-zero mean $[r]$, one can consider the solution w to the Cauchy problem

$$\begin{cases} \partial_t w - \Delta(\mu w) = \Delta f + r - [r] \\ w(0, \cdot) = z_0 \end{cases} \quad (11)$$

Then, $z(t) := w(t) + \int_0^t [r(s)] ds$ is the unique solution of (10). We now prove the duality estimate stated in the lemma. We only need to prove it for the case where μ, f and r are smooth. In this

case, it follows from classical parabolic theory (see [19], Section 7.1.3, Theorem 7) that z is smooth too. Set $\tilde{z}(t) := z(t) - [z(t)]$. Let, by Proposition 4.2 of the appendix, ϕ such that for all t , $\phi(t)$ is the unique mean free solution of the elliptic equation $\Delta\phi(t) = \tilde{z}(t)$ on \mathbb{T}^d . We apply the weak formulation of (10) against ϕ :

$$-\int_{Q_T} z \partial_t \phi - \int_{Q_T} z \mu \Delta \phi + \int_{\mathbb{T}^d} z(T) \phi(T) = \int_{Q_T} (f \Delta \phi + r \phi) + \int_{\mathbb{T}^d} z_0 \phi(0).$$

Replacing z with $[z] + \Delta\phi$, except in $\int_{Q_T} z \mu \Delta \phi$, and remarking that the integral of $[z]$ against a mean free function is zero, we get

$$-\int_{Q_T} \Delta \phi \partial_t \phi - \int_{Q_T} z \mu \tilde{z} + \int_{\mathbb{T}^d} \Delta \phi(T) \phi(T) = \int_{Q_T} (f \Delta \phi + r \phi) + \int_{\mathbb{T}^d} \Delta \phi(0) \phi(0).$$

Integrating by parts and noticing $\|\nabla \phi(t)\|_{L^2(\mathbb{T}^d)} = \|\tilde{z}(t)\|_{H^{-1}(\mathbb{T}^d)}$, we then have

$$\int_0^T \frac{1}{2} \frac{d}{dt} \|\tilde{z}(t)\|_{H^{-1}(\mathbb{T}^d)}^2 - \int_{Q_T} z \mu \tilde{z} - \|\tilde{z}(T)\|_{H^{-1}(\mathbb{T}^d)}^2 = \int_{Q_T} (f \Delta \phi + r \phi) - \|\tilde{z}(0)\|_{H^{-1}(\mathbb{T}^d)}^2.$$

Integrating the first term and multiplying by -2 on both sides yield

$$\|\tilde{z}(T)\|_{H^{-1}(\mathbb{T}^d)}^2 + 2 \int_{Q_T} z \mu \tilde{z} = \|\tilde{z}_0\|_{H^{-1}(\mathbb{T}^d)}^2 - 2 \int_{Q_T} (f \Delta \phi + r \phi).$$

We now add $2 \int_{Q_T} \mu z [z]$ on both sides to complete $2 \int_{Q_T} \mu z \tilde{z}$ in $2 \int_{Q_T} \mu z^2$ and apply Young's inequality to the f term and get, recalling that $\Delta\phi = \tilde{z}$,

$$\begin{aligned} \|\tilde{z}(T)\|_{H^{-1}(\mathbb{T}^d)}^2 + 2 \int_{Q_T} \mu z^2 &\leq \|\tilde{z}_0\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_{Q_T} \left(\frac{f^2}{\mu} + \mu \tilde{z}^2 \right) \\ &\quad + 2 \int_0^T \langle r(s), -\Delta^{-1} \tilde{z}(s) \rangle_{L^2(\mathbb{T}^d)} ds + 2 \int_{Q_T} \mu z [z], \end{aligned}$$

with

$$2 \int_{Q_T} \mu z [z] + \int_{Q_T} \mu \tilde{z}^2 = 2 \int_{Q_T} \mu z [z] + \int_{Q_T} \mu (z - [z])^2 = \int_{Q_T} \mu [z]^2 + \int_{Q_T} \mu z^2,$$

so

$$\|\tilde{z}(T)\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_{Q_T} \mu z^2 \leq \int_{Q_T} \mu [z]^2 + \|\tilde{z}_0\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_{Q_T} \frac{f^2}{\mu} + 2 \int_0^T \langle r(s), \tilde{z}(s) \rangle_{H^{-1}(\mathbb{T}^d)}. \quad (12)$$

To complete $\|\tilde{z}(T)\|_{H^{-1}(\mathbb{T}^d)}^2$ into $\|z(T)\|_{H^{-1}(\mathbb{T}^d)}^2$, there remains to compute $[z(T)]^2$. Integrating (10) in space one gets

$$\partial_t [z] = [r].$$

This implies

$$\partial_t ([z]^2) = 2[z] \partial_t [z] = 2[z][r],$$

and then

$$[z(t)]^2 = [z_0]^2 + 2 \int_0^T [z(s)][r(s)] \, ds. \quad (13)$$

By Pythagorean theorem,

$$\|\tilde{z}(T)\|_{H^{-1}(\mathbb{T}^d)}^2 + [z(T)]^2 = \|z(T)\|_{H^{-1}(\mathbb{T}^d)}^2 \quad \text{and} \quad \|\tilde{z}_0\|_{H^{-1}(\mathbb{T}^d)}^2 + [z_0]^2 = \|z_0\|_{H^{-1}(\mathbb{T}^d)}^2.$$

Moreover, by orthogonality of mean-free functions with constants in $H^{-1}(\mathbb{T}^d)$,

$$\begin{aligned} \langle r(s), \tilde{z}(s) \rangle_{H^{-1}(\mathbb{T}^d)} + [r(s)][z(s)] &= \langle r(s), \tilde{z}(s) \rangle_{H^{-1}(\mathbb{T}^d)} + \langle [r(s)], [z(s)] \rangle_{H^{-1}(\mathbb{T}^d)} \\ &= \langle r(s), \tilde{z}(s) \rangle_{H^{-1}(\mathbb{T}^d)} + \langle r(s), [z(s)] \rangle_{H^{-1}(\mathbb{T}^d)} = \langle r(s), z(s) \rangle_{H^{-1}(\mathbb{T}^d)}, \end{aligned}$$

so adding (13) with $t = T$ to (12) we get

$$\|z(T)\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_{Q_T} \mu z^2 \leq \int_{Q_T} \mu [z]^2 + \|z_0\|_{H^{-1}(\mathbb{T}^d)}^2 + \int_{Q_T} \frac{f^2}{\mu} + 2 \int_0^T \langle r(s), z(s) \rangle_{H^{-1}(\mathbb{T}^d)} \, ds,$$

which, using Cauchy-Schwarz inequality and $\mu \geq \alpha$, gives the announced estimate. \square

Proof of Theorem 1. Set $z = u - \bar{u}$, $w = v - \bar{v}$, $L_i = \|\mu'_i\|_{L^\infty(\mathbb{R}_+)}$. The strategy is to use the previous lemma to then apply Grönwall's lemma to z and w . In order to get closer from the structure of equation (10), we remark that z satisfies

$$\partial_t z - \Delta(\mu_1(\bar{v})z) = \Delta((\mu_1(v) - \mu_1(\bar{v}))u) + R_1(u, v) - R_1(\bar{u}, \bar{v}).$$

Setting $f = (\mu_1(v) - \mu_1(\bar{v}))u$, $\mu = \mu_1(v)$ and $r = R_1(u, v) - R_1(\bar{u}, \bar{v})$, Lemma 2.1 applies and using $\alpha_1 \leq \mu_1$ on its left-hand side we get

$$\begin{aligned} \|z(T)\|_{H^{-1}}^2 + \alpha_1 \int_{Q_T} z^2 & \leq \|z_0\|_{H^{-1}}^2 + \int_0^T [\mu_1(\bar{v}(s))][z(s)]^2 \, ds + 2 \int_0^T \|r(s)\|_{H^{-1}} \|z(s)\|_{H^{-1}} \, ds + \frac{1}{\alpha_1} \|f\|_{L^2(Q_T)}^2, \end{aligned} \quad (14)$$

where H^{-1} stands for $H^{-1}(\mathbb{T}^d)$. Let us analyse the right-hand side of this inequality, starting with $r(s)$. Writing C_1 is the Lipschitz constant of R_1 , $|r(s, x)| \leq C_1(|z(s, x)| + |w(s, x)|)$ and

$$\|r(s)\|_{H^{-1}} \leq \|r(s)\|_{L^2} \leq C_1(\|z(s)\|_{L^2} + \|w(s)\|_{L^2}).$$

For the f term, we use Hölder's inequality and the Lipschitz constant L_1 of μ_1 :

$$\|f\|_{L^2(Q_T)}^2 \leq L_1^2 \int_{Q_T} w^2 u^2 \leq L_1^2 \|u\|_{L^\infty(Q_T)}^2 \int_{Q_T} w^2.$$

Besides $[\mu_1(\bar{v}(s))]$ is upper-bounded by $\|\mu_1\|_{L^\infty(\mathbb{R}_+)}$. Using these estimates, (14) becomes

$$\begin{aligned} \|z(T)\|_{H^{-1}}^2 + \alpha_1 \int_{Q_T} z^2 & \leq \|z_0\|_{H^{-1}}^2 + \|\mu_1\|_{L^\infty(\mathbb{R}_+)} \int_0^T [z(s)]^2 \, ds \\ & \quad + 2C_1 \int_0^T (\|z(s)\|_{L^2} + \|w(s)\|_{L^2}) \|z(s)\|_{H^{-1}} \, ds + \frac{L_1^2}{\alpha_1} \|u\|_{L^\infty(Q_T)}^2 \int_{Q_T} w^2. \end{aligned} \quad (15)$$

We then use Young's inequality $ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$, with ε to be determined later, and get:

$$\begin{aligned} \int_0^T (\|z(s)\|_{L^2} + \|w(s)\|_{L^2}) \|z(s)\|_{H^{-1}} \, ds &\leq \frac{\varepsilon}{2} \int_0^T (\|z(s)\|_{L^2} + \|w(s)\|_{L^2})^2 \, ds + \frac{1}{2\varepsilon} \int_0^T \|z(s)\|_{H^{-1}}^2 \, ds \\ &\leq \varepsilon \left(\int_{Q_T} w^2 + \int_{Q_T} z^2 \right) + \frac{1}{2\varepsilon} \int_0^T \|z(s)\|_{H^{-1}}^2 \, ds. \end{aligned}$$

Dividing the estimate (15) by α_1 , we then get

$$\begin{aligned} \frac{1}{\alpha_1} \|z(T)\|_{H^{-1}}^2 + \int_{Q_T} z^2 &\leq \frac{1}{\alpha_1} \|z_0\|_{H^{-1}}^2 + \frac{\|\mu_1\|_{L^\infty(\mathbb{R}_+)}}{\alpha_1} \int_0^T [z(s)]^2 \, ds + \frac{1}{\alpha_1 \varepsilon} \int_0^T \|z(s)\|_{H^{-1}}^2 \, ds \\ &\quad + \frac{2\varepsilon}{\alpha_1} \left(\int_{Q_T} w^2 + \int_{Q_T} z^2 \right) + \frac{L_1^2}{\alpha_1^2} \|u\|_{L^\infty(Q_T)}^2 \int_{Q_T} w^2. \end{aligned} \quad (16)$$

Passing the $\int_{Q_T} z^2$ term to the left-hand side, then using $[z] \lesssim \|z\|_{H^{-1}}$ and setting $C_1^\varepsilon = (\|\mu_1\|_{L^\infty(\mathbb{R}_+)} + 1/\varepsilon)/\alpha_1$, we obtain

$$\left(1 - \frac{2\varepsilon}{\alpha_1}\right) \int_{Q_T} z^2 \leq \frac{1}{\alpha_1} \|z_0\|_{H^{-1}}^2 + C_1^\varepsilon \int_0^T \|z(s)\|_{H^{-1}}^2 \, ds + \left(\frac{L_1^2}{\alpha_1^2} \|u\|_{L^\infty(Q_T)}^2 + \frac{2\varepsilon}{\alpha_1}\right) \int_{Q_T} w^2.$$

We also have a similar inequality on w :

$$\left(1 - \frac{2\varepsilon}{\alpha_2}\right) \int_{Q_T} w^2 \leq \frac{1}{\alpha_1} \|w_0\|_{H^{-1}}^2 + C_2^\varepsilon \int_0^T \|w(s)\|_{H^{-1}}^2 \, ds + \left(\frac{L_2^2}{\alpha_2^2} \|v\|_{L^\infty(Q_T)}^2 + \frac{2\varepsilon}{\alpha_2}\right) \int_{Q_T} z^2,$$

in which one can inject the one on $\int_{Q_T} z^2$:

$$\begin{aligned} \left(1 - \frac{2\varepsilon}{\alpha_2}\right) \int_{Q_T} w^2 &\leq \frac{1}{\alpha_1} \|w_0\|_{H^{-1}}^2 + C_2^\varepsilon \int_0^T \|w(s)\|_{H^{-1}}^2 \, ds \\ &\quad + \frac{1}{\alpha_1 - 2\varepsilon} \|z_0\|_{H^{-1}}^2 + \frac{C_1^\varepsilon}{1 - 2\varepsilon/\alpha_1} \int_0^T \|z(s)\|_{H^{-1}}^2 \, ds \\ &\quad + \left(\frac{L_2^2}{\alpha_2^2} \|v\|_{L^\infty(Q_T)}^2 + \frac{2\varepsilon}{\alpha_2}\right) \frac{1}{1 - \frac{2\varepsilon}{\alpha_1}} \left(\frac{L_1^2}{\alpha_1^2} \|u\|_{L^\infty(Q_T)}^2 + \frac{2\varepsilon}{\alpha_1}\right) \int_{Q_T} z^2. \end{aligned}$$

The smallness condition then ensures the existence of $\varepsilon > 0$ such that

$$\left(\frac{L_2^2}{\alpha_2^2} \|v\|_{L^\infty(Q_T)}^2 + \frac{2\varepsilon}{\alpha_2}\right) \frac{1}{1 - \frac{2\varepsilon}{\alpha_1}} \left(\frac{L_1^2}{\alpha_1^2} \|u\|_{L^\infty(Q_T)}^2 + \frac{2\varepsilon}{\alpha_1}\right) < 1 - \frac{2\varepsilon}{\alpha_1}.$$

This allows to absorb the $\int_{Q_T} w^2$ term of the left-hand side and get:

$$\int_{Q_T} w^2 \lesssim \|w_0\|_{H^{-1}}^2 + \int_0^T \|w(s)\|_{H^{-1}}^2 \, ds + C \|z_0\|_{H^{-1}}^2 + \int_0^T \|z(s)\|_{H^{-1}}^2 \, ds,$$

where ε and thus the constant in this inequality depend only on the difference between the two members of (4).

One can also get a similar estimate on $\int_{Q_T} z^2$. Back to the inequality (16) with $\|z(T)\|_{H^{-1}}^2$, we inject the previous bound on $\|w\|_{L^2(Q_T)}^2$ to see

$$\|z(T)\|_{H^{-1}}^2 + \|z\|_{L^2(Q_T)}^2 \lesssim \|w_0\|_{H^{-1}}^2 + \|z_0\|_{H^{-1}}^2 + \int_0^T \|w(s)\|_{H^{-1}}^2 ds + \int_0^T \|z(s)\|_{H^{-1}}^2 ds.$$

Summing this inequality with its analogue for w and setting

$$\psi(t) = \|z(t)\|_{H^{-1}}^2 + \|w(t)\|_{H^{-1}}^2, \quad \theta(t) = \|z\|_{L^2(Q_t)}^2 + \|w\|_{L^2(Q_t)}^2,$$

we get

$$\psi(T) + \theta(T) \lesssim \|w_0\|_{H^{-1}}^2 + \|z_0\|_{H^{-1}}^2 + \int_0^T \psi(s) ds.$$

Grönwall's lemma then gives the wanted result. \square

2.2 Space $\ell^2(\mathbb{T}_M)$ and discretization of the continuous solution

Let $M \in \mathbb{N}^*$. In this subsection, we restrict ourselves to dimension $d = 1$ and study the functional spaces over \mathbb{T}_M and the discrete laplacian operator Δ_M . We first recall that Δ_M is a circulant matrix so its eigenvalues are Fourier modes and we can compute explicitly its spectrum, as explained in [26] and already used in [4]. We then complement by some elementary estimates on discrete norms which will be useful to deal with the jumps of the martingales and the reaction terms.

Proposition 2.2. *The operator $-\Delta_M$ defined on the finite dimensional space $\ell^2(\mathbb{T}_M)$ is symmetric. Its eigenvalues are all contained in $\{0\} \cup [16, 4M^2]$ and 0 has multiplicity 1, the kernel of $-\Delta_M$ being composed of constant functions.*

This proposition implies that $-\Delta_M$ is a positive-definite symmetric operator on the orthogonal of constant functions, i.e. mean free functions. This allows to define $\Delta_M^{-1}\mathbf{u}$ for all mean free \mathbf{u} . We can then define the discrete H^{-1} norm

Definition 2.1. *The scalar product $\langle u, v \rangle_{-1, M}$ is defined for all $\mathbf{u}, \mathbf{v} : \mathbb{T}_M \rightarrow \mathbb{R}$ by*

$$\langle \mathbf{u}, \mathbf{v} \rangle_{-1, M} = [\mathbf{u}]_M [\mathbf{v}]_M + \langle \mathbf{u} - [\mathbf{u}]_M, -\Delta_M^{-1}(\mathbf{v} - [\mathbf{v}]_M) \rangle_{2, M}.$$

Its associated norm is defined by

$$\|\mathbf{u}\|_{-1, M}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{-1, M}.$$

The fact that $\langle \cdot, \cdot \rangle_{-1, M}$ is a scalar product follows immediately from $(-\Delta_M)^{-1}$ being a positive-definite symmetric operator on mean free functions in $\ell^2(\mathbb{T}_M)$. The spectral gap (independent on M , see Proposition 2.2) and the upper bound on the spectrum of $-\Delta_M$ imply the following comparisons between norms:

Corollary 2.3. *For $\mathbf{u} \in \ell^2(\mathbb{T}_M)$, $\|\mathbf{u}\|_{-1, M} \leq \|\mathbf{u}\|_{2, M} \lesssim M \|\mathbf{u}\|_{-1, M}$. If u is mean free, then $\|\Delta_M^{-1}\mathbf{u}\|_{2, M} \leq \|\mathbf{u}\|_{2, M}$.*

The following Proposition is a discrete version of the Sobolev embedding $L^1(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$, obtained usually as the dual embedding of $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. It is only true in dimension 1 and is the main use of one-dimensionality in this paper, specially involved in the comparison between the semi-discrete system and the PDE.

Proposition 2.4. For $\mathbf{u} \in \ell^2(\mathbb{T}_M)$, $\|\mathbf{u}\|_{-1,M} \lesssim \|\mathbf{u}\|_{1,M}$.

Proof. Denote $\tilde{\mathbf{u}} := \mathbf{u} - [\mathbf{u}]_M$. We first prove that $\|\Delta_M^{-1}\tilde{\mathbf{u}}\|_\infty \lesssim \|\mathbf{u}\|_{1,M}$. Set

$$\Phi_k = \frac{1}{M^2} \sum_{j=1}^k \sum_{i=1}^{j-1} \tilde{u}_i.$$

One can check that $(\Phi_k)_{k \in \mathbb{N}}$ is M -periodic and thus defines an element Φ of $\ell^2(\mathbb{T}_M)$. Moreover, $(\Delta_M \Phi)_k = \tilde{u}_k$ so $\Phi - [\Phi]_M = \Delta_M^{-1}\tilde{\mathbf{u}}$, hence

$$\|\Delta_M^{-1}\tilde{\mathbf{u}}\|_\infty \leq 2 \|\Phi\|_\infty \leq \frac{1}{M^2} \sum_{j=1}^M \sum_{i=1}^M |\tilde{u}_i| = \|\tilde{\mathbf{u}}\|_{1,M} \leq 2 \|\mathbf{u}\|_{1,M},$$

which proves that $\|\Delta_M^{-1}\tilde{\mathbf{u}}\|_\infty \lesssim \|\mathbf{u}\|_{1,M}$. Then,

$$\|\mathbf{u}\|_{-1,M}^2 = \langle \tilde{\mathbf{u}}, -\Delta_M^{-1}\tilde{\mathbf{u}} \rangle_{2,M} + [\mathbf{u}]_M^2 \leq \|\tilde{\mathbf{u}}\|_{1,M} \|\Delta_M^{-1}\tilde{\mathbf{u}}\|_\infty + \|\mathbf{u}\|_{1,M}^2 \lesssim \|\mathbf{u}\|_{1,M}^2,$$

which ends the proof. \square

Corollary 2.5. Let $\mathbf{u} : [0, T] \rightarrow \ell^2(\mathbb{T}_M)$. Then,

$$\|\mathbf{u}\|_{T,M} \lesssim \sqrt{1 + TM} \sup_{s \leq T} \|\mathbf{u}(s)\|_{1,M}.$$

Proof. First, as a consequence of Proposition 2.4 we have

$$\sup_{s \leq T} \|\mathbf{u}(s)\|_{-1,M}^2 \lesssim \sup_{s \leq T} \|\mathbf{u}(s)\|_{1,M}^2.$$

Then, bounding $|u_j(s)|$ by $\sum_{j=1}^M |u_j(s)| = M \|\mathbf{u}(s)\|_{1,M}$, we get

$$\begin{aligned} \int_{Q_{T,M}} \mathbf{u}^2 &= \frac{1}{M} \int_0^T \sum_{j=1}^M |u_j(s)|^2 ds \\ &\leq \int_0^T \sum_{j=1}^M |u_j(s)| \|\mathbf{u}(s)\|_{1,M} ds = \int_0^T M \|\mathbf{u}(s)\|_{1,M}^2 ds \leq TM \sup_{s \leq T} \|\mathbf{u}(s)\|_{1,M}^2. \end{aligned}$$

Assembling these two bounds yields the result. \square

Proposition 2.6. Denoting by $(\mathbf{e}_i)_{1 \leq i \leq M}$ the canonical basis of $\mathbb{R}^{\mathbb{T}_M}$, the following estimates hold for all $M \in \mathbb{N}^*$,

$$\|\mathbf{e}_i\|_{-1,M}^2 \lesssim \frac{1}{M} \quad ; \quad \|\mathbf{e}_{i+1} - \mathbf{e}_i\|_{-1,M}^2 \lesssim \frac{1}{M^3}.$$

Proof. The first one is a direct consequence of Corollary 2.3 and the fact that $\|\mathbf{e}_i\|_{2,M}^2 = \frac{1}{M}$. For the second one, note that by translation invariance of the $\|\cdot\|_{-1,M}$ norm, $\|\mathbf{e}_{i+1} - \mathbf{e}_i\|_{-1,M} = \|\mathbf{e}_1 - \mathbf{e}_0\|_{-1,M}$. Moreover, $\mathbf{e}_1 - \mathbf{e}_0$ is mean free and

$$\Delta_M^{-1}(\mathbf{e}_1 - \mathbf{e}_0) = \Phi, \quad \text{where } \Phi_j = \frac{M+1-2j}{2M^3} \quad (1 \leq j \leq M).$$

Then

$$\|\mathbf{e}_1 - \mathbf{e}_0\|_{-1,M}^2 = \langle \mathbf{e}_1 - \mathbf{e}_0, -\Delta_M^{-1}(\mathbf{e}_1 - \mathbf{e}_0) \rangle_{2,M} = \frac{1}{M} \left(\frac{M+1-2}{2M^3} - \frac{M+1-2M}{2M^3} \right) \lesssim \frac{1}{M^3},$$

which ends the proof. \square

We observe that this computation shows the order of magnitude of respectively birth/death and motion to neighborhood for the M - discretized H^{-1} norm. In particular, the normalization M^3 of motion event will be able to absorb the acceleration of time M^2 involved in the convergence of random walk to brownian motion. It will be useful in following martingale estimate to obtain convergence of the particle systems soon as N and M go to infinity.

We assume the existence of a bounded non-negative solution $(u, v) \in (L^2([0, T]; H^3(\mathbb{T})) \cap L^\infty(Q_T))^2$ to (2) satisfying the smallness condition (6). Thanks to Corollary 1.1, such a solution is unique. Let us first precise the regularity of $\mu_1(u), \mu_2(v)$:

Lemma 2.7. *If $u \in L^2([0, T]; H^3(\mathbb{T})) \cap L^\infty(Q_T)$ and $\mu \in C^3(\mathbb{R})$, then $\mu(u) \in L^2([0, T]; H^3(\mathbb{T}))$.*

Proof. Applying Theorem 2.87 from [22]:

$$\|\mu(u(t))\|_{H^3(\mathbb{T})} \leq \phi_\mu(\|u\|_{L^\infty(Q_T)}) \|u(t)\|_{H^3(\mathbb{T})}.$$

Thus, integrating in time, $\|\mu(u)\|_{L^2([0, T]; H^3(\mathbb{T}))} \leq \phi_\mu(\|u\|_{L^\infty(Q_T)}) \|u\|_{L^2([0, T]; H^3(\mathbb{T}))} < \infty$. \square

In order to compare (u, v) with $(\mathbf{u}^M, \mathbf{v}^M)$, we begin by putting these two objects in the same space by discretizing (u, v) . We thus discretize (u, v) by restricting it to \mathbb{T}_M and therefore set, for all $t \in [0, T]$,

$$\widehat{u}^M(t) := u|_{\mathbb{T}_M}(t).$$

Proposition 2.8. *Let (u, v) the solution introduced just above. Then*

$$\begin{cases} \frac{d}{dt} \widehat{u}^M = \Delta_M[\widehat{u}^M \mu_1(\widehat{v}^M)] + R_1(\widehat{u}^M, \widehat{v}^M) + \mathbf{r}_1^M \\ \frac{d}{dt} \widehat{v}^M = \Delta_M[\widehat{v}^M \mu_2(\widehat{u}^M)] + R_2(\widehat{u}^M, \widehat{v}^M) + \mathbf{r}_2^M \end{cases}$$

where the error term \mathbf{r} satisfies $\mathbf{r}_1^M, \mathbf{r}_2^M \xrightarrow{M \rightarrow \infty} 0$ for the $L^1([0, T]; L^\infty(\mathbb{T}_M))$ norm.

Proof. If $f \in H^3(\mathbb{T})$, then using Proposition 4.3 from the appendix we get the following convergence in $H^1(\mathbb{T})$:

$$\frac{\tau_h f + \tau_{-h} f - 2f}{h^2} \xrightarrow{h \rightarrow 0} f'' = \Delta f.$$

By Sobolev embedding, this convergence also holds in $(C(\mathbb{T}), \|\cdot\|_{L^\infty})$ so, applying the restriction application $f \mapsto \widehat{f}$;

$$\Delta_M(\widehat{f}) = M^2(\tau_{\frac{1}{M}} \widehat{f} + \tau_{-\frac{1}{M}} \widehat{f} - 2\widehat{f}) = \left(\frac{\tau_{\frac{1}{M}} f + \tau_{-\frac{1}{M}} f - 2f}{1/M^2} \right) |_{\mathbb{T}_M} \xrightarrow{M \rightarrow \infty} (\widehat{\Delta f}).$$

Since $u, \mu_1(v) \in H^3(\mathbb{T})$ and $H^3(\mathbb{T})$ is an algebra, one can apply this convergence to $u\mu_1(v) \in H^3(\mathbb{T})$ and get for almost every $t \in [0, T]$:

$$\begin{aligned} \|\mathbf{r}_1(t)\|_{L^\infty(\mathbb{T}_M)} &= \left\| \frac{d}{dt} \widehat{u}(t) - \Delta_M[\widehat{u}\mu_1(\widehat{v})](t) - R_1(\widehat{u}, \widehat{v})(t) \right\|_{L^\infty(\mathbb{T}_M)} \\ &= \left\| (\partial_t u - R_1(u, v) - \Delta[u\mu_1(v)] + \Delta[u\mu_1(v)])|_{\mathbb{T}_M}(t) - \Delta_M[\widehat{u}\mu_1(\widehat{v})](t) \right\|_{L^\infty(\mathbb{T}_M)} \\ &= \left\| (\Delta[u\mu_1(v)])|_{\mathbb{T}_M}(t) - \Delta_M[\widehat{u}\mu_1(\widehat{v})](t) \right\|_{L^\infty(\mathbb{T}_M)} \xrightarrow{M \rightarrow \infty} 0. \end{aligned}$$

Moreover, using the notation $\Delta_M := M^2(\tau_{1/M} + \tau_{-1/M} - 2)$ on the continuous space $L^2(\mathbb{T})$, the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ and Proposition 4.3 from the appendix gives

$$\begin{aligned} &\left\| (\Delta[u\mu_1(v)])|_{\mathbb{T}_M}(t) - \Delta_M[\widehat{u}\mu_1(\widehat{v})](t) \right\|_{L^\infty(\mathbb{T}_M)} \\ &= \left\| (\Delta[u\mu_1(v)] - \Delta_M[u\mu_1(v)])|_{\mathbb{T}_M}(t) \right\|_{L^\infty(\mathbb{T}_M)} \\ &\leq \left\| (\Delta[u\mu_1(v)] - \Delta_M[u\mu_1(v)])(t) \right\|_{L^\infty(\mathbb{T})} \\ &\leq C \left\| (\Delta[u\mu_1(v)] - \Delta_M[u\mu_1(v)])(t) \right\|_{H^1(\mathbb{T})} \\ &\leq C \|u(t)\mu_1(v(t))\|_{H^3(\mathbb{T})} \leq C \|u(t)\|_{H^3(\mathbb{T})} \|\mu_1(v(t))\|_{H^3(\mathbb{T})}. \end{aligned}$$

Adding that the right hand side belongs to $L^1([0, T])$, bounded convergence concludes the proof. \square

2.3 Stability of pertubated semi-discrete system and proof of Theorem 2

We study now the semi-discrete system $(\mathbf{u}^M, \mathbf{v}^M)$ defined in (5) and will consider a perturbation. First, we control the norm of the original (non pertubated) system :

Proposition 2.9. *Under Hypotheses 1, suppose the initial data $(\mathbf{u}_0^M, \mathbf{v}_0^M)$ is non-negative. Then there exists a global non-negative solution $(\mathbf{u}^M, \mathbf{v}^M)$ of (5), which satisfies for all $T \geq 0$,*

$$\begin{aligned} \sup_{t \leq T} \|\mathbf{u}^M(t)\|_{1,M} + \int_0^T [\mathbf{u}^M(s)\lambda_1^-(\mathbf{u}^M(s), \mathbf{v}^M(s))]_M ds &\lesssim e^{\rho_0 T} \|\mathbf{u}_0^M\|_{1,M}, \\ \sup_{t \leq T} \|\mathbf{v}^M(t)\|_{1,M} + \int_0^T [\mathbf{v}^M(s)\lambda_2^-(\mathbf{u}^M(s), \mathbf{v}^M(s))]_M ds &\lesssim e^{\rho_0 T} \|\mathbf{v}_0^M\|_{1,M}. \end{aligned}$$

Proof. Let $T \in \mathbb{R}_+ \cup \{\infty\}$ the maximal time for which the solution is well-defined. Since $\ell^2(\mathbb{T}_M)$ is finite dimensional, Cauchy-Lipschitz theory proves that $T > 0$.

Let $C = \{(\mathbf{f}, \mathbf{g}) \in \ell^2(\mathbb{T}_M)^2 \mid \mathbf{f}, \mathbf{g} \geq 0\}$ be the closed convex cone of positive functions. We first prove that (\mathbf{u}, \mathbf{v}) stays in C until the explosion time T . To do so, we prove that for any $t < T$, if $(\mathbf{u}(t), \mathbf{v}(t)) \in \partial C$ then $(\mathbf{u}'(t), \mathbf{v}'(t))$ either vanishes or points towards the interior of C . Let $t < T$ be such that $(\mathbf{u}(t), \mathbf{v}(t)) \in \partial C$ and $I = \{1 \leq j \leq M \mid u_j(t) = 0\}$. For all $j \in I$, the structure $R_1(u, v) = u\lambda_1(u, v)$ implies

$$u'_j(t) = M^2(\mu_1(v_{j-1}(t))u_{j-1}(t) + \mu_1(v_{j+1}(t))u_{j+1}(t)).$$

Since all quantities in the right hand side of this equation are non-negative, we have $u'_j(t) \geq 0$. To prove $(\mathbf{u}'(t), \mathbf{v}'(t))$ points towards the interior of C , it is therefore sufficient to see that $u'_j(t) > 0$ for

at least one $j \in I$. The above expression of $u'_j(t)$ shows that it is the case if $j \in I$ and $j+1 \notin I$ so either such a j exists or $I = \{1, \dots, M\}$ and $\mathbf{u}'(t)$ vanishes. Repeating this argument on \mathbf{v} proves that (\mathbf{u}, \mathbf{v}) stays in C .

The fact that the explosion time T is infinite will follow from the estimate in the proposition, which we now prove for all $t < T$. First, one has $\|\mathbf{u}(t)\|_{1,M} = \langle \mathbf{1}, \mathbf{u}(t) \rangle_{2,M}$ where $\mathbf{1} \in \ell^2(\mathbb{T}_M)$ is the constant function taking value 1. Since $\Delta_M \mathbf{1} = 0$, for all $\mathbf{z} \in \ell^2(\mathbb{T}_M)$ one has $\langle \mathbf{1}, \Delta_M \mathbf{z} \rangle_{2,M} = \langle \Delta_M \mathbf{1}, \mathbf{z} \rangle_{2,M} = 0$. Taking the scalar product of (17) with $\mathbf{1}$ and using these facts, one gets

$$\begin{aligned} \|\mathbf{u}(t)\|_{1,M} &= \|\mathbf{u}_0\|_{1,M} + \int_0^t \langle \mathbf{1}, \Delta_M(\mu_1(\mathbf{v})\mathbf{u}) + R_1(\mathbf{u}, \mathbf{v}) \rangle_{2,M}(s) \, ds \\ &= \|\mathbf{u}_0\|_{1,M} + \int_0^t \langle \mathbf{1}, R_1(\mathbf{u}(s), \mathbf{v}(s)) \rangle_{2,M} \, ds \\ &= \|\mathbf{u}_0\|_{1,M} + \int_0^t \langle \mathbf{1}, \mathbf{u}(s) \lambda_1(\mathbf{u}(s), \mathbf{v}(s)) \rangle_{2,M} \, ds. \end{aligned}$$

From Hypotheses 1 ii), we get $\lambda_1^+(\mathbf{u}, \mathbf{v}) \leq \rho_0$ so splitting $\lambda_1 = \lambda_1^+ - \lambda_1^-$ and passing λ_1^- to the right-hand side,

$$\|\mathbf{u}(t)\|_{1,M} + \int_0^t [\mathbf{u} \lambda_1^-(\mathbf{u}, \mathbf{v})](s) \, ds \leq \|\mathbf{u}_0\|_{1,M} + \rho_0 \int_0^t \|\mathbf{u}(s)\|_{1,M} \, ds$$

The Grönwall lemma concludes the proof. \square

Now, we study the semi-discrete system $(\mathbf{u}^M, \mathbf{v}^M)$ defined in (5) with a perturbation term \mathbf{m}^M :

$$\begin{cases} \mathbf{u}^M(t) = \mathbf{u}_0^M + \int_0^t [\Delta_M(\mu_1(\mathbf{v}^M)\mathbf{u}^M) + R_1(\mathbf{u}^M, \mathbf{v}^M)](s) \, ds + \mathbf{m}_1^M(t) \\ \mathbf{v}^M(t) = \mathbf{v}_0^M + \int_0^t [\Delta_M(\mu_2(\mathbf{u}^M)\mathbf{v}^M) + R_2(\mathbf{u}^M, \mathbf{v}^M)](s) \, ds + \mathbf{m}_2^M(t) \end{cases} \quad (17)$$

The perturbation term \mathbf{m}^M will become stochastic in the next section. More precisely, it will be a martingale coming from compensation of random jump process due to birth, deaths and motions. This motivates the following class of deterministic perturbations, which mix absolute continuous component and jumps :

$$\mathbf{m}_i^M(t) = \sum_{k \in \mathbb{N}} \mathbb{1}_{t_{i,k} \leq t} \mathbf{a}_{i,k}^M - \int_0^t \phi_i^M(s) \, ds,$$

where $\phi_i^M : \mathbb{R}_+ \rightarrow \ell^2(\mathbb{T}_M)$ is a function, $(\mathbf{a}_{i,k})_{i \in \{1,2\}, k \in \mathbb{N}}$ is an $\ell^2(\mathbb{T}_M)$ -valued sequence and for $i \in \{1,2\}$, $(t_{i,k})_k$ is an increasing sequence of times which is also divergent : $t_{i,k} \rightarrow \infty$ as $k \rightarrow \infty$.

The following proposition explains how the pertubated system (17) can be compared to discretization $\widehat{u}^M(t) := u_{|\mathbb{T}_M}(t)$ of the continuous limiting system (and also to (5) recalling Proposition 2.8).

Proposition 2.10. *Assume Hypotheses 1 and 2 and let $(\mathbf{u}^M, \mathbf{v}^M)$ a non-negative solution of (17) and set*

$$\mathbf{z}^M = \mathbf{u}^M - \widehat{u}^M, \quad \mathbf{w}^M = \mathbf{v}^M - \widehat{v}^M.$$

Then, for any $T \geq 0$ and $M \geq 1$,

$$\|\mathbf{z}^M\|_{T,M}^2 + \|\mathbf{w}^M\|_{T,M}^2 \lesssim e^{\Lambda^M(T)} \left(\|\mathbf{z}_0^M\|_{-1,M}^2 + \|\mathbf{w}_0^M\|_{-1,M}^2 + \|h_1^M\|_{L_T^\infty} + \|h_2^M\|_{L_T^\infty} + \delta_M \right),$$

where the constants behind this inequality are independent of T, M and \mathbf{m} , and

$$\delta_M = \|\mathbf{r}_1^M\|_{L^1([0,T];L^\infty(\mathbb{T}_M))}^2 + \|\mathbf{r}_2^M\|_{L^1([0,T];L^\infty(\mathbb{T}_M))}^2$$

goes to 0 as M tends to infinity from Proposition 2.8, and Λ^M satisfies

$$\Lambda^M(T) \lesssim \int_{Q_{T,M}} 1 + \mathbf{u}^M + \mathbf{v}^M + \lambda_1^-(\mathbf{u}^M, \mathbf{v}^M)^2 + \lambda_2^-(\mathbf{u}^M, \mathbf{v}^M)^2,$$

and h^M is defined by:

$$\begin{aligned} h_1^M(t) &:= 2 \sum_{k \in \mathbb{N}} \mathbb{1}_{t_1, k \leq t} \langle \mathbf{z}(t_k^-), a_{1,k} \rangle_{-1,M} - 2 \int_0^t \langle \mathbf{z}(s), \phi_1(s) \rangle_{-1,M} ds + \sum_{k \in \mathbb{N}} \mathbb{1}_{t_1, k \leq t} \|a_{1,k}\|_{-1,M}^2, \\ h_2^M(t) &:= 2 \sum_{k \in \mathbb{N}} \mathbb{1}_{t_2, k \leq t} \langle \mathbf{w}(t_k^-), a_{2,k} \rangle_{-1,M} - 2 \int_0^t \langle \mathbf{w}(s), \phi_2(s) \rangle_{-1,M} ds + \sum_{k \in \mathbb{N}} \mathbb{1}_{t_2, k \leq t} \|a_{2,k}\|_{-1,M}^2. \end{aligned}$$

Taking $\mathbf{m} = 0$, this result implies that the interpolations $(\pi_M(\mathbf{u}^M), \pi_M(\mathbf{v}^M))$ converge to (u, v) when $M \rightarrow \infty$. This will yield Theorem 2, see below, after the proof. It will also play a central role in the next section for the convergence of the sequence of stochastic processes by considering the good perturbation term. We recall that in the proofs, we alleviate notation and do not write exponent M in \mathbf{z}, Λ, h_i , etc.

Proof. As in the continuous case, in order to get this kind of estimate we first remark that \mathbf{z} satisfies

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^t (\Delta_M[\mathbf{z}(s)\mu(s) + f(s)] + x(s)) ds + x_J(t),$$

with

$$\mu = \mu_1(\mathbf{v}), \quad f = \widehat{u}(\mu_1(\mathbf{v}) - \mu_1(\widehat{v})), \quad x = R_1(\mathbf{u}, \mathbf{v}) - R_1(\widehat{u}, \widehat{v}) - \phi_1 - \mathbf{r}_1, \quad x_J(t) = \sum_{k \in \mathbb{N}} \mathbb{1}_{t_i, k \leq t} \mathbf{a}_{i,k}.$$

This allows to apply the duality lemma to set up for a Grönwall lemma. Therefore, we can invoke [4] and more precisely we use $\mu_1 \geq \alpha_1$ and Lemma 4.9 in Appendix to get

$$\begin{aligned} \|\mathbf{z}(t)\|_{-1,M}^2 + \alpha_1 \int_{Q_{t,M}} \mathbf{z}^2 &\leq \|z_0\|_{-1,M}^2 + \int_0^t [\mathbf{z}(s)]_M^2 [\mu(s)]_M ds + h_1(t) \\ &\quad + \frac{1}{\alpha_1} \int_{Q_{t,M}} f^2 + 2 \int_0^t \langle \mathbf{z}(s), y(s) \rangle_{-1,M} ds, \end{aligned} \quad (18)$$

where $h_1(t)$ is as defined in the above statement and

$$y = R_1(\mathbf{u}, \mathbf{v}) - R_1(\widehat{u}, \widehat{v}) - \mathbf{r}_1 = x + \phi_1.$$

Let us inspect bounds for the terms on the right-hand side in their order of appearance.

First, for the $[z]_M^2$ term, we use $[z(s)]_M^2 \leq \|z(s)\|_{-1,M}^2$, which is a direct consequence of the definition of $\|\cdot\|_{-1,M}^2$. Then, since μ_1 is Lipschitz-continuous,

$$\int_{Q_{t,M}} f^2 \leq L_1^2 \int_{Q_{t,M}} \widehat{u}^2 |\mathbf{v} - \widehat{v}|^2 \leq L_1^2 \|u\|_{L^\infty(Q_T)}^2 \int_{Q_{t,M}} \mathbf{w}^2.$$

For the last term, since $\|\cdot\|_{-1,M} \leq \|\cdot\|_{2,M} \leq \|\cdot\|_\infty$ and $\|\cdot\|_{-1,M} \lesssim \|\cdot\|_{1,M}$ from Corollary 2.3 and Proposition 2.4,

$$\begin{aligned} \langle \mathbf{z}(s), y(s) \rangle_{-1,M} &\leq \|\mathbf{z}(s)\|_{-1,M} \|y(s)\|_{-1,M} \\ &\leq \|\mathbf{z}(s)\|_{-1,M} \|R_1(\mathbf{u}, \mathbf{v})(s) - R_1(\widehat{u}, \widehat{v})(s)\|_{-1,M} + \|\mathbf{z}(s)\|_{-1,M} \|\mathbf{r}_1(s)\|_{-1,M} \\ &\lesssim \|\mathbf{z}(s)\|_{-1,M} \|R_1(\mathbf{u}, \mathbf{v})(s) - R_1(\widehat{u}, \widehat{v})(s)\|_{1,M} + \|\mathbf{z}(s)\|_{-1,M} \|\mathbf{r}_1(s)\|_\infty, \end{aligned}$$

with, since λ_1 is Lipschitz-continuous,

$$\begin{aligned} |R_1(\mathbf{u}, \mathbf{v}) - R_1(\widehat{u}, \widehat{v})| &= |\mathbf{u}\lambda_1(\mathbf{u}, \mathbf{v}) - \widehat{u}\lambda_1(\widehat{u}, \widehat{v})| \\ &\leq |\mathbf{u}\lambda_1(\mathbf{u}, \mathbf{v}) - \widehat{u}\lambda_1(\mathbf{u}, \mathbf{v})| + |\widehat{u}\lambda_1(\mathbf{u}, \mathbf{v}) - \widehat{u}\lambda_1(\widehat{u}, \widehat{v})| \\ &\lesssim |\mathbf{z}\lambda_1(\mathbf{u}, \mathbf{v})| + \widehat{u}(|\mathbf{z}| + |\mathbf{w}|). \end{aligned}$$

It implies, since \widehat{u} is smaller than $\|u\|_\infty$ which is a constant,

$$\begin{aligned} \|R_1(\mathbf{u}(s), \mathbf{v}(s)) - R_1(\widehat{u}(s), \widehat{v}(s))\|_{1,M} &\lesssim \|\lambda_1(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M} \|\mathbf{z}(s)\|_{2,M} + \|\widehat{u}\|_{2,M} (\|\mathbf{z}(s)\|_{2,M} + \|\mathbf{w}(s)\|_{2,M}) \\ &\lesssim (1 + \|\lambda_1(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}) (\|\mathbf{z}(s)\|_{2,M} + \|\mathbf{w}(s)\|_{2,M}). \end{aligned}$$

Thus, using Young's inequality with $\varepsilon > 0$ to choose later,

$$\begin{aligned} \langle \mathbf{z}(s), x(s) \rangle_{-1,M} &\lesssim \|\mathbf{z}(s)\|_{-1,M} (1 + \|\lambda_1(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}) (\|\mathbf{z}(s)\|_{2,M} + \|\mathbf{w}(s)\|_{2,M}) + \|\mathbf{z}(s)\|_{-1,M} \|\mathbf{r}_1(s)\|_\infty \\ &\lesssim \varepsilon^{-1} (1 + \|\lambda_1(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}^2) \|\mathbf{z}(s)\|_{-1,M}^2 + \varepsilon (\|\mathbf{z}(s)\|_{2,M}^2 + \|\mathbf{w}(s)\|_{2,M}^2) + \|\mathbf{z}(s)\|_{-1,M} \|\mathbf{r}_1(s)\|_\infty. \end{aligned}$$

Plugging all these bounds in (18) and gathering terms involving $\|\mathbf{z}(s)\|_{-1,M}$, we get

$$\begin{aligned} \|\mathbf{z}(t)\|_{-1,M}^2 + \alpha_1 \int_{Q_{t,M}} \mathbf{z}^2 &\tag{19} \\ &\lesssim \frac{L_1^2}{\alpha_1} \|u\|_{L^\infty(Q_T)}^2 \int_{Q_{t,M}} \mathbf{w}^2 + \varepsilon \int_{Q_{t,M}} (\mathbf{z}^2 + \mathbf{w}^2) + g_1(t) + h_1(t), \end{aligned}$$

where

$$\begin{aligned} g_1(t) &= \|\mathbf{z}(0)\|_{-1,M}^2 + \int_0^t \|\mathbf{z}(s)\|_{-1,M}^2 [\mu_1(\mathbf{v}(s))]_M ds \\ &\quad + \int_0^t \left(\varepsilon^{-1} (1 + \|\lambda_1(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}^2) \|\mathbf{z}(s)\|_{-1,M}^2 + \|\mathbf{z}(s)\|_{-1,M} \|\mathbf{r}_1(s)\|_\infty \right) ds. \end{aligned} \tag{20}$$

In particular, we have

$$(\alpha_1 - \varepsilon) \int_{Q_{t,M}} \mathbf{z}^2 \lesssim \left(\frac{L_1^2}{\alpha_1} \|u\|_{L^\infty(Q_T)}^2 + \varepsilon \right) \int_{Q_{t,M}} \mathbf{w}^2 + g_1(t) + h_1(t). \tag{21}$$

There is a similar inequality for \mathbf{w} :

$$(\alpha_2 - \varepsilon) \int_{Q_{t,M}} \mathbf{w}^2 \lesssim \left(\frac{L_2^2}{\alpha_2} \|v\|_{L^\infty(Q_T)}^2 + \varepsilon \right) \int_{Q_{t,M}} \mathbf{z}^2 + g_2(t) + h_2(t).$$

Injecting (21) in it to bound $\int_{Q_{t,M}} \mathbf{z}^2$, one gets

$$(\alpha_2 - \varepsilon) \int_{Q_{t,M}} \mathbf{w}^2 \lesssim A_\varepsilon \int_{Q_{t,M}} \mathbf{w}^2 + h_1(t) + g_2(t) + h_2(t),$$

where

$$A_\varepsilon = \left(\frac{L_2^2}{\alpha_2} \|v\|_{L^\infty(Q_T)}^2 + \varepsilon \right) \frac{1}{\alpha_1 - \varepsilon} \left(\frac{L_1^2}{\alpha_1} \|u\|_{L^\infty(Q_T)}^2 + \varepsilon \right).$$

By the smallness Assumption (6), for ε small enough, $A_\varepsilon < \alpha_2 - \varepsilon$. Therefore term $\int_{Q_{t,M}} \mathbf{w}^2$ from the right-hand side can be sent to the left-hand side to get

$$\int_{Q_{t,M}} \mathbf{w}^2 \lesssim g_2(t) + g_1(t) + h_1(t) + h_2(t).$$

The choice of ε depends only on the parameters of the problem and (u, v) (through the dependence on the difference of the two members of the smallness condition), so the constant behind \lesssim depends only on these same objects. A similar inequality holds for $\int_{Q_{t,M}} \mathbf{z}^2$. Back to (19), we get by summing it with its analogue for \mathbf{w} and using the previous bounds on $\int_{Q_{t,M}} \mathbf{z}^2$ and $\int_{Q_{t,M}} \mathbf{w}^2$ that

$$\|\mathbf{z}(t)\|_{-1,M}^2 + \int_{Q_{t,M}} \mathbf{z}^2 + \|\mathbf{w}(t)\|_{-1,M}^2 + \int_{Q_{t,M}} \mathbf{w}^2 \lesssim g_1(t) + g_2(t) + h_1(t) + h_2(t). \quad (22)$$

Recalling from (20) the expression of g_1 and that g_2 has a similar form and setting

$$\psi(t) := \|\mathbf{z}(t)\|_{-1,M}^2 + \|\mathbf{w}(t)\|_{-1,M}^2, \quad \phi(t) := \int_{Q_{t,M}} \mu_1(\mathbf{v}) \mathbf{z}^2 + \int_{Q_t} \mu_2(\mathbf{u}) \mathbf{w}^2,$$

$$a(t) := 1 + [\mu_2(\mathbf{u}(t))]_M + [\mu_1(\mathbf{v}(t))]_M + \|\lambda_1(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}^2 + \|\lambda_2(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}^2,$$

inequality (22) rewrites

$$\psi(t) + \phi(t) \lesssim \psi(0) + \int_0^t [a(s)\psi(s) + (\|\mathbf{r}_1(s)\|_\infty + \|\mathbf{r}_2(s)\|_\infty)\psi(s)^{1/2}] ds + h_1(t) + h_2(t).$$

Bounding $h_1(t) + h_2(t)$ by $\|h_1\|_{L_T^\infty} + \|h_2\|_{L_T^\infty}$ and using non-linear Grönwall lemma (see Lemma 4.4), we get

$$\begin{aligned} & \sup_{t \leq T} \psi(t) + \phi(T) \\ & \lesssim e^{c \int_0^T a(s) ds} \left(\psi(0) + \|h_1\|_{L_T^\infty} + \|h_2\|_{L_T^\infty} + \|\mathbf{r}_1\|_{L^1([0,T]; L^\infty(\mathbb{T}_M))}^2 + \|\mathbf{r}_2\|_{L^1([0,T]; L^\infty(\mathbb{T}_M))}^2 \right). \end{aligned}$$

It remains to control a and conclude. Using Hypotheses 1, we get $\mu_i(x) \lesssim 1 + x$ and $|\lambda_i(u, v)| \leq |\lambda_i^+(u, v)| + |\lambda_i^-(u, v)| \leq \rho_0 + |\lambda_i^-(u, v)|$,

$$\begin{aligned} \int_0^T a(s) ds & \lesssim \int_0^T 1 + [\mathbf{u}(s)]_M + [\mathbf{v}(s)]_M + \|\lambda_1^-(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}^2 + \|\lambda_2^-(\mathbf{u}(s), \mathbf{v}(s))\|_{2,M}^2 ds \\ & \lesssim \int_{Q_{T,M}} (1 + \mathbf{u} + \mathbf{v} + \lambda_1^-(\mathbf{u}, \mathbf{v})^2 + \lambda_2^-(\mathbf{u}, \mathbf{v})^2). \end{aligned}$$

It ends the proof. \square

We are now in position to prove Theorem 2 stated in the first section.

Proof of Theorem 2. In the case of the semi-discrete system with no perturbations (5), $\mathbf{m} = 0$ so h_1 and h_2 from Proposition 2.10 vanish and Hypothesis 1 iii) and Proposition 2.9 yield

$$\begin{aligned} \Lambda(T) &= \int_{Q_{T,M}} 1 + \mathbf{u} + \mathbf{v} + \lambda_1^-(\mathbf{u}, \mathbf{v})^2 + \lambda_2^-(\mathbf{u}, \mathbf{v})^2 \\ &\lesssim \int_{Q_{T,M}} 1 + \mathbf{u} + \mathbf{v} + \mathbf{u}\lambda_1^-(\mathbf{u}, \mathbf{v}) + \mathbf{v}\lambda_2^-(\mathbf{u}, \mathbf{v}) \lesssim e^{cT}(\|\mathbf{u}_0\|_{1,M} + \|\mathbf{v}_0\|_{1,M}) \lesssim \exp(cT). \end{aligned}$$

Thus, the estimate from Proposition 2.10 rewrites

$$\|\mathbf{z}\|_{T,M}^2 + \|\mathbf{w}\|_{T,M}^2 \lesssim e^{\exp(cT)}(\|\mathbf{z}_0\|_{-1,M}^2 + \|\mathbf{w}_0\|_{-1,M}^2 + \delta_M),$$

where δ_M converges to 0 according to Proposition 2.8. Note that the constant in front of $\exp(cT)$ has been by changing the value of c . Then, forthcoming Proposition 4.5 in Appendix ensures that

$$\begin{aligned} \|\pi_M(\mathbf{u}) - u\|_T &\lesssim \|\mathbf{z}\|_{T,M} + \delta'_M, \\ \|\mathbf{z}_0\|_{-1,M} &\lesssim \|\pi_M(\mathbf{u}_0) - u_0\|_{H^{-1}(\mathbb{T})} + \delta'_M, \end{aligned}$$

where δ'_M goes to 0 and can be absorbed in δ_M . Combining the previous inequalities ends the proof. \square

3 Convergence of stochastic processes

In this chapter, we consider an individual based model, living on the discrete space \mathbb{T}_M . We prove that when the local population scale N goes to infinity, the stochastic process is close to the semi discrete system. We obtain quantitative estimates which are uniform with respect to the number of sites M . It will prove that when M and N become large, the individual based model converges to the crossed diffusion system. The norm is inherited from the duality estimates. It relies also on the control of fluctuations through martingales estimates in H^{-1} and large deviations estimates for births and deaths terms. We first justify the existence of stochastic processes and provide preliminary moment estimates. We then derive the martingales inequalities (Subsection 3.2) and study the large deviations (tail estimates) in Subsection 3.3. We will then combine the results to prove convergence of the stochastic process in Subsection 3.4 (Theorem 3).

3.1 Construction of the process and first estimates

We consider $p \geq 1$ and assume here that initial conditions satisfy

$$\mathbb{E} \left[\|\mathbf{U}^{M,N}(0)\|_{1,M}^p + \|\mathbf{V}^{M,N}(0)\|_{1,M}^p \right] < \infty. \quad (23)$$

It will cover our framework. Indeed, Hypothesis 3 *i*) involved for convergence will ensure that (23) is satisfied for $p < p_0$.

For the proofs, we proceed by localization of processes to keep them bounded and work on a finite state space. More precisely, we first define the càdlàg jump Markov process $(\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ satisfying (7), (8) until time

$$T^K = \inf\{t \geq 0 : \|\mathbf{U}^{M,N}(t)\|_\infty + \|\mathbf{V}^{M,N}(t)\|_\infty \geq K\} \in [0, \infty].$$

Indeed, the process $(\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ remains on finite state space $[0, K+1]^{2M}$ in the time interval $[0, T^K] \cap \mathbb{R}_+$. The process is now simply a jump Markov process on a finite state space and its construction and characterization are classical. Thus, it can be constructed by iteration, considering the successive jumps, which provides a strong solution to (7) – (8) on the time interval $[0, T^K] \cap \mathbb{R}_+$. Let us also note that pathwise uniqueness on this time interval immediately holds.

Let us now prove that T^K goes to infinity as K goes to infinity, using the the control the growth rate (Hypothesis 1 *iii*) and our moment condition above. First, we observe that compensating the Poisson random measure in (7) gives the following semi-martingale decomposition (see Section II-3 [24]):

$$\mathbf{U}(t) = \mathbf{A}(t) + \mathcal{M}_1(t), \quad (24)$$

for $t \in \mathbb{R}_+ \cap T^K$, where

$$\begin{aligned} \mathbf{A}^{M,N}(t) &= \mathbf{U}_0^{M,N} + \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}^{M,N}(s^-), \mathbf{V}^{M,N}(s^-), \theta)} \theta \, ds d\rho d\theta \\ &= \mathbf{U}_0^{M,N} + \int_0^t \phi_1(\mathbf{U}^{M,N}(s), \mathbf{V}^{M,N}(s)) ds \end{aligned}$$

and

$$\phi_1(\mathbf{u}, \mathbf{v}) = \sum_{\theta \in \mathcal{T}} \frac{1}{N} \nu_1(\mathbf{u}, \mathbf{v}, \theta) \theta = [\Delta_M(\mu_1(\mathbf{v})\mathbf{u}) + R_1(\mathbf{u}, \mathbf{v})] \quad (25)$$

and $(\mathcal{M}_1(t \wedge T^K))_{t \geq 0}$ is a martingale defined for $t \in \mathbb{R}_+ \cap [0, T^K]$ by

$$\mathcal{M}_1(t) = \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \theta \tilde{\mathcal{N}}_1(ds, d\rho, d\theta),$$

where $\tilde{\mathcal{N}}_1$ is the compensated Poisson measure of \mathcal{N}_1 . This semi-martingale decomposition is the stochastic counterpart of the semi-discrete problem (17) where the source \mathbf{m}_1 is random and given by the local martingale \mathcal{M}_1 .

We can now prove the following moment estimates.

Proposition 3.1. *Assume that $b_i - d_i$ is bounded and b_i is globally Lipschitz for $i \in \{1, 2\}$. Let $p \geq 1$ and assume also that (23) holds. Then there exists $c \in \mathbb{R}$ such that for all $K \in \mathbb{N}$ and $M, N \geq 1$ and $t \geq 0$,*

$$\mathbb{E} \left[\|\mathbf{U}^{M,N}(t \wedge T^K)\|_{1,M}^p + \|\mathbf{V}^{M,N}(t \wedge T^K)\|_{1,M}^p \right] \lesssim e^{ct} \left(\mathbb{E} \left[\|\mathbf{U}^{M,N}(0)\|_{1,M}^p + \|\mathbf{V}^{M,N}(0)\|_{1,M}^p \right] + \frac{t}{N} \right).$$

Proof. Fix $K \in \mathbb{N}$ and $p \geq 1$. We take the scalar product of SDE (7) with the constant function $\mathbf{1}$ and for any $t \leq T^K$,

$$\|\mathbf{U}(t)\|_{1,M} = \|\mathbf{U}_0\|_{1,M} + \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \langle \mathbf{1}, \theta \rangle \mathcal{N}_1(ds, d\rho, d\theta).$$

For a jump of $\|\mathbf{U}(t)\|_{1,M}$ given by $\frac{1}{N}\langle \mathbf{1}, \theta \rangle$, the corresponding jump of $\|\mathbf{U}(t)\|_{1,M}^p$ is

$$\left(\|\mathbf{U}(t^-)\|_{1,M} + \frac{1}{N}\langle \mathbf{1}, \theta \rangle\right)^p - \|\mathbf{U}(t^-)\|_{1,M}^p,$$

thus we have

$$\begin{aligned} \|\mathbf{U}(t)\|_{1,M}^p &= \|\mathbf{U}_0\|_{1,M}^p \\ &+ \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \left[\left(\|\mathbf{U}(s^-)\|_{1,M} + \frac{1}{N}\langle \mathbf{1}, \theta \rangle \right)^p - \|\mathbf{U}(s^-)\|_{1,M}^p \right] \mathcal{N}_1(ds, d\rho, d\theta). \end{aligned}$$

Taking the expectation using that T^K is a stopping time for the natural filtration,

$$\begin{aligned} &\mathbb{E} \left[\|\mathbf{U}(t \wedge T^K)\|_{1,M}^p \right] \\ &= \mathbb{E} \left[\|\mathbf{U}_0\|_{1,M}^p + \int_0^{t \wedge T^K} \sum_{\theta \in \mathcal{T}} \left(\left(\|\mathbf{U}(s)\|_{1,M} + \frac{1}{N}\langle \mathbf{1}, \theta \rangle \right)^p - \|\mathbf{U}(s)\|_{1,M}^p \right) \nu_1(\mathbf{U}(s), \mathbf{V}(s), \theta) ds \right]. \end{aligned}$$

The mobility transitions have zero contribution, i.e. $\langle \mathbf{1}, \mathbf{e}_{i+\epsilon} - \mathbf{e}_i \rangle = 0$. Then for any $\mathbf{u}, \mathbf{v} \in \ell^2(\mathbb{T}_M)$,

$$\begin{aligned} &\sum_{\theta \in \mathcal{T}} \left(\left(\|\mathbf{u}\|_{1,M} + \frac{1}{N}\langle \mathbf{1}, \theta \rangle \right)^p - \|\mathbf{u}\|_{1,M}^p \right) \nu_1(\mathbf{u}, \mathbf{v}, \theta) \\ &= \sum_{j=1}^M N u_j \left[\left(\left(\|\mathbf{u}\|_{1,M} + \frac{1}{MN} \right)^p - \|\mathbf{u}\|_{1,M}^p \right) b_1(u_j, v_j) \right. \\ &\quad \left. + \left(\left(\|\mathbf{u}\|_{1,M} - \frac{1}{MN} \right)^p - \|\mathbf{u}\|_{1,M}^p \right) d_1(u_j, v_j) \right]. \end{aligned}$$

We then use the inequalities $(x + \varepsilon)^p - x^p \leq x^p - (x - \varepsilon)^p + C_p \varepsilon^2 x^{p-2}$ and $x^p - (x - \varepsilon)^p \leq C_p \varepsilon x^{p-1}$, which are given in Proposition 4.1 of forthcoming Appendix, together with the upper bounds of $b_1 - d_1$ and b_1 coming from assumptions. We get

$$\begin{aligned} &\sum_{\theta \in \mathcal{T}} \left(\left(\|\mathbf{u}\|_{1,M} + \frac{1}{N}\langle \mathbf{1}, \theta \rangle \right)^p - \|\mathbf{u}\|_{1,M}^p \right) \nu_1(\mathbf{u}, \mathbf{v}, \theta) \\ &\leq \sum_{j=1}^M N u_j \left[\left(\|\mathbf{u}\|_{1,M}^p - \left(\|\mathbf{u}\|_{1,M} - \frac{1}{MN} \right)^p \right) (b_1 - d_1)(u_j, v_j) + C_p \frac{1}{(MN)^2} \|\mathbf{u}\|_{1,M}^{p-2} b_1(u_j, v_j) \right] \\ &\lesssim \sum_{j=1}^M N u_j \left[\frac{\rho_0}{MN} \|\mathbf{u}\|_{1,M}^{p-1} + \frac{1}{(MN)^2} \|\mathbf{u}\|_{1,M}^{p-2} (1 + u_j + v_j) \right] \\ &\lesssim \frac{1}{M} \sum_{j=1}^M u_j \left[\|\mathbf{u}\|_{1,M}^{p-1} + \frac{1}{MN} \|\mathbf{u}\|_{1,M}^{p-2} (1 + M \|\mathbf{u}\|_{1,M} + M \|\mathbf{v}\|_{1,M}) \right] \\ &\leq \|\mathbf{u}\|_{1,M} \left[\|\mathbf{u}\|_{1,M}^{p-1} + \frac{1}{N} \|\mathbf{u}\|_{1,M}^{p-2} (1 + \|\mathbf{u}\|_{1,M} + \|\mathbf{v}\|_{1,M}) \right]. \end{aligned}$$

Using Young's inequality, $\|u\|_{1,M}^{p-1} (1 + \|u\|_{1,M} + \|v\|_{1,M}) \lesssim 1 + \|u\|_{1,M}^p + \|v\|_{1,M}^p$, we obtain

$$\begin{aligned} & \mathbb{E}[\|\mathbf{U}(t \wedge T^K)\|_{1,M}^p] \\ & \leq \mathbb{E}[\|\mathbf{U}(0)\|_{1,M}^p] + C \int_0^t \mathbb{E}[\|\mathbf{U}(s \wedge T^K)\|_{1,M}^p] + \frac{1}{N} \left(1 + \mathbb{E}[\|\mathbf{V}(s \wedge T^K)\|_{1,M}^p]\right) ds, \end{aligned}$$

for some $c > 0$. Going through similar arguments for estimating \mathbf{V}^K , we sum the two estimates and get

$$\psi(t) \leq \psi(0) + c \int_0^t \left(\psi(s) + \frac{1}{N}\right) ds,$$

where

$$\psi(t) = \mathbb{E}[\|\mathbf{U}(t \wedge T^K)\|^p] + \mathbb{E}[\|\mathbf{V}(t \wedge T^K)\|^p].$$

This implies, by Grönwall's lemma, that

$$\psi(t) \leq e^{ct} \left(\psi(0) + \frac{ct}{N}\right),$$

that is to say

$$\mathbb{E} \left[\|\mathbf{U}(t \wedge T^K)\|_{1,M}^p + \|\mathbf{V}(t \wedge T^K)\|_{1,M}^p \right] \leq e^{ct} \left(\mathbb{E} \left[\|\mathbf{U}(0)\|_{1,M}^p + \|\mathbf{V}(0)\|_{1,M}^p \right] + \frac{ct}{N} \right).$$

□

The previous proposition ensures that

$$\lim_{K \rightarrow \infty} T^K = \infty \quad \text{a.s.},$$

since for any $t \geq 0$ and a fixed initial condition with bounded first moment,

$$\mathbb{P}(T^K \leq t) \leq \frac{\mathbb{E} \left[\mathbb{1}_{T^K \leq t} \left(\|\mathbf{U}^{M,N}(t \wedge T^K)\|_{1,M} + \|\mathbf{V}^{M,N}(t \wedge T^K)\|_{1,M} \right) \right]}{K} \lesssim \frac{C e^{ct}}{K}$$

and the right hand side goes to 0 as K tends to infinity. It allows to construct the process $(\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ for any time $t \in \mathbb{R}_+$, and $(\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ is the unique strong solution of (7) in $\mathbb{D}(\mathbb{R}^+, \mathbb{N}^{2M})$. Let us observe that existence and pathwise uniqueness can then be extended to any finite initial conditions by truncation procedure on the initial conditions, while we are using the moment estimates obtained above for the proofs of convergence.

3.2 Martingales inequalities

We have shown in the previous part that $T^K \rightarrow \infty$ a.s. as $K \rightarrow \infty$. Then for $i \in \{1, 2\}$,

$$\mathcal{M}_i^{M,N}(t) = \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_i(\mathbf{U}^{M,N}(s^-), \mathbf{V}^{M,N}(s^-), \theta)} \theta \tilde{\mathcal{N}}_i(ds, d\rho, d\theta) \quad (26)$$

is a local martingale well defined for any $t \in \mathbb{R}_+$. Reading the right-hand side of the estimate in Proposition 2.10, we see that the approximation results will rely on bounds for the quantities

h^M . We denote $H^{M,N}$ the stochastic counterpart, arising when the source term \mathbf{m}^M is the local martingale $\mathcal{M}^{M,N}$. Thus, $H_1^{M,N}$ is given by the following semi-martingale decomposition

$$H_1^{M,N}(t) = 2\mathcal{Q}_1^{M,N}(t) + \sum_{0 \leq s \leq t} \left\| \mathcal{M}_1^{M,N}(s) - \mathcal{M}_1^{M,N}(s^-) \right\|_{-1,M}^2, \quad (27)$$

where $\mathcal{Q}_1^{M,N}$ is the following local martingale

$$\begin{aligned} \mathcal{Q}_1^{M,N}(t) &= - \int_0^t \langle \mathbf{Z}^{M,N}(s), \phi_1(\mathbf{U}^{M,N}(s), \mathbf{V}^{M,N}(s)) \rangle_{-1,M} ds \\ &\quad + \sum_{0 \leq s \leq t} \langle \mathbf{Z}^{M,N}(s^-), \mathcal{M}_1^{M,N}(s) - \mathcal{M}_1^{M,N}(s^-) \rangle_{-1,M} \\ &= \frac{1}{N} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}^{M,N}(s^-), \mathbf{V}^{M,N}(s^-), \theta)} \langle \mathbf{Z}^{M,N}(s^-), \theta \rangle_{-1,M} \tilde{\mathcal{N}}_1(ds, d\rho, d\theta) \end{aligned} \quad (28)$$

and ϕ_1 has been defined in (25) and

$$\mathbf{Z}^{M,N} := \mathbf{U}^{M,N} - \hat{u}^M.$$

The local martingale $\mathcal{Q}_2^{M,N}$ and semimartingale $H_2^{M,N}$ for the second species are defined similarly. We prove now that these local martingales are square integrable martingales whose moments are bounded by powers of $1/N$ in H^{-1} .

Proposition 3.2. *For $i \in \{1, 2\}$, assume that $b_i - d_i$ is bounded and b_i and d_i are globally Lipschitz. i) Assume that initial conditions satisfy (23) with $p = 2$. Then, for each $i \in \{1, 2\}$, $\mathcal{M}_i^{M,N}$ is a square-integrable martingale in $(\mathbb{R}_+)^{2M}$ and for any $T \geq 0$ and $M, N \geq 1$,*

$$\mathbb{E} \left[\sup_{t \leq T} \left\| \mathcal{M}_i^{M,N}(t) \right\|_{-1,M}^2 \right] \lesssim \frac{1}{N} \mathbb{E} \left[\int_{Q_{T,M}} 1 + (\mathbf{U}^{M,N})^2 + (\mathbf{V}^{M,N})^2 \right].$$

ii) Let $p \geq 2$ and assume that initial conditions satisfy (23). Then, for each $i \in \{1, 2\}$, $\mathcal{Q}_i^{M,N}$ is a square-integrable martingale in \mathbb{R} and for any $T \geq 0$, $M, N \geq 1$, for any $T \geq 0$,

$$\mathbb{E} \left[\sup_{t \leq T} |\mathcal{Q}_i^{M,N}(t)|^{p/2} \right] \lesssim \left(\frac{1+T}{N} \right)^{\frac{p}{4}} \mathbb{E} [1 + \|\mathbf{U}^{M,N}\|_{T,M}^p + \|\mathbf{V}^{M,N}\|_{T,M}^p].$$

Proof. Let us first control the jumps of the local martingales in H^{-1} . Jumps can be birth, deaths (BD) and motion (M) and we recall that they are given by \mathcal{T} :

$$\mathcal{T} = \mathcal{T}_{BD} \cup \mathcal{T}_M, \quad \mathcal{T}_{BD} = \bigcup_{j=1}^M \left\{ \frac{1}{N} \mathbf{e}_j, -\frac{1}{N} \mathbf{e}_j \right\}, \quad \mathcal{T}_M = \bigcup_{j=1}^M \left\{ \frac{1}{N} (\mathbf{e}_{j+1} - \mathbf{e}_j), \frac{1}{N} (\mathbf{e}_{j-1} - \mathbf{e}_j) \right\}.$$

We first deal with births and deaths and use the first part of Proposition 2.6 to for $\|\theta\|_{-1,M}$ and get

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathcal{T}_{BD}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{u}, \mathbf{v}, \theta)} \left\| \frac{1}{N} \theta \right\|_{-1,M}^2 d\rho d\theta \\ &= \frac{1}{N^2 M} \int_{\mathbb{R}_+ \times \mathcal{T}_{BC}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{u}, \mathbf{v}, \theta)} d\rho d\theta = \frac{1}{N^2 M} \sum_{\theta \in \mathcal{T}_{BC}} \nu_1(\mathbf{u}, \mathbf{v}, \theta) \\ &= \frac{1}{N^2 M} \sum_{j=1}^M N u_j b_1(u_j, v_j) + N u_j d_1(u_j, v_j) \lesssim \frac{1}{NM} \sum_{j=1}^M 1 + u_j^2 + v_j^2 = \frac{1}{N} (1 + \|\mathbf{u}\|_{2,M}^2 + \|\mathbf{v}\|_{2,M}^2), \end{aligned}$$

the last inequality coming from the fact that b_1 and d_1 are Lipschitz and Young's inequality. Similarly, the second part of Proposition 2.6 yields a control of the motion part :

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathcal{T}_\mu} \mathbb{1}_{\rho \leq \nu_1(\mathbf{u}, \mathbf{v}, \theta)} \left\| \frac{1}{N} \theta \right\|_{-1,M}^2 d\rho d\theta \\ &= \frac{1}{N^2 M^3} \int_{\mathbb{R}_+ \times \mathcal{T}_\mu} \mathbb{1}_{\rho \leq \nu_1(\mathbf{u}, \mathbf{v}, \theta)} d\rho d\theta = \frac{1}{N^2 M^3} \sum_{\theta \in \mathcal{T}_\mu} \nu_1(\mathbf{u}, \mathbf{v}, \theta) = \frac{1}{N^2 M^3} \sum_{j=1}^M N M^2 u_j \mu_1(v_j) \\ &\lesssim \frac{1}{NM} \sum_{j=1}^M u_j (1 + v_j) \lesssim \frac{1}{NM} \sum_{j=1}^M 1 + u_j^2 + v_j^2 = \frac{1}{N} (1 + \|\mathbf{u}\|_{2,M}^2 + \|\mathbf{v}\|_{2,M}^2). \end{aligned}$$

Proceeding similarly for the second species and assembling these two bounds for the jumps yields

$$\int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_i(\mathbf{u}, \mathbf{v}, \theta)} \left\| \frac{1}{N} \theta \right\|_{-1,M}^2 d\rho d\theta \lesssim \frac{1}{N} (1 + \|\mathbf{u}\|_{2,M}^2 + \|\mathbf{v}\|_{2,M}^2) \quad (29)$$

for any $\mathbf{u}, \mathbf{v} \in \ell^2(\mathbb{T}_M)$ and $i \in \{1, 2\}$. Replacing (\mathbf{u}, \mathbf{v}) by $(\mathbf{U}(s), \mathbf{V}(s))$ in this expression and integrating in time and taking the expectation, we obtain for any $M, N \geq 1$ and $t \in [0, T]$,

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \left\| \frac{1}{N} \mathbb{1}_{\rho \leq \nu_i(\mathbf{U}(s), \mathbf{V}(s), \theta)} \theta \right\|_{-1,M}^2 d\rho d\theta \right] \lesssim \frac{1}{N} \mathbb{E} \left[\int_{Q_{T,M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \right]. \quad (30)$$

Moreover,

$$\int_{Q_{T,M}} \mathbf{U}^2 + \mathbf{V}^2 = \int_0^T \|\mathbf{U}(s)\|_{2,M}^2 + \|\mathbf{V}(s)\|_{2,M}^2 ds \lesssim M \int_0^T \|\mathbf{U}(s)\|_{1,M}^2 + \|\mathbf{V}(s)\|_{1,M}^2 ds,$$

and we know from we know from Proposition 3.1 that the second moment on the left hand side is finite. It ensures that for each $i \in \{1, 2\}$,

$$\|\mathcal{M}_i(t)\|_{-1,M}^2 - \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \left\| \frac{1}{N} \mathbb{1}_{\rho \leq \nu_i(\mathbf{U}(s), \mathbf{V}(s), \theta)} \theta \right\|_{-1,M}^2 ds d\rho d\theta$$

is a real martingale for any $t \geq 0$ and \mathcal{M}_i is a square-integrable martingale in $(\mathbb{R}_+)^{2M}$. We work here with $\|\cdot\|_{-1,M}$ rather than $\|\cdot\|_{2,M}$, which doesn't change the integrability properties of \mathcal{M}_i since

\mathbb{R}_+^{2M} is finite-dimensional and refer to Section II-3 of [24]. Using (30), we can bound the second moment of $\mathcal{M}_i(t)$:

$$\mathbb{E} \left[\|\mathcal{M}_i(t)\|_{-1,M}^2 \right] \lesssim \frac{1}{N} \mathbb{E} \left[\int_{Q_{T,M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \right].$$

Since \mathcal{M}_i is a martingale, $\|\mathcal{M}_i(t)\|_{-1,M}$ is a submartingale and Doob's inequality proves part *i*) of the lemma.

Let us now consider the local martingale \mathcal{Q}_1 . The domination of \mathbf{Z} by \mathbf{U} and expression (28) ensure that \mathcal{Q}_1 is a square-integrable martingale whose quadratic variation is

$$\langle \mathcal{Q}_1 \rangle (t) = \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s), \mathbf{V}(s), \theta)} \left\langle \mathbf{Z}(s), \frac{1}{N} \theta \right\rangle_{-1,M}^2 ds d\rho d\theta.$$

More precisely, using successively Cauchy-Schwarz inequality and (29)

$$\begin{aligned} \langle \mathcal{Q}_1 \rangle (t) &\leq \sup_{0 \leq s \leq t} \|\mathbf{Z}(s)\|_{-1,M}^2 \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s), \mathbf{V}(s), \theta)} \left\| \frac{1}{N} \theta \right\|_{-1,M}^2 d\rho d\theta ds \\ &\lesssim \sup_{0 \leq s \leq t} (1 + \|\mathbf{U}(s)\|_{-1,M}^2) \int_0^t \frac{1}{N} (1 + \|\mathbf{U}(s)\|_{2,M}^2 + \|\mathbf{V}(s)\|_{2,M}^2) ds, \\ &\lesssim \frac{1}{N} (1 + \|\mathbf{U}\|_{t,M}^2) (t + \|\mathbf{U}\|_{t,M}^2 + \|\mathbf{V}\|_{t,M}^2) \lesssim (1+t)(1 + \|\mathbf{U}\|_{t,M}^4 + \|\mathbf{V}\|_{t,M}^4). \end{aligned}$$

Raising to the power $p/4$ and taking the expectation give

$$\mathbb{E}[\langle \mathcal{Q}_1 \rangle (T)^{\frac{p}{4}}] \lesssim \frac{1}{N^{\frac{p}{4}}} (1+T)^{\frac{p}{4}} \mathbb{E}[1 + \|\mathbf{U}\|_{T,M}^p + \|\mathbf{V}\|_{T,M}^p].$$

Since \mathcal{Q}_1 is a martingale starting from 0, Burkholder-Davis-Gundy inequality yields the second part of the proposition. \square

We can now obtain the expected estimations on $H_i^{M,N}$, allowing for convergence of martingale perturbations to 0 as N tends to infinity, uniformly with respect to M .

Lemma 3.3. *For $i \in \{1, 2\}$, assume that $b_i - d_i$ is bounded and b_i and d_i are globally Lipschitz. Let $p \geq 2$ and assume that the initial conditions satisfy (23). Then for any $T \geq 0$ and $N, M \geq 1$,*

$$\mathbb{E} \left[\sup_{t \leq T} |H_i^{M,N}(t)|^{p/2} \right] \lesssim \left(\frac{1+T^2}{N} \right)^{\frac{p}{4}} \mathbb{E}[1 + \|\mathbf{U}^{M,N}\|_{T,M}^p + \|\mathbf{V}^{M,N}\|_{T,M}^p].$$

Proof. We write $q = p/2$. Proposition 3.2 allows to control \mathcal{Q}_1^q and we need now to estimate the second term of H_1^q , i.e. \mathcal{P}_1^q , where

$$\begin{aligned} \mathcal{P}_1(t) &:= \sum_{0 \leq s \leq t} \|\mathcal{M}_1(s) - \mathcal{M}_1(s^-)\|_{-1,M}^2 = \sum_{0 \leq s \leq t} \|\mathbf{U}(s) - \mathbf{U}(s^-)\|_{-1,M}^2 \\ &= \frac{1}{N^2} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \|\theta\|_{-1,M}^2 \mathcal{N}_1(ds, d\rho, d\theta). \end{aligned}$$

First,

$$\mathcal{P}_1(t)^q = \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \left[\left(Q(s^-) + \frac{1}{N^2} \|\theta\|_{-1, M}^2 \right)^q - Q(s^-)^q \right] \mathcal{N}_1(ds, d\rho, d\theta).$$

Using the inequality $(x+a)^q - x^q \lesssim ax^{q-1} + a^q$ from Proposition 4.1 in the appendix we deduce

$$\mathcal{P}_1^q(t) \leq \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \left[\frac{1}{N^2} \|\theta\|_{-1, M}^2 \mathcal{P}_1(s^-)^{q-1} + \frac{1}{N^{2q}} \|\theta\|_{-1, M}^{2q} \right] \mathcal{N}_1(ds, d\rho, d\theta).$$

Bounding $\|\theta\|_{-1, M}^{2(q-1)}$ from the last term by $1/M^{q-1}$ thanks to Proposition 2.6,

$$\begin{aligned} \mathcal{P}_1(t)^q &\lesssim \frac{1}{N^2} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \|\theta\|_{-1, M}^2 \mathcal{P}_1(s^-)^{q-1} \mathcal{N}_1(ds, d\rho, d\theta) \\ &\quad + \frac{1}{N^{2q} M^{q-1}} \int_0^t \int_{\mathbb{R}_+ \times \mathcal{T}} \mathbb{1}_{\rho \leq \nu_1(\mathbf{U}(s^-), \mathbf{V}(s^-), \theta)} \|\theta\|_{-1, M}^{2q} \mathcal{N}_1(ds, d\rho, d\theta). \end{aligned}$$

Taking the expectation and using (29) and $M \geq 1$,

$$\begin{aligned} \mathbb{E}[\mathcal{P}_1(t)^q] &\lesssim \frac{1}{N} \mathbb{E} \left[\int_0^t (1 + \|\mathbf{U}(s)\|_{2, M}^2 + \|\mathbf{V}(s)\|_{2, M}^2) \mathcal{P}_1(s)^{q-1} ds \right] \\ &\quad + \frac{1}{N^{2q-1}} \mathbb{E} \left[\int_0^t (1 + \|\mathbf{U}(s)\|_{2, M}^2 + \|\mathbf{V}(s)\|_{2, M}^2) ds \right]. \end{aligned}$$

Since \mathcal{P}_1 is non-decreasing,

$$\mathbb{E}[\mathcal{P}_1(t)^q] \lesssim \frac{1}{N} \mathbb{E} \left[\mathcal{P}_1(t)^{q-1} \int_{Q_{t, M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \right] + \frac{1}{N^{2q-1}} \mathbb{E} \left[\int_{Q_{t, M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \right].$$

This ends the proof for $q = 1$ (i.e. $p = 2$). For $q > 1$, using Young's inequality $xy \lesssim (\varepsilon x)^{\frac{q}{q-1}} + (y/\varepsilon)^q$,

$$\mathbb{E}[\mathcal{P}_1(t)^q] \lesssim \varepsilon^{\frac{q}{q-1}} \mathbb{E}[\mathcal{P}_1(t)^q] + \left(\frac{\varepsilon^{-q}}{N^q} + \frac{1}{N^q} \right) \mathbb{E} \left[\left(\int_{Q_{t, M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \right)^q \right] + \frac{1}{N^{(2q-2)\frac{q}{q-1}}}.$$

Fixing $\varepsilon > 0$ small enough to absorb $\mathbb{E}[\mathcal{P}_1(t)^p]$ from the right-hand side in the left-hand side, we get

$$\mathbb{E}[\mathcal{P}_1(t)^q] \lesssim \frac{1}{N^q} \mathbb{E} \left[\left(\int_{Q_{t, M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \right)^q \right] + \frac{1}{N^{2q}}$$

Since $N \geq 1$ and

$$\int_{Q_{t, M}} 1 + \mathbf{U}^2 + \mathbf{V}^2 \leq t + \|\mathbf{U}\|_{t, M}^2 + \|\mathbf{V}\|_{t, M}^2,$$

this concludes the proof. \square

3.3 Large deviations estimates for births and deaths

In this section, we control the number of births and deaths along time together with the corresponding rates cumulated over time. The first step is to compare the number of jumps of interest to their cumulative intensity over time, when time is becoming large. We provide large deviations estimates for the gap, for general jump Markov processes, which can be of independent interest. We then apply it for our framework.

We consider a general jump process Markov X on a discrete countable space E and write $\nu(x, y)$ the jump rate from x to y , with $\nu(x, x) = 0$ by convention. Our aim is to control terms of the form

$$I_A(t) = \int_0^t \nu_A(X_s) ds,$$

where A is a subset of the set of transitions $\{(x, y) : x \in E, y \in E, x \neq y\}$ and

$$\nu_A(x) = \sum_{(x, y) \in A} \nu(x, y)$$

is the total rate of transitions of type A when the process is in state x . The process I_A term quantifies the cumulative transition rates of type A along trajectories of X . It appears in the semi martingale decomposition of jump Markov processes. The purpose of this subsection is to prove that this term is close to the cumulative number of transitions of type A , defined by :

$$N_A(t) = \sum_{s \leq t} 1_{(X_{s-}, X_s) \in A}.$$

More precisely, we prove that I_A and N_A have to be comparable with a great probability as soon as one of them is large, in the following sense. We consider the times when N_A or I_A is larger than a given value K :

$$\mathfrak{J}_A^K = \{t \in \mathbb{R}_+ : \max(N_A(t), I_A(t)) \geq K\}$$

and prove the following large deviation estimate. This estimate does not need assumption on the ergodicity and asymptotic behavior of the process.

Proposition 3.4. *There exists two universal constants c, C such that for all $\varepsilon \in (0, \frac{1}{2})$ and $K > 0$,*

$$\mathbb{P}(\exists t \in \mathfrak{J}_A^K : |N_A(t) - I_A(t)| \geq \varepsilon I_A(t)) \leq C \left(1 + \frac{1}{\varepsilon^3 K}\right) e^{-c\varepsilon^2 K}.$$

We first prove this result for Poisson process and then extend it by using the suitable time change. Indeed, the identity

$$Y_{I_A(t)} = X(t)$$

defines a jump Markov process Y as soon as X does not reach a point where ν_A vanishes. Moreover the process counting the jumps of Y in A is a Poisson process with parameter 1. Authorizing ν_A to vanish requires an additional approximation argument and a more involved time change, see the proof below.

Proof. We first consider the case when $X = P$ is a Poisson process with parameter 1. The jumps of P are of size 1 and counting all its jumps $J = \{(n, n+1) : n \geq 0\}$ until time t yields the value of the process at time t : $P(t) = N_J(t)$. Moreover $\nu_J = 1$ and $I_J(t) = t$ and $\mathfrak{J}_J^K = \{t \in \mathbb{R}_+ : P(t) \geq K\} \cup [K, \infty)$. We first estimate

$$\mathbb{P}(\mathfrak{J}_J^K \cap [0, (1-\varepsilon)K] \neq \emptyset) \leq \mathbb{P}(P((1-\varepsilon)K) \geq K) \leq e^{-c\varepsilon^2 K} \quad (31)$$

using a classical large deviation estimate for Poisson random variable, see for instance Theorem 1.1 in [3]. Second, we split the remaining time interval

$$[(1-\varepsilon)K, \infty) = \bigcup_{j=0}^{\infty} [t_j, t_{j+1}], \quad \text{where } t_j := \beta^j(1-\varepsilon)K$$

and

$$\beta = \frac{1}{2} \left\{ 1 + \min \left(\frac{1+\varepsilon}{1+\varepsilon/2}, \frac{1-\varepsilon/2}{1-\varepsilon} \right) \right\} \in \left(1, \min \left(\frac{1+\varepsilon}{1+\varepsilon/2}, \frac{1-\varepsilon/2}{1-\varepsilon} \right) \right).$$

This choice of β and monotonicity of P implies for any $t \in [t_j, t_{j+1}]$,

$$\{P(t) \leq (1-\varepsilon)t\} \subset \{P(t_j) \leq (1-\varepsilon)t_{j+1}\} \subset \{P(t_j) - t_j \leq \varepsilon t_j/2\}$$

and

$$\{P(t) \geq (1+\varepsilon)t\} \subset \{P(t_{j+1}) \geq (1+\varepsilon)t_j\} \subset \{P(t_{j+1}) - t_{j+1} \geq \varepsilon t_j/2\}.$$

Combining these inclusions of events ensures that

$$\{\exists t \in [(1-\varepsilon)K, \infty) : |P(t) - t| \geq \varepsilon t\} \subset \bigcup_{j=0}^{\infty} \{|P(t_j) - t_j| \geq \varepsilon t_j/2\}.$$

We can use again large deviation (tail) estimate for Poisson law and get

$$\begin{aligned} & \mathbb{P}(\exists t \in [(1-\varepsilon)K, \infty) : |P(t) - t| \geq \varepsilon t) \\ & \leq \sum_{j=0}^{\infty} \mathbb{P}(|P(t_j) - t_j| \geq \varepsilon t_j/2) \leq \sum_{j=0}^{\infty} e^{-c\varepsilon^2 t_j} \lesssim \frac{1}{\varepsilon^2 t_0 \ln \beta} e^{-c\varepsilon^2 t_0}, \end{aligned}$$

where \lesssim means here that the constant is universal (it depends neither on K nor on ε) and this inequality comes from the following computation for $\alpha > 0$ and $\beta > 1$, using the change of variables $y = \alpha\beta^x$,

$$\sum_{j=0}^{\infty} e^{-\alpha\beta^j} \leq \int_{-1}^{\infty} e^{-\alpha\beta^x} dx = \frac{1}{\ln \beta} \int_{\alpha/\beta}^{\infty} \frac{e^{-y}}{y} dy \leq \frac{\beta}{\alpha \ln \beta} \int_{\alpha/\beta}^{\infty} e^{-y} dy = \frac{\beta}{\alpha \ln \beta} e^{-\alpha/\beta}.$$

Combining this estimate with (31) and using $t_0 \ln \beta \gtrsim \varepsilon K$ yields the expected result for Poisson process :

$$\begin{aligned} & \mathbb{P}(\exists t \in \mathfrak{J}_J^K : |P(t) - t| \geq \varepsilon t) \\ & \leq \mathbb{P}(\mathfrak{J}_J^K \cap [0, (1-K)\varepsilon] \neq \emptyset) + \mathbb{P}(\exists t \in [(1-\varepsilon)K, \infty) : |P(t) - t| \geq \varepsilon t) \\ & \lesssim \left(1 + \frac{1}{\varepsilon^3 K} \right) e^{-c\varepsilon^2 K}. \end{aligned} \quad (32)$$

We now prove the result for any jump process X and set of transitions A such that $\inf_{x \in E} \nu_A(x) > 0$. In that case, $I_A(t) = \int_0^t \nu_A(X_s) ds$ increases continuously and at least linearly, thus realizing a bijection from $[0, \infty)$ into $[0, \infty)$ and we define its inverse function

$$\tau_A(t) = I_A^{-1}(t) = \inf\{u \in \mathbb{R}_+ : \int_0^u \nu_A(X_s) ds \geq t\}.$$

The proof now relies on the fact that the process P defined for $t \geq 0$ by

$$P(t) = N_A(\tau_A(t))$$

is a Poisson process with parameter 1, see for instance Chapter 6 in [18]. Then $P(u) = N_A(t)$ with $u = I_A(t)$ and

$$\{\exists u \in [0, K] \cup P^{-1}([K, \infty)) : |P(u) - u| \geq \varepsilon u\} = \{\exists t \in \mathfrak{J}_A^K : |N_A(t) - I_A(t)| \geq \varepsilon I_A(t)\}.$$

Then we can use (32) for the left hand side and control the right hand side and get

$$\mathbb{P}(\exists t \in \mathfrak{J}_A^K : |N_A(t) - I_A(t)| \geq \varepsilon I_A(t)) \lesssim \left(1 + \frac{1}{\varepsilon^3 K}\right) e^{-c\varepsilon^2 K}. \quad (33)$$

To conclude the proof, we need to deal with the case when $\inf_{x \in E} \nu_A(x) = 0$, which will be useful latter. For that purpose, we introduce a new Poisson process P^α with parameter $\alpha > 0$, independent of X . We consider the jump Markov process (X, P_α) in $E \times \mathbb{N}$ and consider the following set of transitions

$$B = \{((x, n), (y, n)) : (x, y) \in A, n \geq 0\} \cup \{((x, n), (x, n+1)) : x \in E, n \geq 0\},$$

which means that either the first coordinate X makes a jump in A or the second coordinate P_α jumps. We observe that $\nu_B(x, n) = \nu_A(x) + \alpha$ for any $(x, n) \in E \times \mathbb{N}$. Thus, for this jump Markov process (X, P_α) , $\inf_{(x, n) \in E \times \mathbb{N}} \nu_B(x, n) \geq \alpha > 0$ and we can invoke (33). We get for any $T \geq 0$,

$$\mathbb{P}(\exists t \in \mathfrak{J}_B^K \cap [0, T] : |N_B(t) - I_B(t)| \geq \varepsilon I_B(t)) \lesssim \left(1 + \frac{1}{\varepsilon^3 K}\right) e^{-c\varepsilon^2 K}. \quad (34)$$

Observe that the right hand side does not depend on α or $T > 0$. We also recall that $N_A(t) = \sum_{s \leq t} \mathbf{1}_{(X_{s-}, X_s) \in A}$ and here $N_B(t) = \sum_{s \leq t} \mathbf{1}_{(X_{s-}, X_s) \in A} + \mathbf{1}_{P^\alpha(s-) \neq P^\alpha(s)}$,

$$N_B(t) = N_A(t) + P^\alpha(t), \quad I_B(t) = I_A(t) + \alpha t$$

and

$$\mathfrak{J}_B^K \cap [0, T] = \{t \in [0, T] : \max(N_A(t) + P^\alpha(t), I_A(t) + \alpha t) \geq K\}.$$

Adding that P^α converges to 0 in $\mathbb{D}([0, T], \mathbb{N})$ and similarly $(\alpha t)_{t \in [0, T]}$ converges uniformly to 0 in $[0, T]$ as α tends to 0, we can let α go to zero in (34) and get

$$\mathbb{P}(\exists t \in \mathfrak{J}_A^K \cap [0, T] : |N_A(t) - I_A(t)| \geq \varepsilon I_A(t)) \lesssim \left(1 + \frac{1}{\varepsilon^3 K}\right) e^{-c\varepsilon^2 K}.$$

Letting T go to infinity ends the proof. □

A direct consequence of this estimate is that the distribution of N_A can be controlled by I_A , which induces a bound on the moments of N_A expressed with analogue moments of I_A .

We are now using the previous estimates for the jump Markov process $X = (\mathbf{U}^{M,N}, \mathbf{V}^{M,N})$ describing the evolution of species. We recall from the first part of Hypothesis 3 *ii*), that there exists $a, c > 0$ and $\alpha \in (0, 1)$ such that for any $u, v \geq 0$,

$$b_1(u, v) \leq \rho_0 + \alpha d_1(u, v). \quad (35)$$

It will allow us to control births terms by deaths terms. Keeping the notation above, we consider the renormalized cumulated intensity of births of the first species

$$I_B^{M,N}(t) = \frac{1}{M} \int_0^t \left(\sum_{i=1}^M U_i^{M,N}(s) b_1(U_i^{M,N}(s), V_i^{M,N}(s)) \right) ds.$$

Similarly the deaths are evaluated by

$$I_D^{M,N}(t) = \frac{1}{M} \int_0^t \left(\sum_{i=1}^M U_i^{M,N}(s) d_1(U_i^{M,N}(s), V_i^{M,N}(s)) \right) ds.$$

We observe that the death term I_D dominates the birth term I_B , due to the domination of b by d coming from (35), so the delicate part is the control of I_D . Using the previous result, I_D can be related to the number of deaths, while there cannot be too many deaths. Indeed this would require a comparable number of births in the first place but for large populations the hypothesis on birth and death rates prevents the births to compensate the deaths. Combining these arguments, we can prove the following large deviation estimate. We only focus here on the first species. An analogous result holds for the second species and more general structured population models.

Proposition 3.5. *Assume (35). Then, there exists constants $t_1 > 0$ and $\kappa > 1$ and $\gamma \in (0, 1)$ such that for all $M, N \geq 1$ and for any $K_0 \geq 1$ and for any initial condition satisfying $[\mathbf{U}_0^{M,N}]_M \leq K_0$ a.s., we have*

$$\mathbb{P} \left(I_B^{M,N}(t_1) + I_D^{M,N}(t_1) \geq \kappa K_0 \right) + \mathbb{P} \left(\sup_{s \leq t_1} [\mathbf{U}^{M,N}(s)]_M \geq \kappa K_0 \right) \lesssim \gamma^{K_0 MN}.$$

Moreover, the constant behind \lesssim depends only on ρ_0, α and not on K_0, M or N .

We observe in the proof that κ can be chosen large enough, and the constant behind \lesssim decreases with respect to κ .

Proof. Let us consider N_B (resp. N_D) the process counting the number of births (resp. deaths) along time for the first species \mathbf{U} . To be alive or die at some time, an individual has to be born before or to already exist at the initial time, so for any $t \geq 0$,

$$\max(N_D(t), MN[\mathbf{U}(t)]_M) \leq N_B(t) + MN[\mathbf{U}_0]_M \leq N_B(t) + MNK_0 \quad \text{a.s.} \quad (36)$$

for any initial condition satisfying $[\mathbf{U}_0]_M \leq K_0$. Moreover, applying (35) to the process of population densities and integrating it along time yields

$$I_B(t) \leq \alpha I_D(t) + \rho_0 \int_0^t [\mathbf{U}(s)]_M ds \leq \alpha I_D(t) + \rho_0 t \left(\frac{N_B(t)}{MN} + K_0 \right) \quad \text{a.s.} \quad (37)$$

since N_B is non decreasing. We need to control the tail of N_B and the other bounds will follow using the two previous inequalities, together with the link between I_D and N_D obtained above via large deviations estimates. To control N_B , we split its tail as follows

$$\{N_B(t) \geq \kappa K_0 MN\} \subset A_{t,\varepsilon,\kappa} \cup B_{t,\varepsilon,\varrho,\kappa} \cup C_{t,\varrho,\varepsilon} \cup D_{t,\varepsilon,\kappa} \quad (38)$$

where t, ϱ, κ are positive real numbers and $\varepsilon \in (0, 1)$ that will be fixed later and

$$\begin{aligned} A_{t,\varepsilon,\kappa} &= \{N_B(t) \geq \kappa K_0 MN; N_B(t) \geq (1 + \varepsilon) MNI_B(t)\} \\ B_{t,\varepsilon,\varrho,\kappa} &= \{N_B(t) \geq \kappa K_0 MN; N_B(t) \leq (1 + \varepsilon) MNI_B(t); I_D(t) < \varrho K_0\} \\ C_{t,\varepsilon,\kappa} &= \{N_B(t) \geq \kappa K_0 MN; N_B(t) \leq (1 + \varepsilon) MNI_B(t); MNI_D(t) \leq (1 + \varepsilon) N_D(t)\} \\ D_{t,\varrho,\varepsilon} &= \{I_D(t) \geq \varrho K_0; MNI_D(t) \geq (1 + \varepsilon) N_D(t)\}. \end{aligned}$$

Let us deal with these four terms. Roughly, the estimates for A and D will come from large deviations estimates of Proposition 3.4, while the two other will be due to (36) and (37). First, using Proposition 3.4 for births of \mathbf{U} with $K = \kappa K_0 MN$ ensures that for any $t \geq 0$, $\kappa > 1$ and $N, M \geq 1$ and $K_0 > 0$,

$$\mathbb{P}(A_{t,\varepsilon,\kappa}) \leq c_{\varepsilon,\kappa,K_0} \gamma_{\varepsilon,\kappa}^{K_0 MN},$$

where $\gamma_{\varepsilon,\kappa} \in (0, 1)$ and $c_{\varepsilon,\kappa,K_0} = C(1 + 1/(\varepsilon^3 \kappa K_0))$. Furthermore, using (37), we observe that

$$\begin{aligned} &\{N_B(t) \leq (1 + \varepsilon) MNI_B(t)\} \\ &\subset \{N_B(t)(1 - (1 + \varepsilon)t\rho_0) \leq (1 + \varepsilon)(\alpha MNI_D(t) + t\rho_0 MNK_0)\}. \end{aligned}$$

Then

$$B_{t,\varepsilon,\varrho,\kappa} \subset \{\kappa(1 - t\rho_0 - \varepsilon t\rho_0) \leq (1 + \varepsilon)(\alpha\varrho + t\rho_0)\},$$

which ensures that

$$B_{t_1,\varepsilon,\varrho,\kappa} = \emptyset, \quad \text{for } t_1 = \frac{1 - \sqrt{\alpha}}{\rho_0}, \quad \varrho = \frac{\kappa \frac{\sqrt{\alpha} - \varepsilon(1 - \sqrt{\alpha})}{1 + \varepsilon} - 1 + \sqrt{\alpha}}{2\alpha}$$

soon as $\varrho > 0$, i.e. for any $\kappa \in (0, \infty)$ large enough and $\varepsilon \in (0, 1)$ small enough. Furthermore, combining (36) and (37) yields

$$C_{t_1,\varepsilon,\kappa} \subset \left\{ N_B(t_1) \geq \kappa K_0 MN; \left(\frac{1}{1 + \varepsilon} - \rho_0 t_1 - \alpha(1 + \varepsilon) \right) N_B(t_1) \leq (\varrho_0 t_1 + \alpha(1 + \varepsilon)) MNK_0 \right\}.$$

Since $\alpha < 1$, $1 - \rho_0 t_1 = \sqrt{\alpha} > \alpha$ and we can also choose $\varepsilon \in (0, 1)$ (small enough) and then $\kappa > 0$ (large enough) such that

$$\frac{1}{1 + \varepsilon} - \rho_0 t_1 - \alpha(1 + \varepsilon) > \frac{\rho_0 t_1 + \alpha(1 + \varepsilon)}{\kappa},$$

which forces

$$C_{t_1,\varepsilon,\kappa} = \emptyset.$$

Finally using again Proposition 3.4, but now for death events, ensures that

$$\mathbb{P}(D_{t_1,\varrho,\varepsilon}) \leq c_{\varepsilon,\varrho,K_0} \gamma_{\varepsilon,\varrho}^{K_0 MN}.$$

Recalling (38) and gathering the previous estimates for A, B, C, D proves the expected result for $N_B(t_1)$ and ends the proof. \square

Corollary 3.6. *Let $K_0 \geq 1$. Then, there exists $c, d, C > 0$ and $\gamma \in (0, 1)$ such that for all $M, N \geq 1$ and any initial condition $\mathbf{U}_0^{M,N}$,*

$$\mathbb{P} \left(\sup_{s \geq 0} \left\{ \left(I_B^{M,N}(s) + I_D^{M,N}(s) + [\mathbf{U}^{M,N}(s)]_M \right) e^{-cs} \right\} \geq d \right) \leq C \left(\mathbb{P}([\mathbf{U}_0^{M,N}]_M \geq K_0) + \gamma^{MN} \right).$$

Moreover, the constants c, d, C, γ depend only on ρ_0, α, K_0 .

Proof. We use $t_1 > 0$ and $\kappa > 1$ coming from the previous proposition and consider the increments of I_B and I_D :

$$\Delta_k = I_B((k+1)t_1) - I_B(kt_1) + I_D((k+1)t_1) - I_D(kt_1).$$

The proof involves the first time when a deviation is observed. By Markov property at time kt_1 for $k \geq 0$, Proposition 3.5 ensures that for any $k \geq 0$,

$$\mathbb{P} \left(\left\{ \Delta_k \geq \kappa^{k+1} K_0 \right\} \cup \left\{ \sup_{s \in [kt_1, (k+1)t_1]} [\mathbf{U}(s)]_M \geq \kappa^{k+1} K_0 \right\} \mid [\mathbf{U}(kt_1)]_M \leq \kappa^k K_0 \right) \lesssim \gamma^{\kappa^k K_0 MN}.$$

As $\kappa > 1$ and $\gamma < 1$, the right hand is summable for $k \geq 0$ and its sum is dominated by $\gamma^{K_0 MN}$. We get, by total probability formula,

$$\begin{aligned} & \mathbb{P} \left(\exists k \geq 0 \text{ such that } \Delta_k \geq \kappa^{k+1} K_0 \text{ or } \sup_{s \in [kt_1, (k+1)t_1]} [\mathbf{U}(s)]_M \geq \kappa^{k+1} K_0 \right) \\ & \leq \mathbb{P}([\mathbf{U}_0]_M \geq K_0) \\ & \quad + \sum_{k \geq 0} \mathbb{P} \left(\left\{ \Delta_k \geq \kappa^{k+1} K_0 \right\} \cup \left\{ \sup_{s \in [kt_1, (k+1)t_1]} [\mathbf{U}(s)]_M \geq \kappa^{k+1} K_0 \right\} ; [\mathbf{U}(kt_1)]_M \leq \kappa^k K_0 \right) \\ & \lesssim \mathbb{P}([\mathbf{U}_0]_M \geq K_0) + \gamma^{K_0 MN}. \end{aligned}$$

Adding that for any $n \geq 0$ and $t \in [nt_1, (n+1)t_1]$, by monotonicity of I_B, I_D ,

$$(I_B(t) + I_D(t))e^{-ct} \leq \sum_{k=0}^n \Delta_k e^{-ckt_1}, \quad \sup_{s \geq 0} [\mathbf{U}(s)]_M e^{-cs} \leq \max_{k=0, \dots, n} \sup_{s \in [kt_1, (k+1)t_1]} [\mathbf{U}(s)]_M e^{-ckt_1}$$

yields the result. \square

3.4 Quantitative estimates and proof of Theorem 3

We can now quantify the gap between the stochastic process and the crossed diffusion PDE :

$$\mathbf{Z}^{M,N} = \mathbf{U}^{M,N} - \hat{u}^M, \quad \mathbf{W}^{M,N} = \mathbf{V}^{M,N} - \hat{v}^M,$$

where $(u, v) \in L^\infty(Q_T) \cap L^2([0, T]; H^3(\mathbb{T}))$ is solution of (2) satisfying the smallness condition (6) and (\hat{u}, \hat{v}) is given by (17). The semi martingale decomposition of $\mathbf{U}^{M,N}$ in (24)-(25) can be written as follows

$$\begin{aligned} \mathbf{U}^{M,N}(t) &= \mathbf{U}_0^{M,N} + \int_0^t [\Delta_M(\mu_1(\mathbf{U}^{M,N}(s), \mathbf{V}^{M,N}(s))\mathbf{U}^{M,N}(s))] \\ & \quad + R_1(\mathbf{U}^{M,N}(s), \mathbf{V}^{M,N}(t))ds + \mathcal{M}_1^{M,N}(t), \end{aligned}$$

where we recall that $\mathcal{M}_1^{M,N}$ is a (square integrable) martingale given in (26). We are now in position to apply our stability result (Proposition 2.10) and use martingale inequalities and large deviations estimates of the previous subsection to evaluate the gaps $\mathbf{Z}^{M,N}$ and $\mathbf{W}^{M,N}$.

Proposition 3.7. *Assume that Hypothesis 1, 2 and 3 hold and let $p \in [2, 2p_0/3]$. Then there exists $c > 0$ and $(\delta'_M)_M$ going to 0 such that for any $T \geq 0$ and $M, N \geq 1$,*

$$\begin{aligned} \mathbb{E}[\|\mathbf{Z}^{M,N}\|_{T,M}^p + \|\mathbf{W}^{M,N}\|_{T,M}^p] &\lesssim e^{\exp(cT)} \mathbb{E} \left[\|\mathbf{Z}_0^{M,N}\|_{-1,M}^p + \|\mathbf{W}_0^{M,N}\|_{-1,M}^p \right] \\ &\quad + e^{\exp(cT)} \left(\delta'_M + \frac{1}{N^{\frac{p}{4}}} \mathbb{E}[1 + \|\mathbf{U}^{M,N}\|_{T,M}^p + \|\mathbf{V}^{M,N}\|_{T,M}^p] \right). \end{aligned} \quad (39)$$

Furthermore,

$$\mathbb{E}[\|\mathbf{U}^{M,N}\|_{T,M}^p + \|\mathbf{V}^{M,N}\|_{T,M}^p] \lesssim e^{\exp(cT)} \mathbb{E} \left[\|\mathbf{U}_0^{M,N}\|_{-1,M}^p + \|\mathbf{V}_0^{M,N}\|_{-1,M}^p \right].$$

Proof. Proposition 2.10 applied to \mathbf{Z} and \mathbf{U} yields

$$\|\mathbf{Z}\|_{T,M}^2 + \|\mathbf{W}\|_{T,M}^2 \lesssim e^{\Lambda(T)} \left(\|\mathbf{Z}_0\|_{-1,M}^2 + \|\mathbf{W}_0\|_{-1,M}^2 + \delta_M + \|H_1\|_{L_T^\infty} + \|H_2\|_{L_T^\infty} \right), \quad (40)$$

where

$$\Lambda(T) \lesssim \int_{Q_{T,M}} 1 + \mathbf{U} + \mathbf{V} + d_1(\mathbf{U}, \mathbf{V})^2 + d_2(\mathbf{U}, \mathbf{V})^2,$$

and H_1 (resp. H_2) have been defined in (27). Let us first control $\Lambda(T)$ using Hypotheses 3 ii):

$$\begin{aligned} \Lambda(T) &\lesssim \int_{Q_{T,M}} 1 + \mathbf{U}(1 + d_1(\mathbf{U}, \mathbf{V})) + \mathbf{V}(1 + d_2(\mathbf{U}, \mathbf{V})) \\ &\lesssim T + T \sup_{0 \leq s \leq T} [\mathbf{U}(s)]_M + T \sup_{0 \leq s \leq T} [\mathbf{V}(s)]_M + (I_D^1(T) + I_D^2(T)), \end{aligned}$$

where

$$I_D^1(T) = \int_{Q_{T,M}} \mathbf{U} d_1(\mathbf{U}, \mathbf{V}) \quad \text{and} \quad I_D^2(T) = \int_{Q_{T,M}} \mathbf{V} d_2(\mathbf{U}, \mathbf{V}).$$

Let K_0 be given by Hypotheses 3 i). We can invoke Corollary 3.6 both for the processes (\mathbf{U}, I_D^1) and for (\mathbf{V}, I_D^2) to control the four terms on the right hand side with high probability. This proves the existence of $c, C > 0$ and $\gamma \in (0, 1)$ such that for any $T \geq 0$ and $M, N \geq 1$,

$$\mathbb{P}(\Lambda(T) \geq C \exp(cT)) \lesssim \mathbb{P}([\mathbf{U}_0^{M,N}]_M \geq K_0) + \gamma^{MN}.$$

Since K_0 is constant and $\gamma^{MN} \lesssim (MN)^{-p_0} \lesssim M^{-p_0}$, Hypotheses 3 i) yields

$$\mathbb{P}(\Lambda(T) \geq C \exp(cT)) \lesssim \frac{1}{M^{p_0}}.$$

Then, by Hölder's inequality, for all $q \geq 1$,

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\Lambda(T) \geq C \exp(cT)} \|\mathbf{Z}\|_{T,M}^p \right] &\leq \mathbb{P}(\Lambda(T) \geq C \exp(cT))^{1/q'} \mathbb{E}[\|\mathbf{Z}\|_{T,M}^{qp}]^{1/q} \\ &\lesssim \frac{1}{M^{p_0/q'}} \left(\|\widehat{u}\|_{T,M}^{qp} + \mathbb{E}[\|\mathbf{U}\|_{T,M}^{qp}] \right)^{1/q}, \end{aligned}$$

where $1/q' + 1/q = 1$. The last term of the right hand side can be bounded thanks to Corollary 2.5 and Proposition 3.1 when $qp < p_0$. Using also $\|\mathbf{U}(s)\|_{1,M} = [\mathbf{U}(s)]_M$, we obtain

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\Lambda(T) \geq C \exp(cT)} \|\mathbf{Z}\|_{T,M}^p] &\lesssim \frac{1}{M^{p_0/q'}} \left(\|u\|_\infty^{qp} + (1+TM)^{qp/2} \mathbb{E}[\sup_{s \leq T} \|\mathbf{U}(s)\|_{1,M}^{qp}] \right)^{1/q} \\ &\lesssim M^{p/2 - p_0/q'} e^{cT}, \end{aligned}$$

for $c > 0$ which allows to absorb the $T^{p/2}$. Since $p < 2p_0/3$, we can now choose $q \in (p_0/p, 1/(1 - p/2p_0))$. Then $qp < p_0$ and $p/2 < p_0/q'$, so that the right hand side of the last estimate goes to zero. This provides the expected estimate on the gap \mathbf{Z} on the event $\{\Lambda(T) \geq C \exp(cT)\}$ and we know consider the complementary event.

We elevate the a.s. estimate (40) to the power $p/2$. Taking then expectation and bounding $\mathbb{E}(\|H_i\|_{L_T^\infty})$ thanks to Lemma 3.3, we obtain

$$\begin{aligned} &\mathbb{E} \left[\mathbb{1}_{\Lambda(T) < C \exp(cT)} (\|\mathbf{Z}\|_{T,M}^p + \|\mathbf{W}\|_{T,M}^p) \right] \\ &\lesssim e^{\frac{p}{2} C \exp(cT)} \left(\mathbb{E}[\|\mathbf{Z}_0\|_{-1,M}^p + \|\mathbf{W}_0\|_{-1,M}^p] + \delta_M^{p/2} + \left(\frac{1+T^2}{N} \right)^{\frac{p}{4}} \mathbb{E}[1 + \|\mathbf{U}\|_{T,M}^p + \|\mathbf{V}\|_{T,M}^p] \right). \end{aligned}$$

Gathering the bounds for the event A and its complementary event and observing that $(1+T^2)^{\frac{p}{4}} \lesssim \exp(\exp(cT))$ ends the proof of (39).

We can now apply (39) to compare (\mathbf{U}, \mathbf{V}) the null solution $(u, v) = (0, 0)$. In this case, $\mathbf{r} = 0$, i.e. $\delta_M = 0$, and $(\mathbf{Z}, \mathbf{W}) = (\mathbf{U}, \mathbf{V})$. We obtain

$$\mathbb{E}[\|\mathbf{U}\|_{T,M}^p + \|\mathbf{V}\|_{T,M}^p] \lesssim e^{\exp(cT)} \left(\mathbb{E}[\|\mathbf{U}_0\|_{-1,M}^p + \|\mathbf{V}_0\|_{-1,M}^p] + \frac{1}{N^{\frac{p}{4}}} \mathbb{E}[1 + \|\mathbf{U}\|_{T,M}^p + \|\mathbf{V}\|_{T,M}^p] \right).$$

For N large enough, the $\|\cdot\|_{T,M}$ norms from the right-hand side can be absorbed in the left-hand side. Recalling that Hypothesis 3 i) bounds the moments of the initial condition concludes the proof. \square

We are now in position to prove the following approximation result, which implies Theorem 3 for $p = 2$.

Theorem 4. *Assume that Hypotheses 1, 2 and 3 hold and let $p \in [2, 2p_0/3)$. Then, there exists a sequence $(\delta_M)_M$ going to 0 and a constant $c > 0$ such that the following estimate holds for any $T \geq 0$ and $N, M \geq 1$,*

$$\mathbb{E} \left[\|\pi_M(\mathbf{U}^{M,N}) - u\|_T^p + \|\pi_M(\mathbf{V}^{M,N}) - v\|_T^p \right] \lesssim e^{\exp(cT)} \left(D_{0,p}^{M,N} + \delta_M + \frac{1}{N^{p/4}} \right),$$

where δ'_M goes to 0 as M tends to infinity and

$$D_{0,p}^{M,N} = \mathbb{E} \left[\left\| \pi_M(\mathbf{U}_0^{M,N}) - u_0 \right\|_{H^{-1}(\mathbb{T})}^p + \left\| \pi_M(\mathbf{V}_0^{M,N}) - v_0 \right\|_{H^{-1}(\mathbb{T})}^p \right].$$

Proof. Proposition 3.7 ensures that

$$\begin{aligned} & \mathbb{E}[\|\mathbf{Z}\|_{T,M}^p + \|\mathbf{W}\|_{T,M}^p] \\ & \lesssim e^{\exp(cT)} \left(\mathbb{E} \left[\|\mathbf{Z}_0\|_{-1,M}^p + \|\mathbf{W}_0\|_{-1,M}^p \right] + \delta'_M + \frac{1}{N^{\frac{p}{4}}} \mathbb{E}[1 + \|\mathbf{U}_0\|_{-1,M}^p + \|\mathbf{V}_0\|_{-1,M}^p] \right), \end{aligned}$$

which can be reduced using the inequality $\|\cdot\|_{-1,M} \lesssim \|\cdot\|_{1,M}$ from Proposition 2.4 and the uniform bound on moments of the $\|\cdot\|_{1,M}$ norm implied by Hypothesis 3 i) to

$$\mathbb{E}[\|\mathbf{Z}\|_{T,M}^p + \|\mathbf{W}\|_{T,M}^p] \lesssim e^{\exp(cT)} \left(\mathbb{E} \left[\|\mathbf{Z}_0\|_{-1,M}^p + \|\mathbf{W}_0\|_{-1,M}^p \right] + \delta'_M + \frac{1}{N^{\frac{p}{4}}} \right).$$

Then, Proposition 4.5 from the appendix gives

$$\begin{aligned} \|\pi_M(\mathbf{U}) - u\|_T & \lesssim \|\mathbf{Z}\|_{T,M} + \delta''_M, \\ \|\mathbf{Z}_0\|_{-1,M} & \lesssim \|\pi_M(\mathbf{U}_0) - u_0\|_{H^{-1}(\mathbb{T})} + \delta''_M, \end{aligned}$$

with $\delta''_M \xrightarrow{M \rightarrow \infty} 0$ and similar inequalities for the second species. It concludes the proof. \square

4 Appendix

4.1 Elementary inequalities

Proposition 4.1. *Let $p \geq 1$. There exists a constant C_p depending only on p such that for all $x, \varepsilon \geq 0$ the following inequalities hold:*

- (i) *If $x \geq \varepsilon$, $x^p - (x - \varepsilon)^p \leq C_p \varepsilon x^{p-1}$,*
- (ii) *$(x + \varepsilon)^p - x^p \leq C_p \varepsilon x^{p-1} + C_p \varepsilon^p$,*
- (iii) *If $x \geq \varepsilon$ and moreover $p \geq 2$, then $(x + \varepsilon)^p - x^p \leq x^p - (x - \varepsilon)^p + C_p \varepsilon^2 x^{p-2}$.*

Proof. For the first inequality, write

$$x^p - (x - \varepsilon)^p = p \int_{x-\varepsilon}^x y^{p-1} dy \leq p \int_{x-\varepsilon}^x x^{p-1} dy \leq p \varepsilon x^{p-1}.$$

The second one can then be deduced from the first one. For the third, if $p \geq 2$ then using the first inequality

$$\begin{aligned} (x + \varepsilon)^p - x^p - (x^p - (x - \varepsilon)^p) & = p \int_{x-\varepsilon}^x ((y + \varepsilon)^{p-1} - y^{p-1}) dy \\ & \leq p \int_{x-\varepsilon}^x (p-1)\varepsilon(y + \varepsilon)^{p-2} dy \\ & \leq p(p-1)\varepsilon \int_{x-\varepsilon}^x (x + \varepsilon)^{p-2} dy \leq p(p-1)2^{p-2}\varepsilon^2 x^{p-2}, \end{aligned}$$

which ends the proof in the case $p \geq 2$. If $p \in [1, 2)$, the proof is similar: instead of $y + \varepsilon)^{p-1} - y^{p-1} \leq (p-1)\varepsilon(y + \varepsilon)^{p-2}$ we have $y + \varepsilon)^{p-1} - y^{p-1} \leq (1-p)\varepsilon y^{p-2}$, which can then be dragged through the previous successive inequalities to get

$$(x + \varepsilon)^p - x^p - (x^p - (x - \varepsilon)^p) \leq C_p \varepsilon^2 (x - \varepsilon)^{p-2}.$$

If $x \geq 2\varepsilon$, this implies

$$(x + \varepsilon)^p - x^p - (x^p - (x - \varepsilon)^p) \leq C_p \varepsilon^2 \left(x - \frac{x}{2}\right)^{p-2} = C_p \varepsilon^2 x^{p-2}.$$

If $\varepsilon \leq x \leq 2\varepsilon$, we actually have

$$(x + \varepsilon)^p - x^p - (x^p - (x - \varepsilon)^p) \leq (x + \varepsilon)^p + (x - \varepsilon)^p \leq (3\varepsilon)^p + \varepsilon^p \leq (3^p + 1)\varepsilon^2 x^{p-2},$$

which concludes the proof. \square

4.2 Sobolev spaces and Poisson equation on the torus

We collect in this short paragraph classical results about Sobolev spaces and the Poisson equation considered on the torus. Recall that any periodic distribution $f \in \mathcal{D}'(\mathbb{T}^d)$ decomposes

$$f = \sum_{k \in \mathbb{Z}^d} c_k(f) e_k,$$

where $e_k : x \mapsto e^{2i\pi k \cdot x}$ and $c_k(f) := \langle f, e_k \rangle$.

Definition 4.1. For $s \in \mathbb{R}$, we define $H^s(\mathbb{T}^d)$ as the set of periodic distributions f such that

$$\|f\|_{H^s(\mathbb{T}^d)} := \left(\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 (1 + |k|^2)^s \right)^{1/2} < +\infty.$$

Equipped with $\|\cdot\|_{H^s(\mathbb{T}^d)}$, $H^s(\mathbb{T}^d)$ is a Hilbert space. The closed subspace of mean-free elements of $H^s(\mathbb{T}^d)$ is the kernel of the linear form $f \mapsto c_0(f) := [f]$ and is denoted $H_m^s(\mathbb{T}^d)$.

Note that $\|\cdot\|_{H^0} = \|\cdot\|_{L^2}$ and $\|\cdot\|_{H^s} \leq \|\cdot\|_{H^r}$ for $s \leq r$. In particular, $\|\cdot\|_{H^{-1}} \leq \|\cdot\|_{L^2}$. With this functional setting at hand we recall the following standard facts for Poisson's equation:

Proposition 4.2. Let $f \in H_m^{-1}(\mathbb{T}^d)$ a mean-free function. Then, there exists a unique mean-free function $\phi \in H_m^1(\mathbb{T}^d)$, also denoted $\Delta^{-1}f$, solution of $\Delta\phi = f$. Moreover, $\|\nabla\phi\|_{L^2(\mathbb{T}^d)} = \|f\|_{H^{-1}(\mathbb{T}^d)}$. If $f \in H_m^k(\mathbb{T}^d)$ for a $k \in \mathbb{N}$, then $\Delta^{-1}f \in H_m^{k+2}(\mathbb{T}^d)$ and the application $f \mapsto \Delta^{-1}f$ is continuous $H_m^k(\mathbb{T}^d) \rightarrow H_m^{k+2}(\mathbb{T}^d)$. In particular, if f is smooth then so is $\Delta^{-1}f$.

From this proposition we see that $\nabla\Delta^{-1}$ induces an isometry from $H_m^{-1}(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$, so that using a straightforward integration by parts we have the following expression for the $H^{-1}(\mathbb{T}^d)$ inner-product of two elements f and g of $H^{-1}(\mathbb{T}^d)_m$

$$\langle f, g \rangle_{H^{-1}} = \langle \nabla\Delta^{-1}f, \nabla\Delta^{-1}g \rangle_{L^2} = -\langle f, \Delta^{-1}g \rangle_{L^2}.$$

Of course, this formula extends to the whole $H^{-1}(\mathbb{T}^d)$ after adding the constant components so that for any $f, g \in H^{-1}(\mathbb{T}^d)$ there holds,

$$\langle f, g \rangle_{H^{-1}(\mathbb{T}^d)} = [f][g] - \langle f - [f], \Delta^{-1}(g - [g]) \rangle_{L^2}.$$

This last expression of the $H^{-1}(\mathbb{T}^d)$ inner-product motivates the following definition of inner-product on $\ell^2(\mathbb{T}_M^d)$

$$\langle \mathbf{u}, \mathbf{v} \rangle_{-1, M} = [\mathbf{u}]_M [\mathbf{v}]_M - \langle \mathbf{u} - [\mathbf{u}]_M, \Delta_M^{-1}(\mathbf{v} - [\mathbf{v}]_M) \rangle_{-1, M}.$$

Lastly, for $d = 1$, defining for $h \neq 0$ and $f \in \mathcal{D}'(\mathbb{T})$ the operator $\Delta_h f := h^{-2}(\tau_h f + \tau_{-h} f - 2f)$, we have (see for instance [4, Subsection 6.2]) that:

Proposition 4.3. *For $f \in H^2(\mathbb{T})$, we have $\|\Delta_h f\|_{L^2(\mathbb{T})} \leq \|f\|_{H^2(\mathbb{T})}$ and in $L^2(\mathbb{T})$,*

$$\Delta_h f \xrightarrow{h \rightarrow 0} \Delta f.$$

4.3 A non-linear Grönwall lemma

Lemma 4.4. *Let $\psi, \phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous by parts, $K_0 \geq 0$, $a \in L_{loc}^\infty(\mathbb{R}_+)$ et $b \in L_{loc}^1(\mathbb{R}_+)$:*

$$\psi(t) + \phi(t) \leq K_0 + \int_0^t [a(s)\psi(s) + b(s)\psi(s)^{1/2}] ds$$

Then:

$$\sup_{s \leq t} \psi(s) + \phi(t) \leq 2e^{\|a\|_{L^1([0, t])}} \left(K_0 + \|b\|_{L^1([0, t])}^2 \right).$$

Proof. Set $f(t) = K_0 + \int_0^t [a(s)\psi(s) + b(s)\psi(s)^{1/2}] ds$. Then $\psi(t) \leq f(t)$, so $f'(t) \leq a(t)f(t) + b(t)f(t)^{1/2}$. Dividing on both sides by $y(t) := f(t)^{1/2}$,

$$2y'(t) \leq a(t)y(t) + b(t).$$

The classical Grönwall lemma then gives

$$y(t) \leq y(0)e^{\frac{1}{2} \int_0^t a} + \frac{1}{2} \int_0^t e^{\frac{1}{2} \int_s^t a} b(s) ds,$$

$$(\psi(t) + \phi(t))^{1/2} \leq f(t)^{1/2} \leq y(t) \leq e^{\frac{1}{2} \|a\|_{L^1([0, t])}} \left(K_0^{1/2} + \int_0^t b(s) ds \right).$$

Squaring this inequality,

$$\psi(t) + \phi(t) \leq 2e^{\|a\|_{L^1([0, t])}} \left(K_0 + \|b\|_{L^1([0, t])}^2 \right),$$

hence the result by taking the supremum. □

4.4 Interpolation operators

The goal of this appendix is to prove the following lemma, which is used to translate estimates on the discrete problem to estimates in the continuous space at the end of the proofs of Theorems 2 and 3:

Proposition 4.5. *Let $u \in L^2([0, T]; H^3(\mathbb{T}))$ a solution of the SKT system (2), $\mathbf{u}^M \in \mathcal{C}([0, T], \ell^2(\mathbb{T}_M))$ a sequence of functions such that $\sup_{M \in \mathbb{N}} \|\mathbf{u}^M(0)\|_{2,M} < \infty$. Setting $\zeta^M = \pi_M(\mathbf{u}^M) - u$ and $\mathbf{z}^M = \mathbf{u}^M - \widehat{u}^M$, there exists a sequence δ_M depending only on u and $\sup_{M \in \mathbb{N}} \|\mathbf{u}^M(0)\|_{2,M}$ such that $\delta_M \xrightarrow{M \rightarrow \infty} 0$ and*

$$\begin{aligned} \|\zeta^M\|_T &\lesssim \|\mathbf{z}^M\|_{T,M} + \delta_M. \\ \|\mathbf{z}^M(0)\|_{-1,M} &\lesssim \|\zeta^M(0)\|_{H^{-1}(\mathbb{T})} + \delta_M \end{aligned}$$

To do so, we will rely on estimates for the operator π_M , which has already been studied in [4], Propositions 2 and 5:

Proposition 4.6. *For all $u \in \ell^2(\mathbb{T}_M)$,*

$$\|\pi_M(u)\|_{L^2(\mathbb{T})} \leq \|u\|_{2,M}.$$

Moreover, we have a uniform equivalence

$$M \|\pi_M(u)\|_{H^{-1}(\mathbb{T})} + \|\pi_M(u)\|_{L^2(\mathbb{T})} \sim M \|u\|_{-1,M} + \|\pi_M(u)\|_{L^2(\mathbb{T})}.$$

More specifically, we will use the following consequence of these inequalities:

Corollary 4.7. *For all $u \in L^2([0, T] \times \mathbb{T}_M)$,*

$$\|\pi_M(u)\|_T \lesssim \|u\|_{T,M}.$$

Proof. The uniform equivalence gives, for all $v \in \ell^2(\mathbb{T}_M)$,

$$\|\pi_M(v)\|_{H^{-1}(\mathbb{T})} \lesssim \|v\|_{-1,M} + M^{-1} \|\pi_M(v)\|_{L^2(\mathbb{T})}.$$

We then use $\|\pi_M(v)\|_{L^2(\mathbb{T})} \leq \|v\|_{2,M} \leq M \|v\|_{-1,M}$, which gives $\|\pi_M(v)\|_{H^{-1}(\mathbb{T})} \lesssim \|v\|_{-1,M}$. Therefore,

$$\sup_{s \leq T} \|\pi_M(u(s))\|_{H^{-1}(\mathbb{T})}^2 \lesssim \sup_{s \leq T} \|u(s)\|_{-1,M}^2 \leq \|u\|_{T,M}^2.$$

For the L^2 part,

$$\int_0^T \|\pi_M(u(s))\|_{L^2(\mathbb{T})}^2 ds \leq \int_0^T \|u(s)\|_{2,M}^2 ds \leq \|u\|_{T,M}^2,$$

hence the result. \square

Proof of Proposition 4.5. We decompose

$$\zeta^M = \pi_M(\mathbf{u}^M - \widehat{u}^M) + \pi_M(\widehat{u}^M) - u = \pi_M(\mathbf{z}^M) + \pi_M(\widehat{u}^M) - u.$$

Corollary 4.7 yields the following upperbound:

$$\|\zeta^M\|_T \lesssim \|\mathbf{z}^M\|_{T,M} + \|\pi_M(\widehat{u}^M) - u\|_{T,M}.$$

To prove the first part of the proposition, there remains to see that the last term converges to 0. This is ensured by Lemma 6.2.10 of [1]:

Lemma 4.8. *Let $f \in H^2(\mathbb{T})$. The two following inequalities hold:*

$$\left\| f - \pi_M(\widehat{f}^M) \right\|_{L^2(\mathbb{T})} \lesssim M^{-2} \|f\|_{H^2(\mathbb{T})},$$

$$\left\| f - \pi_M(\widehat{f}^M) \right\|_{H^{-1}(\mathbb{T})} \lesssim M^{-2} \|f\|_{H^2(\mathbb{T})}.$$

This lemma implies $\|\pi_M(\widehat{u}^M) - u\|_{T,M} \lesssim M^{-2} \|u\|_{L_T^\infty H^2}$. Adding that $u \in L^\infty([0, T]; H^2(\mathbb{T}))$, this ends the proof of the first part.

For the second part, by Proposition 4.6

$$\begin{aligned} \|\mathbf{z}^M(0)\|_{-1,M} &\lesssim \|\pi_M(\mathbf{z}^M(0))\|_{H^{-1}(\mathbb{T})} + M^{-1} \|\pi_M(\mathbf{z}^M(0))\|_{L^2(\mathbb{T})} \\ &\leq \|\zeta^M(0)\|_{H^{-1}(\mathbb{T})} + \|\pi_M(\widehat{u}_0^M) - u_0\|_{H^{-1}(\mathbb{T})} + M^{-1} \|\mathbf{z}^M(0)\|_{2,M}. \end{aligned}$$

Moreover, Lemma 4.8 ensures that $\|\pi_M(\widehat{u}_0^M) - u_0\|_{H^{-1}(\mathbb{T})}$ goes to 0. Bounding $\|\mathbf{z}^M(0)\|_{2,M}$ by $\|u\|_\infty + \sup_{M \in \mathbb{N}} \|\mathbf{u}^M(0)\|_{2,M}$ ends the proof. \square

4.5 A discrete duality lemma

The following duality lemma is used for the proof of Lemma 2.1 in Section 2. It is proved in [4], Lemma 4.14, writing $x = x'_S$:

Lemma 4.9. *Let $\mu, f, x \in L^\infty([0, T] \times \mathbb{T}_M)$ with μ uniformly lower bounded in space and time by some constant $\alpha > 0$, and $x_J(t) = \sum_{k=1}^m \mathbb{1}_{t \geq t_k} a_k$ piecewise constant function on $[0, T]$. Let $\mathbf{z}_0 : \mathbb{T}_M \rightarrow \mathbb{R}$. Assume that there is a C^1 piecewise function \mathbf{z} taking values in values in \mathbb{R}^M and solution for $t \geq 0$ of*

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^t (\Delta_M[\mathbf{z}(s)\mu(s) + f(s)] + x(s)) \, ds + x_J(t).$$

Then, for any $T \geq 0$,

$$\begin{aligned} \|\mathbf{z}(t)\|_{-1,M}^2 + \int_{Q_{T,M}} \mu \mathbf{z}^2 &\leq \|\mathbf{z}_0\|_{-1,M}^2 + \int_0^t [\mathbf{z}(s)]^2 [\mu(s)] \, ds + \frac{1}{\alpha} \int_{Q_{T,M}} f^2 \\ &\quad + 2 \int_0^t \langle \mathbf{z}(s), x(s) \rangle_{-1,M} \, ds + \sum_{k=1}^m \|a_k\|_{-1,M}^2 + 2 \sum_{k=1}^m \langle \mathbf{z}(t_k^-), a_k \rangle_{-1,M}. \end{aligned}$$

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References

- [1] Grégoire Allaire. *Analyse numérique et optimisation : une introduction à la modélisation mathématique et à la simulation numérique / Grégoire Allaire*. fre. 2e édition. Mathématiques appliquées. Palaiseau: Éditions de l'École Polytechnique, 2012. ISBN: 978-2-7302-1255-7.
- [2] Herbert Amann. “Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems”. In: *Function Spaces, Differential Operators and Nonlinear Analysis*. Ed. by Hans-Jürgen Schmeisser and Hans Triebel. Wiesbaden, 1993, pp. 9–126.
- [3] Heshan Aravinda, Arnaud Marsiglietti, and James Melbourne. *Concentration Inequalities for Ultra Log-Concave Distributions*. 2021. arXiv: [2104.05054](https://arxiv.org/abs/2104.05054) [math.PR]. URL: <https://arxiv.org/abs/2104.05054>.
- [4] Vincent Bansaye, Ayman Moussa, and Felipe Muñoz-Hernández. “Stability of a cross-diffusion system and approximation by repulsive random walks: A duality approach”. In: *Journal of the European Mathematical Society* (Nov. 2024). ISSN: 1435-9863. DOI: [10.4171/jems/1540](https://doi.org/10.4171/jems/1540). URL: <http://dx.doi.org/10.4171/JEMS/1540>.
- [5] Douglas Blount. “Comparison of Stochastic and Deterministic Models of a Linear Chemical Reaction with Diffusion”. In: *The Annals of Probability* 19.4 (1991), pp. 1440–1462. DOI: [10.1214/aop/1176990219](https://doi.org/10.1214/aop/1176990219). URL: <https://doi.org/10.1214/aop/1176990219>.
- [6] Douglas Blount. “Law of Large Numbers in the Supremum Norm for a Chemical Reaction with Diffusion”. In: *The Annals of Applied Probability* 2.1 (1992), pp. 131–141. DOI: [10.1214/aoap/1177005774](https://doi.org/10.1214/aoap/1177005774). URL: <https://doi.org/10.1214/aoap/1177005774>.
- [7] Maxime Breden, Christian Kuehn, and Cinzia Soresina. “On the influence of cross-diffusion in pattern formation”. In: *Journal of Computational Dynamics* 8.2 (2021), pp. 213–240. ISSN: 2158-2491. DOI: [10.3934/jcd.2021010](https://www.aims sciences.org/article/id/bb2cabc1-02ea-49b4-82a5-0a9125f26513). URL: <https://www.aims sciences.org/article/id/bb2cabc1-02ea-49b4-82a5-0a9125f26513>.
- [8] L. Chen and A. Jüngel. “Analysis of a parabolic cross-diffusion population model without self-diffusion”. In: *J. Differential Equations* 224.1 (2006), pp. 39–59. ISSN: 0022-0396. DOI: [10.1016/j.jde.2005.08.002](http://dx.doi.org/10.1016/j.jde.2005.08.002). URL: <http://dx.doi.org/10.1016/j.jde.2005.08.002>.
- [9] Li Chen et al. “Rigorous Derivation of Population Cross-Diffusion Systems from Moderately Interacting Particle Systems”. In: 31.6 (Sept. 2021). DOI: [10.1007/s00332-021-09747-9](https://doi.org/10.1007/s00332-021-09747-9). URL: <https://doi.org/10.1007/s00332-021-09747-9>.
- [10] Xiuqing Chen and Ansgar Jüngel. “A note on the uniqueness of weak solutions to a class of cross-diffusion systems”. In: *Journal of Evolution Equations* 18.2 (Nov. 2017), pp. 805–820. DOI: [10.1007/s00028-017-0420-4](https://doi.org/10.1007/s00028-017-0420-4). URL: <https://doi.org/10.1007/s00028-017-0420-4>.
- [11] Xiuqing Chen and Ansgar Jüngel. “Weak–strong uniqueness of renormalized solutions to reaction–cross-diffusion systems”. In: *Mathematical Models and Methods in Applied Sciences* 29.02 (Feb. 2019), pp. 237–270. DOI: [10.1142/s0218202519500088](https://doi.org/10.1142/s0218202519500088). URL: <https://doi.org/10.1142/s0218202519500088>.
- [12] Esther S. Daus, Laurent Desvillettes, and Helge Dietert. “About the entropic structure of detailed balanced multi-species cross-diffusion equations”. In: *Journal of Differential Equations* 266.7 (Mar. 2019), pp. 3861–3882. DOI: [10.1016/j.jde.2018.09.020](https://doi.org/10.1016/j.jde.2018.09.020). URL: <https://doi.org/10.1016/j.jde.2018.09.020>.

- [13] Esther S. Daus, Laurent Desvillettes, and Helge Dietert. “About the entropic structure of detailed balanced multi-species cross-diffusion equations”. In: *Journal of Differential Equations* 266.7 (2019), pp. 3861–3882. ISSN: 0022-0396. DOI: <https://doi.org/10.1016/j.jde.2018.09.020>. URL: <https://www.sciencedirect.com/science/article/pii/S0022039618305412>.
- [14] L Desvillettes and A Moussa. *Perturbative global solutions of a large class of cross diffusion systems in any dimension*. 2024. arXiv: [2403.00391](https://arxiv.org/abs/2403.00391) [math.AP]. URL: <https://arxiv.org/abs/2403.00391>.
- [15] L. Desvillettes et al. “On the Entropic Structure of Reaction–Cross Diffusion Systems”. In: *Communications in Partial Differential Equations* 40.9 (Sept. 2015), pp. 1705–1747.
- [16] Laurent Desvillettes and Ariane Trescases. “New results for triangular reaction cross diffusion system”. In: *Journal of Mathematical Analysis and Applications* 430.1 (2015), pp. 32–59.
- [17] Helge Dietert and Ayman Moussa. “Persisting entropy structure for nonlocal cross-diffusion systems”. en. In: *Annales de la Faculté des sciences de Toulouse : Mathématiques* Ser. 6, 33.1 (2024), pp. 69–104. DOI: [10.5802/afst.1762](https://afst.centre-mersenne.org/articles/10.5802/afst.1762/). URL: <https://afst.centre-mersenne.org/articles/10.5802/afst.1762/>.
- [18] S.N. Ethier and T.G. Kurtz. *Markov processes, Characterization and Convergence*. John Wiley & Sons, New York., 1986.
- [19] Lawrence C. Evans. *Partial differential equations / Lawrence C. Evans,...* eng. 2nd edition [updated, new chapter 12, added 80 exercises]. Graduate studies in mathematics. Providence (R.I.): American Mathematical Society, 2010. ISBN: 978-0-8218-4974-3.
- [20] J. Fontbona and S. Méléard. “Non local Lotka-Volterra system with cross-diffusion in an heterogeneous medium”. In: *Journal of mathematical biology* 70.4 (2015), pp. 829–854.
- [21] Isabelle Gallagher and Ayman Moussa. *The Cauchy problem for quasi-linear parabolic systems revisited*. 2024. arXiv: [2407.08226](https://arxiv.org/abs/2407.08226) [math.AP]. URL: <https://arxiv.org/abs/2407.08226>.
- [22] Raphaël Danchin Hajer Bahouri Jean-Yves Chemin. *Fourier Analysis and Nonlinear Partial Differential Equations*. 2011. DOI: <https://doi.org/10.1007/978-3-642-16830-7>. URL: <https://link.springer.com/book/10.1007/978-3-642-16830-7>.
- [23] M. Iida, M. Mimura, and H. Ninomiya. “Diffusion, cross-diffusion and competitive interaction”. In: *Journal of mathematical biology* 53.4 (2006), pp. 617–641.
- [24] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*. eng. North-Holland mathematical library ; volume 24. Amsterdam ; North-Holland Publishing Company, 1981. ISBN: 0-08-096012-X.
- [25] Jong Uhn Kim. “Smooth solutions to a quasi-linear system of diffusion equations for a certain population model”. In: *Nonlinear analysis* 8.10 (1984), pp. 1121–1144.
- [26] Irwin Kra and Santiago R. Simanca. “On Circulant Matrices”. In: *Notices of the AMS* 59.3 (2012). DOI: [10.1090/noti804](https://doi.org/10.1090/noti804). URL: <https://www.ams.org/journals/notices/201203/rtx120300368p.pdf>.
- [27] Thomas Lepoutre and Ayman Moussa. “Entropic structure and duality for multiple species cross-diffusion systems”. In: *Nonlinear Analysis* 159 (Aug. 2017), pp. 298–315. ISSN: 0362-546X. DOI: [10.1016/j.na.2017.02.008](https://doi.org/10.1016/j.na.2017.02.008). URL: <http://dx.doi.org/10.1016/j.na.2017.02.008>.

- [28] R. H. Martin and Michel Pierre. “Nonlinear Reaction-Diffusion Systems”. In: *Mathematics in science and engineering* 185 (1992), pp. 363–398. URL: <https://api.semanticscholar.org/CorpusID:115539273>.
- [29] Ayman Moussa. “From Nonlocal to Classical Shigesada–Kawasaki–Teramoto Systems: Triangular Case with Bounded Coefficients”. In: *SIAM Journal on Mathematical Analysis* 52.1 (Jan. 2020), pp. 42–64. DOI: [10.1137/19m1243294](https://doi.org/10.1137/19m1243294). URL: <https://doi.org/10.1137/19m1243294>.
- [30] Michel Pierre and Didier Schmitt. *Blow-Up in Reaction Diffusion Systems with Dissipation of Mass*. Research Report RR-2652. INRIA, 1995, p. 17. URL: <https://inria.hal.science/inria-00074038>.
- [31] Nanako Shigesada, Kohkichi Kawasaki, and Ei Teramoto. “Spatial segregation of interacting species”. In: *Journal of Theoretical Biology* 79.1 (1979), pp. 83–99. ISSN: 0022-5193. DOI: [https://doi.org/10.1016/0022-5193\(79\)90258-3](https://doi.org/10.1016/0022-5193(79)90258-3). URL: <https://www.sciencedirect.com/science/article/pii/0022519379902583>.

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